On the twisted cobar construction

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Introduction

The classical cobar construction ΩC for a coalgebra C (first introduced by Adams [1]) is an important algebraic concept motivated by the singular chain complex of a loop space ΩX. If X is a 1-reduced simplicial set with realisation $|X|$ Adams proved that there is a natural isomorphism of homology groups

$$H_*(\Omega C(X), A) \cong H_*(\Omega|X|, A) \qquad (*)$$

where $C(X)$ is the coalgebra given by the chain complex on X and the Alexander-Whitney diagonal. Here the homology has coefficients in an abelian group A. The purpose of this paper is the extension of this result to the case of twisted coefficients given by $\pi_1\Omega|X|$-modules A, with $\pi_1\Omega|X| = H_2X$.

We introduce the new algebraic concepts of a twisted coalgebra C and a twisted cobar construction ΩC which extend the classical notions. We are able to define for any 1-reduced simplicial set X a twisted coalgebra $\widehat{C}(X)$ together with a natural projection $\widehat{C}(X) \to C(X)$, such that there is a natural isomorphism

$$H_*(\Omega\widehat{C}(X), A) \cong H_*(\Omega|X|, A) \qquad (**)$$

for all twisted coefficients A. For this we prove that there is a natural homology equivalence of differential algebras between $\Omega\widehat{C}(X)$ and $C\widehat{\Omega|X|}$ where $\widehat{\Omega|X|}$ is the universal cover of the loop space $\Omega|X|$. We show

$$\Omega\widehat{C}(X)\otimes_{\mathbb{Z}[H_2X]}\mathbb{Z} \cong \Omega C(X)$$

and hence recover from $(**)$ the result $(*)$ of Adams.

Iterated loop spaces and the problem of iterating the cobar construction lead to the theory of operads in which there has been much recent interest [16, 17, 18, 19, 20]. The twisted cobar construction therefore yields a new problem of iteration corresponding to the sequence of *simply-connected* spaces

$$|X|,\ \widehat{\Omega}|X|,\ \widehat{\Omega}\widehat{\Omega}|X|, \dots$$

with $\widehat{\Omega}(Y) = \widehat{\Omega Y}$. For this an extension of the structure of the twisted coalgebra $\widehat{C}(X)$ is needed to allow iteration of the twisted cobar construction.

The proof of the main theorem relies on the geometric cobar construction introduced in [2] and the computation of its crossed chain complex. The theory of crossed chain complexes goes back to Whitehead [23] and has been developed in, for example, [5, 11, 13]. Here we also need the associated theory of crossed chain algebras [8, 22]; first examples of such algebras were studied in [5, 6, 7, 10, 21].

1 The twisted cobar construction

Algebras, coalgebras and twisted coalgebras

We begin by recalling some elementary definitions, and introduce the notion of a twisted differential coalgebra.

A *(graded) module* $M = (M, R)$ is a family of R-modules M_i, $i \in \mathbb{Z}$, for R a commutative ring with unit $1 = 1_R$. For $x \in M_i$ we write $|x| = i$, and we denote the action of $\alpha \in R$ on x by x^α or $x\alpha$. A module is termed *positive* if $M_i = 0$ for $i < 0$. For $n \in \mathbb{Z}$ a *map of degree n* of modules $(f, g) : (M, R) \to (M', R')$ is a family of group homomorphisms $f_i : M_i \to M'_{i+n}$ together with a ring homomorphism $g : R \to R'$ satisfying $f_i(x^\alpha) = (f_i x)^{g\alpha}$ for $\alpha \in R$, $x \in M_i$, $i \in \mathbb{Z}$. We have a suspension functor s on the category of modules, with $(sM)_{n+1} = M_n$, and natural isomorphisms $s^n : M \to s^n M$ of degree n for $n \in \mathbb{Z}$.

A *chain complex* is an R-module M together with a differential $d : M \to M$ of degree -1 satisfying $dd = 0$. A chain map is a map of degree 0 which commutes with the differentials. The homology of a chain complex M is the graded module HM with $(HM)_n = H_n(M) = \ker d_n / \operatorname{Im} d_{n+1}$. The tensor product of R-chain complexes is given by the tensor product of modules, with $(M \otimes M')_n = \bigoplus_{i+j=n} M_i \otimes_R M_j$, and the differential

$$d_\otimes(x \otimes y) \;=\; (d \otimes 1 + 1 \otimes d)(x \otimes y) \;=\; dx \otimes y + (-1)^{|x|} x \otimes dy$$

An *R-chain algebra* (or a *differential algebra* over R) consists of a positive chain complex A over R together with R-chain maps

$$R \xrightarrow{\ \eta\ } A, \qquad A \otimes A \xrightarrow{\ \mu\ } A$$

with R concentrated in dimension zero, which yield an associative multiplication $x \cdot y = \mu(x \otimes y)$ for $x, y \in A$ with neutral element $* = \eta(1)$. Morphisms of chain algebras are chain maps which respect the multiplications and the units. We write $\widehat{\mathbf{Alg}}$ for the category of chain algebras. An R-chain algebra A is *augmented* if a chain algebra morphism $\varepsilon : A \to R$ is given with $\varepsilon\eta = 1$; morphisms of augmented chain algebras must respect the augmentations.

An *R-coalgebra* consists of a positive R-module C together with maps of degree zero

$$C \xrightarrow{\varepsilon} R, \qquad C \xrightarrow{\Delta} C \otimes C$$

where Δ is coassociative and ε is a counit for the comultiplication Δ. Morphisms of coalgebras are maps of degree 0 which respect the comultiplications and counits. A coalgebra C is *augmented* if a morphism of coalgebras $\eta : R \to C$ is given with $\varepsilon\eta = 1$.

For C an augmented coalgebra, let $\widetilde{C}$ be the quotient $C/\eta(R)$, so that we have $C \cong R \oplus \widetilde{C}$ as modules. Let $\widetilde{\Delta}$ be the map

$$\widetilde{C} \xrightarrow{\widetilde{\Delta}} \widetilde{C} \otimes \widetilde{C}$$

induced by Δ.

Definition 1.1 A *twisted coalgebra* over R is an augmented R-coalgebra C together with R-module maps

$$\partial : \widetilde{C} \longrightarrow \widetilde{C} \quad \text{of degree } -1$$
$$\delta : \widetilde{C} \longrightarrow R \quad \text{of degree } -2$$

such that $\delta\partial = 0$ and

$$\widetilde{\Delta}(\partial x) = (1 \otimes \partial + \partial \otimes 1)\widetilde{\Delta} x \qquad (*)$$
$$\partial\partial x = (1 \otimes \delta - \delta \otimes 1)\widetilde{\Delta} x \qquad (**)$$

Note that in (1.1)(**) we use $\widetilde{C} \otimes_R R \cong \widetilde{C} \cong R \otimes_R \widetilde{C}$. Let $\widehat{\mathbf{Coalg}}$ be the category of twisted coalgebras, with morphisms $(f, g) : (C, R) \to (C', R')$ given by morphisms of augmented coalgebras which commute with ∂ and with δ.

Remark 1.2 The map δ on C_2 is to be thought of as giving the twisted structure; if $\delta = 0$ definition 1.1 reduces to the usual definition of an *augmented differential coalgebra*.

Definition 1.3 Suppose R is augmented by a ring homomorphism $\varepsilon : R \to \mathbb{Z}$. Then we say that C is an *ε-twisted coalgebra* if $\varepsilon\delta = 0$. In this case we get a projection

$$(C, R) \xrightarrow{(p,\varepsilon)} (C \otimes_R \mathbb{Z}, \mathbb{Z})$$

where $C \otimes_R \mathbb{Z}$ is a differential coalgebra with augmentation $\mathbb{Z} \to C \otimes_R \mathbb{Z}$, $n \mapsto 1 \otimes n$.

The twisted cobar construction

Let M be an R-module and let

$$M^{\otimes n} = M \otimes M \otimes \ldots \otimes M$$

be the n-fold tensor product of M over R. Then the *tensor algebra*

$$T(M) = \bigoplus_{n \geq 0} M^{\otimes n}$$

is the sum of all the graded R-modules $M^{\otimes n}$. The algebra multiplication and unit are given by the canonical isomorphisms

$$M^{\otimes n} \otimes M^{\otimes m} \cong M^{\otimes (n+m)} \quad \text{and} \quad R \cong M^{\otimes 0}$$

respectively.

We say that a chain algebra A is *free* if forgetting the differentials there is an isomorphism $A \cong T(M)$ of algebras for some M. In this case we write $\imath^n$, $n \geq 0$, for the inclusion of $M^{\otimes n}$ in A. The differential on A is determined by its restriction to M

$$d\imath^1 : M \longrightarrow A$$

Definition 1.4 Given a twisted R-coalgebra C we define the *twisted cobar construction*

$$\Omega C = (T(s^{-1}\widetilde{C}), d_\Omega)$$

to be the free R-chain algebra generated by the desuspension $s^{-1}\widetilde{C}$ with the differential given by

$$d_\Omega \imath^1 = \imath^0 \delta s - \imath^1 s^{-1} \partial s + \imath^2 (s^{-1} \otimes s^{-1}) \widetilde{\Delta} s$$

This will give a functor

$$\widehat{\mathbf{C}}\mathbf{oalg} \xrightarrow{\Omega} \widehat{\mathbf{A}}\mathbf{lg}$$

which reduces to the classical cobar construction of Adams [1] in the case $\delta = 0$. Moreover the chain algebra ΩC is augmented by the projection $\Omega C \to R$ if and only if $\delta = 0$.

Lemma 1.5 *ΩC is a well defined R-chain algebra.*

Proof: Let

$$d = d_\Omega s^{-1} : \widetilde{C} \longrightarrow T(s^{-1}\widetilde{C}) \tag{1}$$

$$d = \delta - s^{-1}\partial + (s^{-1} \otimes s^{-1})\widetilde{\Delta} \tag{2}$$

We have to show $d_\Omega d = 0$. We have

$$d_\Omega d \;=\; d_\Omega\delta - d_\Omega s^{-1}\partial + d_\Omega(s^{-1}\otimes s^{-1})\widetilde{\Delta} \tag{3}$$

where $d_\Omega\delta = 0$ since $d_\Omega\iota^0 = 0$. Hence we get

$$d_\Omega d \;=\; -d\partial + (d\otimes s^{-1})\widetilde{\Delta} - (s^{-1}\otimes d)\widetilde{\Delta} \tag{4}$$

with

$$-d\partial \;=\; -\delta\partial + s^{-1}\partial\partial - (s^{-1}\otimes s^{-1})\widetilde{\Delta}\partial \tag{5}$$

where $\delta\partial = 0$. Moreover

$$\begin{aligned}
(d\otimes s^{-1})\widetilde{\Delta} &= (\delta\otimes s^{-1})\widetilde{\Delta} - (s^{-1}\partial\otimes s^{-1})\widetilde{\Delta} && (6)\\
&+ ((s^{-1}\otimes s^{-1})\widetilde{\Delta}\otimes s^{-1})\widetilde{\Delta} && (7)\\
-(s^{-1}\otimes d)\widetilde{\Delta} &= -(s^{-1}\otimes\delta)\widetilde{\Delta} + (s^{-1}\otimes s^{-1}\partial)\widetilde{\Delta} && (8)\\
&- (s^{-1}\otimes(s^{-1}\otimes s^{-1})\widetilde{\Delta})\widetilde{\Delta} && (9)
\end{aligned}$$

Here we have (7) $= (s^{-1}\otimes s^{-1}\otimes s^{-1})(\widetilde{\Delta}\otimes 1)\widetilde{\Delta}$ and (9) $= -(s^{-1}\otimes s^{-1}\otimes s^{-1})(1\otimes\widetilde{\Delta})\widetilde{\Delta}$ so that (7) and (9) cancel by the coassociativity of Δ.

Moreover we have

$$\begin{aligned}
&s^{-1}\partial\partial + (\delta\otimes s^{-1})\widetilde{\Delta} - (s^{-1}\otimes\delta)\widetilde{\Delta}\\
&= s^{-1}\left(\partial\partial + (\delta\otimes 1)\widetilde{\Delta} - (1\otimes\delta)\widetilde{\Delta}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
&-(s^{-1}\otimes s^{-1})\widetilde{\Delta}\partial - (s^{-1}\partial\otimes s^{-1})\widetilde{\Delta} + (s^{-1}\otimes s^{-1}\partial)\widetilde{\Delta}\\
&= (s^{-1}\otimes s^{-1})\left(-\widetilde{\Delta}\partial + (\partial\otimes 1)\widetilde{\Delta} + (1\otimes\partial)\widetilde{\Delta}\right) = 0
\end{aligned}$$

This completes the proof. □

Lemma 1.6 *If C is an ε-twisted coalgebra over R then there is a natural isomorphism of augmented chain algebras over $\mathbb{Z}$*

$$(\Omega C)\otimes_R\mathbb{Z} \;\cong\; \Omega(C\otimes_R\mathbb{Z})$$

where the right hand side is the classical cobar construction.

Proof: We have $(M\otimes M')\otimes_R\mathbb{Z} \cong (M\otimes_R\mathbb{Z})\otimes_{\mathbb{Z}}(M'\otimes_R\mathbb{Z})$ for R-modules M, M', and so

$$\Omega C\otimes_R\mathbb{Z} \;\cong\; \bigoplus_{n\geq 0}(s^{-1}\widetilde{C})^{\otimes n}\otimes_R\mathbb{Z} \;\cong\; \bigoplus_{n\geq 0}(s^{-1}\widetilde{C}\otimes_R\mathbb{Z})^{\otimes n}$$

Since $s^{-1}\widetilde{C}\otimes_R\mathbb{Z} \cong s^{-1}(\widetilde{C\otimes_R\mathbb{Z}})$ we have the result at the level of free algebras. Also $\delta\otimes_R\mathbb{Z} = 0$, so under these isomorphisms we have

$$d_\Omega \iota^1\otimes_R\mathbb{Z} \;\cong\; -s^{-1}(\partial\otimes_R\mathbb{Z})s + (s^{-1}\otimes s^{-1})(\widetilde{\Delta\otimes_R\mathbb{Z}})s$$

and the lemma is proved. □

The twisted chain coalgebra

Let Δ be the simplicial category, with objects the ordered sets $\underline{n} = \{0, 1, \ldots, n\}$ and morphisms the monotonic increasing functions. A simplicial set X is a contravariant functor from Δ to the category of sets; equivalently it is a family of sets $(X_n)_{n\geq 0}$ with degeneracy and face maps

$$X_n \xrightarrow{s_i} X_{n+1} \qquad\qquad X_n \xrightarrow{d_i} X_{n-1}$$

for $0 \leq i \leq n$, satisfying the usual relations. Simplices in the image of some s_i are termed *degenerate*. For an n-simplex $\sigma \in X_n$ and a monotonic function $a : \underline{m} \to \underline{n}$ we also write $\sigma(a_0 \ldots a_m)$ for $a^*\sigma \in X_m$ and $\sigma(0 \ldots \hat{i} \ldots n)$ for $d_i\sigma$. If X is a simplicial set, then the $\mathbb{Z}$-chain complex $C(X)$ is defined as follows. Let F be the chain complex with F_n the free abelian group on X_n and differential $d\sigma = \sum_0^n (-1)^i d_i\sigma$. Let D be the subchain complex generated by the degenerate simplices. Then $C(X)$ is the quotient F/D. The homology $H(X)$ of X is given by the homology of the chain complex $C(X)$.

Let G be a group with unit 1_G, and IG its augmentation module given by the kernel of the ring homomorphism $\mathbb{Z}G \to \mathbb{Z}$, $\sum n_i g_i \mapsto \sum n_i$. Then IG is a right $\mathbb{Z}G$-module which is generated as an abelian group by $g - 1_G$, $1_G \neq g \in G$.

Suppose H is an abelian group and $\phi : G \to H$ is a group homomorphism. Then the *derived module* D_ϕ of ϕ is the $\mathbb{Z}H$-module

$$D_\phi \;=\; IG \otimes_{\mathbb{Z}G} \mathbb{Z}H$$

where G acts on the left on $\mathbb{Z}H$ via ϕ. The function $h_\phi : G \to D_\phi$, $x \mapsto (x - 1_G)\otimes 1_H$, is the universal ϕ-derivation; it satisfies

$$h_\phi(xy) \;=\; h_\phi(x)^{\phi(y)} + h_\phi(y)$$

and any other function h from G to a $\mathbb{Z}H$-module V with such a property factors as $h = fh_\phi$ for a unique $\mathbb{Z}H$-homomorphism $f : D_\phi \to V$.

Definition 1.7 Suppose X is a 1-reduced simplicial set, that is, $X_0 = X_1 = \{*\}$, and let R be the commutative ring given by the group ring $\mathbb{Z}[H_2X]$. Let ϕ be the quotient map

$$\langle X_2 \rangle \twoheadrightarrow C_2X \twoheadrightarrow H_2X$$

from the free group $\langle X_2 \rangle$ on X_2, with the universal ϕ-derivation

$$\langle X_2 \rangle \xrightarrow{h_\phi} D_\phi$$

Let $D_\phi{}' \subset D_\phi$ be the submodule generated by the image $h_\phi(s_0*)$ of the degenerate 2-simplex. We define the *twisted chain R-coalgebra* $\widehat{C}(X)$ associated to X by

$$\begin{aligned}
\widehat{C}_0(X) &= R \\
\widehat{C}_1(X) &= 0 \\
\widehat{C}_2(X) &= D_\phi / D_\phi{}' \\
\widehat{C}_n(X) &= C_n(X) \otimes_{\mathbf{Z}} R \quad \text{for } n \geq 3
\end{aligned}$$

For each $i \geq 0$ we have functions

$$X_i \longrightarrow \widehat{C}_i(X)$$

which are defined for $\sigma_i \in X_i$ by $\sigma_0 \mapsto 1$, $\sigma_1 \mapsto 0$, $\sigma_2 \mapsto h_\phi \sigma_2$ and $\sigma_n \mapsto \sigma_n \otimes 1$ for $n \geq 3$. We will identify non-degenerate simplices of X with their images in $\widehat{C}(X)$ and degenerate simplices with 0. The coaugmentation and counit η, ε are given by $R \cong \widehat{C}_0(X)$ and the comultiplication

$$\widehat{C}(X) \xrightarrow{\Delta} \widehat{C}(X) \otimes \widehat{C}(X)$$

is the Alexander-Whitney diagonal

$$\begin{aligned}
\Delta(x) &= 1 \otimes x + x \otimes 1 \quad \text{for } |x| \leq 2 \\
\Delta(\sigma) &= \sum_{i=0}^{n} \sigma(0 \ldots i) \otimes \sigma(i \ldots n) \quad \text{for } \sigma \in X_n,\ n \geq 3
\end{aligned}$$

Moreover, let

$$\widehat{C}_2(X) \xrightarrow{\delta} R$$

be the $\mathbb{Z}[H_2 X]$-homomorphism defined by $\delta h_\phi(x) = \phi x - 1_{H_2 X}$ for $x \in \langle X_2 \rangle$, and let

$$\widehat{C}_n(X) \xrightarrow{\partial} \widehat{C}_{n-1}(X)$$

be defined on generators $\sigma \in X_n$, $n \geq 3$, by

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i (d_i \sigma)^{z_i(\sigma)}$$

where $z_i(\sigma) \in H_2 X$ is $\phi(\sigma(i-1, i, i+1))$ for $1 \leq i \leq n-1$ and trivial for $i = 0, n$.

This will give a functor

$$\mathbf{sSet}_1 \xrightarrow{\widehat{C}} \widehat{\mathbf{C}}\mathbf{oalg}$$

where $\mathbf{sSet}_1$ is the category of 1-reduced simplicial sets. Note that $\widehat{C}(X)$ is an ε-twisted coalgebra for $\varepsilon : \mathbb{Z}[H_2X] \to \mathbb{Z}$ the usual augmentation homomorphism, and that $\operatorname{coker}\delta = \mathbb{Z}$.

Lemma 1.8 *For $\sigma \in X_3$ we have*

$$\partial\sigma \;=\; h_\phi(-d_3\sigma - d_1\sigma + d_2\sigma + d_0\sigma)$$

Proof: Let $w = -d_3\sigma - d_1\sigma + d_2\sigma + d_0\sigma \in \langle X_2\rangle$. Then by the derivation property we may expand $h_\phi(w)$ as

$$-h_\phi(d_3\sigma)^{\phi(w)} - h_\phi(d_1\sigma)^{\phi(d_3\sigma+w)} + h_\phi(d_2\sigma)^{\phi(d_0\sigma)} + h_\phi(d_0\sigma)$$

But w is a boundary in C_2X and hence trivial in H_2X, so we have

$$h_\phi(w) \;=\; h_\phi(d_0\sigma) - h_\phi(d_1\sigma)^{z_1(\sigma)} + h_\phi(d_2\sigma)^{z_2(\sigma)} - h_\phi(d_3\sigma)$$

Since we identify simplices in X_2 with their images under h_ϕ, this agrees with the formula for $\partial\sigma$ in the definition. □

Lemma 1.9 *$\widehat{C}(X)$ is a well defined twisted coalgebra over $\mathbb{Z}[H_2X]$.*

Proof: The Alexander-Whitney map defines a coassociative comultiplication. To show (1.1)($*$) is straightforward in dimensions ≤ 4 since all terms vanish. For $\sigma \in X_n$, $n \geq 5$, we have

$$(1\otimes\partial)\tilde{\Delta}\sigma \;=\; (1\otimes\partial)\sum_{j=0}^{n}\sigma(0\ldots j)\otimes\sigma(j\ldots n) \;=$$

$$\sum_{j=0}^{n-1}(-1)^j\sigma(0\ldots j)\otimes\left(\sum_{i=j}^{n}(-1)^{i-j}\sigma(j\ldots\hat{\imath}\ldots n)^{\tau_j^n(i)}\right) \qquad (10)$$

$$(\partial\otimes 1)\tilde{\Delta}\sigma \;=\; (\partial\otimes 1)\sum_{j=0}^{n}\sigma(0\ldots j)\otimes\sigma(j\ldots n) \;=$$

$$\sum_{j=1}^{n}\left(\sum_{i=0}^{j}(-1)^i\sigma(0\ldots\hat{\imath}\ldots j)^{\tau_0^j(i)}\right)\otimes\sigma(j\ldots n) \qquad (11)$$

where $\tau_p^q(i) = \phi\sigma(i-1, i, i+1)$ for $i \notin \{p, q\}$, trivial otherwise. Since the terms for $i = j = k$ in (10) cancel with those for $i = j = k+1$ in (11), we can write (10) + (11) as

$$\sum_{i=0}^{n}(-1)^i \left(\sum_{j=0}^{i-1} \sigma(0\ldots j)\otimes\sigma(j\ldots\hat{\imath}\ldots n) + \sum_{j=i+1}^{n} \sigma(0\ldots\hat{\imath}\ldots j)\otimes\sigma(j\ldots n)\right)^{z_i\sigma}$$

$$= \tilde{\Delta}\left(\sum_{i=0}^{n}(-1)^i (d_i\sigma)^{z_i(\sigma)}\right) = \tilde{\Delta}\partial\sigma$$

as required. We get $\delta\partial = 0$ since for $\sigma \in X_3$ we have by lemma 1.8

$$\begin{aligned} \delta\partial\sigma &= \delta h_\phi(-d_3\sigma - d_1\sigma + d_2\sigma + d_0\sigma) \\ &= \phi(-d_3\sigma - d_1\sigma + d_2\sigma + d_0\sigma) - 1_{H_2X} = 0 \end{aligned}$$

since $-d_3\sigma - d_1\sigma + d_2\sigma + d_0\sigma$ is a boundary in C_2X and so is mapped to the trivial element in homology. It remains to check (1.1)(**). This is trivial in dimensions ≤ 3. For $\sigma \in X_n$, $n \geq 4$ we have

$$\begin{aligned} \partial\partial\sigma &= \partial\sum_{i=0}^{n}(-1)^i\sigma(0\ldots\hat{\imath}\ldots n)^{z_i\sigma} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i-1}(-1)^{i+j}\sigma(0\ldots\hat{\jmath}\ldots\hat{\imath}\ldots n)^{z_j(d_i\sigma)+z_i\sigma} \\ &+ \sum_{i=0}^{n}\sum_{j=i+1}^{n}(-1)^{i+j-1}\sigma(0\ldots\hat{\imath}\ldots\hat{\jmath}\ldots n)^{z_{j-1}(d_i\sigma)+z_i\sigma} \end{aligned}$$

Now for $i - j \geq 2$ we have $z_j(d_i\sigma) + z_i\sigma = z_{i-1}(d_j\sigma) + z_j\sigma$. This also holds for $i - j = 1$, $2 \leq i \leq n-1$, since then their difference is the boundary of $\sigma(i-2, i-1, i, i+1)$ in $C_2(X)$ and so is zero in homology. Thus all the terms in $\partial\partial\sigma$ cancel except

$$\begin{aligned} &-\sigma(2\ldots n)^{z_1\sigma} - \sigma(0\ldots n-2) + \sigma(2\ldots n) + \sigma(0\ldots n-2)^{z_{n-1}\sigma} \\ &= \sigma(0\ldots n-2)^{\delta h_2\sigma(n-2,n-1,n)} - \sigma(2\ldots n-2)^{\delta h_2\sigma(0,1,2)} \end{aligned}$$

But this is just $(1\otimes\delta - \delta\otimes 1)\tilde{\Delta}\sigma$. □

Lemma 1.10 *There is a natural isomorphism of augmented differential coalgebras*

$$\hat{C}(X)\otimes_{\mathbb{Z}[H_2X]}\mathbb{Z} \cong C(X)$$

where the right hand side is the $\mathbb{Z}$-chain complex on X with the Alexander-Whitney diagonal.

Proof: Let F be the free group $\langle X_2 - s_0 * \rangle$ and note that $\widehat{C}_2(X)$ may be regarded as the derived module of the map

$$F \xrightarrow{\phi'} H_2X$$

Thus we have $\widehat{C}_2(X) \otimes_{\mathbb{Z}[H_2X]} \mathbb{Z} \cong IF \otimes_{\mathbb{Z}F} \mathbb{Z}$. But this is the derived module of the homomorphism $F \to 1$ and so is just the abelianisation $F^{\mathrm{ab}} \cong C_2(X)$. We in fact have $\widehat{C}_i(X) \otimes_{\mathbb{Z}[H_2X]} \mathbb{Z} \cong C_i(X)$ for all i, and the composite

$$X_i \longrightarrow \widehat{C}_i(X) \longrightarrow \widehat{C}_i(X) \otimes_{\mathbb{Z}[H_2X]} \mathbb{Z} \cong C_i(X)$$

is the inclusion of simplices as generators of the chain complex, mapping degenerate simplices to zero. The formulæ for $\Delta \otimes_{\mathbb{Z}[H_2X]} \mathbb{Z}$ and $\partial \otimes_{\mathbb{Z}[H_2X]} \mathbb{Z}$ in $\widehat{C}_i(X) \otimes_{\mathbb{Z}[H_2X]} \mathbb{Z}$ are then precisely the classical formulæ for Δ and ∂ in $C(X)$. □

Proposition 1.11 *For X a 1-reduced simplicial set, there is a natural isomorphism of augmented $\mathbb{Z}$-chain algebras*

$$\Omega\widehat{C}(X) \otimes_{\mathbb{Z}[H_2X]} \mathbb{Z} \quad \cong \quad \Omega C(X)$$

Proof: Lemmas 1.6 and 1.10. □

The main theorem

In the above we introduced the twisted cobar construction, giving a chain algebra ΩC from a twisted coalgebra C, and we have examples of twisted coalgebras $\widehat{C}(X)$ arising from 1-reduced simplicial sets X. Let $|X|$ be the realisation of X. We now state the connection between the construction $\Omega\widehat{C}(X)$ and the singular chain complex $C\widehat{\Omega|X|}$ on the universal cover $\widehat{\Omega|X|}$ of the loop space of $|X|$. In fact these constructions yield functors:

$$\mathbf{sSet}_1 \xrightarrow{\Omega\widehat{C},\ C\widehat{\Omega|\ |}} \widehat{\mathbf{Alg}}$$

Theorem 1.12 *For 1-reduced simplicial sets X there is a natural homology equivalence in $\widehat{\mathbf{Alg}}$*

$$\Omega\widehat{C}(X) \sim C\widehat{\Omega|X|}$$

Here *natural homology equivalence* of functors $F, G : \mathbf{sSet}_1 \to \widehat{\mathbf{Alg}}$ is the equivalence relation generated by the relation that $F \sim G$ if there is a natural transformation $F \to G$ in $\widehat{\mathbf{Alg}}$ which induces homology isomorphisms.

The functor $C\widehat{\Omega|\ |}$ above is obtained by composing the following functors

$$\mathbf{sSet}_1 \xrightarrow{\Omega|\ |} \mathbf{Mon}_0 \xrightarrow{\widehat{u}} \widehat{\mathbf{Mon}} \xrightarrow{C} \widehat{\mathbf{Alg}}$$

Let $\mathbf{Mon}_0$ be the category of path-connected topological monoids M which admit a universal covering $\widehat{M}$. Then $\Omega|\ |$ carries a 1-reduced simplicial set X to the space of Moore loops on $|X|$ with the monoid structure given by composition of loops.

A *twisted monoid* (M, G) is a path-connected topological monoid $(M, \cdot)$ together with an abelian group G such that M is also a G-space with

$$x^\alpha \cdot y^\beta \;=\; (x \cdot y)^{\alpha\beta}$$

for $x, y \in M$, $\alpha, \beta \in G$, where x^α denotes the action of α on x. Morphisms $(f, \theta) : (M, G) \to (M', G')$ consist of group homomorphisms $\theta : G \to G'$ and θ-equivariant topological monoid maps $f : M \to M'$. We write $\widehat{\mathbf{Mon}}$ for the category of twisted monoids.

We define the functor $\widehat{u}$ by $\widehat{u}(M) = (\widehat{M}, \pi_1 M)$. For this choose a basepoint $* \in \widehat{M}$ covering 1_M. Then $\widehat{M}$ is a monoid with $1_{\widehat{M}} = *$ and multiplication

$$\widehat{M} \times \widehat{M} \;\cong\; \widehat{M \times M} \xrightarrow{\;\;\widehat{m}\;\;} \widehat{M}$$

where $m : M \times M \to M$ is the multiplication on M. Note that the map

$$\pi_1 M \times \pi_1 M \;\cong\; \pi_1(M \times M) \xrightarrow{\;\;\pi_1(m)\;\;} \pi_1 M$$

is the group law of the abelian group $\pi_1 M$ and therefore $(\widehat{M}, \pi_1 M)$ is a twisted monoid.

Given a twisted monoid (M, G) let $C(M)$ be the singular chain complex of M and let $R = \mathbb{Z}G$ be the group ring of the abelian group G. The action of G on M gives an action of R on $C(M)$. A unit $* \in C_0(M)$ is given by 1_M. The $\mathbb{Z}$-bilinear map

$$C(M) \otimes_{\mathbb{Z}} C(M) \longrightarrow C(M \times M) \xrightarrow{C(\mu)} C(M)$$

induces an R-bilinear multiplication

$$C(M) \otimes_R C(M) \longrightarrow C(M)$$

since $x^\alpha \cdot y = (x \cdot y)^\alpha = x \cdot y^\alpha$ in M. Hence we can define the functor C above by $C(M, G) = (CM, R)$.

2 The crossed cobar construction

Simplicial strings and interval categories

We start by describing the category $\Omega\Delta$ of *simplicial strings*, and the associated monoidal functors ΩX, L, first introduced in [2]. We introduce the notion of a category with an *interval object*; any such category serves as the target for L.

Let $\Delta_\bullet \subset \Delta$ be the subcategory of the simplicial category Δ containing only those morphisms $a : \underline{n} \to \underline{m}$ with $a(0) = 0$ and $a(n) = m$. Recall that $\Delta_\bullet$ is generated by the maps

$$s_i : \underline{n+1} \to \underline{n}, \quad (0 \leq i \leq n), \qquad d_i : \underline{n} \to \underline{n+1}, \quad (1 \leq i \leq n)$$

which repeat and omit the value i respectively.

Next consider the category $\{0,1\}/\mathbf{Set}$ of double-pointed sets (A, a_0, a_1) and functions preserving the basepoints. We can regard $\Delta_\bullet$ as a subcategory of $\{0,1\}/\mathbf{Set}$ with objects $[n] = (\underline{n}, 0, n)$. Note that $\{0,1\}/\mathbf{Set}$ has a monoidal structure given by

$$(A, a_0, a_1)\Box(B, b_0, b_1) \quad = \quad \left(\frac{A \amalg B}{a_1 \sim b_0}, a_0, b_1\right)$$

and unit element $* = [0]$.

Definition 2.1 The *category of simplicial strings* $\Omega\Delta$ is the monoidal subcategory of $\{0,1\}/\mathbf{Set}$ generated by $\Delta_\bullet$ and the functions

$$[n]\Box[m] \xrightarrow{v_{n,m}} [n+m]$$

defined by $i \mapsto i$ on $[n]$ and $i \mapsto n+i$ on $[m]$.

Let $(\mathbf{C}, \otimes)$ be a monoidal category. Using the above presentation of $\Omega\Delta$, we see that to define a monoidal functor $C : \Omega\Delta \to \mathbf{C}$ it is necessary and sufficient to give the following data in $\mathbf{C}$:

1. objects C_n for $n \geq 1$, with $C_0 = *$,
2. morphisms $s_i : C_{n+1} \to C_n$ for $0 \leq i \leq n$,
3. morphisms $d_i : C_n \to C_{n+1}$ for $1 \leq i \leq n$,
4. morphisms $v_{n,m} : C_n \otimes C_m \to C_{n+m}$ for $n, m \geq 0$, with $v_{0,n} = v_{n,0} = 1_{C_n}$,

such that the following relations hold

$$\begin{array}{rcl}
s_j s_i & = & s_i s_{j+1} \quad \text{for } i \leq j \\
d_j d_i & = & d_i d_{j-1} \quad \text{for } i < j \\
s_j d_i & = & \begin{cases} d_i s_{j-1} & \text{for } i < j \\ \text{id} & \text{for } i = j \text{ or } i = j+1 \\ d_{i-1} s_j & \text{for } i > j \end{cases} \\
s_i v_{n,m} & = & \begin{cases} v_{n-1,m}(s_i \otimes 1) & \text{for } i < n \\ v_{n,m-1}(1 \otimes s_{i-n}) & \text{for } i \geq n \end{cases} \\
d_i v_{n,m} & = & \begin{cases} v_{n+1,m}(d_i \otimes 1) & \text{for } i \leq n \\ v_{n,m+1}(1 \otimes d_{i-n}) & \text{for } i > n \end{cases} \\
v_{n,m+l}(1 \otimes v_{m,l}) & = & v_{n+m,l}(v_{n,m} \otimes 1)
\end{array}$$

To define a contravariant monoidal functor on $\Omega\Delta$ the data and relations needed are dual to these.

Definition 2.2 Let **Set** be the category of sets with the cartesian monoidal structure. Then given a 0-reduced simplicial set X, $X_0 = \{*\}$, the monoidal functor

$$(\Omega\Delta)^{\text{op}} \xrightarrow{\Omega X} \mathbf{Set}$$

is defined on the generating objects of $\Omega\Delta$ by $(\Omega X)_n = X_n$ and on the generating morphisms s_i, d_i, $v_{n,m}$ by

$$s_i : X_n \to X_{n+1},$$
$$d_i : X_{n+1} \to X_n,$$
$$v_{n,m} = (d^m_{n+1}, d_0{}^n) : X_{n+m} \to X_n \times X_m$$

respectively; cf. I.2.12 of [2].

We may also write $v_{n,m}(\sigma)$ as $(\sigma(0,\ldots,n), \sigma(n,\ldots,n+m))$ for $\sigma \in X_{n+m}$.

A map $X \to X'$ of 0-reduced simplicial sets induces a natural transformation $\Omega X \to \Omega X'$ of monoidal functors in the obvious way.

Definition 2.3 An *interval object* in a monoidal category $(\mathbf{C}, \otimes)$ is an object $\mathcal{I}$ of $\mathbf{C}$ together with morphisms $d^{\pm} : * \to \mathcal{I}$, $e : \mathcal{I} \to *$ and $m : \mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$ satisfying the following relations:

1. $m(1 \otimes d^-) = m(d^- \otimes 1) = 1_{\mathcal{I}}$
2. $m(1 \otimes d^+) = m(d^+ \otimes 1) = d^+ e$
3. $m(1 \otimes m) = m(m \otimes 1)$

An *interval category* is a monoidal category with a specified interval object. Two examples of interval categories are the following:

1. Let $\mathbf{C}$ be the category $\mathbf{FTop}$ of filtered spaces $X = (X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots)$. The tensor product is the product with the compactly generated topology and the filtration $(X \otimes Y)_n = \bigcup_{i+j=n} X_i \times Y_j$. Then $\mathbf{C}$ has an interval object $\mathcal{I}$ with $\mathcal{I}_0 = \{0, 1\}$ and $\mathcal{I}_n$ the unit interval $[0, 1]$ for $n \geq 1$. The maps d^- and d^+ take $*$ to 0 and 1 respectively, e is the identification to a single point, and m is the maximum function $(t_1, t_2) \mapsto \max(t_1, t_2)$. Then, for example, the relation $m(d^+ \otimes 1) = d^+ e$ becomes $\max(1, t) = 1$. Note that the n-cube $\mathcal{I}^{\otimes n}$ has a natural CW-complex structure, such that the filtration agrees with the skeletal filtration.

2. Let $\mathbf{C}$ be the cartesian monoidal category $\mathbf{sSet}$ of simplicial sets. This has an interval object given by the standard 1-simplex $\Delta[1]$. Regarding elements of $\Delta[1]_n$ as monotonic functions $a : \underline{n} \to \underline{1}$, the multiplication m is given by $m(a, b)(i) = \max(a(i), b(i))$. The maps d^-, d^+, e are defined from d_1, d_0, s_0 respectively.

On the n-cubes $\mathcal{I}^{\otimes n}$ in any interval category we have coface maps

$$\mathcal{I}^{\otimes n} \xrightarrow{d_i^\pm} \mathcal{I}^{\otimes (n+1)}$$

given by $1_{\mathcal{I}^{\otimes(i-1)}} \otimes d^\pm \otimes 1_{\mathcal{I}^{\otimes(n-i+1)}}$ for $1 \leq i \leq n+1$, and codegeneracy maps

$$\mathcal{I}^{\otimes n} \xrightarrow{m_i} \mathcal{I}^{\otimes (n-1)}$$

given by $1_{\mathcal{I}^{\otimes(i-1)}} \otimes m \otimes 1_{\mathcal{I}^{\otimes(n-i-1)}}$ for $1 \leq i \leq n-1$, or by $e \otimes 1_{\mathcal{I}^{\otimes(n-1)}}$, $1_{\mathcal{I}^{\otimes(n-1)}} \otimes e$ for $i = 0, n$.

Definition 2.4 The *standard simplicial string model functor* in an interval category $\mathbf{C}$ is the monoidal functor $L : \Omega\Delta \to \mathbf{C}$ given on the generating objects by $L_n = \mathcal{I}^{\otimes(n-1)}$ and on the generating morphisms s_i, d_i, $v_{n,m}$ by

$$\begin{aligned} m_i &: \mathcal{I}^{\otimes n} \to \mathcal{I}^{\otimes(n-1)} \\ d_i^- &: \mathcal{I}^{\otimes(n-1)} \to \mathcal{I}^{\otimes n} \\ d_n^+ &: \mathcal{I}^{\otimes(n-1)} \otimes \mathcal{I}^{\otimes(m-1)} \to \mathcal{I}^{\otimes(m+n-1)} \end{aligned}$$

respectively.[1]

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[1] There is a misprint in the definition of L on p.9 of [2]; either a_1 needs to be changed to reverse the roles of d^+ and d^-, or δ should be 'min' rather than 'max'.

Coends and the geometric cobar construction

Suppose $\mathbf{C}$ is an arbitrary cocomplete category, $\mathbf{D}$ a small category, and F a functor $\mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$. Then the *coend* of F over $\mathbf{D}$, written $\int^d F(d,d)$, is given by the equaliser in $\mathbf{C}$ of the morphisms:

$$\coprod_{f \in \mathbf{D}(d_1, d_2)} F(d_2, d_1) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{d \in \mathrm{Ob}(\mathbf{D})} F(d,d)$$

which are given componentwise on the coproduct by

$$ai_f \;=\; i_{d_2}F(d_2, f) \quad \text{and} \quad bi_f \;=\; i_{d_1}F(f, d_1)$$

In suitable categories $\mathbf{C}$ we can define coends more explicitly in terms of elements and relations. Let A be the $\mathrm{Ob}(\mathbf{D})$-indexed coproduct of the objects $F(d,d)$ in $\mathbf{C}$. Then $\int^d F(d,d)$ is the quotient object of A given by imposing the relations $F(d_1, f)(x) \sim F(f, d_2)(x)$ for each $f : d_2 \to d_1$ in D and x in $F(d_1, d_2)$.

Suppose now that $\mathbf{C}$, $\mathbf{D}$ are monoidal categories and F is a monoidal functor. Also we assume that $\otimes$ preserves colimits in $\mathbf{C}$; this is the case for example if $\mathbf{C}$ is monoidal closed. Then the coend of F has the structure of a monoid object in $\mathbf{C}$, with identity $F(*,*) = *$ and multiplication induced by the maps

$$F(d_1, d_1) \otimes F(d_2, d_2) \cong F(d_1 \otimes d_2, d_1 \otimes d_2)$$

If $\mathbf{C}$ is an interval category, and X is a 0-reduced simplicial set, then we have monoidal functors

$$(\Omega\Delta)^{\mathrm{op}} \xrightarrow{\Omega X} \mathbf{Set} \qquad\qquad \Omega\Delta \xrightarrow{L} \mathbf{C}$$

from the previous section. Using the 'copower' functor $\mathbf{Set} \times \mathbf{C} \longrightarrow \mathbf{C}$ given by taking set-indexed coproducts in $\mathbf{C}$, one obtains the monoidal functor

$$(\Omega\Delta)^{\mathrm{op}} \times \Omega\Delta \xrightarrow{\Omega X \cdot L} \mathbf{C}$$

Definition 2.5 The *(geometric) cobar construction* on a 0-reduced simplicial set X is the $\mathbf{C}$-monoid $\underline{\Omega}_{\mathbf{C}}(X)$ given by the coend of $\Omega X \cdot L$ over $\Omega\Delta$.

$$\underline{\Omega}_{\mathbf{C}}(X) \;=\; \int^A (\Omega X)(A) \cdot L(A)$$

This yields the functor

$$\mathbf{sSet}_0 \xrightarrow{\underline{\Omega}_{\mathbf{C}}} \mathbf{C}\text{-}\mathbf{Monoids}$$

where $\mathbf{sSet}_0$ is the category of 0-reduced simplicial sets.

Since we have a nice presentation for $\Omega\Delta$ we can give a more explicit description of the cobar construction than the coend definition above.

Proposition 2.6 *The cobar construction $\underline{\Omega}_{\mathbf{C}}X$ on a simplicial set X, $X_0 = *$, is given by a coproduct in* **C** *indexed by words in $X_{\geq 1}$*

$$\coprod_{r\geq 0}\coprod_{(x_1,\ldots,x_r)} \mathcal{I}^{\otimes(n_1-1)}\otimes\cdots\otimes\mathcal{I}^{\otimes(n_r-1)}$$

which has 'generating' elements

$$(x_1,\ldots,x_r;y)$$

for $y\in\mathcal{I}^{\otimes(n_1-1)}\otimes\cdots\otimes\mathcal{I}^{\otimes(n_r-1)}$, $x_k\in X_{n_k}$, $n_k\geq 1$, $k=1,\ldots,r$, quotiented by the relations

$$\begin{aligned}
(x_1,\ldots,x_{k-1},s_ix_k,x_{k+1},\ldots,x_r;y) &\sim ((x_j)_1^r;(1_{<k}\otimes m_i\otimes 1_{>k})(y))\\
(x_1,\ldots,x_{k-1},d_ix_k,x_{k+1},\ldots,x_r;y) &\sim ((x_j)_1^r;(1_{<k}\otimes d_i^-\otimes 1_{>k})(y))\\
(x_1,\ldots,x_{k-1},d_{i+1}^{n_k-i}x_k,d_0^ix_k,x_{k+1},\ldots,x_r;y) &\sim ((x_j)_1^r;(1_{<k}\otimes d_i^+\otimes 1_{>k})(y))
\end{aligned}$$

where $1_{<k}$ is the identity map on $\mathcal{I}^{\otimes\Sigma_{j<k}(n_j-1)}$, and $1_{>k}$ similarly. Note that $i\neq 0, n_k$ in the second relation.

The monoid structure on $\underline{\Omega}_{\mathbf{C}}X$ is given by the unit $(\ ;*)$ and the multiplication

$$(w_1,\ldots,w_s;y)\otimes(x_1,\ldots,x_r;z) = (w_1,\ldots,w_s,x_1,\ldots,x_r;y\otimes z).$$

The importance of the geometric cobar construction is that it provides a model for the loop space on the realisation of a simplicial set. In fact from [2] we have the following result (compare also [9]):

Theorem 2.7 *For 1-reduced simplicial sets X there is a natural homotopy equivalence of path-connected topological monoids*

$$\underline{\Omega}_{\mathbf{FTop}}X \simeq \Omega|X|$$

Also $\underline{\Omega}_{\mathbf{FTop}}X$ has a natural CW-complex structure and its filtration in **FTop** *coincides with the skeletal filtration.*

Here *natural homotopy equivalence* of functors $F,G:\mathbf{sSet}_1\to\mathbf{Mon}_0$ is the equivalence relation generated by the relation that $F\simeq G$ if there is a natural transformation $F\to G$ in $\mathbf{Mon}_0$ which for each object is a homotopy equivalence in the category of pointed topological spaces.

The crossed cobar construction

Let **C** be the monoidal closed category **Crs** of crossed complexes (see for example [11, 13]). The tensor product $C\otimes D$ of crossed complexes is defined in terms of generators $c\otimes c'\in(C\otimes D)_{n+m}$ for $c\in C_n$, $c'\in D_m$ together with certain relations which may be found in [13]. A monoid object C in **Crs** is termed a *crossed algebra*, or a *crossed chain algebra* if $C_0 = \{*\}$.

An interval object $\mathcal{I}$ in **Crs** is given by the crossed complex on generators $0, 1 \in \mathcal{I}_0$, $\iota \in \mathcal{I}_1$, with $s\iota = 0$, $t\iota = 1$. The maps $d^-, d^+ : * \to \mathcal{I}$ are given by $* \mapsto 0$, $* \mapsto 1$ respectively, $e : \mathcal{I} \to *$ is the unique map to the terminal object and the map $m : \mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$ is given on the standard generators by

$$a \otimes b \mapsto \begin{cases} 0 & \text{if } a = b = 0 \\ \iota & \text{if } \{a, b\} = \{0, \iota\} \\ 1 & \text{otherwise.} \end{cases}$$

Alternatively this may be obtained by applying the fundamental crossed complex functor $\pi : \mathbf{FTop} \to \mathbf{Crs}$ to the interval object structure in **FTop** defined above.

If $\iota^{\otimes n}$ is the n-dimensional generator of $\mathcal{I}^{\otimes n}$ then from the tensor product relations we can obtain

$$\begin{aligned} s(\iota) &= 0 \\ t(\iota) &= 1 \\ \beta(\iota^{\otimes n}) &= 1^{\otimes n} \quad \text{for } n \geq 1 \\ \delta(\iota^{\otimes 2}) &= -1 \otimes \iota - \iota \otimes 0 + 0 \otimes \iota + \iota \otimes 1 \\ \delta(\iota^{\otimes 3}) &= -\iota \otimes \iota \otimes 1 - \iota \otimes 0 \otimes \iota^{1 \otimes \iota \otimes 1} - 1 \otimes \iota \otimes \iota \\ &\quad + \iota \otimes \iota \otimes 0^{1 \otimes 1 \otimes \iota} + \iota \otimes 1 \otimes \iota + 0 \otimes \iota \otimes \iota^{\iota \otimes 1 \otimes 1} \\ \delta(\iota^{\otimes n}) &= \sum_{i=1}^{n} (-1)^i \left(d_i^+ \iota^{\otimes (n-1)} - \left(d_i^- \iota^{\otimes (n-1)} \right)^{z_i} \right) \quad \text{for } n \geq 4 \end{aligned}$$

where $z_i \in (\mathcal{I}^{\otimes (n-1)})_1$ is given by $(d_{i+1}^+)^{n-i-1} (d_1^+)^{i-1} (\iota)$.

For $\lambda = (\lambda_k)_1^r$ an ordered subset of $\{1 < 2 < \ldots < n\}$ and $\alpha \in \{-, +\}^r$, let d_λ^α be the morphism

$$d_{\lambda_r}^{\alpha_r} \ldots d_{\lambda_1}^{\alpha_1} : \mathcal{I}^{\otimes (n-r)} \to \mathcal{I}^{\otimes n}$$

Then the 3^n generators of $\mathcal{I}^{\otimes n}$ may be written as $d_\lambda^\alpha \iota^{\otimes (n-r)}$ for $0 \leq r \leq n$, and the relations on these generators are obtained by applying d_λ^α to the terms in the relations above.

By proposition 2.6, we can now give a presentation for the *crossed cobar construction* $\Omega_{\mathbf{Crs}}(X)$ on a 1-reduced simplicial set X. For an $x \in X_n$ only the top-dimensional generator of $\mathcal{I}^{\otimes (n-1)}$ needs to be considered since the lower-dimensional ones can be obtained by applying $d_i^\pm$ and so are identified with generators coming from (products of) faces of x. Since m maps top-dimensional generators to an identity we can also throw out degenerate simplices. The resulting monoid C in **Crs** has $C_0 = \{*\}$ since we are treating the 1-reduced case only, and is in fact a *free* crossed chain algebra [22].

Theorem 2.8 *Let X be a simplicial set with $X_0 = X_1 = \{*\}$. For $x_n \in X_n$, $n \geq 4$, set $z_i(x_n) = d_0^{i-1} d_{i+2}^{n-i-1} x_n = x_n(i-1, i, i+1) \in X_2$ for $1 \leq i \leq n-1$. Then $C = \underline{\Omega}_{\mathbf{Crs}}(X)$ is the crossed chain algebra with generators $x_n \in C_{n-1}$ for $x_n \in X_n$, $n \geq 2$, subject to the relations*

$$\begin{aligned} x_n &= * \quad \text{if } x_n \text{ is degenerate} \\ \delta_2(x_3) &= -d_0x_3 - d_2x_3 + d_1x_3 + d_3x_3 \\ \delta_3(x_4) &= -d_4x_4 - d_2x_4{}^{z_2(x_4)} - d_0x_4 \\ &\quad + d_3x_4{}^{z_3(x_4)} + d_3d_4x_4 \otimes d_0d_1x_4 + d_1x_4{}^{z_1(x_4)} \\ \delta_{n-1}(x_n) &= -d_0x_n + \sum_{i=2}^{n-2} (-1)^i d_{i+1}^{n-i} x_n \otimes d_0^i x_n - (-1)^n d_n x_n \\ &\quad - \sum_{i=1}^{n-1} (-1)^i d_i x_n{}^{z_i(x_n)} \qquad \text{for } n \geq 5 \end{aligned}$$

together with the usual relations on tensor products of crossed complexes.

Recall from [4, 14, 23] that there is a functor $\mathcal{D}$ from crossed complexes to R-chain complexes. Given a crossed complex of groups

$$\cdots \longrightarrow C_4 \xrightarrow{\delta_4} C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

let $\pi_1 = \pi_1 C = \operatorname{coker} \delta_2$ and let ϕ be the quotient map $C_1 \to \pi_1$, with $h_\phi : C_1 \to D_\phi$ the universal ϕ-derivation. Then $\mathcal{D}(C)$ is the $\mathbb{Z}\pi_1$-chain complex

$$\cdots \longrightarrow C_4 \xrightarrow{\delta_4} C_3 \xrightarrow{\delta_3} C_2^{\text{ab}} \xrightarrow{d_2} D_\phi \xrightarrow{d_1} \mathbb{Z}\pi_1$$

where $d_2x = h_\phi\delta_2x$ and $d_1h_\phi x = \phi x - 1_{\pi_1}$.

Lemma 2.9 *$\mathcal{D}$ induces a functor*

$$\mathbf{CrsAlg} \xrightarrow{\quad\mathcal{D}\quad} \widehat{\mathbf{Alg}}$$

from crossed chain algebras to chain algebras

Proof: If A, B are crossed complexes, then $\pi_1(A \otimes B)$ is $\pi_1 A \times \pi_1 B$ and from [14] we know that $\mathcal{D}(A \otimes B)$ is the chain complex $\mathcal{D}A \otimes_{\mathbb{Z}} \mathcal{D}B$ with the action of $\pi_1 A \times \pi_1 B$ given by $(x \otimes y)^{(a,b)} = x^a \otimes y^b$. A morphism

$$A \otimes B \xrightarrow{m} C$$

of pointed crossed complexes induces a multiplication $\pi_1 A \times \pi_1 B \to \pi_1 C$ via $a \cdot b = m(a \otimes *).m(* \otimes b)$. Moreover the $\mathbb{Z}$-chain map

$$\mathcal{D}A \otimes_{\mathbb{Z}} \mathcal{D}B \xrightarrow{\mathcal{D}m} \mathcal{D}C$$

satisfies $(\mathcal{D}m)(x^a \otimes y^b) = (\mathcal{D}m)(x \otimes y)^{a \cdot b}$. In particular if $A = B = C$ and m is a monoid structure on C then $\mathcal{D}m$ induces a $\mathbb{Z}\pi_1$-chain algebra structure

$$\mathcal{D}C \otimes_{\mathbb{Z}\pi_1} \mathcal{D}C \xrightarrow{\mathcal{D}m} \mathcal{D}C$$

where π_1 acts on $\mathcal{D}C \otimes_{\mathbb{Z}\pi_1} \mathcal{D}C$ by $(x \otimes y)^a = x^a \otimes y = x \otimes y^a$. □

We can now relate the crossed and twisted cobar constructions.

Proposition 2.10 *For 1-reduced simplicial sets X, there is a natural isomorphism of $\mathbb{Z}H_2X$-chain algebras*

$$\mathcal{D}\underline{\Omega}_{\mathbf{Crs}} X \;\cong\; \Omega \widehat{C} X$$

Proof: Let A, B be the chain algebras $\mathcal{D}\underline{\Omega}X$, $\Omega\widehat{C}X$ respectively, and recall that $B_0 = R = \mathbb{Z}H_2X$ and

$$B_n \;=\; \bigoplus_{i_1 + \ldots + i_r = n} C_{i_1} \otimes_R C_{i_2} \otimes_R \ldots \otimes_R C_{i_r}$$

where C_1 is the derived module of $\phi' : \langle X_2 - s_0 * \rangle \to H_2X$ and $C_i = C_{i+1}(X; R)$ for $i \geq 2$. Now $(\underline{\Omega}X)_1$ is the free group on $X_2 - s_0*$, and $(\underline{\Omega}X)_2$ is the free crossed $(\underline{\Omega}X)_1$-module with generators σ_3 and $\sigma_2 \otimes \sigma_2'$ and boundary relations

$$\begin{aligned} \delta_2 \sigma_3 &= -d_0\sigma_3 - d_2\sigma_3 + d_1\sigma_3 + d_3\sigma_3 \\ \delta_2(\sigma_2 \otimes \sigma_2') &= -\sigma_2' - \sigma_2 + \sigma_2' + \sigma_2 \end{aligned}$$

where as usual we quotient out degenerate simplices. Thus

$$(\underline{\Omega}X)_2 \xrightarrow{\delta_2} \langle X_2 - s_0 * \rangle \xrightarrow{\phi'} H_2X \longrightarrow 0$$

is exact and we have $A_0 = B_0 = R$ and $A_1 = B_1 = D_{\phi'}$, with $d_1 h_{\phi'} x = \phi' x - 1_{H_2X}$ in A and B. In general $\underline{\Omega}X$ is generated as a crossed complex by $\sigma_1 \otimes \ldots \otimes \sigma_r$ in dimension $\sum(\dim \sigma_i - 1)$. Since tensor products of pointed crossed complexes satisfy the relations

$$\begin{aligned} (c_1 + c_1') \otimes d_j &= c_1' \otimes d_j + (c_1 \otimes d_j)^{c_1'} \\ c_i \otimes (d_1 + d_1') &= (c_i \otimes d_1)^{d_1'} + c_i \otimes d_1' \\ (c_i + c_i') \otimes d_j &= c_i \otimes d_j + c_i' \otimes d_j \quad \text{for } i \geq 2 \\ c_i \otimes (d_j + d_j') &= c_i \otimes d_j + c_i \otimes d_j' \quad \text{for } j \geq 2 \\ c_i^{c_1} \otimes d_j &= (c_i \otimes d_j)^{c_1} \quad \text{for } i \geq 2 \\ c_i \otimes d_j^{d_1} &= (c_i \otimes d_j)^{d_1} \quad \text{for } j \geq 2 \end{aligned}$$

we obtain $A_2 = (\underline{\Omega}X)_2^{\mathrm{ab}} = C_3(X; R) \oplus D_{\phi'} \otimes_R D_{\phi'}$, and similarly for $n \geq 3$ we find that $A_n = (\underline{\Omega}X)_n$ agrees with B_n above. Note that for X 2-dimensional the result $A_n = D_{\phi'}{}^{\otimes n}$ was proved in [7]. For $\sigma \in X_{\geq 4}$ the differentials in A, B agree by

$$\begin{aligned} d_A\sigma &= \sum_2^{n-2}(-1)^i\sigma(0\ldots i)\otimes\sigma(i\ldots n) - \sum_0^n(-1)^i d_i\sigma^{\sigma(i-1,i,i+1)} \\ &= \tilde{\Delta}\sigma - \partial\sigma \quad = \quad d_B\sigma \end{aligned}$$

and for $\sigma \in X_3$ we have $d_A\sigma = h_{\phi'}\delta_2\sigma$ which agrees with $d_B\sigma = -\partial\sigma$ by lemma 1.8. □

Proof of the main theorem

We now complete the proof of theorem 1.12, that for X a simplicial set with $X_0 = X_1 = \{*\}$ there is a natural homology equivalence between the cobar construction $\Omega\widehat{C}(X)$ of the twisted chain coalgebra on X, and the singular chain algebra $C\widehat{\Omega|X|}$ of the universal cover of the loops on X. We have just seen in 2.10 that $\Omega\widehat{C}$ is given by applying $\mathcal{D}$ to the crossed cobar construction $\underline{\Omega}_{\mathbf{Crs}}$. Also by 2.7 we know that the loop space on X is given up to homotopy by the geometric cobar construction, and so there is a natural homology equivalence of chain algebras $C\widehat{\Omega|X|} \sim C\widehat{\underline{\Omega}}_{\mathbf{FTop}}X$. The main theorem thus follows from the following:

Proposition 2.11 *For 1-reduced simplicial sets X, there is a natural homology equivalence of chain algebras*

$$\mathcal{D}\underline{\Omega}_{\mathbf{Crs}}X \sim C\widehat{\underline{\Omega}}_{\mathbf{FTop}}X$$

Proof: Let Y be the monoid in **FTop** given by $\underline{\Omega}_{\mathbf{FTop}}X$. Since the fundamental crossed complex functor π preserves colimits and tensor products of the spaces involved we note that $\underline{\Omega}_{\mathbf{Crs}}X$ is just πY. It therefore remains to show that there is a natural homology equivalence $\mathcal{D}\pi Y \sim C\widehat{Y}$. Let $\widehat{Y}$ have the filtration given by the the inverse image under the covering map of the (skeletal) filtration on Y. Then by [23], or proposition 5.2 of [14], we can identify $\mathcal{D}\pi Y$ with the cellular chain complex $\mathcal{H}\widehat{Y}$ given by the relative homology groups:

$$\cdots \longrightarrow H_3(\widehat{Y}_3, \widehat{Y}_2) \xrightarrow{\delta_3} H_2(\widehat{Y}_2, \widehat{Y}_1) \xrightarrow{\delta_2} H_1(\widehat{Y}_1, \widehat{Y}_0) \xrightarrow{\delta_1} H_0(\widehat{Y}_0)$$

Finally we note that there is a natural equivalence $\mathcal{H}\widehat{Y} \sim C\widehat{Y}$ given via

$$\mathcal{H}\widehat{Y} \xleftarrow{\tau} C_{\mathrm{cell}}\widehat{Y} \subseteq C\widehat{Y}$$

where $C_{cell}\widehat{Y}$ is the subchain complex of the singular chain complex $C\widehat{Y}$ generated by all singular simplices $\sigma : \Delta^n \to \widehat{Y}$ which are cellular maps. The map τ carries σ to $\sigma_*[\Delta^n]$ where $[\Delta^n] \in H_n(\Delta^n, \partial\Delta^n)$ is the fundamental class. □

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