COLLAPSING AND DIRAC-TYPE OPERATORS

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Abstract. We analyze the limit of the spectrum of a geometric Dirac-type operator under a collapse with bounded diameter and bounded sectional curvature. In the case of a smooth limit space B, we show that the limit of the spectrum is given by the spectrum of a certain first-order differential operator on B, which can be constructed using superconnections. In the case of a general limit space X, we express the limit operator in terms of a transversally elliptic operator on a G-space $\check{X}$ with $X = \check{X}/G$. As an application, we give a characterization of manifolds which do not admit uniform upper bounds, in terms of diameter and sectional curvature, on the k-th eigenvalue of the square of a Dirac-type operator. We also give a formula for the essential spectrum of the square of a Dirac-type operator on a finite-volume manifold with pinched negative sectional curvature.

1. Introduction

In a previous paper we analyzed the limit of the spectrum of the differential form Laplacian on a manifold, under a collapse with bounded diameter and bounded sectional curvature [17]. In the present paper, we extend the analysis of [17] to geometric Dirac-type operators. As the present paper is a sequel to [17], we refer to the introduction of [17] for background information about collapsing with bounded curvature and its relation to analytic questions.

Let M be a connected closed oriented Riemannian manifold of dimension $n > 0$. If M is spin then we put $G = \mathrm{Spin}(n)$ and if M is not spin then we put $G = \mathrm{SO}(n)$. The spinor-type fields that we consider are sections of a vector bundle E^M associated to a G-Clifford module V, the latter being in the sense of Definition 2 of Section 2. The ensuing Dirac-type operator D^M acts on sections of E^M. We will think of the spectrum $\sigma(D^M)$ of D^M as a set of real numbers with multiplicities, corresponding to possible multiple eigenvalues. For simplicity, in this introduction we will sometimes refer to the Dirac-type operators as acting on spinors, even though the results are more general.

We first consider a collapse in which the limit space is a smooth Riemannian manifold. The model case is that of a Riemannian affine fiber bundle.

Definition 1. *A Riemannian affine fiber bundle is a smooth fiber bundle $\pi : M \to B$ whose fiber Z is an infranilmanifold and whose structure group is reduced from* $\mathrm{Diff}(Z)$ *to* $\mathrm{Aff}(Z)$, *along with*

- *A horizontal distribution $T^H M$ whose holonomy lies in* $\mathrm{Aff}(Z)$,
- *A family of vertical Riemannian metrics g^{TZ} which are parallel with respect to the flat affine connections on the fibers Z_b and*
- *A Riemannian metric g^{TB} on B.*

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Given a Riemannian affine fiber bundle $\pi : M \to B$, there is a Riemannian metric g^{TM} on M constructed from T^HM, g^{TZ} and g^{TB}. Let R^M denote the Riemann curvature tensor of (M, g^{TM}), let Π denote the second fundamental forms of the fibers $\{Z_b\}_{b\in B}$ and let $T \in \Omega^2(M; TZ)$ be the curvature of T^HM. Given $b \in B$, there is a natural flat connection on $E^M\big|_{Z_b}$ which is constructed using the affine structure of Z_b. We define a Clifford bundle E^B on B whose fiber over $b \in B$ consists of the parallel sections of $E^M\big|_{Z_b}$. The operator D^M restricts to a first-order differential operator D^B on $C^\infty(B; E^B)$. If V happens to be the spinor module then we show that D^B is the "quantization" of a certain superconnection on B. For general V, there is an additional zeroth-order term in D^B which depends on Π and T.

We show that the spectrum of D^M coincides with that of D^B up to a high level, which depends on the maximum diameter $\mathrm{diam}(Z)$ of the fibers $\{Z_b\}_{b\in B}$.

Theorem 1. *There are positive constants A, A' and C which only depend on n and V such that if $\| R^Z \|_\infty \, \mathrm{diam}(Z)^2 \leq A'$ then the intersection of $\sigma(D^M)$ with the interval*

$$\begin{aligned}[-\big(A\,\mathrm{diam}(Z)^{-2} - C\,\big(\| R^M \|_\infty + \| \Pi \|_\infty{}^2 + \| T \|_\infty{}^2\big)\big)^{1/2}, \\ \big(A\,\mathrm{diam}(Z)^{-2} - C\,\big(\| R^M \|_\infty + \| \Pi \|_\infty{}^2 + \| T \|_\infty{}^2\big)\big)^{1/2}]\end{aligned} \tag{1.1}$$

equals the intersection of $\sigma(D^B)$ with (1.1).

If $Z = S^1$, $\Pi = 0$ and V is the spinor module then we recover some results of [1, Section 4]; see also [12, Theorem 1.5]. The proof of Theorem 1 follows the same strategy as the proof of the analogous [17, Theorem 1]. Consequently, in the proof of Theorem 1, we only indicate the changes that need to be made in the proof of [17, Theorem 1] and refer to [17] for details.

Given B, Cheeger, Fukaya and Gromov showed that under some curvature bounds, any Riemannian manifold M which is sufficiently Gromov-Hausdorff close to B can be well approximated by a Riemannian affine fiber bundle [11]. Using this fact, we show that the spectrum of D^M can be uniformly approximated by that of a certain first-order differential operator D^B on B, at least up to a high level which depends on the Gromov-Hausdorff distance between M and B.

Given $\epsilon > 0$ and two collections of real numbers $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$, we say that $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are ϵ-close if there is a bijection $\alpha : I \to J$ such that for all $i \in I$, $|b_{\alpha(i)} - a_i| \leq \epsilon$.

Theorem 2. *Let B be a fixed smooth connected closed Riemannian manifold. Given $n \in \mathbb{Z}^+$, take $G \in \{\mathrm{SO}(n), \mathrm{Spin}(n)\}$ and let V be a G-Clifford module. Then for any $\epsilon > 0$ and $K > 0$, there are positive constants $A(B,n,V,\epsilon,K)$, $A'(B,n,V,\epsilon,K)$, and $C(B,n,V,\epsilon,K)$ so that the following holds. Let M be an n-dimensional connected closed oriented Riemannian manifold with a G-structure such that $\| R^M \|_\infty \leq K$ and $d_{GH}(M,B) \leq A'$. Then there are a Clifford module E^B on B and a certain first-order differential operator D^B on $C^\infty(B; E^B)$ such that*

1. $\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{2K}}\right) : \lambda \in \sigma(D^M) \text{ and } \lambda^2 \leq A\, d_{GH}(M,B)^{-2} - C\right\}$ is ϵ-close to a subset of $\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{2K}}\right) : \lambda \in \sigma(D^B)\right\}$, and

2. $\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{2K}}\right) : \lambda \in \sigma(D^B) \text{ and } \lambda^2 \leq A\, d_{GH}(M,B)^{-2} - C\right\}$ *is* ϵ*-close to a subset of* $\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{2K}}\right) : \lambda \in \sigma(D^M)\right\}$.

The other results in this paper concern collapsing to a possibly-singular space. Let X be a limit space of a sequence $\{M_i\}_{i=1}^\infty$ of n-dimensional connected closed oriented Riemannian manifolds with uniformly bounded diameter and uniformly bounded sectional curvature. In general, X is not homeomorphic to a manifold. However, Fukaya showed that X is homeomorphic to $\check{X}/G$, where $\check{X}$ is a manifold and G is a compact Lie group which acts on $\check{X}$ [14]. This comes from writing $M_i = P_i/G$, where $G = \mathrm{SO}(n)$ and P_i is the oriented orthonormal frame bundle of M_i. There is a canonical Riemannian metric on P_i. Then $\{P_i\}_{i=1}^\infty$ has a subsequence which Gromov-Hausdorff converges to a manifold $\check{X}$. As the convergence argument can be done G-equivariantly, the corresponding subsequence of $\{M_i\}_{i=1}^\infty$ converges to $X = \check{X}/G$. In general, $\check{X}$ is a smooth manifold with a metric which is $C^{1,\alpha}$ regular for all $\alpha \in (0,1)$.

In [17] we dealt with the limit of the spectra of the differential form Laplacians $\{\triangle^{M_i}\}_{i=1}^\infty$ on the manifolds $\{M_i\}_{i=1}^\infty$. We defined a limit operator $\triangle^X$ which acts on the "differential forms" on X, coupled to a superconnection. In order to make this precise, we defined the "differential forms" on X to be the G-basic differential forms on $\check{X}$. We constructed the corresponding differential form Laplacian $\triangle^X$ and showed that its spectrum described the limit of the spectra of $\{\triangle^{M_i}\}_{i=1}^\infty$. We refer to [17] for the precise statements.

In the case of geometric Dirac-type operators D^{M_i}, there is a fundamental problem in extending this approach. Namely, if $\check{X}$ is a spin manifold on which a compact Lie group G acts isometrically and preserving the spin structure then there does not seem to be a notion of G-basic spinors on $\check{X}$. In order to get around this problem, we take a different approach. For a given n-dimensional Riemannian spin manifold M, put $G = \mathrm{Spin}(n)$, let P be the principal $\mathrm{Spin}(n)$-bundle of M and let V be the spinor module. One can identify the spinor fields on M with $(C^\infty(P) \otimes V)^G$, the G-invariant subspace of $C^\infty(P) \otimes V$. There are canonical horizontal vector fields $\{\mathfrak{Y}_j\}_{j=1}^n$ on P and the Dirac operator takes the form $D^M = -i \sum_{j=1}^n \gamma^j\, \mathfrak{Y}_j$. Furthermore, $(D^M)^2$ can be written in a particularly simple form. As in equation (4.2) below, when acting on $(C^\infty(P) \otimes V)^G$, $(D^M)^2$ becomes the scalar Laplacian on P (acting on V-valued functions) plus a zeroth-order term.

Following this viewpoint, it makes sense to define the limiting "spinor fields" on X to be the elements of $\left(C^\infty(\check{X}) \otimes V\right)^G$. We can then extend Theorem 1 to the setting of G-equivariant Riemannian affine fiber bundles. Namely, the limit operator D^X turns out to be a G-invariant first-order differential operator on $C^\infty(\check{X}) \otimes V$, transversally elliptic in the sense of Atiyah [2], which one then restricts to the G-invariant subspace $\left(C^\infty(\check{X}) \otimes V\right)^G$. In Theorem 6 below, we show that the analog of Theorem 1 holds, in which D^B is replaced by D^X.

Theorem 6 refers to a given G-equivariant Riemannian affine fiber bundle. In order to deal with arbitrary collapsing sequences, we use the aforementioned representation of $(D^M)^2$ as a Laplace-type operator on P. If $\{M_i\}_{i=1}^\infty$ is a sequence of n-dimensional Riemannian manifolds with uniformly bounded diameter and uniformly bounded sectional curvature then we show that after taking a subsequence, the spectra of $\{(D^{M_i})^2\}_{i=1}^\infty$ converge to the spectrum of a Laplace-type operator on a limit space. Let $\{\lambda_k(|D^M|)\}_{k=1}^\infty$ denote the eigenvalues of $|D^M|$, counted with multiplicity.

Theorem 3. *Given $n \in \mathbb{Z}^+$ and $G \in \{\mathrm{SO}(n), \mathrm{Spin}(n)\}$, let $\{M_i\}_{i=1}^\infty$ be a sequence of connected closed oriented n-dimensional Riemannian manifolds with a G-structure. Let V be a G-Clifford module. Suppose that for some D, $K > 0$ and for each $i \in \mathbb{Z}^+$, we have $\mathrm{diam}(M_i) \le D$ and $\| R^{M_i} \|_\infty \le K$. Then there are*
1. A subsequence of $\{M_i\}_{i=1}^\infty$, which we relabel as $\{M_i\}_{i=1}^\infty$,
2. A smooth closed G-manifold $\check{X}$ with a G-invariant Riemannian metric $g^{T\check{X}}$ which is $C^{1,\alpha}$-regular for all $\alpha \in (0,1)$,
3. A positive G-invariant function $\chi \in C(\check{X})$ with $\int_{\check{X}} \chi \, \mathrm{dvol} = 1$ and
4. A G-invariant function $\mathcal{V} \in L^\infty(\check{X}) \otimes \mathrm{End}(V)$
such that if $\triangle^{\check{X}}$ denotes the Laplacian on $L^2(\check{X}, \chi \, \mathrm{dvol}) \otimes V$ [13, (0.8)] and $|D^X|$ denotes the operator $\sqrt{\triangle^{\check{X}} + \mathcal{V}}$ acting on $\left(L^2(\check{X}, \chi \, \mathrm{dvol}) \otimes V\right)^G$ then for all $k \in \mathbb{Z}^+$,

$$\lim_{i \to \infty} \lambda_k \left(|D^{M_i}| \right) = \lambda_k \left(|D^X| \right). \tag{1.2}$$

In the special case of the signature operator, the proof of Theorem 3 is somewhat simpler than that of the analogous [17, Proposition 11], in that we essentially only have to deal with scalar Laplacians. However, [17, Proposition 11] gives more detailed information. In particular, it expresses the limit operator in terms of a basic flat degree-1 superconnection on $\check{X}$. This seems to be necessary in order to prove the results of [17] concerning small eigenvalues. Of course, one does not expect to have analogous results concerning the small eigenvalues of general geometric Dirac-type operators, as their zero-eigenvalues have no topological meaning.

As an application of Theorem 3, we give a characterization of manifolds which do not have a uniform upper bound on the k-th eigenvalue of $|D^M|$, in terms of diameter and sectional curvature.

Theorem 4. *Let M be a connected closed oriented manifold with a G-structure. Let V be a G-Clifford module. Suppose that for some $K > 0$ and $k \in \mathbb{Z}^+$, there is no uniform upper bound on $\lambda_k(|D^M|)$ among Riemannian metrics on M with $\mathrm{diam}(M) = 1$ and $\| R^M \|_\infty \le K$. Then M admits a possibly-singular fibration $M \to X$ whose generic fiber is an infranilmanifold Z such that the restriction of E^M to Z does not have any nonzero affine-parallel sections.*

More precisely, the possibly-singular fibration $M \to X$ of Theorem 4 is the G-quotient of a G-equivariant Riemannian affine fiber bundle $P \to \check{X}$. Theorem 4 is an analog of [17, Theorem 5.2]. A simple example of Theorem 4 comes from considering spinors on $M = S^1 \times N$, where N is a spin manifold and the spin structure on S^1 is the one that does not admit a harmonic spinor. Upon shrinking the S^1-fiber, the eigenvalues of D_M go off to $\pm\infty$. More generally, let $\pi : M \to B$ be an affine fiber bundle. Theorem 1 implies that if $E^M\big|_Z$ does not have any nonzero affine-parallel sections then upon collapsing M to B as in [15, Section 6], the eigenvalues of D_M go off to $\pm\infty$.

Finally, we state a result about the essential spectrum of the square of a geometric Dirac-type operator on a finite-volume manifold of pinched negative curvature, which is an analog of [18, Theorem 2]. Let M be a complete connected oriented n-dimensional Riemannian manifold with a G-structure. Suppose that M has finite volume and its sectional curvatures satisfy $-b^2 \leq K \leq -a^2$, with $0 < a \leq b$. Let V be a G-Clifford module. Label the ends of M by $I \in \{1, \ldots, B\}$. An end of M has a neighborhood U_I whose closure is diffeomorphic to $[0, \infty) \times Z_I$, where the first coordinate is the Busemann function corresponding to a ray exiting the end, and Z_I is an infranilmanifold. Let E^M be the vector bundle on M associated to the pair (G, V) and let D^M be the corresponding Dirac-type operator. If U_I lies far enough out the end then for each $s \in [0, \infty)$, $C^\infty \left(\{s\} \times Z_I; E^M\big|_{\{s\} \times Z_I} \right)$ decomposes as the direct sum of a finite-dimensional space $E^B_{I,s}$, consisting of "bounded energy" sections, and its orthogonal complement, consisting of "high energy" sections. The vector spaces $\{E^B_{I,s}\}_{s \in [0,\infty)}$ fit together to form a vector bundle E^B_I on $[0, \infty)$. Let P_0 be orthogonal projection from $\bigoplus_{I=1}^B C^\infty \left(\overline{U_I}; E^M\big|_{\overline{U_I}} \right)$ to $\bigoplus_{I=1}^B C^\infty \left([0, \infty); E^B_I \right)$. Let $(D^M)^2_0$ be the restriction of $(D^M)^2$ to $\bigoplus_{I=1}^B C^\infty \left(\overline{U_I}; E^M\big|_{\overline{U_I}} \right)$, say with Atiyah-Patodi-Singer boundary conditions. Then $P_0 \, (D^M)^2_0 \, P_0$ is a second-order ordinary differential operator on $\bigoplus_{I=1}^B C^\infty \left([0, \infty); E^B_I \right)$.

Theorem 5. *The essential spectrum of $(D^M)^2$ is the same as that of $P_0 \, (D^M)^2_0 \, P_0$.*

Using Propositions 1 and 2, the proof of Theorem 5 is similar in structure to that of [18, Theorem 2], after one replaces the operator $d + d^*$ of [18] by the operator D^M. Consequently, we do not give the details here.

There is some intersection between Theorem 5 and the results of [4, Theorem 0.1], concerning the essential spectrum of D^M when $n = 2$ and under an additional curvature assumption, and [5, Theorem 1], concerning the essential spectrum of D^M when M is hyperbolic and V is the spinor module.

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2. Dirac-type Operators and Infranilmanifolds

Given $n \in \mathbb{Z}^+$, let G be either $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$.

Definition 2. *A G-Clifford module consists of a finite-dimensional Hermitian G-vector space V and a G-equivariant linear map $\gamma : \mathbb{R}^n \to \mathrm{End}(V)$ such that $\gamma(v)^2 = |v|^2 \, \mathrm{Id.}$ and $\gamma(v)^* = \gamma(v)$.*

Let M be a connected closed oriented smooth n-dimensional Riemannian manifold. Put $G = \mathrm{Spin}(n)$ or $G = \mathrm{SO}(n)$, according as to whether or not M is spin. If M is spin, fix a spin structure. Let P be the corresponding principal G-bundle, covering the oriented orthonormal frame bundle. Its topological isomorphism class is independent of the choice of Riemannian metric. Given the Riemannian metric, there is a canonical $\mathbb{R}^n$-valued 1-form θ on P, the soldering form.

With respect to the standard basis $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$, we write $\gamma^j = \gamma(e_j)$. We also take a basis $\{\sigma^{ab}\}_{a,b=1}^n$ for the representation of the Lie algebra $\mathfrak{g}$ on V, so that $\left(\sigma^{ab}\right)^* = -\sigma^{ab}$ and

$$[\sigma^{ab}, \sigma^{cd}] = \delta^{ad} \, \sigma^{bc} - \delta^{ac} \, \sigma^{bd} + \delta^{bc} \, \sigma^{ad} - \delta^{bd} \, \sigma^{ac}. \tag{2.1}$$

The G-equivariance of γ implies

$$[\gamma^a, \sigma^{bc}] = \delta^{ab} \gamma^c - \delta^{ac} \gamma^b. \tag{2.2}$$

Examples :
1. If $G = \mathrm{Spin}(n)$ and V is the spinor representation of G then $\sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]$.
2. If $G = \mathrm{SO}(n)$ and $V = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, let E^j and I^j denote exterior and interior multiplication by e^j, respectively. Put $\gamma^j = i\,(E^j - I^j)$ and $\widehat{\gamma}^j = E^j + I^j$. Then $\sigma^{ab} = \frac{1}{4} \left([\gamma^a, \gamma^b] + [\widehat{\gamma}^a, \widehat{\gamma}^b] \right)$.

Put $E^M = P \times_G V$. The Dirac-type operator D^M acts on the space $C^\infty(M; E^M)$. As the topological vector space $C^\infty(M; E^M)$ is independent of any choice of Riemannian metric on M, it makes sense to compare Dirac-type operators for different Riemannian metrics on M; see [16, Section 2] for further discussion.

Let g^{TM} be the Riemannian metric on M. Let ω be the Levi-Civita connection on P. Let $\{e_j\}_{j=1}^n$ be a local oriented orthonormal basis of TM, with dual basis $\{\tau^j\}_{j=1}^n$. Then we can write ω locally as a matrix-valued 1-form $\omega^a{}_b = \sum_{j=1}^n \omega^a{}_{bj} \tau^j$, and

$$D^M = -i \sum_{j=1}^n \gamma^j \nabla_{e_j} = -i \sum_{j=1}^n \gamma^j \left(e_j + \frac{1}{2} \sum_{a,b=1}^n \omega_{abj}\, \sigma^{ab} \right). \tag{2.3}$$

We have the Bochner-type equation

$$(D^M)^2 = \nabla^* \nabla - \frac{1}{8} \sum_{a,b,i,j=1}^n R^M_{abij} \, (\gamma^i \gamma^j - \gamma^j \gamma^i)\, \sigma^{ab}. \tag{2.4}$$

As the set of Riemannian metrics on M is an affine space modeled on a Fréchet space, it makes sense to talk about an analytic 1-parameter family $\{c(t)\}_{t \in [0,1]}$ of metrics. Then for $t \in [0,1]$, $\dot{c}(t)$ is a symmetric 2-tensor on M. Let $\parallel \dot{c}(t) \parallel_{c(t)}$ denote the norm of $\dot{c}(t)$ with respect to $c(t)$, i.e.

$$\parallel \dot{c}(t) \parallel_{c(t)} = \sup_{v \in TM - 0} \frac{|\dot{c}(t)(v,v)|}{c(t)(v,v)}. \tag{2.5}$$

Put $l(c) = \int_0^1 \parallel \dot{c}(t) \parallel_{c(t)} dt$. We extend the definition of $l(c)$ to piecewise-analytic families of metrics in the obvious way. Given $K > 0$, let $\mathcal{M}(M,K)$ be the set of Riemannian metrics on M with $\parallel R^M \parallel_\infty \leq K$. Let d be the corresponding length metric on $\mathcal{M}(M,K)$, computed using piecewise-analytic paths in $\mathcal{M}(M,K)$. Let $\sigma(D^M, g^{TM})$ denote the spectrum of D^M as computed with g^{TM}, a discrete subset of $\mathbb{R}$ which is counted with multiplicity.

Proposition 1. *There is a constant $C = C(n,V) > 0$ such that for all $K > 0$ and $g_1^{TM}, g_2^{TM} \in \mathcal{M}(M,K)$,*

$$\left\{ \sinh^{-1} \left(\frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, g_1^{TM}) \right\} \tag{2.6}$$

and

$$\left\{ \sinh^{-1} \left(\frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, g_2^{TM}) \right\} \tag{2.7}$$

are $C\, d(g_1^{TM}, g_2^{TM})$-close.

Proof. It is enough to show that there is a number C such that if $\{c(t)\}_{t\in[0,1]}$ is an analytic 1-parameter family of metrics contained in $\mathcal{M}(M,K)$ then $\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{K}}\right)\ :\ \lambda\in\sigma(D^M,c(0))\right\}$ and $\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{K}}\right)\ :\ \lambda\in\sigma(D^M,c(1))\right\}$ are $Cd(c(0),c(1))$-close. By eigenvalue perturbation theory [19, Chapter XII], the subset $\bigcup_{t\in[0,1]}\{t\}\times\sigma(D^M,c(t))$ of $\mathbb{R}^2$ is the union of the graphs of functions $\{\lambda_j(t)\}_{j\in\mathbb{Z}}$ which are analytic in t. Thus it is enough to show that for each $j\in\mathbb{Z}$,

$$\left|\sinh^{-1}\left(\frac{\lambda_j(1)}{\sqrt{K}}\right)-\sinh^{-1}\left(\frac{\lambda_j(0)}{\sqrt{K}}\right)\right|\ \le\ C\,l(c). \tag{2.8}$$

Let $D(t)$ denote the Dirac-type operator constructed with the metric $c(t)$. It is self-adjoint when acting on $L^2(E^M,d\mathrm{vol}(t))$. In order to have all of the operators $\{D(t)\}_{t\in[0,1]}$ acting on the same Hilbert space, define $f(t)\in C^\infty(M)$ by $f(t)=\frac{dvol(t)}{dvol(0)}$. Then the spectrum of $D(t)$, acting on $L^2(E^M,d\mathrm{vol}(t))$, is the same as the spectrum of the self-adjoint operator $f(t)^{1/2}\,D(t)\,f(t)^{-1/2}$ acting on $L^2(E^M,d\mathrm{vol}(0))$. One can now compute $\frac{d\lambda_j}{dt}$ using eigenvalue perturbation theory, as in [19, Chapter XII]. Let $\psi_j(t)$ be a smoothly-varying unit eigenvector whose eigenvalue is $\lambda_j(t)$. Define a quadratic form $T(t)$ on TM by

$$T(t)(X,Y)\ =\ \langle\psi_j,-i\,(\gamma(X)\nabla_Y\psi_j\ +\ \gamma(Y)\nabla_X\psi_j)\rangle\ +\ \langle-i\,(\gamma(X)\nabla_Y\psi_j\ +\ \gamma(Y)\nabla_X\psi_j),\psi_j\rangle. \tag{2.9}$$

Using the metric $c(t)$ to convert the symmetric tensors $\dot{c}(t)$ and $T(t)$ to self-adjoint sections of $\mathrm{End}(TM)$, one finds

$$\frac{d\lambda_j}{dt}\ =\ -\ \frac{1}{8}\int_M\mathrm{Tr}\left(\dot{c}(t)\,T(t)\right)\ d\mathrm{vol}(t). \tag{2.10}$$

(This equation was shown for the pure Dirac operator, by different means, in [10].) Then

$$\left|\frac{d\lambda_j}{dt}\right|\ \le\ \text{const.}\ \parallel\dot{c}(t)\parallel_{c(t)}\ \int_M\mathrm{Tr}(|T(t)|)\,d\mathrm{vol}(t). \tag{2.11}$$

Letting $\{x_i\}_{i=1}^n$ be an orthonormal basis of eigenvectors of $T(t)$ at a point $m\in M$, we have $\mathrm{Tr}(|T(t)|)\ =\ \sum_{i=1}^n|T(t)(x_i,x_i)|$. Then from (2.9), we obtain

$$\int_M\mathrm{Tr}(|T(t)|)\,d\mathrm{vol}(t)\ \le\ \text{const.}\left(\int_M|\nabla\psi_j|^2\,d\mathrm{vol}(t)\right)^{1/2}. \tag{2.12}$$

From (2.4),

$$\int_M|\nabla\psi_j|^2\,d\mathrm{vol}(t)\ \le\ \lambda_j^2\ +\ \text{const.}\,K. \tag{2.13}$$

In summary, from (2.11), (2.12) and (2.13), there is a positive constant C such that

$$\left|\frac{d\lambda_j}{dt}\right|\ \le\ C\ \parallel\dot{c}(t)\parallel_{c(t)}\ \left(\lambda_j^2\ +K\right)^{1/2}. \tag{2.14}$$

Integration gives equation (2.8). The proposition follows. □

For some basic facts about infranilmanifolds, we refer to [17, Section 3]. Let N be a simply-connected connected nilpotent Lie group. Let Γ be a discrete subgroup of $\mathrm{Aff}(N)$ which acts freely and cocompactly on N. Put $Z\ =\ \Gamma\backslash N$, an infranilmanifold. There is a canonical flat linear connection ∇^{aff} on TZ. Put $\widehat{\Gamma} = \Gamma \cap N$, a cocompact subgroup of N. There is a short exact sequence

$$1 \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow F \longrightarrow 1, \tag{2.15}$$

with F a finite group. Put $\widehat{Z} = \widehat{\Gamma} \backslash N$, a nilmanifold which finitely covers Z with covering group F.

Let g^{TZ} be a Riemannian metric on Z which is parallel with respect to ∇^{aff}. Let us discuss the condition for Z to be spin. Suppose first that Z is spin. Choose a spin structure on Z. Fix the basepoint $z_0 = \Gamma\, e \in Z$. As ∇^{aff} preserves g^{TZ}, its holonomy lies in $\mathrm{SO}(n)$. Hence ∇^{aff} lifts to a flat connection on the principal $\mathrm{Spin}(n)$-bundle, which we also denote by ∇^{aff}. There is a corresponding holonomy representation $\Gamma \rightarrow \mathrm{Spin}(n)$.

Conversely, suppose that we do not know *a priori* if Z is spin. Suppose that the affine holonomy $\Gamma \rightarrow F \rightarrow \mathrm{SO}(n)$ lifts to a homomorphism $\Gamma \rightarrow \mathrm{Spin}(n)$. Naturally, the existence of this lifting is independent of the particular choice of g^{TZ}. Then there is a corresponding spin structure on Z with principal bundle $\Gamma \backslash (N \times \mathrm{Spin}(n))$. The different spin structures on Z correspond to different lifts of $\Gamma \rightarrow \mathrm{SO}(n)$ to $\Gamma \rightarrow \mathrm{Spin}(n)$. These are labelled by $\mathrm{H}^1(\Gamma; \mathbb{Z}_2) \cong \mathrm{H}^1(Z; \mathbb{Z}_2)$. Note that there are examples of nonspin flat manifolds [3]. Also, even if Z is spin and has a fixed spin structure, the action of $\mathrm{Aff}(Z)$ on Z generally does not lift to the principal $\mathrm{Spin}(n)$-bundle, as can be seen for the $\mathrm{SL}(n, \mathbb{Z})$-action on $Z = T^n$.

Now let G be either $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$. Let V be a G-Clifford module. Suppose that Z has a G-structure. If $G = \mathrm{SO}(n)$ then we have the affine holonomy homomorphism $\rho : \Gamma \rightarrow \mathrm{SO}(n)$. If $G = \mathrm{Spin}(n)$ then we have a given lift of it to $\rho : \Gamma \rightarrow \mathrm{Spin}(n)$. In either case, there is an action of Γ on V coming from $\Gamma \stackrel{\rho}{\rightarrow} G \rightarrow \mathrm{Aut}(V)$. The vector bundle E^Z can now be written as $E^Z = \Gamma \backslash (N \times V)$. We see that the vector space of sections of E^Z which are parallel with respect to ∇^{aff} is isomorphic to V^Γ, the subspace of V which is fixed by the action of Γ.

If V is the spinor representation of $G = \mathrm{Spin}(n)$ then let us consider the conditions for V^Γ to be nonzero. First, as the restriction of $\rho : \Gamma \rightarrow \mathrm{Spin}(n)$ to $\widehat{\Gamma}$ maps $\widehat{\Gamma}$ to ± 1, we must have $\rho\big|_{\widehat{\Gamma}} = 1$. Given this, the homomorphism ρ factors through a homomorphism $F \rightarrow \mathrm{Spin}(n)$. Then we have $V^\Gamma = V^F$. This may be nonzero even if the homomorphism $F \rightarrow \mathrm{Spin}(n)$ is nontrivial.

Returning to the case of general V, as g^{TZ} is parallel with respect to ∇^{aff}, the operator D^Z preserves the space V^Γ of affine-parallel sections of E^Z. Let D^{inv} be the restriction of D^Z to V^Γ.

Proposition 2. *There are positive constants A, A' and C depending only on* $\dim(Z)$ *and V such that if $\parallel R^Z \parallel_\infty \mathrm{diam}(Z)^2 \leq A'$ then the spectrum $\sigma(D^Z)$ of D^Z satisfies*

$$\begin{aligned} &\sigma(D^Z) \cap [-\left(A\, \mathrm{diam}(Z)^{-2} - C \parallel R^Z \parallel_\infty\right)^{1/2}, \left(A\, \mathrm{diam}(Z)^{-2} - C \parallel R^Z \parallel_\infty\right)^{1/2}] = \\ &\sigma(D^{inv}) \cap [-\left(A\, \mathrm{diam}(Z)^{-2} - C \parallel R^Z \parallel_\infty\right)^{1/2}, \left(A\, \mathrm{diam}(Z)^{-2} - C \parallel R^Z \parallel_\infty\right)^{1/2}]. \end{aligned} \tag{2.16}$$

Proof. As D^Z is diagonal with respect to the orthogonal decomposition

$$C^\infty(Z; E^Z) = V^\Gamma \oplus \left(V^\Gamma\right)^\perp, \tag{2.17}$$

it is enough to show that there are constants A, A' and C as in the statement of the proposition such that the eigenvalues of $(D^Z)^2\big|_{(V^\Gamma)^\perp}$ are greater than $A\,\mathrm{diam}(Z)^{-2} - C \parallel R^Z \parallel_\infty$. As in the proof of [17, Proposition 2], we can reduce to the case when $F = \{e\}$, i.e. Z is a nilmanifold $\Gamma\backslash N$. Then

$$C^\infty(Z; E^Z) \cong (C^\infty(N) \otimes V)^\Gamma . \tag{2.18}$$

Using an orthonormal frame $\{e_i\}_{i=1}^{dim(Z)}$ for the Lie algebra $\mathfrak{n}$ as in the proof of [17, Proposition 2], we can write

$$\nabla^{aff}_{e_i} = e_i \otimes \mathrm{Id}. \tag{2.19}$$

and

$$\nabla^Z_{e_i} = (e_i \otimes \mathrm{Id.}) + \left(\mathrm{Id.} \otimes \frac{1}{2} \sum_{a,b=1}^{dim(Z)} \omega_{abi}\, \sigma^{ab}\right). \tag{2.20}$$

The rest of the proof now proceeds as in that of [17, Proposition 2], to which we refer for details. ☐

3. Collapsing to a Smooth Base

For background information about superconnections and their applications, we refer to [7]. Let M be a connected closed oriented Riemannian manifold which is the total space of a Riemannian submersion $\pi : M \to B$. Suppose that M has a G^M-structure and that V^M is a G^M-Clifford module, as in Section 2. If $G^M = \mathrm{SO}(n)$, put $G^Z = \mathrm{SO}(\dim(Z))$ and $G^B = \mathrm{SO}(\dim(B))$. If $G^M = \mathrm{Spin}(n)$, put $G^Z = \mathrm{Spin}(\dim(Z))$ and $G^B = \mathrm{Spin}(\dim(B))$. As a fiber Z_b has a trivial normal bundle in M, it admits a G^Z-structure. Fixing an orientation of T_bB fixes the G^Z-structure of Z_b. Note, however, that B does not necessarily have a G^B-structure. For example, if M is oriented then B is not necessarily oriented, as is shown in the example of $S^1 \times_{\mathbb{Z}_2} S^2 \to \mathbb{R}P^2$, where the generator of $\mathbb{Z}_2$ acts on S^1 by complex conjugation and on S^2 by the antipodal map. And if M is spin then B is not necessarily spin, as is shown in the example of $S^5 \to \mathbb{C}P^2$. What is true is that if the vertical tangent bundle TZ, a vector bundle on M, has a G^Z-structure then B has a G^B-structure.

Put $E^M = P \times_{G^M} V^M$. There is a Clifford bundle C on B with the property that $C^\infty(B; C) \cong C^\infty(M; E^M)$ [7, Section 9.2]. If $\dim(Z) > 0$ then $\dim(C) = \infty$. To describe C more explicitly, let $V^M = \bigoplus_{l\in L} V_l^B \otimes V_l^Z$ be the decomposition of V_M into irreducible representations of $G^B \times G^Z \subset G^M$.

Examples :
1. If $G^M = \mathrm{Spin}(n)$ and V^M is the spinor representation then V^B and V^Z are spinor representations.
2. If $G^M = \mathrm{SO}(n)$ and $V^M = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ then $V^B = \Lambda^*(\mathbb{R}^{dim(B)}) \otimes_{\mathbb{R}} \mathbb{C}$ and $V^Z = \Lambda^*(\mathbb{R}^{dim(Z)}) \otimes_{\mathbb{R}} \mathbb{C}$.

Let U be a contractible open subset of B. Choose an orientation on U. For $b \in U$, let $E^Z_{b,l}$ be the vector bundle on Z_b associated to the pair (G^Z, V_l^Z). Then $E^M\big|_{Z_b} \cong \bigoplus_{l\in L} V_l^B \otimes E^Z_{b,l}$. The vector bundles $\{E^Z_{b,l}\}_{b\in U}$ are the fiberwise restrictions of a vector bundle E^Z_l on $\pi^{-1}(U)$, a vertical "spinor" bundle. There is a pushforward vector bundle W_l on U whose fiber $W_{l,b}$ over $b \in U$ is $C^\infty(Z_b; E^Z_{b,l})$. If $\dim(Z) > 0$ then $\dim(W_l) = \infty$. There are Hermitian inner products $\{h^{W_l}\}_{l \in L}$ on $\{W_l\}_{l \in L}$ induced from the vertical Riemannian metric g^{TZ}. Furthermore, there are Clifford bundles $\{C_l\}_{l \in L}$ on U for which the fiber $C_{l,b}$ of C_l over $b \in U$ is isomorphic to $V^B_l \otimes W_{l,b}$. By construction, $C^\infty\left(Z_b; E^M\big|_{Z_b}\right) \cong \bigoplus_{l \in L} C_{l,b}$. The Clifford bundles $\{C_l\}_{l \in L}$ exist globally on B and $C = \bigoplus_{l \in L} C_l$. The Dirac-type operator D^M decomposes as $D^M = \bigoplus_{l \in L} D^M_l$, where D^M_l acts on $C^\infty(B; C_l)$.

In order to write D^M_l explicitly, let us recall the Bismut superconnection on W_l. We will deal with each $l \in L$ separately and so we drop the subscript l for the moment. We use the notation of [9, Section III(c)] to describe the local geometry of the fiber bundle $M \to B$, and the Einstein summation convention. Let ∇^{TZ} denote the Bismut connection on TZ [7, Proposition 10.2], which we extend to a connection on E^Z_l. The Bismut superconnection on W [7, Proposition 10.15] is of the form

$$A = D^W + \nabla^W - \frac{1}{4} c(T). \tag{3.1}$$

Here D^W is the fiberwise Dirac-type operator and has the form

$$D^W = - i \gamma^j \nabla^{TZ}_{e_j} = - i \gamma^j \left(e_j + \frac{1}{2} \omega_{pqj} \sigma^{pq} \right). \tag{3.2}$$

Next, ∇^W is a Hermitian connection on W given by

$$\nabla^W = \tau^\alpha \left(\nabla^{TZ}_{e_\alpha} - \frac{1}{2} \omega_{\alpha jj} \right) = \tau^\alpha \left(e_\alpha + \frac{1}{2} \omega_{jk\alpha} \sigma^{jk} - \frac{1}{2} \omega_{\alpha jj} \right). \tag{3.3}$$

Finally,

$$c(T) = i \omega_{\alpha\beta j} \gamma^j \tau^\alpha \tau^\beta. \tag{3.4}$$

The superconnection A can be "quantized" into an operator D^A on $C^\infty(B; V^B \otimes W)$. Explicitly,

$$\begin{aligned} D^A = & - i \gamma^j \left(e_j + \frac{1}{2} \omega_{pqj} \sigma^{pq} \right) \\ & - i \gamma^\alpha \left(e_\alpha + \frac{1}{2} \omega_{\beta\gamma\alpha} \sigma^{\beta\gamma} + \frac{1}{2} \omega_{jk\alpha} \sigma^{jk} - \frac{1}{2} \omega_{\alpha jj} \right) \\ & + i \frac{1}{2} \omega_{\alpha\beta j} \gamma^j \sigma^{\alpha\beta}. \end{aligned} \tag{3.5}$$

Let $\mathcal{V} \in \mathrm{End}(C_l)$ be the self-adjoint operator given by

$$\mathcal{V} = - i \left(\omega_{\alpha jk} \gamma^k \sigma^{\alpha j} + \frac{1}{2} \omega_{\alpha jj} \gamma^\alpha + \omega_{\alpha\beta j} \left(\gamma^j \sigma^{\alpha\beta} + \gamma^\alpha \sigma^{j\beta} \right) \right). \tag{3.6}$$

Then restoring the index l everywhere,

$$D^M_l = D^{A_l} + \mathcal{V}_l. \tag{3.7}$$

Examples :
1. If $G^M = \mathrm{Spin}(n)$ and V^M is the spinor representation then $\mathcal{V} = 0$.

2. If $G^M = \mathrm{SO}(n)$ and $V^M = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ then

$$\mathcal{V} = -\frac{1}{4} i \left(\omega_{\alpha jk} \gamma^k [\widehat{\gamma}^\alpha, \widehat{\gamma}^j] + \omega_{\alpha\beta j}(\gamma^j [\widehat{\gamma}^\alpha, \widehat{\gamma}^\beta] + \gamma^\alpha [\widehat{\gamma}^j, \widehat{\gamma}^\beta])\right). \tag{3.8}$$

Now suppose that $\pi : M \to B$ is a Riemannian affine fiber bundle. Then $E^M\big|_{Z_b}$ inherits a flat connection from the flat affine connections on $\{E^Z_{b,l}\}_{l\in L}$. Let E^B be the Clifford bundle on B whose fiber over $b \in B$ is the space of parallel sections of $E^M\big|_{Z_b}$. Then D^M restricts to a first-order differential operator D^B on $C^\infty(B; E^B)$.

Given $b \in U$ and $l \in L$, let $W^{inv}_{l,b}$ be the finite-dimensional subspace of $W_{l,b}$ consisting of affine-parallel elements of $C^\infty(Z_b; E^Z_{b,l})$. From the discussion in Section 2, $W^{inv}_{l,b}$ is isomorphic to $\left(V^Z_l\right)^\Gamma$. The vector spaces $W^{inv}_{l,b}$ fit together to form a finite-dimensional subbundle W^{inv}_l of W_l. There is a corresponding finite-dimensional Clifford subbundle C^{inv}_l of C_l whose fiber over $b \in U$ is isomorphic to $V^B_l \otimes W^{inv}_{l,b}$. Again, C^{inv}_l exists globally on B. Then $E^B = \bigoplus_{l\in L} C^{inv}_l$. Let D^B_l be the restriction of D^M_l to $C^\infty(B; C^{inv}_l)$. Then

$$D^B = \bigoplus_{l\in L} D^B_l. \tag{3.9}$$

The superconnection A_l restricts to an superconnection A^{inv}_l on W^{inv}_l, the endomorphism $\mathcal{V}_l$ restricts to an endomorphism of C^{inv}_l and D^M_l restricts to the first-order differential operator

$$D^B_l = D^{A^{inv}_l} + \mathcal{V}^{inv}_l \tag{3.10}$$

on $C^\infty(B; C^{inv}_l)$.

Proof of Theorem 1 :
The operator D^M_l is diagonal with respect to the orthogonal decomposition

$$C_l = C^{inv}_l \oplus \left(C^{inv}_l\right)^\perp. \tag{3.11}$$

Thus it suffices to show that there are constants A, A' and C such that the spectrum of $\sigma(D^M_l)$, when restricted to $(C^{inv}_l)^\perp$, is disjoint from (1.1).

For simplicity, we drop the subscript l. Given $\eta \in C^\infty\left(B; (C^{inv})^\perp\right) \subset C^\infty(M; E^M)$, it is enough to show that for suitable constants,

$$\langle D^M\eta, D^M\eta\rangle \geq \left(\text{const.}\,\mathrm{diam}(Z)^{-2} - \text{const.}\,\left(\| R^M \|_\infty + \| \Pi \|^2_\infty + \| T \|^2_\infty\right)\right) \langle \eta, \eta\rangle. \tag{3.12}$$

Using (2.4), it is enough to show that

$$\langle \nabla^M\eta, \nabla^M\eta\rangle \geq \left(\text{const.}\,\mathrm{diam}(Z)^{-2} - \text{const.}\,\left(\| R^M \|_\infty + \| \Pi \|^2_\infty + \| T \|^2_\infty\right)\right) \langle \eta, \eta\rangle. \tag{3.13}$$

We can write $\nabla^M = \nabla^V + \nabla^H$, where

$$\nabla^V : C^\infty(M; E^M) \to C^\infty(M; T^*Z \otimes E^M) \tag{3.14}$$

denotes covariant differentiation in the vertical direction and

$$\nabla^H : C^\infty(M; E^M) \to C^\infty(M; \pi^*T^*B \otimes E^M) \tag{3.15}$$

denotes covariant differentiation in the horizontal direction. Then

$$\begin{aligned}\langle \nabla^M \eta, \nabla^M \eta \rangle &= \langle \nabla^V \eta, \nabla^V \eta \rangle + \langle \nabla^H \eta, \nabla^H \eta \rangle \\ &\geq \langle \nabla^V \eta, \nabla^V \eta \rangle \\ &= \int_B \int_{Z_b} \left| \nabla^V \eta \right|^2 (z) \, d\mathrm{vol}_{Z_b} \, d\mathrm{vol}_B. \end{aligned} \tag{3.16}$$

On a given fiber Z_b, we have

$$E^M \Big|_{Z_b} \cong V^B \otimes E^Z_b. \tag{3.17}$$

Hence we can also use the Bismut connection ∇^{TZ} to vertically differentiate sections of E^M. That is, we can define

$$\nabla^{TZ} : C^\infty(M; E^M) \to C^\infty(M; T^*Z \otimes E^M). \tag{3.18}$$

Explicitly, with respect to a local framing,

$$\nabla^{TZ}_{e_j} = e_j \, \eta + \frac{1}{2} \, \omega_{pqj} \, \sigma^{pq} \, \eta \tag{3.19}$$

and

$$\nabla^V_{e_j} = e_j \, \eta + \frac{1}{2} \, \omega_{pqj} \, \sigma^{pq} \, \eta + \omega_{\alpha k j} \, \sigma^{\alpha k} \, \eta + \frac{1}{2} \, \omega_{\alpha \beta j} \, \sigma^{\alpha \beta} \, \eta. \tag{3.20}$$

Then from (3.16), (3.19) and (3.20),

$$\langle \nabla^M \eta, \nabla^M \eta \rangle \geq \int_B \left[\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) - \text{const.} \left(\parallel T_b \parallel^2 + \parallel \Pi_b \parallel^2 \right) \left| \eta(z) \right|^2 \right] d\mathrm{vol}_{Z_b} \, d\mathrm{vol}_B. \tag{3.21}$$

Thus it suffices to bound $\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) \, d\mathrm{vol}_{Z_b}$ from below on a given fiber Z_b in terms of $\langle \eta, \eta \rangle_{Z_b}$, under the assumption that $\eta \in (W_b^{inv})^\perp$. Using the Gauss-Codazzi equation, we can estimate $\parallel R^{Z_b} \parallel_\infty$ in terms of $\parallel R^M \parallel_\infty$ and $\parallel \Pi \parallel_\infty^2$. Then the desired bound on $\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) \, d\mathrm{vol}_{Z_b}$ follows from Proposition 2. □

Proof of Theorem 2 :
Let g_0^{TM} denote the Riemannian metric on M. From Proposition 1, if a Riemannian metric g_1^{TM} on M is close to g_0^{TM} in $(\mathcal{M}(M, 2K), d)$ then applying the function $x \to \sinh^{-1}\left(\frac{x}{\sqrt{2K}}\right)$ to $\sigma(D^M, g_0^{TM})$ gives a collection of numbers which is close to that obtained by applying $x \to \sinh^{-1}\left(\frac{x}{\sqrt{2K}}\right)$ to $\sigma(D^M, g_1^{TM})$. We will use the geometric results of [11] to find a metric g_2^{TM} on M which is close to g_0^{TM} and to which we can apply Theorem 1.

First, as in [11, (2.4.1)], by the smoothing results of Abresch and others [11, Theorem 1.12], for any $\epsilon > 0$ we can find metrics on M and B which are ϵ-close in the C^1-topology to the original metrics such that the new metrics satisfy $\parallel \nabla^i R \parallel_\infty \leq A_i(n, \epsilon)$ for some appropriate sequence $\{A_i(n, \epsilon)\}_{i=0}^\infty$. Let g_1^{TM} denote the new metric on M. In the proof of the smoothing result, such as using the Ricci flow [20, Proposition 2.5], one obtains an explicit smooth 1-parameter family of metrics on M in $\mathcal{M}(M, K')$, for some $K' > K$, going from g_0^{TM} to g_1^{TM}. We can approximate this family by a piecewise-analytic family. Hence one obtains an upper bound on $d\left(g_0^{TM}, g_1^{TM}\right)$ in $\mathcal{M}(M, K')$, for some $K' > K$, which depends

on K and is proportionate to ϵ. (Note that d is essentially the same as the C^0-metric on $\mathcal{M}(M, K')$.) By rescaling, we may assume that $\| R^M \|_\infty \leq 1$, $\| R^B \|_\infty \leq 1$ and $\mathrm{inj}(B) \geq 1$. We now apply [11, Theorem 2.6], with B fixed. It implies that there are positive constants $\lambda(n)$ and $c(n, \epsilon)$ so that if $d_{GH}(M, B) \leq \lambda(n)$ then there is a fibration $f : M \to B$ such that
1. $\mathrm{diam}\,(f^{-1}(b)) \leq c(n, \epsilon)\, d_{GH}(M, B)$.
2. f is a $c(n, \epsilon)$-almost Riemannian submersion.
3. $\| \Pi_{f^{-1}(b)} \|_\infty \leq c(n, \epsilon)$.
As in [15], the Gauss-Codazzi equation, the curvature bound on M and the second fundamental form bound on $f^{-1}(b)$ imply a uniform bound on $\left\{ \| R^{f^{-1}(b)} \|_\infty \right\}_{b \in B}$. Along with the diameter bound on $f^{-1}(b)$, this implies that if $d_{GH}(M, B)$ is sufficiently small then $f^{-1}(b)$ is almost flat.

From [11, Propositions 3.6 and 4.9], we can find another metric g_2^{TM} on M which is ϵ-close to g_1^{TM} in the C^1-topology so that the fibration $f : M \to B$ gives M the structure of a Riemannian affine fiber bundle. Furthermore, by [11, Proposition 4.9], there is a sequence $\{A_i'(n, \epsilon)\}_{i=0}^\infty$ so that we may assume that g_1^{TM} and g_2^{TM} are close in the sense that

$$\| \nabla^i \left(g_1^{TM} - g_2^{TM} \right) \|_\infty \leq A_i'(n, \epsilon)\, d_{GH}(M, B), \tag{3.22}$$

where the covariant derivative in (3.22) is that of the Levi-Civita connection of g_2^{TM}. Then we can interpolate linearly between g_1^{TM} and g_2^{TM} within $\mathcal{M}(M, K'')$ for some $K'' > K'$, and obtain an upper bound on $d\left(g_1^{TM}, g_2^{TM}\right)$ in $\mathcal{M}(M, K'')$ which is proportionate to ϵ. From [20, Theorem 2.1], we can take $K'' = 2K$ (or any number greater than K).

We now apply Theorem 1 to the Riemannian affine fiber bundle with metric g_2^{TM}. It remains to estimate the geometric terms appearing in (1.1). We have an estimate on $\| \Pi \|_\infty$ as above. Applying O'Neill's formula [8, (9.29)] to the Riemannian affine fiber bundle, we can estimate $\| T \|_\infty^2$ in terms of $\| R^M \|_\infty$ and $\| R^B \|_\infty$. Putting this together, the theorem follows. □

4. Collapsing to a Singular Base

Let $\mathfrak{p} : P \to M$ be the principal G-bundle of Section 2. Let $\{\mathfrak{Y}_j\}_{j=1}^n$ be the horizontal vector fields on P such that $\theta(\mathfrak{Y}_j) = e_j$. Put $D^P = -i \sum_{j=1}^n \gamma^j \mathfrak{Y}_j$, acting on $C^\infty(P) \otimes V$.

There is an isomorphism $C^\infty(M; E^M) \cong \left(C^\infty(P) \otimes V\right)^G$. Under this isomorphism, $D^M \cong D^P\big|_{(C^\infty(P)\otimes V)^G}$. The Bochner-type equation (2.4) becomes

$$(D^M)^2 \cong -\sum_{j=1}^n \mathfrak{Y}_j^2 - \frac{1}{8} \sum_{a,b,i,j=1}^n (\mathfrak{p}^* R^M)_{abij} \left(\gamma^i \gamma^j - \gamma^j \gamma^i\right) \sigma^{ab} \tag{4.1}$$

when acting on $\left(C^\infty(P) \otimes V\right)^G$.

Let $\{x_a\}_{a=1}^{\dim(G)}$ be a basis for the Lie algebra $\mathfrak{g}$ which is orthonormal with respect to the negative of the Killing form. Let $\{\mathfrak{Y}_a\}_{a=1}^{\dim(G)}$ be the corresponding vector fields on P. Then $-\sum_{a=1}^{\dim(G)} \mathfrak{Y}_a^2$ acts on $\left(C^\infty(P) \otimes V\right)^G$ as $c_V \in \mathbb{R}$, the Casimir of the G-module V. Give P the Riemannian metric g^{TP} with the property that $\{\mathfrak{Y}_j, \mathfrak{Y}_a\}$ forms an orthonormal basis of vector fields. Let $\triangle^P$ denote the corresponding (nonnegative) scalar Laplacian on P, extended to act on $C^\infty(P) \otimes V$. Then when acting on $\left(C^\infty(P) \otimes V\right)^G$, equation (4.1) is equivalent to

$$(D^M)^2 \;\cong\; \triangle^P \;-\; \frac{1}{8} \sum_{a,b,i,j=1}^{n} (\mathfrak{p}^* R^M)_{abij} \, (\gamma^i \, \gamma^j \;-\; \gamma^j \, \gamma^i) \, \sigma^{ab} \;-\; c_V \, \mathrm{Id}. \tag{4.2}$$

Definition 3. *A G-equivariant Riemannian affine fiber bundle structure on P consists of a Riemannian affine fiber bundle structure $\check{\pi} : P \to \check{X}$ which is G-equivariant.*

Given a G-equivariant Riemannian affine fiber bundle, let $\check{Z}$ be the fiber of $\check{\pi} : P \to \check{X}$, an infranilmanifold. For collapsing purposes it suffices to take $\check{Z}$ to be a nilmanifold $\Gamma \backslash N$ [11, (7.2)]. We assume hereafter that this is the case. Put $X = \check{X}/G$, a possibly singular space. As N acts isometrically in a neighborhood of a given fiber $\check{Z}$ and preserves the horizontal subspaces of $P \to M$, it follows that the vector fields $\{\mathfrak{Y}_j\}_{j=1}^n$ are projectable with respect to $\check{\pi}$ and push forward to vector fields $\{\mathcal{X}_j\}_{j=1}^n$ on $\check{X}$. Put $D^{\check{X}} \;=\; -\, i \, \sum_{j=1}^n \gamma^j \, \mathcal{X}_j$, acting on $C^\infty(\check{X}) \otimes V$. Let $v \in C^\infty(\check{X})$ be given by $v(\check{x}) \;=\; \mathrm{vol}(\check{Z}_{\check{x}})$. We give $C^\infty(\check{X}) \otimes V$ the weighted L^2-inner product with respect to the weight function v.

We recall that there is a notion of a pseudodifferential operator being transversally elliptic with respect to the action of a Lie group G [2, Definition 1.3].

Lemma 1. *$D^{\check{X}}$ is transversally elliptic on $\check{X}$.*

Proof. Let $s(D^{\check{X}}) \in C^\infty(T^*\check{X}) \otimes \mathrm{End}(V)$ denote the symbol of $D^{\check{X}}$. Suppose that $\xi \in T^*_{\check{x}}\check{X}$ satisfies $\xi(\check{v}) \;=\; 0$ for all $\check{v} \in T_{\check{x}}\check{X}$ which lie in the image of the representation of $\mathfrak{g}$ by vector fields on $\check{X}$. Then if $p \in \check{\pi}^{-1}(\check{x})$, we have that $(\check{\pi}^*\xi)(r) \;=\; 0$ for all $r \in T_pP$ which lie in the image of the representation of $\mathfrak{g}$ by vector fields on P. In other words, $\check{\pi}^*\xi$ is horizontal. Suppose in addition that $s(D^{\check{X}})(\xi) \;=\; 0$. Then $s(D^P)(\check{\pi}^*\xi) \;=\; 0$. As D^P is horizontally elliptic, it follows that $\check{\pi}^*\xi \;=\; 0$. Thus $\xi \;=\; 0$, which proves the lemma. □

Definition 4. *For notation, write $C^\infty(X; E^X) \;=\; \left(C^\infty(\check{X}) \otimes V\right)^G$. Let D^X be the restriction of $D^{\check{X}}$ to $C^\infty(X; E^X)$.*

It will follow from the proof of the next theorem that D^X is self-adjoint on the Hilbert space completion of $C^\infty(X; E^X)$ with respect to the (weighted) inner product. As $D^{\check{X}}$ is transversally elliptic, it follows that D^X has a discrete spectrum [2, Proof of Theorem 2.2].

Let $\check{\Pi}$ denote the second fundamental forms of the fibers $\{\check{Z}_{\check{x}}\}_{\check{x} \in \check{X}}$. Let $\check{T} \in \Omega^2(P; T\check{Z})$ be the curvature of the horizontal distribution on the affine fiber bundle $P \to \check{X}$.

Theorem 6. *There are positive constants A, A' and C which only depend on n and V such that if $\parallel R^{\check{Z}} \parallel_\infty \, \mathrm{diam}(\check{Z})^2 \;\le\; A'$ then the intersection of $\sigma(D^M)$ with*

$$\begin{aligned} [-\Big(A \, \mathrm{diam}(\check{Z})^{-2} -\; C \, \Big(1 + \; \parallel R^M \parallel_\infty \; + \parallel \check{\Pi} \parallel_\infty{}^2 \; + \; \parallel \check{T} \parallel_\infty{}^2\Big)\Big)^{1/2}, \\ \Big(A \, \mathrm{diam}(\check{Z})^{-2} \;-\; C \, \Big(1 + \; \parallel R^M \parallel_\infty \; + \parallel \check{\Pi} \parallel_\infty{}^2 \; + \; \parallel \check{T} \parallel_\infty{}^2\Big)\Big)^{1/2}] \end{aligned} \tag{4.3}$$

equals the intersection of $\sigma(D^X)$ with (4.3).

Proof. Let us write

$$C^\infty(P) \otimes V \;=\; \left(C^\infty(\check{X}) \otimes V\right) \oplus \left(C^\infty(\check{X}) \otimes V\right)^\perp, \tag{4.4}$$

where we think of $C^\infty(\check{X}) \otimes V$ as the elements of $C^\infty(P) \otimes V$ which are constant along the fibers of the fiber bundle $\check{\pi} : P \to \check{X}$. Taking G-invariant subspaces, we have an orthogonal decomposition

$$C^\infty(M; E^M) \; = \; C^\infty(X; E^X) \oplus \left(C^\infty(X; E^X)\right)^\perp, \tag{4.5}$$

with respect to which D^M decomposes as

$$D^M \; = \; D^X \oplus D^M\big|_{(C^\infty(X;E^X))^\perp}. \tag{4.6}$$

As in the proof of Theorem 1, it suffices to obtain a lower bound on the spectrum of $(D^M)^2\big|_{(C^\infty(X;E^X))^\perp}$. As $\left(C^\infty(X; E^X)\right)^\perp \subset \left(C^\infty(\check{X}) \otimes V\right)^\perp$, using (4.2) it suffices to obtain a lower bound on the spectrum of $\triangle^P\big|_{\left(C^\infty(\check{X})\otimes V\right)^\perp}$. This follows from the arguments of the proof of Theorem 1, using the fact that $\parallel R^P \parallel_\infty \leq$ const. $\left(1 + \parallel R^M \parallel_\infty\right)$. We omit the details. In fact, it is somewhat easier than the proof of Theorem 1, since we are now only dealing with the scalar Laplacian and so can replace Proposition 2 by standard eigenvalue estimates (which just involve a lower Ricci curvature bound); see [6] and references therein. □

Proof of Theorem 3 :
Everything in the proof will be done in a G-equivariant way, so we may omit to mention this explicitly. Let P_i be the principal G-bundle of M_i, equipped with a Riemannian metric as in the beginning of the section. From the G-equivariant version of Gromov's compactness theorem, we obtain a subsequence $\{P_i\}_{i=1}^\infty$ which converges in the equivariant Gromov-Hausdorff topology to a G-Riemannian manifold $\left(\check{X}, g^{T\check{X}}\right)$ with a $C^{1,\alpha}$-regular metric. As in [13, Section 3], the measure $\chi\, d\mathrm{vol}_{\check{X}}$ is a weak-$*$ limit point of the pushforwards of the normalized Riemannian measures on $\{P_i\}_{i=1}^\infty$. As in [13, p. 535], after smoothing we may assume that we have G-equivariant Riemannian affine fiber bundles $\check{\pi}_i : P_i' \to \check{X}_i$, with G acting freely on P_i', along with G-diffeomorphisms $\check{\phi}_i \; : \; P_i \to P_i'$ and $\Phi_i \; : \; \check{X} \to \check{X}_i$. Put $M_i' = P_i'/G$. Then $\check{\phi}_i$ descends to a diffeomorphism $\phi_i : M_i \to M_i'$ and we may also assume, as in the proof of Theorem 2, that
1. $\phi_i^* g^{TM_i'} \in \mathcal{M}(M_i, \text{const.}\, K)$,
2. $d(\phi_i^* g^{TM_i'}, g^{TM_i}) \leq 2^{-i}$ in $\mathcal{M}(M_i, \text{const.}\, K)$ and
3. $\lim_{i\to\infty} \Phi_i^* g^{T\check{X}_i} = g^{T\check{X}}$ in the $C^{1,\alpha}$-topology.

Using Proposition 1, we can effectively replace M_i by M_i' for the purposes of the argument. For simplicity, we relabel M_i' as M_i and P_i' as P_i. For the purposes of the limiting argument, using Theorem 6 and (4.2), we may replace the spectrum of $|D^{M_i}|$ by the spectrum of the operator $|D^{X_i}| \equiv \sqrt{\triangle^{\check{X}_i} + \mathcal{V}_i}$ acting on $C^\infty(X_i, E^{X_i}) = \left(C^\infty(\check{X}_i) \otimes V\right)^G$, where $\mathcal{V}_i$ is the restriction of

$$- \frac{1}{8} \sum_{a,b,i,j=1}^{n} (\mathfrak{p}^* R^{M_i})_{abij} \left(\gamma^i \gamma^j - \gamma^j \gamma^i\right) \sigma^{ab} - c_V \,\mathrm{Id}. \tag{4.7}$$

to the elements of $\left(C^\infty(P_i) \otimes V\right)^G$ which are constant along the fibers of $\check{\pi}_i : P_i \to \check{X}_i$, i.e. to $C^\infty(X_i, E^{X_i})$.

From the curvature bound, we have a uniform bound on $\{\| \mathcal{V}_i \|_\infty\}_{i=1}^\infty$. Using the weak-$*$ compactness of the unit ball, let $\mathcal{V}$ be a weak-$*$ limit point of $\{\Phi_i^* \mathcal{V}_i\}_{i=1}^\infty$ in $L^\infty(\check{X}) \otimes \mathrm{End}(V) = \left(L^1(\check{X}) \otimes \mathrm{End}(V)\right)^*$. We claim that with this choice of $\check{X}$, χ and $\mathcal{V}$, equation (1.2) holds.

To see this, we use the minimax characterization of eigenvalues as in [13, Section 5]. Using the diffeomorphisms $\{\Phi_i\}_{i=1}^\infty$, we identify each $\check{X}_i$ with $\check{X}$. We denote by $\langle \cdot, \cdot \rangle_{X_i}$ an L^2-inner product constructed using $\Phi_i^* g^{T\check{X}_i}$ and the weight function $(\check{\pi}_i)_*(d\mathrm{vol}_{P_i}) / \int_{\check{X}_i} (\check{\pi}_i)_*(d\mathrm{vol}_{P_i})$. We denote by $\langle \cdot, \cdot \rangle_X$ an L^2-inner product constructed using $g^{T\check{X}}$ and the weight function $\chi \, d\mathrm{vol}_{\check{X}}$. As $\triangle^{\check{X}}$ has a compact resolvent, it follows that $|D^X|^2$ has a compact resolvent. Then

$$\lambda_k(|D^X|)^2 \;=\; \inf_W \sup_{\psi \in W - 0} \frac{\langle d\psi, d\psi \rangle_X \;+\; \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X}, \tag{4.8}$$

where W ranges over the k-dimensional subspaces of the Sobolev space $H^1(X; E^X)$. Given $\epsilon > 0$, let W_∞ be a k-dimensional subspace such that

$$\sup_{\psi \in W_\infty - 0} \frac{\langle d\psi, d\psi \rangle_X \;+\; \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \;\le\; \lambda_k(|D^X|)^2 \;+ \epsilon. \tag{4.9}$$

As $\psi \otimes \psi^*$ lies in the finite-dimensional subspace $W_\infty \otimes W_\infty^*$ of $L^1(\check{X}) \otimes \mathrm{End}(V)$, it follows that

$$\lim_{i \to \infty} \langle \psi, \mathcal{V}_i \psi \rangle_X \;=\; \langle \psi, \mathcal{V}\psi \rangle_X \tag{4.10}$$

uniformly on $\{\psi \in W_\infty \,:\, \langle \psi, \psi \rangle_X \;=\; 1\}$. Then

$$\lim_{i \to \infty} \sup_{\psi \in W_\infty - 0} \frac{\langle d\psi, d\psi \rangle_{X_i} \;+\; \langle \psi, \mathcal{V}_i\psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}} \;=\; \sup_{\psi \in W_\infty - 0} \frac{\langle d\psi, d\psi \rangle_X \;+\; \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X}. \tag{4.11}$$

As

$$\lambda_k(|D^{X_i}|)^2 \;=\; \inf_W \sup_{\psi \in W - 0} \frac{\langle d\psi, d\psi \rangle_{X_i} \;+\; \langle \psi, \mathcal{V}_i\psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}}, \tag{4.12}$$

it follows that

$$\limsup_{i \to \infty} \lambda_k(|D^{X_i}|) \;\le\; \lambda_k(|D^X|). \tag{4.13}$$

We now show that

$$\liminf_{i \to \infty} \lambda_k(|D^{X_i}|) \;\ge\; \lambda_k(|D^X|). \tag{4.14}$$

Along with (4.13), this will prove the theorem. Suppose that (4.14) is not true. Then there is some $\epsilon > 0$ and some infinite subsequence of $\{M_i\}_{i=1}^\infty$, which we relabel as $\{M_i\}_{i=1}^\infty$, such that for all $i \in \mathbb{Z}^+$,

$$\lambda_k(|D^{X_i}|)^2 \;\le\; \lambda_k(|D^X|)^2 \;-\; 2\,\epsilon. \tag{4.15}$$

For each $i \in \mathbb{Z}^+$, let W_i be a k-dimensional subspace of $H^1(X; E^X)$ such that

$$\sup_{\psi \in W_i - 0} \frac{\langle d\psi, d\psi \rangle_{X_i} \;+\; \langle \psi, \mathcal{V}_i\psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}} \;\le\; \lambda_k(|D^{X_i}|)^2 \;+\; \epsilon. \tag{4.16}$$

Let $\{f_{i,j}\}_{j=1}^k$ be a basis for W_i which is orthonormal with respect to $\langle \cdot, \cdot \rangle_X$. Then for a given j, the sequence $\{f_{i,j}\}_{i=1}^\infty$ is bounded in $H^1(X; E^X)$. After taking a subsequence, which we relabel as $\{f_{i,j}\}_{i=1}^\infty$, we can assume that $\{f_{i,j}\}_{i=1}^\infty$ converges weakly in $H^1(X; E^X)$ to some $f_{\infty,j}$. Doing this successively for $j \in \{1, \ldots, k\}$, we can assume that for each j, $\lim_{i \to \infty} f_{i,j} = f_{\infty,j}$ weakly in $H^1(X; E^X)$. Then from the compactness of the embedding $H^1(X; E^X) \to L^2(X; E^X)$, we have strong convergence in $L^2(X; E^X)$. In particular, $\{f_{\infty,j}\}_{j=1}^k$ are orthonormal. Put $W_\infty = \mathrm{span}(f_{\infty,1}, \ldots, f_{\infty,k})$.

If $w_\infty = \sum_{j=1}^k c_j f_{\infty,j}$ is a nonzero element of W_∞, put $w_i = \sum_{j=1}^k c_j f_{i,j}$. Then $\{w_i\}_{i=1}^\infty$ converges weakly to w_∞ in $H^1(X; E^X)$ and hence converges strongly to w_∞ in $L^2(X; E^X)$. From a general result about weak limits, we have

$$\langle w_\infty, w_\infty \rangle_{H^1} \leq \limsup_{i \to \infty} \langle w_i, w_i \rangle_{H^1}. \tag{4.17}$$

Along with the L^2-convergence of $\{w_i\}_{i=1}^\infty$ to w_∞, this implies that

$$\langle dw_\infty, dw_\infty \rangle_X \leq \limsup_{i \to \infty} \langle dw_i, dw_i \rangle_{X_i}. \tag{4.18}$$

As $w_i \otimes w_i^*$ converges in $L^1(\check{X}) \otimes \mathrm{End}(E)$ to $w_\infty \otimes w_\infty^*$, we have

$$\begin{aligned} \lim_{i \to \infty} \langle w_i, \mathcal{V}_i w_i \rangle_X &= \lim_{i \to \infty} \left(\langle w_\infty, \mathcal{V}_i w_\infty \rangle_X + \left(\langle w_i, \mathcal{V}_i w_i \rangle_X - \langle w_\infty, \mathcal{V}_i w_\infty \rangle_X \right) \right) \\ &= \langle w_\infty, \mathcal{V} w_\infty \rangle_X. \end{aligned} \tag{4.19}$$

Then

$$\sup_{\psi \in W_\infty - 0} \frac{\langle d\psi, d\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \leq \limsup_{i \to \infty} \sup_{\psi \in W_i - 0} \frac{\langle d\psi, d\psi \rangle_{X_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}}. \tag{4.20}$$

Thus from (4.15), (4.16) and (4.20),

$$\inf_W \sup_{\psi \in W - 0} \frac{\langle d\psi, d\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \leq \lambda_k(|D^X|)^2 - \epsilon, \tag{4.21}$$

which is a contradiction. This proves the theorem. □

Proof of Theorem 4 :

Let $\{g_i^{TM}\}_{i=1}^\infty$ be a sequence of Riemannian metrics on M as in the statement of the theorem, with respect to which $\lambda_k(|D^M|)$ goes to infinity. Let P be the principal G-bundle of M and let $\check{X}$ be the limit space of Theorem 3, a smooth manifold with a $C^{1,\alpha}$-regular metric. As the limit space $X = \check{X}/G$ has diameter 1, it has positive dimension. As in the proof of Theorem 3, after slightly smoothing the metric on $\check{X}$, there is a G-equivariant Riemannian affine fiber bundle $\check{\pi} : P \to \check{X}$ whose fiber is a nilmanifold $\check{Z}$. Let $\check{x}$ be a point in a principal orbit for the G-action on $\check{X}$, with isotropy group $H \subset G$. Then H acts affinely on the nilmanifold fiber $\check{Z}_{\check{x}}$. In particular, H is virtually abelian. The quotient $Z = \check{Z}_{\check{x}}/H$ is the generic fiber of the possibly-singular fiber bundle $\pi : M \to X$, the G-quotient of $\check{\pi} : P \to \check{X}$. Then $E^M\big|_Z = \check{Z}_{\check{x}} \times_H V$. In particular, the vector space of affine-parallel sections of $E^M\big|_Z$ is isomorphic to V^H. On the other hand, if $C^\infty(X; E^X) \neq 0$ then $|D^X|$ has an infinite discrete spectrum. Theorem 3 now implies that $C^\infty(X; E^X) \cong \left(C^\infty(\check{X}) \otimes V\right)^G$ must be the zero space. As the orbit $\check{x} \cdot G$ has a neighborhood consisting of principal orbits, the restriction map from $\left(C^\infty(\check{X}) \otimes V\right)^G$ to $(C^\infty(\check{x} \cdot G) \otimes V)^G$ is surjective. However, $(C^\infty(\check{x} \cdot G) \otimes V)^G$ is isomorphic to V^H. Thus $V^H = 0$. This proves the theorem. $\square$

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