A NOTE ON THE CYCLE MAP MODULO TORSION FOR POINTS

by

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In [B], theorem (6.1), S. Bloch proves that on a surface X with $p_g = q = 0$, the degree map on the Chow group of points $CH^2(X)$ is realized as
$\deg : CH^2(X) = H^2_{zar}(X,\mathcal{K}_2) \longrightarrow \mathbb{Z} \subset H^2_{an}(X,\mathcal{K}_2^{an})$, where $\mathcal{K}_2$ and $\mathcal{K}_2^{an}$ are the sheaves of K_2 groups in the Zariski and the classical topology, and where the map comes from the change of topology.

In this note we generalize this using the Deligne-Beilinson cohomology.

Let X be an algebraic proper smooth scheme of dimension n over $\mathbb{C}$. We consider the morphism of Zariski sheaves

$$f_{p,q} : \mathcal{H}^q_{\mathcal{D}}(p) \longrightarrow \mathcal{H}^q_{\mathcal{D},an}(p)$$

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where $\mathcal{H}^q_{\mathcal{D}}(p)$ (resp. $\mathcal{H}^q_{\mathcal{D},an}(p)$) is the Zariski sheaf associated to the Deligne-Beilinson cohomology $H^q_{\mathcal{D}}(U,p)$ (resp. to the analytic Deligne cohomology $H^q_{\mathcal{D},an}(p)$) modulo torsion, and f_{pq} is the forgetful morphism (1.2), (1.3).

We prove that $H^n_{zar}(f_{nn})$ is the cycle map for points (modulo torsion) with values in the Deligne-Beilinson cohomology $H^{2n}_{\mathcal{D}}(X,n)$ (2.6). As f_{pp} is injective (1.3), this implies that the kernel of the Albanese map (modulo torsion) is coming from

$$H^{n-1}_{zar}(X,\mathcal{H}^n_{\mathcal{D},an}(n)/\mathcal{H}^n_{\mathcal{D}}(n)). \qquad (2.7.1)$$

Moreover if the Betti cohomology $H^{2n-2}(X,\mathbb{Q})$ is generated by the Betti classes of algebraic cycles (which implies that $H^{n-2}(X,\Omega^n_X) = 0$ and is equivalent to it on a surface), then the kernel of the Albanese map (modulo torsion) is exactly $H^{n-1}_{zar}(X,\mathcal{H}^n_{\mathcal{D},an}(n)/\mathcal{H}^n_{\mathcal{D}}(n))$ (2.8).

We define a sheaf $\mathcal{H}^{q,an}_{\mathcal{D},an}(p)$ in the classical topology (1.6) with a natural morphism $H^n_{zar}(X,\mathcal{H}^n_{\mathcal{D},an}(n)) \to H^n_{an}(X,\mathcal{H}^{n,an}_{\mathcal{D},an}(n))$ which we show to be the degree map (3.2).

I thank U. Jannsen for several useful conversations.

§1. Definition of the sheaves

Through this section X is an algebraic smooth scheme over $\mathbb{C}$ of dimension n.

1.1 For each Zariski open set U, of good compactification $j : U \longrightarrow \bar{U}$ such that $\bar{U} - U$ is a normal crossing divisor, we consider on $\bar{U}$ the complexes

$$Rj_*^m\Omega_U^\bullet = j_*^m\Omega_U^\bullet = \varinjlim_{\ell\geq 0} \Omega_{\bar{U}}^\bullet \ (\ell\cdot(\bar{U} - U))$$

and

$$Rj_*^m\Omega_U^{\geq p} = j_*^m\Omega_U^{\geq p} = \varinjlim_{\ell\geq 0} \Omega_{\bar{U}}^{\geq p} \ (\ell\cdot(\bar{U} - U))$$

where $\Omega_U^\bullet$ is the holomorphic de Rham complex on U and j^m is the meromorphic extension. If $\Omega_{\bar{U}}^\bullet$ $(\log(\bar{U} - U))$ denotes the holomorphic de Rham complex on U with logarithmic poles along $\bar{U} - U$, and $F^p(\log(\bar{U} - U))$ denotes its Hodge-Deligne F filtration $\Omega_{\bar{U}}^{\geq p}(\log(\bar{U} - U))$, one has morphisms

$$\begin{aligned} \Omega_{\bar{U}}^\bullet(\log(\bar{U} - U)) &\longrightarrow j_*^m\Omega_U^\bullet \\ F^p(\log(\bar{U} - U)) &\longrightarrow j_*^m\Omega_U^{\geq p} \end{aligned} \quad (1.1.1)$$

by forgetting the growth condition at infinity.

1.2 We consider the following presheaves in the Zariski topology: $U \longrightarrow$

α) $F^pH^q(U,p) := H^q(U,\mathbb{Q}(p)) \cap F^pH^q(U,\mathbb{C})$

α') $F^pH^q(U,\mathbb{C})$

β) $Hol_{\mathbb{Q}}^{pq}(U) := \{\omega \in H^q(\overline{U}, j_*^m\Omega_U^{\geq p})$ such that the cohomology class of ω in $H^q(U,\mathbb{C})$ lies in $H^q(U,\mathbb{Q}(p))\}$

γ) $H^q(U,p) := H^q(U,\mathbb{Q}(p))$

γ') $H^q(U,\mathbb{C})$, $H^q(U,\mathbb{C}/\mathbb{Q}(p)) = H^q(U,\mathbb{C})/H^q(U,p)$

δ) $H_{\mathfrak{D}}^q(U,p) := \mathbb{H}^q(\overline{U}, \text{cone}(Rj_*\mathbb{Q}(p) + F^p(\log(\overline{U} - U)) \longrightarrow Rj_*\mathbb{C})[-1])$, the Deligne-Beilinson cohomology of U

ε) $H_{\mathfrak{D},an}^q(U,p) := \mathbb{H}^q(\overline{U}, \text{cone}(Rj_*\mathbb{Q}(p) + j_*^m\Omega_U^{\geq p} \longrightarrow Rj_*\mathbb{C})[-1])$, the analytic Deligne cohomology of U.

The morphisms (1.1.1) define forgetful morphisms

$$f_{pq} : H_{\mathfrak{D}}^q(U,p) \longrightarrow H_{\mathfrak{D},an}^q(U,p)$$
$$g_{pq} : F^pH^q(U,p) \longrightarrow Hol_{\mathbb{Q}}^{pq}(U)$$

For $p = q$, f_{pp} and g_{pp} are injective ([E],(1.1)).

1.3 Denote by $\mathcal{F}_{\mathbb{Q}}^{pq}$, $\mathcal{F}_{\mathbb{C}}^{pq}$, $\Omega_{\mathbb{Q}}^{pq}$, $\mathcal{H}^q(p)$, $\mathcal{H}^q(\mathbb{C})$, $\mathcal{H}^q(\mathbb{C}/\mathbb{Q}(p))$, $\mathcal{H}_{\mathfrak{D}}^q(p)$, $\mathcal{H}_{\mathfrak{D},an}^q(p)$ the Zariski sheaves associated to α,α',β,γ,γ',δ,ε.

The morphisms of sheaves

$$f_{pq} : \mathcal{H}_{\mathfrak{D}}^q(p) \longrightarrow \mathcal{H}_{\mathfrak{D},an}^q(p)$$
$$g_{pq} : \mathcal{F}_{\mathbb{Q}}^{pq} \longrightarrow \Omega_{\mathbb{Q}}^{pq}$$

are injective for $p = q$ (1.2).

1.4 By [E], (1.1) one has a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{H}^{p-1}(\mathbb{C}/\mathbb{Q}(p)) & \longrightarrow & \mathcal{H}^{p}_{\mathcal{D}}(p) & \longrightarrow & \mathcal{F}^{pp}_{\mathbb{Q}} & \longrightarrow 0 \\ & \Vert & & \downarrow f_{pp} & & g_{pp}\downarrow & \\ 0 \longrightarrow & \mathcal{H}^{p-1}(\mathbb{C}/\mathbb{Q}(p)) & \longrightarrow & \mathcal{H}^{p}_{\mathcal{D},an}(p) & \longrightarrow & \Omega^{pp}_{\mathbb{Q}} & \longrightarrow 0 \end{array}$$

By [B.O] (2.3), (0.3) one has $H^{p-1+\ell}_{zar}(X,\mathcal{H}^{p-1}(\mathbb{C}/\mathbb{Q}(p)) = 0$ for $\ell > 0$, and by $[\bar{b}]$ one has: $H^{q}_{zar}(\mathcal{H}^{p}_{\mathcal{D}}(p)) = 0$ for $q > p$, $H^{p}_{zar}(X,\mathcal{H}^{p}_{\mathcal{D}}(p)) = CH^{p}(X)_{\mathbb{Q}}$, the Chow group of codimension p cycles modulo torsion. Therefore one has

<u>Lemma</u> 1)

$$\begin{array}{ccc} H^{p}_{zar}(X,\mathcal{H}^{p}_{\mathcal{D}}(p)) & = & H^{p}_{zar}(X,\mathcal{F}^{pp}_{\mathbb{Q}}) \\ \downarrow & H^{p}_{zar}(f_{pp}) = H^{p}_{zar}(g_{pp}) & \downarrow \\ H^{p}_{zar}(X,\mathcal{H}^{p}_{\mathcal{D},an}(p)) & = & H^{p}_{zar}(X,\Omega^{pp}_{\mathbb{Q}}) \end{array}$$

2) $H^{q}_{zar}(X,\mathcal{F}^{pp}_{\mathbb{Q}}) = 0$ for $q > p$.

1.5

<u>Lemma</u> One has

$$\mathcal{H}^{n+\ell}(k) = \mathcal{H}^{n+\ell}(\mathbb{C}/\mathbb{Q}(k)) = 0 \qquad \ell \geq 1$$

$$\mathcal{F}^{n,n+\ell}_{\mathbb{Q}} = \mathcal{F}^{n,n+\ell}_{\mathbb{C}} = 0 \qquad \ell \geq 1$$

$$\mathcal{H}^{n+1}_{\mathcal{D}}(n) = \mathcal{H}^{n}(\mathbb{C}/\mathbb{Q}(n))/\mathcal{F}^{n,n}_{\mathbb{C}}$$

$$\mathcal{H}^{n+\ell}_{\mathcal{D}}(n) = 0 \qquad \ell \geq 2$$

$$\Omega^{n,n+\ell}_{\mathbb{Q}} = \mathcal{H}^{n+\ell}_{\mathcal{D},an}(n) = 0 \qquad \ell \geq 1$$

Proof. As each point has a fundamental system of affine neighbourhoods, one has $\mathcal{H}^{n+\ell}(k) = \mathcal{H}^{n+\ell}(\mathbb{C}/\mathbb{Q}(k)) = \mathcal{F}_{\mathbb{Q}}^{n,n+\ell} = \mathcal{F}_{\mathbb{C}}^{n,n+\ell} = 0$ for $\ell \geq 1$. This implies, via the exact sequence on each Zariski open set U

$$0 \longrightarrow H^{n+\ell-1}(\mathbb{C}/\mathbb{Q}(n))/F^n \longrightarrow H_{\mathfrak{D}}^{n+\ell}(n) \longrightarrow F^n H^{n+\ell}(n) \longrightarrow 0$$

that $\mathcal{H}_{\mathfrak{D}}^{n+1}(n) = \mathcal{H}^n(\mathbb{C}/\mathbb{Q}(n))/\mathcal{F}_{\mathbb{C}}^{n,n}$

and that $\mathcal{H}_{\mathfrak{D}}^{n+\ell}(n) = 0$ for $\ell \geq 2$.

For U affine, one has $\mathbb{H}^{n+\ell}(\bar{U}, j_*^m \Omega^{\geq n}) = H^{\ell}(\bar{U}, j_*^m \Omega^n) = 0$ for $\ell \geq 1$, and therefore $\Omega_{\mathbb{Q}}^{n,n+\ell} = 0$ $\ell \geq 1$. On the other hand, for U affine, one has surjections

$$\mathbb{H}^n(\bar{U}, j_*^m \Omega^{\geq n}) = H^0(\bar{U}, j_*^m \Omega^n) \twoheadrightarrow H^n(U, \mathbb{C}) \twoheadrightarrow H^n(U, \mathbb{C}/\mathbb{Q}(n))$$

This implies, via the exact sequence on each Zariski open set U

$$\begin{array}{ccc} 0 \longrightarrow H^{n+\ell-1}(\mathbb{C}/\mathbb{Q}(n))/\mathbb{H}^{n+\ell-1}(j_*^m \Omega^{\geq n}) & \longrightarrow & H_{\mathfrak{D},an}^{n+\ell}(n) \\ & & \downarrow \\ & & \mathrm{Hol}_{\mathbb{Q}}^{n,n+\ell} \\ & & \downarrow \\ & & 0 \end{array}$$

that $\mathcal{H}_{\mathfrak{D},an}^{n+\ell}(n) = 0$ for $\ell \geq 1$.

1.6 If U is a Zariski open set, we may also define $H^{q,an}_{\mathscr{D},an}(U,p) := \mathbb{H}^q(U, \text{cone } (\mathbb{Q}(p) + \Omega^{\geq p}_U) \longrightarrow \mathbb{C})\ [-1])$. The morphism $j^m_*\Omega^{\geq p}_U \longrightarrow j_*\Omega^{\geq p}_U$ defines a morphism $H^q_{\mathscr{D},an}(U,p) \longrightarrow H^{q,an}_{\mathscr{D},an}(U,p)$, and therefore a morphism from $\mathscr{H}^q_{\mathscr{D},an}(p)$ to the Zariski sheaf associated to $H^{q,an}_{\mathscr{D},an}(U,p)$.

As $H^{q,an}_{\mathscr{D},an}(U,p)$ is also defined in the classical topology, we denote by $\mathscr{H}^{q,an}_{\mathscr{D},an}(p)$ the associated sheaf in the classical topology. Recall that $\dim X = n$.

Lemma.

If $n = 1$, one has $\mathscr{H}^{1,an}_{\mathscr{D},an}(1) = \mathcal{O}^*/\text{torsion}$, the sheaf of holomorphic invertible functions modulo torsion, quasi isomorphic to $\mathcal{O}/\mathbb{Q}(1)$ via the exponential map. If $n > 1$, one has $\mathscr{H}^{n,an}_{\mathscr{D},an}(n) = \Omega^n$, the sheaf of holomorphic n-forms. Especially if X is proper one has if $n = 1$ $H^1_{an}(X,\mathscr{H}^{1,an}_{\mathscr{D},an}(1)) = CH^1(X)_{\mathbb{Q}}$, the Chern group of points modulo torsion, if $n > 1$ $H^n_{an}(X,\mathscr{H}^{n,an}_{\mathscr{D},an}(n)) = \mathbb{C} = H^n_{an}(X,\Omega^n)$.

Proof. For each analytic open set U, one has an exact sequence

$$0 \longrightarrow H^{n-1}(\mathbb{C}/\mathbb{Q}(n-1)) \longrightarrow H^{n,an}_{\mathscr{D},an}(n) \longrightarrow \{\omega \in H^0(U,\Omega^n), \text{ whose cohomology class in } H^n(U,\mathbb{C}) \text{ lies in } H^n(U,\mathbb{Q}(n))\} \longrightarrow 0$$

As each point has a fundamental system of neighbourhoods U for which $H^n(U,\mathbb{C}) = 0$, the sheaf associated to the cokernel is Ω^n. If $n > 1$, then $H^{n-1}(U,\mathbb{C}/\mathbb{Q}(n-1)) = 0$ for a fundamental system of good neighbourhoods U of each point, and therefore $\mathcal{H}^{n,an}_{\mathfrak{D},an}(n) = \Omega^n$. If $n = 1$, one has on each open set

$$H^{1,an}_{\mathfrak{D},an}(U,1) = H^1_{an}(U,\mathbb{Q}(1) \longrightarrow \mathcal{O}) \xrightarrow[\exp]{\simeq} H^0(U,\mathcal{O}^*/\text{torsion})$$

§2 Description of the cycle map for points modulo torsion

Through this section, X is a proper smooth algebraic scheme of dimension n over $\mathbb{C}$.

2.1 Proposition. One has

$$H^n_{zar}(X,\mathcal{H}^n_{\mathcal{D},an}(n)) = H^{2n}_{\mathcal{D}}(X,n)$$

Proof. By (1.3), one has

$$H^n_{zar}(X,\mathcal{H}^n_{\mathcal{D},an}(n)) = H^n_{zar}(X,\Omega^{n,n}_{\mathbb{Q}}).$$

As each point has a fundamental system of affine neighbourhoods, one has an exact sequence of Zariski sheaves,

$$(2.1.1) \qquad 0 \longrightarrow \Omega^{p,p}_{\mathbb{Q}} \longrightarrow \Omega^{p,p}_{\mathbb{C}} \longrightarrow \mathcal{H}^p(\mathbb{C}/\mathbb{Q}(p)) \longrightarrow 0$$

where $\Omega^{p,p}_{\mathbb{Q}}$ is the Zariski sheaf associated to the presheaf $\mathbb{H}^p(\bar{U},j^m_*\Omega^{\geq p}_U) = \{\omega \in H^0(\bar{U},j^m_*\Omega^p_U), \ d\omega = 0\}$. This is the Zariski sheaf of closed holomorphic p forms on U which are meromorphic at infinity. By (1.4), $\mathcal{H}^{n+\ell}(\mathbb{C}/\mathbb{Q}(k)) = 0$ for $\ell \geq 1$, and by [B.O], (6.2), (6.3), (3.9), (2.3) one has

$$H^{n-1}_{zar}(X,\mathcal{H}^n(\mathbb{C}/\mathbb{Q}(n))) = H^{2n-1}(X,\mathbb{C}/\mathbb{Q}(n))$$

$$H^n_{zar}(X,\mathcal{H}^n(\mathbb{C}/\mathbb{Q}(n))) = H^{2n}(X,\mathbb{C}/\mathbb{Q}(n)) = \mathbb{C}/\mathbb{Q}(n)$$

and $H^{n-2}_{zar}(X,\mathcal{H}^n(\mathbb{C}/\mathbb{Q}(n))$ is a quotient of $H^{2n-2}(X,\mathbb{C}/\mathbb{Q}(n))$.

2.1.2. Lemma

α) One has

$\Omega^n_{alg} = \Omega^{n,n}_{\mathbb{C}}$, where Ω^n_{alg} is the Zariski sheaf of algebraic n forms and

$$\begin{aligned} H^q_{zar}(X,\Omega^n_{alg}) &= H^q_{zar}(X,\Omega^{n,n}_{\mathbb{C}}) \\ &= F^nH^{n+q}(X,\mathbb{C}) \end{aligned}$$

β) The map

$$\begin{array}{ccc} H^q_{zar}(X,\Omega^{n,n}_{\mathbb{C}}) & \longrightarrow & H^q_{zar}(X,\mathcal{H}^n(\mathbb{C}/\mathbb{Q}(n)) \\ \| & & \| \\ F^nH^{n+q}(X,\mathbb{C}) & \longrightarrow & \text{quotient of } H^{n+q}(X,\mathbb{C}/\mathbb{Q}(n)) \end{array}$$

for $q = n-2,\ n-1,\ n$ arising from (2.1.1) is the natural one.

Proof α) By [S], one has

$$H^q_{zar}(X,\Omega^n_{alg}) = F^nH^{n+q}(X,\mathbb{C}).$$

Applying a simplified argument a la Grothendieck, we have for the presheaves on each Zariski open set

$$\Gamma(U,\Omega^n_{alg}) = \varinjlim_{\ell \geq 0} \Gamma(\bar{U},\Omega^n_{alg}(\ell\cdot(\bar{U} - U))$$

$$= \varinjlim_{\ell \geq 0} \Gamma(\bar{U},\Omega^{n,n}_{\mathbb{C}}(\ell\cdot(\bar{U} - U)) \quad \text{(GAGA)}$$

$$= \Gamma(U,\Omega^{n,n}_{\mathbb{C}}).$$

Therefore the two Zariski sheaves Ω^n_{alg} and $\Omega^{n,n}_{\mathbb{C}}$ are the same.

β) The spectral sequence $E_2^{pq} = H^p_{zar}(X,\mathcal{H}^q(\mathbb{C}))$ is coming from the second spectral sequence of hypercohomology for the algebraic de Rham cohomology [B.O], (6.9). Therefore for each Zariski open set the map $H^0(U,\Omega^n_{alg}) \longrightarrow \mathbb{H}^n(U,\Omega^{\bullet}_{alg})$, where $\Omega^{\bullet}_{alg}$ is the algebraic de Rham complex, defines the map of sheaves $\Omega^n_{alg} \longrightarrow \mathcal{H}^n(\mathbb{C})$ via Grothendieck's isomorphism $\mathbb{H}^n(U,\Omega^{\bullet}_{alg}) = H^n(U,\mathbb{C})$ [G]. This proves β).

Go back to the proof of (2.1).

The sequence (2.1.1) gives rise to an exact sequence

$$F^nH^{2n-1}(\mathbb{C}) \xrightarrow{\alpha} H^{2n-1}(\mathbb{C}/\mathbb{Q}(n)) \longrightarrow H^n_{zar}(\Omega^{n,n}_{\mathbb{Q}}) \longrightarrow$$
$$\longrightarrow F^nH^{2n}(\mathbb{C}) \xrightarrow{\beta} H^{2n}(\mathbb{C}/\mathbb{Q}(n))$$

Therefore one has

$$H^n_{zar}(X,\Omega^{n,n}_{\mathbb{Q}}) = \mathbb{H}^{2n}(X,\mathrm{cone}(\mathbb{Q}(n) + F^n \longrightarrow \Omega^\bullet)[-1]) \quad (2.1.2)$$
$$= H^{2n}_{\mathcal{D}}(X,n)$$
$$= H^n_{zar}(X,\mathcal{H}^n_{\mathcal{D},an}(n)) \quad (1.4)$$

2.2 Remark

As $F^nH^{2n-1}(\mathbb{C}) \cap H^{2n-1}(\mathbb{Q}(n)) = 0$, α is injective; as β is the natural map (2.1.2), β is surjective.

One has the known extension

$$0 \longrightarrow \frac{\mathrm{Alb}\ X}{\mathrm{torsion}} = H^{2n-1}(\mathbb{C}/\mathbb{Q}(n))/F^nH^{2n-1}(\mathbb{C}) \longrightarrow H^{2n}_{\mathcal{D}}(n) \longrightarrow \mathbb{Q} \longrightarrow 0$$
$$\mathbb{Q} = H^{2n}(\mathbb{Q}(n)) \cap F^n$$

2.3 Denote by alg the subgroup of $H^{2n-2}(X,\mathbb{Q})$ generated by the Betti classes of algebraic cycles in $CH^{n-1}(X)_{\mathbb{Q}}$.

Lemma One has

$$H^{n-2}_{zar}(X,\mathcal{H}^n(\mathbb{C}/\mathbb{Q}(n))) = \left(\frac{H^{2n-2}(X,\mathbb{Q})}{\mathrm{alg}}\right) \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{Q}(n).$$

Proof By [B.O], (6.3), (7.4) and by (1.4) one has an exact sequence

$$0 \longrightarrow H^{n-1}_{zar}(\mathcal{H}^{n-1}(\mathbb{Q})) \longrightarrow H^{2n-2}_{an}(X,\mathbb{Q}) \longrightarrow H^{n-2}_{zar}(\mathcal{H}^n(\mathbb{Q})) \longrightarrow 0$$

with $H^{n-1}_{zar}(\mathcal{H}^{n-1}(\mathbb{Q})) = \dfrac{CH^{n-1}(X)_{\mathbb{Q}}}{\{\text{cycles which are algebraic equivalent to zero}\}_{\mathbb{Q}}}$.

2.4 <u>Proposition</u> One has

$$H^{n-1}_{zar}(X,\Omega^{n,n}_{\mathbb{Q}}) = \left(\frac{H^{2n-2}(X,\mathbb{Q})}{alg}\right) \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{Q}(n)/F^nH^{2n-2}(X,\mathbb{C}).$$

<u>Proof.</u> Take the cohomology of (2.1.1) applying (2.1.2), (2.3) and the fact that α is injective (2.2).

2.5 Consider the diagram of exact sequences

(2.5.1)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^{n,n}_{\mathbb{Q}} & \longrightarrow & \mathcal{F}^{n,n}_{\mathbb{C}} & \longrightarrow & \mathcal{F}^{n,n}_{\mathbb{C}}/\mathcal{F}^{n,n}_{\mathbb{Q}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^{n,n}_{\mathbb{Q}} & \longrightarrow & \Omega^{n,n}_{\mathbb{C}} & \longrightarrow & \mathcal{H}^n(\mathbb{C}/\mathbb{Q}(n)) & \longrightarrow & 0 \qquad (2.1.7) \\
 & & (1.4)\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{H}^n_{\mathfrak{D},an}(n)/\mathcal{H}^n_{\mathfrak{D}}(n) & \to & \Omega^{n,n}_{\mathbb{C}}/\mathcal{F}^{n,n}_{\mathbb{C}} & \longrightarrow & \mathcal{H}^n(\mathbb{C}/\mathbb{Q}(n))/\mathcal{F}^{n,n}_{\mathbb{C}} \underset{(1.5)}{=} \mathcal{H}^{n+1}_{\mathfrak{D}}(n) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The Bloch-Ogus theory for the Deligne-Beilinson cohomology [$\bar{b}$] tells us that there is a spectral sequence $E^{pq}_2 = H^p_{zar}(X,\mathcal{H}^q_{\mathfrak{D}}(n))$ converging to $H^{p+q}_{\mathfrak{D}}(X,n)$. As $\mathcal{H}^{n+\ell}_{\mathfrak{D}}(n) = 0$ for $\ell > 1$ (1.5), one obtains the following exact sequence

(2.5.2)

$$H^{n-1}_{zar}(\mathcal{H}^{n}_{\mathfrak{D}}(n)) \longrightarrow H^{2n-1}_{\mathfrak{D}}(X,n) \longrightarrow H^{n-2}_{zar}(\mathcal{H}^{n+1}_{\mathfrak{D}}(n)) \longrightarrow$$

$$\longrightarrow H^{n}_{zar}(\mathcal{H}^{n}_{\mathfrak{D}}(n)) \xrightarrow[\Psi]{} H^{2n}_{\mathfrak{D}}(n) \longrightarrow H^{n-1}_{zar}(\mathcal{H}^{n+1}_{\mathfrak{D}}(n))$$

By (1.3) and [B.O] (7.4), one has a map

(2.5.3)

$$H^{n-1}_{zar}(\mathcal{H}^{n-1}(\mathbb{C}/\mathbb{Q}(n)) = \frac{CH^{n-1}(X)_{\mathbb{Q}}}{\{\text{cycles which are algebraic equivalent to zero}\}_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{Q}(n)$$

$$\longrightarrow H^{n-1}_{zar}(\mathcal{H}^{n}_{\mathfrak{D}}(n)) \longrightarrow H^{2n-2}(\mathbb{C}/\mathbb{Q}(n))/F^{n}$$

$$||$$

$$H^{2n-1}_{\mathfrak{D}}(n)$$

2.6 The map from (2.5.2)

$$H^{n-2}_{zar}(\mathcal{H}^{n+1}_{\mathfrak{D}}(n) \longrightarrow H^{n}_{zar}(\mathcal{H}^{n}_{\mathfrak{D}}(n)) \underset{(1.4)}{=} H^{n}_{zar}(\mathcal{F}^{nn}_{\mathbb{Q}})$$

is coming from the cohomology of the vertical right sequence and the horizontal top sequence of (2.5.1). This implies

<u>Theorem</u> $H^{n}_{zar}(f_{nn}) = H^{n}_{zar}(g_{nn})$ is the cycle map $\Psi : CH^{n}(X)_{\mathbb{Q}} \longrightarrow H^{2n}_{\mathfrak{D}}(n)$.

<u>Proof.</u> The first equality is (1.4)1. Let I be the image of Ψ in $H^{2n}_{\mathcal{D}}(n)$, I' be the image of $H^{n}_{zar}(\mathcal{G}_{nn})$ in $H^{2n}_{\mathcal{D}}(n)$ (2.1).

One has a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow & \operatorname{im} H^{n-2}_{zar}(\mathcal{K}^{n+1}_{\mathcal{D}}(n)) \subset CH^{n}(X)_{\mathbb{Q}} & \longrightarrow & CH^{n}(X)_{\mathbb{Q}} & \longrightarrow & I & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \operatorname{im} H^{n-1}_{zar}(\mathcal{K}^{n}_{\mathcal{D},an}(n)/\mathcal{K}^{n}_{\mathcal{D}}(n)) \subset CH^{n}(X)_{\mathbb{Q}} & \longrightarrow & CH^{n}(X)_{\mathbb{Q}} & \longrightarrow & I' & \longrightarrow & 0
\end{array}$$

As I and I' are subgroup of $H^{2n}_{\mathcal{D}}(n)$, one has $I = I'$.

2.7 <u>Corollary</u> One has

1) $\operatorname{im} H^{n-2}_{zar}(\mathcal{K}^{n+1}_{\mathcal{D}}(n))$ in $CH^{n}(X)_{\mathbb{Q}}$
$= H^{n-1}_{zar}(\mathcal{K}^{n}_{\mathcal{D},an}(n)/\mathcal{K}^{n}_{\mathcal{D}}(n))$ in $CH^{n}(X)_{\mathbb{Q}}$ is the kernel of Ψ

2) $H^{n-1}_{zar}(\mathcal{K}^{n+1}_{\mathcal{D}}(n))$
$\simeq H^{n}_{zar}(\mathcal{K}^{n}_{\mathcal{D},an}(n)/\mathcal{K}^{n}_{\mathcal{D}}(n))$ and
$H^{n-1}_{zar}(\mathcal{K}^{n}_{\mathcal{D},an}(n)/\mathcal{K}^{n}_{\mathcal{D}}(n))$ surjects into
$H^{n-1}_{zar}(\Omega^{n,n}_{\mathbb{C}}/\mathcal{F}^{n,n}_{\mathbb{C}})$.

<u>Proof</u> 1) is consequence of $I = I'$, and 2) of the surjectivity of $H^{2n}_{\mathcal{D}}(n) \longrightarrow H^{n-1}_{zar}(\mathcal{K}^{n+1}_{\mathcal{D}}(n))$, of $H^{n+1}_{zar}(\mathcal{F}^{n,n}_{\mathbb{Q}}) = 0$ (1.4.) 2. and of (2.6) and (2.5.1).

2.3. <u>Corollary</u> If $H^{2n-2}(X,\mathbb{Q})$ is generated by the Betti class of algebraic cycles in $CH^{n-1}(X)_{\mathbb{Q}}$, then one has

$$H^{n-2}_{zar}(\mathcal{K}^{n+1}_{\mathfrak{D}}(n)) = H^{n-1}_{zar}(\mathcal{K}^{n}_{\mathfrak{D},an}(n)/\mathcal{K}^{n}_{\mathfrak{D}}(n))$$

and this group is the kernel of the Albanese mapping modulo torsion. Moreover in this case one has $H^{n-1}_{zar}(\Omega^{n,n}_{\mathbb{C}}/\mathcal{F}^{n,n}_{\mathbb{C}}) = 0$.

<u>Proof.</u> By (2.4), $H^{n-1}_{zar}(\Omega^{n,n}_{\mathbb{Q}}) = 0$. Therefore by (2.6) $\operatorname{Ker}\Psi = H^{n-1}_{zar}(\mathcal{K}^{n}_{\mathfrak{D},an}(n)/\mathcal{K}^{n}_{\mathfrak{D}}(n))$. By (2.5.3) and (2.5.2), $\operatorname{Ker}\Psi$ is also $H^{n-2}_{zar}(\mathcal{K}^{n+1}_{\mathfrak{D}}(n))$. Apply (2.7.2).

§3 Description of the degree map for points modulo torsion

Through this section X is an algebraic smooth proper scheme over $\mathbb{C}$ of dimension n.

3.1 In (2.1.1) one has defined a map

$$\Omega_{\mathbb{Q}}^{n,n} \longrightarrow \Omega_{\mathbb{C}}^{n,n}$$

Lemma The degree map $H_{\mathfrak{D}}^{2n}(X,n) \xrightarrow{\deg} H^{2n}(X,\mathbb{C}) = \mathbb{C}$ is the map $H_{zar}^{n}(X,\Omega_{\mathbb{Q}}^{n,n}) \longrightarrow H_{zar}^{n}(X,\Omega_{\mathbb{C}}^{n,n})$.

Proof By (2.1) one has $H_{\mathfrak{D}}^{2n}(X,n) = H_{zar}^{n}(X,\Omega_{\mathbb{Q}}^{n,n})$ and by (2.1.2) the map

$$\begin{aligned} H_{zar}^{n}(X,\Omega_{\mathbb{Q}}^{n,n}) \longrightarrow H_{zar}^{n}(X,\Omega_{\mathbb{C}}^{n,n}) &= F^{n}H^{2n}(X,\mathbb{C}) \\ &= H^{2n}(X,\mathbb{C}) = \mathbb{C} \end{aligned}$$

is the natural one.

3.2 Denote by $\mathcal{H}$ just for a moment the Zariski sheaf associated to the presheaf $H_{\mathfrak{D},an}^{n,an}(U,n)$. The map $\mathcal{H}_{\mathfrak{D},an}^{n}(n) \longrightarrow \mathcal{H}$ (1.6) defines a map $H_{zar}^{n}(X,\mathcal{H}_{\mathfrak{D},an}^{n}(n)) \longrightarrow H_{zar}^{n}(X,\mathcal{H})$. There is also the map arising from the change of topology $H_{zar}^{n}(X,\mathcal{H}) \longrightarrow H_{an}^{n}(X,\mathcal{H}_{\mathfrak{D},an}^{n,an}(n))$. This gives a commutative diagram

$$\begin{array}{ccccc}
H^n_{zar}(\mathfrak{X}^n_{\mathfrak{D},an}(n)) & \longrightarrow & H^n_{zar}(\mathfrak{X}) \overset{}{\longrightarrow} & & H^n_{an}(\mathfrak{X}^{n,an}_{\mathfrak{D},an}(n)) \\
{\scriptstyle (1.4)+(2.1.1)}\Big\downarrow & & \delta & & \Big\downarrow{\scriptstyle (1.6)} \\
H^n_{zar}(\Omega^{n,n}_{\mathbb{C}}) & & \xrightarrow[(2.1.2)\alpha]{\sim} & & H^n_{an}(\Omega^n)
\end{array}$$

<u>Theorem</u> The map of change of topology δ is the degree map for $n > 1$ and the identity if $n = 1$.

<u>Proof.</u> Just apply (3.1) on the left vertical arrow and (1.6) on the right vertical arrow for $n > 1$, and (1.6) for $n = 1$.

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