Max-Planck-Institut für Mathematik Bonn

Combinatorics of double Grothendieck polynomials
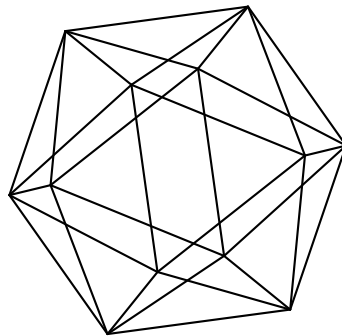
by

Graham Hawkes

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COMBINATORICS OF DOUBLE GROTHENDIECK POLYNOMIALS

GRAHAM HAWKES

ABSTRACT. We give a self-contained treatment of double Grothendieck polynomials including many new combinatorial results such as a combinatorial proof of the k-theoretic Giambelli formula.

1. INTRODUCTION

Grothendieck polynomials [LS82b],[LS83], are a non-homogeneous generalization of Schubert polynomials, the latter a family of polynomials indexed by permutations which are studied among other things in relation to the combinatorics of Coxeter groups. In particular, the lowest degree term of a Grothendieck polynomial is a Schubert polynomial. Combinatorially, Grothendieck polynomials replace the notion of the symmetric group with that of the 0-Hecke monoid [BKS$^+$08]. Double Schubert polynomials are considered in [LS82a] to generalize Schubert polynomials by extending them to two sets of variables in such a way that setting the second variable set to zero returns a regular Schubert polynomial. In turn, double Grothendieck polynomials generalize Grothendieck polynomials by extending them to two sets of variables in such a way that setting the second variable set to zero returns a regular Grothendieck polynomial.

Double Grothendieck polynomials themselves are generally not symmetric in either set of variables. However, there exists a way to derive a (doubly) symmetric function from a double Grothendieck polynomial through a process of letting the number of variables go to infinity and then setting all but finitely many of them to zero. The limit is stable, meaning that once the number of variables is high enough the result of the process is the same for all higher numbers of variables. In fact, what this number is, is not difficult to specify, so to define these so-called stable double Grothendieck polynomials requires no real appeal to any notion of the infinite. We remark that stable (single) Grothendieck polynomials are studied here [BKS$^+$08]. Because of a natural involution on symmetric functions, often referred to as ω (although in this paper it will be denoted Ω since we use ω extensively as a name for the permutation by which the polynomials are indexed) it is natural to study the result of applying this involution to (both sets of variables of) stable double Grothendieck polynomials. We call the result the weak stable double Grothendieck polynomial.

An important special class of stable double Grothendieck polynomials is that which is composed of the polynomials indexed by Grassmannian permutations. In fact, all stable double Grothendieck polynomials expand in terms of these so-called symmetric double Grothendieck polynomials. Moreover, they have a very nice

combinatorial interpretation in terms of certain tableaux. Of course, we will also analyze the weak symmetric double Grothendieck polynomials. There is another (doubly) symmetric polynomial which is useful in our proofs as well as interesting in its own right (for one it is self-dual when the two sets of variables are set equal) that we call the quasi-weak stable double Grothendieck polynomial. It is in some sense between the regular and weak versions of stable double Grothendieck polynomials although it is not obtained by applying Ω to just one set of variables as might be imagined.

The above are the topics covered in this paper. The paper is self-contained to the degree that it should be able to be read and understood without referring to any external sources as long as the reader has a basic understanding of the symmetric group and symmetric functions. Of, course, this means a number of the results of this paper may be found elsewhere. Wherever this is the case we give citations to other appearances of these results for the curious reader and to credit the original works. All of the proofs of such results contained in the paper, are however, original, and generally more combinatorial than other proofs of said results. We give a short list of some of the highlights of the new results of this paper here:

- A combinatorial proof of the Giambelli formula for double Grothendieck polynomials.
- New combinatorial interpretations of and results concerning the
  - double Grothendieck polynomial
  - stable double Grothendieck polynomial
  - weak stable double Grothendieck polynomial
  - symmetric double Grothendieck polynomial
  - weak symmetric double Grothendieck polynomial
- Definitions and results concerning the new
  - quasi-weak stable double Grothendieck polynomial
  - quasi-weak symmetric double Grothendieck polynomial

Here is an overview of the organization of the paper:

- **Section 1: Introduction**–Contains this.
- **Section 2: Single Grothendieck Polynomials**–Contains results which mostly appear other places. The reader familiar with the topic may skim this section looking only for the definitions and notation (that will be used later).
- **Section 3: Double Grothendieck Polynomials**–Contains new combinatorial interpretations of said polynomial and a new combinatorial proof of the more general k-theoretic version of the Giambelli formula for double Schubert polynomials.
- **Section 4: Stable double Grothendieck polynomials**–Contains new combinatorial interpretations of stable, weak stable, symmetric, and weak symmetric double Grothendieck polynomials and related results.
- **Section 5: Quasi-weak stable double Grothendieck polynomials**– Contains the definitions of, and results related to new symmetric functions which we call the quasi-weak stable and quasi-weak symmetric double Grothendieck polynomials.

2. Single Grothendieck Polynomials

2.1. **Operator definition.**

Definition 2.1. Let $f \in \mathbb{Z}[x_1, \ldots x_{n+1}]$. For each $1 \le i \le n$ define the divided difference operator δ_i by $\delta_i(f) = \frac{f - s_i f}{x_i - x_{i+1}}$ where s_i acts by interchanging the variables x_i and x_{i+1}. Define π_i by the formula $\pi_i(f) = \delta_i(f) + \delta_i(x_{i+1} f)$.

Lemma 2.2. *The divided difference operators satisfy the following relations:*

(1) *If $|i - j| > 1$ then $\delta_i \delta_j = \delta_j \delta_i$ and $\pi_i \pi_j = \pi_j \pi_i$.*
(2) *If $i = j + 1$ then $\delta_i \delta_j \delta_i = \delta_j \delta_i \delta_j$ and $\pi_i \pi_j \pi_i = \pi_j \pi_i \pi_j$.*
(3) *$\delta_i^2 = 0$.*
(4) *$\pi_i^2 = -\pi_i$.*

Proof.

(1) Computing $\delta_i \delta_j(f)$ and $\delta_j \delta_i(f)$ shows that the second is obtained from the first by replacing $s_i s_j f$ with $s_j s_i f$. However, since $|i - j| > 1$, s_i and s_j commute so this difference is trivial. From this and the fact that s_i commutes with multiplication by x_{j+1} and vice versa we see $\pi_i \pi_j = \pi_j \pi_i$.
(2) Since the operators are clearly linear it suffices to prove the statement for monomials. Doing this for π will imply the same result for δ by comparing lowest degree terms. To prove the statement for π on a monomial it suffices without loss of generality to show that $\pi_1 \pi_2 \pi_1(x_1^{a_1} x_2^{a_2} x_3^{a_3})$ is the same as $\pi_2 \pi_1 \pi_2(x_1^{a_1} x_2^{a_2} x_3^{a_3})$. Computing both of these directly yields:

$$\sum_{\sigma \in S_3} \frac{x_{\sigma_1}^{a_1} x_{\sigma_2}^{a_2} x_{\sigma_3}^{a_3} (1 + x_{\sigma_2} + 2x_{\sigma_3} + 2x_{\sigma_2} x_{\sigma_3} + x_{\sigma_3}^2 + x_{\sigma_2} x_{\sigma_3}^2)}{(x_{\sigma_1} - x_{\sigma_2})(x_{\sigma_1} - x_{\sigma_3})(x_{\sigma_2} - x_{\sigma_3})}$$

(3) $\delta_i(f)$ is a symmetric function with respect to x_i and x_{i+1}. But δ_i vanishes on any function with such symmetry.
(4) We have:

$$\pi_i^2(f) = \pi_i(\delta_i f + \delta_i x_{i+1} f) = \delta_i^2 f + \delta_i^2 x_{i+1} f + \delta_i x_{i+1} \delta_i f + \delta_i x_{i+1} \delta_i x_{i+1} f =$$

$$\delta_i x_{i+1}(\delta_i f + \delta_i x_{i+1} f) = \frac{x_{i+1}(\delta_i f + \delta_i x_{i+1} f) - x_i s_i(\delta_i f + \delta_i x_{i+1} f)}{x_i - x_{i+1}} =$$

$$\frac{x_{i+1}(\delta_i f + \delta_i x_{i+1} f) - x_i(\delta_i f + \delta_i x_{i+1} f)}{x_i - x_{i+1}} = -(\delta_i f + \delta_i x_{i+1} f) = -\pi_i(f)$$

$\square$

Given a permutation $\omega \in S_n$ one can write down (non-uniquely in general) ω as sequence of adjacent transpositions, i.e.: $\omega = s_{i_1} \cdots s_{i_\ell}$ where ℓ is the inversion number of the permutation. We can then define δ_ω by $\delta_{i_1} \cdots \delta_{i_\ell}$ and π_ω to be $\pi_{i_1} \cdots \pi_{i_\ell}$. By parts 1 and 2 of Lemma 2.2 this procedure is well defined, i.e., the definition of δ_ω and π_ω does not depend on the chosen reduced word. Note that it is important however that the word for ω be reduced as, in particular it cannot have repeated (consecutive) entries since, as noted earlier $\delta_i^2 = 0 \ne \delta_i$.

Definition 2.3. Fix $\omega \in S_n$ and let ω_0 refer to the element of S_n with maximal inversion number. Define the Schubert polynomial for ω by:

$$\mathfrak{S}_\omega = \delta_{(\omega^{-1} \omega_0)}(x_1^n x_2^{n-1} \cdots x_n^1 x_{n+1}^0).$$

Define the Grothendieck polynomial for ω by:

$$\mathfrak{G}_\omega = \pi_{(\omega^{-1} \omega_0)}(x_1^n x_2^{n-1} \cdots x_n^1 x_{n+1}^0).$$

See [LS82b] and [LS83] for original formulations. Note that the Schubert polynomial can always be recovered from the Grothendieck polynomial by taking its lowest degree term. Therefore we will make no further mention of the less general Schubert polynomials.

2.2. **Hecke Insertion.** Consider a new operator $\bar{s}_i$ acting on permutations of the set $\{1, 2, 3, \ldots, n, n+1\}$ where the operation $\bar{s}_i$ is given by interchanging i and $i+1$ if i lies to the left of $i+1$ and by doing nothing otherwise (In particular $\bar{s}_i^2 = \bar{s}_i$ whereas $s_i^2 = e$). In this setting, a *Hecke word* for ω is a sequence $\bar{s}_{i_1} \cdots \bar{s}_{i_k}$ such that applying this sequence (right to left) to the starting arrangement $(1, 2, 3, \ldots, n, n+1)$ gives the permutation ω.

We will give an overview of a simple insertion algorithm [BKS$^+$08] for Hecke words which will be necessary at various stages. First we need to define two types of tableaux:

Definition 2.4. A standard set-valued tableau of shape λ is a filling of a Young diagram of shape λ with exactly one of each of the letters $\{1, \ldots, N\}$ for some integer $N \geq |\lambda|$ such that each box contains at least one entry and such that all entries in a given box are smaller than all the entries in the box below and smaller than all the entries in the box to the right.

Definition 2.5. Given a permutation ω, a Hecke tableau for ω of shape λ or element of $HT_\omega(\lambda)$ is a tableau where each box of λ is filled with exactly one of the symbols $\{\bar{s}_1, \ldots, \bar{s}_n\}$ in such a way that reading the boxes by rows, moving left to right within the rows and moving bottom to top amongst the rows gives a Hecke word for ω, and, such that the rows and columns are strictly increasing in the order $\bar{s}_1 < \cdots < \bar{s}_n$. In actual examples, the boxes of a Hecke tableau will be written as, for instance $\boxed{3}$ instead of $\boxed{\bar{s}_3}$.

To define Hecke insertion, we first show how to insert some $a \in \{\bar{s}_1, \ldots, \bar{s}_n\}$ into some row of a Hecke tableau, say $Y = (y_1, y_2, \ldots, y_j)$ read from left to right. Suppose that the row above Y (if it exists) is X and it has entries $X = (x_1, x_2, \ldots, x_k)$ and the row below it is Z with entries $Z = (z_1, z_2, \ldots, z_i)$. Here all $x, y, z \in \{\bar{s}_1, \ldots, \bar{s}_n\}$. We assume that $a \in [x_h, y_h)$ in the order $\bar{s}_1 < \cdots < \bar{s}_n$ for some h (where possibly $h > j$ and y_h is taken, by convention to be ∞) and that $a > x_1$. There are no restrictions on a if X does not exist that is, if Y is the first row of the tableau.

x_1	x_2	$\cdots$	x_i	$\cdots$	x_j	x_{j+1}	$\cdots$	x_k
y_1	y_2	$\cdots$	y_i	$\cdots$	y_j			
z_1	z_2	$\cdots$	z_i					

$\longleftarrow a$

We insert a into Y as follows:

(1) If $a \geq y_j$ and:
- $a > y_j$ and $a > x_{j+1}$. Then a is appended to the right of y_j.
- $a = y_j$ or $a = x_{j+1}$. Then a simply disappears.

(2) If $a < y_j$. Let h be minimal such that $a \leq y_h$.
- $a = y_h$. Then Y stays the same and y_{h+1} is inserted into Z.
- $a < y_h$ and $a > x_h$ then a replaces y_h and y_h is inserted into row Z.

- $a < y_h$ and $a = x_h$ then Y is unchanged and y_h is inserted into row Z.

Note that the result is strictly decreasing down columns by construction and that our assumption on a guarantees one of the situations above must occur. Moreover, the assumption is maintained moving on to the next insertion. That is, the element (if it exists) to be inserted into row Z exists in some interval $[y_\ell, z_\ell)$ and is greater than y_1.

Now we show how to outsert some a through a row Y (i.e., loosely speaking do the opposite of insert an element into a row). Again we denote the row above Y by X and the row below Y by Z. Here we assume that $a \in [y_h, z_h)$ for some h (where possibly $h > i$ and $z_h = \infty$) and that $a > y_1$.

x_1	x_2	$\cdots$	x_i	$\cdots$	x_j	x_{j+1}	$\cdots$	x_k
y_1	y_2	$\cdots$	y_i	$\cdots$	y_j			
z_1	z_2	$\cdots$	z_i					

$a \longrightarrow$

- $a \notin Y$. Let h be maximal such that $a > y_h$. Then replace y_h by a and outsert y_h through X.
- $a = y_{h+1}$ for some $h \geq 1$. Then Y remains unchanged and $max(y_h, x_{h+1})$ is outserted through X.

Note that our assumption on a ensures one of these is possible and that the result is still strictly increasing down columns. Moreover the assumption is maintained moving on to the next outsertion. That is, the element (if it exists) to be outserted through row X is greater than x_1 and exists in some interval $[x_\ell, y_\ell)$.

Finally, we show how to outsert *from* a row Y (loosely speaking do the opposite of append an element to a row). The row above Y is X and the row below is Z. There is a way to do this both with removing a box from Y and without moving a box from Y. We assume the second can only occur if the length of X is greater than the length of Y, i.e., $x_{j+1} < \infty$. (Which of these options we choose will be "dictated" by the recording tableau as described after the next lemma and its proof.)

x_1	x_2	$\cdots$	x_i	$\cdots$	x_j	x_{j+1}	$\cdots$	x_k
y_1	y_2	$\cdots$	y_i	$\cdots$	y_j $\hookrightarrow$			
z_1	z_2	$\cdots$	z_i					

- If the recording tableau dictates a box be removed from Y then the box containing y_j is removed and y_j is outserted through X.
- If the recording tableau dictates no box is removed from Y, then Y remains the same and $max(y_j, x_{j+1})$ is outserted through X.

Note that the element to be outserted through X satisfies the assumptions for outsertion: it is greater than x_1 and exists in some interval $[x_\ell, y_\ell)$.

Lemma 2.6. *Insertion and outsertion are mutual inverses. More precisely, considering again three rows of a tableau at a time, denoted by X, Y, and Z we have:*

(1) *Suppose a is inserted into Y and changes Y to Y' and an element, a' is produced to be inserted into Z. Then if a' is outserted through Y' it changes Y' back to Y and a comes out to be outserted through X. Conversely, suppose a is outserted through Y and changes Y to Y' and an element, a'*

is produced to be outserted through X. Then if a' is inserted into Y' it
changes Y' back to Y and a comes out to be inserted into Z.

(2) If a is inserted into Y and is appended to form Y' then outserting from
Y' with removal of a box returns Y and sends a to be outserted through
X. Conversely, if outserting from Y with box removal results in Y' and
produces a' to be be outserted through X, then inserting a' to Y' results in
Y.

(3) If a is inserted into Y and disappears then outserting from Y without re-
moval of a box sends a to be outserted through X. Conversely, if a' is
outserted from row Y without box removal then inserting a' into Y causes
it to disappear.

Proof.

(1) $\implies$

Since a is not appended or disappeared we know $a < y_j$.

- Suppose $a = y_h$ for some $h < j$. Then $Y' = Y$ and $a' = y_{h+1}$.
  Moreover the assumption that $a \in [x_\ell, y_\ell)$ for some ℓ ensures that $a \geq$
  x_{h+1}. When a' is outserted through Y', we are outserting y_{h+1} through
  Y. Since $y_{h+1} \in Y$, the row Y remains unchanged and $max(y_h, x_{h+1})$
  is sent to be outserted through X. But this maximum is y_h which is
  equal to a. Hence the original row, Y, and the original element, a are
  returned.
- Suppose $a \notin Y$. Let h be minimal such that $a < y_h$.
  - If $a > x_h$ then a replaces y_h in Y and $a' = y_h$. Now $y_h \notin Y'$,
    which is to say $a \notin Y'$ so to outsert a' through Y we find the
    rightmost element of Y' that a' is greater than. This element is
    a. So a', which equals y_h, replaces a giving back Y and a is sent
    to be outserted through the next row up, X. Hence the original
    row, Y, and the original element, a are returned.
  - If $a = x_h$ then $a' = y_h$ and $Y' = Y$ and since we assume $a > x_1$
    we know $h > 1$. Thus $a' \in Y'$ so outserting a' through Y'
    does not change Y' and $max(y_{h-1}, x_h)$ is sent to be outserted
    through X. By minimality of h, $a > y_{h-1}$, that is $x_h > y_{h-1}$ so
    this maximum is x_h, or a. Hence the original row, Y, and the
    original element, a are returned.

$\impliedby$

- Suppose that $a = y_{h+1}$ for some $h \geq 1$.
  - Suppose that $y_h \geq x_{h+1}$. Then $Y' = Y$ and $a' = y_h$. When we
    insert a' into Y' since $a' \in Y'$, the row $Y' = Y$ stays the same
    and y_{h+1} is the element to be inserted into Z. Hence the original
    row, Y, and the original element, a are returned.
  - Suppose that $y_h < x_{h+1}$. Then $Y' = Y$ and $a' = x_{h+1}$. Since
    $y_h < x_{h+1} < y_{h+1}$ it follows that $x_{h+1} \notin Y'$. Thus when we
    insert a' into Y', y_{h+1} is the leftmost element greater than a'.
    But since $a' = x_{h+1}$ we do not replace y_{h+1} but keep Y' the same
    and then would insert y_{h+1} into Z. Hence the original row, Y,
    and the original element, a are returned.
- Suppose that $a \notin Y$. Let h be maximal such that $a > y_h$. Then Y'
  is obtained from Y by replacing y_h with a and $a' = y_h$. Thus $a' \notin Y'$

and a is the leftmost element of Y' greater than a'. Since the element above a is x_h and $a' = y_h > x_h$ then a' replaces a and a is moved out to be inserted into Z. Hence the original row, Y, and the original element, a are returned.

(2) $\implies$

This is trivial.

$\impliedby$

If a' is removed from the last box of Y to form Y' then a' is larger than all elements of Y'. Thus inserting a' will result it in being appended to reform Y.

(3) $\implies$

- $a = y_j$. In this case the assumption that $a \in [x_\ell, y_\ell)$ for some h implies that $a \geq x_{j+1}$. Hence outsertion from Y without box removal results in $max(y_j, x_{j+1}) = a$ being sent to be outserted through X.
- $a = x_{j+1}$. Since a disappeared it implies that $a \geq y_j$. Thus outsertion from Y without box removal will send $max(y_j, x_{j+1}) = a$ to be outserted through X.

$\impliedby$

If we apply outsertion without box removal to row Y then $a' = max(y_j, x_{j+1})$ comes out. Certainly $a' \geq y_j$ and equal to one of y_j or x_{j+1} so it disappears when inserted.

$\square$

We now describe complete Hecke insertion and its inverse. Given a Hecke word say $w_1 \cdots w_m$ create a sequence of pairs of tableaux of the same shapes $(P_0, Q_0), (P_1, Q_1), \ldots, (P_m, Q_m)$ by setting $P_0 = \emptyset = Q_0$ and creating (P_{i+1}, Q_{i+1}) from (P_i, Q_i) as follows. Insert w_{i+1} into P_i by inserting it into the first row of P_i. As long as there is an output, insert the output into the next row. The algorithm stops when either an element is appended to the end of a row or disappears. The resulting Hecke tableau is P_{i+1}. If the algorithm ends by appending an element, add a box to the corresponding position of Q_i and fill it with the number $i + 1$ to form Q_{i+1}. If the algorithm stops by an element disappearing, take the row where the last insertion occurred and caused this element to disappear and consider its rightmost box b. Now find the lowest box in the same column as b, call it b'. Add an $i + 1$ to the position corresponding to b' in Q_i to form Q_{i+1}. (Of course, it is possible $b' = b$.)

$\longleftarrow$ disappearing element
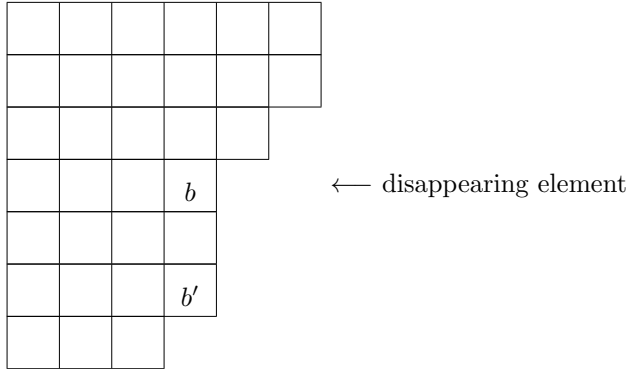
Example 2.7. Suppose that we have

$$P_{16} = \quad$$

1	2	4	5
2	4	6	8
3	5	7	
4	7		
6	8		
9			

$$Q_{16} = \quad$$

1	2	3	4
5	6	7	8
9	10	11	
12	13		
14	15		
16			

$$\omega_{17} = 3$$

Then P_{17} and Q_{17} are computed as follows. First the 3 is inserted into row one of P_{16}. 3 replaces 4 in this row and 4 is sent to be inserted into row two. When 4 is inserted into row two, since a 4 already appears in row two this row does not change and the number to the right of the 4 in row two, which is 6, will be inserted into row three. When 6 is inserted into row three it does not replace the 7 with itself because the number above this 7 in row two is not less than 6 (it is 6). Thus row three remains unchanged and the number 7 is inserted into row four. Row four ends in 7 itself so the inserted 7 is disappeared. Finally the 17 is added to the recording tableau not in the box, b, at the end of row four but to the box b' at the bottom of the column containing b. All in all the only changes are in the first row of the insertion tableau and the fifth row of the recording tableau and the result is:

$$P_{17} = \quad$$

1	2	3	5
2	4	6	8
3	5	7	
4	7		
6	8		
9			

$$Q_{17} = \quad$$

1	2	3	4
5	6	7	8
9	10	11	
12	13		
14	15, 17		
16			

The (claimed) inverse of Hecke insertion is described as follows: Given a pair of a standard set-valued tableau, P with m entries and a Hecke tableau, Q of the same shape create a sequence $(P_m, Q_m), \ldots, (P_1, Q_1), (P_0, Q_0)$ as well as a Hecke word $w_1, \ldots, w_m$ as follows. Set $P_m = P$ and $Q_m = Q$. In general, to obtain w_{i+1} and (P_i, Q_i) from P_{i+1}, Q_{i+1} do as follows.

- If $i + 1$ exists in its own box in Q_{i+1} form Q_i by removing this box and its entry. Form P_i from P_{i+1} by outserting with box removal from the corresponding box of P_{i+1} and continuing to outsert until an element is removed from the tableau. The resulting tableau is P_i and the removed element is w_{i+1}.
- If $i+1$ shares a box b' with another number in Q_{i+1}, form Q_i by just deleting $i+1$ from Q_{i+1}, and then scan the column of b' for the highest box, b, that

is rightmost box its row. Form P_i from P_{i+1} by outserting without box removal from the box of P_{i+1} corresponding to b and continuing to outsert until an element is removed from the tableau. The resulting tableau is P_i and the removed element is w_{i+1}.

Proposition 2.8. *If $w_1 \cdots w_m$ maps under Hecke insertion to (P,Q) then (P,Q) maps under the (claimed) inverse to $w_1 \cdots w_m$. Conversely, if (P,Q) maps under the (claimed) inverse to $w_1 \cdots w_m$ then $w_1 \cdots w_m$ maps to (P,Q) under Hecke insertion. In other words, the claimed inverse is correct.*

Proof. $\Longrightarrow$
It suffices to show that if we insert a into (P,Q) to get (P',Q') then applying one step of the inverse returns (P,Q) and a.

- If insertion of a ends with an element being appended in some new box then this box is the box that contains the highest number in Q'. Thus the inverse removes this number from Q' returning Q and since this box contains only one number, the inverse applies outsertion with box removal to this box of P'. Continuing to apply outsertion through each remaining row then returns P and a by Lemma 2.6.
- If insertion of a ends in an element disappearing into a row ending with a box b, then the highest number in Q' lies below b in a box b' with another number. The inverse dictates to remove this number from b' and then apply outsertion without box removal to box b of P' and then continue outserting through rows. By Lemma 2.6 this returns P and a.

$\Longleftarrow$
It suffices to show that if we apply one step of the inverse to (P,Q) to get (P',Q') and a' then inserting a' into (P',Q') returns (P,Q).

- If the largest number of Q is in its own box then we erase this box from Q to get Q' and apply outsertion with box removal to the corresponding box of P and continue outserting to get P' and a'. By Lemma 2.6 inserting a' into P' returns P and ends in the appending of an element to the box missing from P' but in P. Thus we add back a new largest number to this box of Q' and regain Q.
- If the largest number of Q shares a box, b', then we erase this number from Q to get Q' and apply outsertion without box removal to a box, b of P, above the box corresponding to b' in P. We then continue outserting to get P' and a'. By Lemma 2.6 inserting a' into P' returns P and ends in an element disappearing into the row of b. Thus we add back a new largest number to the box b' of Q' and regain Q.

$\square$

Lemma 2.9. *If the word $w_1 \cdots w_m$ maps to the pair (P,Q), then the Hecke word $w_1 \cdots w_m$ and the Hecke tableau, P, correspond to the same permutation.*

Proof. It suffices to check that if an element a is inserted into some row $Y = (y_1 \cdots y_j)$ (denote the row above it, if it exists, by $X = (x_1 \cdots x_j \cdots x_k)$) and changes it to Y' possibly producing an a' be to inserted into the next row then the Hecke words $[(a')Y'X]$ and $[YaX]$ represent the same permutation (where the

a' in quotations is to be ignored if this insertion results in an appension or disappearance). We write $u \sim v$ to mean the Hecke words u and v represent the same permutation.

- If a is appended to Y this just says that $[(Ya)X]$ and $[YaX]$ represent the same permutation.
- Now suppose that a disappears when inserted into Y. Then either:
    - $a = y_j$. In this case it simply says that the words $[y_1 \cdots y_j y_j X]$ and $[y_1 \cdots y_j X]$ represent the same permutation, which is true since $y_j^2 \sim y_j$.
    - $a > y_j$. In this case $a = x_{j+1}$ and it just says that the words $[y_1 \cdots y_j x_{j+1} x_1 \cdots x_j x_{j+1} \cdots x_k]$ and $[y_1 \cdots y_j x_1 \cdots x_j x_{j+1} \cdots x_k]$ represent the same permutation. But $x_j < y_j < x_{j+1}$ so that $x_{j+1} x_1 \cdots x_j \sim x_1 \cdots x_j x_{j+1}$. This along with the fact that $x_{j+1}^2 \sim x_{j+1}$ implies that $x_{j+1} x_1 \cdots x_j x_{j+1} \sim x_1 \cdots x_j x_{j+1}$.
- Now suppose that a is inserted into Y and results in an element a' being produced to be inserted to the next row.
    - If $a = y_h$ for some $h < j$ then note that $y_h y_{h+1} y_h \sim y_{h+1} y_h y_{h+1}$ either by the Coxeter relation $\bar{s}_i \bar{s}_{i+1} \bar{s}_i \sim \bar{s}_{i+1} \bar{s}_i \bar{s}_{i+1}$ if y_{h+1} covers y_h in the order $\bar{s}_1 < \cdots < \bar{s}_n$ or if not, because both are equivalent to to $y_h y_{h+1}$. Thus

    $$[Ya] = y_1 \cdots y_h y_{h+1} \cdots y_j y_h \sim y_1 \cdots y_h y_{h+1} y_h \cdots y_j \sim$$
    $$y_1 \cdots y_{h+1} y_h y_{h+1} \cdots y_j \sim y_{h+1} y_1 \cdots y_h y_{h+1} \cdots y_j = [a'Y]$$

    which shows that $[a'Y'X] = [YaX]$ since $Y = Y'$ in this case.
    - Now suppose that $a \notin Y$, h is minimal such that $a < y_h$ and $x_h < a$. Then

    $$[Ya] = y_1 \cdots y_{h-1} y_h y_{h+1} \cdots y_j a = y_1 \cdots y_{h-1} y_h a y_{h+1} \cdots y_j =$$
    $$y_h y_1 \cdots y_{h-1} a y_{h+1} \cdots y_j = [a'Y']$$

    by the regular commutation relations among nonadjacent transpositions.
    - Now suppose that $a \notin Y$, h is minimal such that $a < y_h$ and $x_h = a$. Then $h > 1$ and $x_h = a > y_{h-1}$ while $x_{h-1} < y_{h-1}$ so that $x_h x_1 \cdots x_{h-1} \sim x_1 \cdots x_{h-1} x_h$ which along with $x_h^2 \sim x_h$ implies $aX \sim X$ so that $YaX \sim YX$. On the other hand, $y_{h-1} < x_h < y_h$ so that $y_h y_1 \cdots y_{h-1} \sim y_1 \cdots y_{h-1} y_h$ which along with $y_h^2 \sim y_h$ implies that $y_h Y \sim Y$. But $a' = y_h$ and $Y' = Y$ so that $a'Y'X \sim YX$. Hence $[(a')Y'X] \sim [YaX]$.

$\square$

Corollary 2.10 ([BKS$^+$08])**.** *Fix $\omega \in S_{n+1}$. Hecke insertion is a bijection from Hecke words for ω to pairs (P, Q) of tableaux of the same shape where P is a Hecke tableau for ω and Q is a standard set-valued tableau.*

Proof. This follows from Proposition 2.8 and Lemma 2.9. $\square$

2.3. **A property of Hecke Insertion.**

Lemma 2.11. *If the word $w_1 \cdots w_m$ maps to (P, Q) and $w_i > w_{i+1}$ the row of Q containing $i+1$ (abusive notation for $\bar{s}_{i+1}$) is strictly below the row of Q containing i (abusive notation for $\bar{s}_i$) .*

Proof. Suppose some a is inserted into some Hecke tableau P and then some $b < a$ is inserted to the result, P'. Let ℓ denote the row in which an element is appended or disappears after insertion of a into P, and ℓ' the row of the recording tableau to which a new number is added. Let r denote the row in which an element is appended or disappears after insertion of b to P', and r' the row of the recording tableau to which new number is added. By definition we have $\ell \leq \ell'$ and $r \leq r'$. We must show that $\ell' < r'$. This will follow from $\ell < r$ unless $\ell < \ell'$ where rows $\ell, \ldots, \ell'$ of P' all have the same length. But this makes it impossible for $r \in \{\ell+1, \ldots, \ell'\}$ because appending or disappearing can only occur in a row that is shorter than the row above it. Hence $r > \ell'$ and so also $r' > \ell'$. Thus it suffices to show $\ell < r$.

Suppose that during the insertion of a into P the elements that are inserted into rows $1, \ldots, \ell$ are $a = a_1, a_2, \ldots a_\ell$. Suppose that during the insertion of b into P' the elements that are inserted into rows $1, \ldots, r$ are $b = b_1, b_2, \ldots b_r$. Now $r \geq 1$ and $b_1 < a_1$. Now let $1 < i \leq \ell$ and suppose that $r \geq i$ and $b_i < a_i$. Denote row $i-1$ of P' by $X = [x_1, \ldots, x_j, \ldots, x_k]$ and row i of of P' by $Y = [y_1, \ldots, y_j]$. If $a_i \in Y$ then since $b_i < a_i$ it follows $r \geq i+1$ and that $b_{i+1} \leq a_i$. Further, if $i < \ell$ then $a_i < a_{i+1}$ so $b_{i+1} < a_{i+1}$. If $a_i \notin Y$ then $a_i \in X$ say $a_i = x_h$ with $h > 1$ and $a_i > y_{h-1}$ (so $a_{i+1} = y_h$ if $i < \ell$). Now any element sent to be inserted to a new row must have come from the row above it so we must have that $b_i \in X$. Since $b_i < a_i$ this now means $b_i = x_{h'}$ for some $h' < h$ and in particular $b_i < y_{h-1}$ (so for one, $r \geq i+1$). From this it follows that $b_{i+1} \leq y_{h-1}$ which implies that, if $i < \ell$ then $b_{i+1} < y_h = a_{i+1}$.

Thus by induction we see that for all $i \leq \ell$ we have $r \geq i$ and $b_i < a_i$, and by one more step that in fact $r > \ell$. $\square$

Lemma 2.12. *If the word $w_1 \cdots w_m$ maps to (P, Q) and $w_i \leq w_{i+1}$ the row of Q containing $i + 1$ is weakly above the row of Q containing i.*

Proof. Suppose some a is inserted into some Hecke tableau P and then some $b \geq a$ is inserted to the result, P'. Let ℓ denote the row in which an element is appended or disappears after insertion of a into P, and ℓ' the row of the recording tableau to which new number is added. Let r denote the row in which an element is appended or disappears after insertion of b to P', and r' the row of the recording tableau to which a new number is added. By definition we have $\ell \leq \ell'$ and $r \leq r'$. We must show that $\ell' \geq r'$. Since in P' row ℓ' has more boxes than the row below it, it will suffice to show that $r \leq \ell'$ which itself follows from showing $r \leq \ell$.

Suppose that during the insertion of a into P the elements that are inserted into rows $1, \ldots, \ell$ are $a = a_1, a_2, \ldots a_\ell$. Suppose that during the insertion of b into P' the elements that are inserted into rows $1, \ldots, r$ are $b = b_1, b_2, \ldots b_r$. Moreover suppose that $r > \ell$.

First, $b_1 \geq a_1$. Now let $1 < i < \ell$ and suppose that and $b_i \geq a_i$. Denote row $i-1$ of P' by $X = [x_1, \ldots, x_j, \ldots x_k]$ and row i of of P' by $Y = [y_1, \ldots, y_j]$. If $a_i \in Y$ say $a_i = y_h$ then $a_{i+1} \leq y_{h+1}$ while $b_i \geq a_i$ implies that $b_{i+1} \geq y_{h+1}$. If $a_i \notin Y$ then $a_i \in X$ say $a_i = x_h$ with $h > 1$ and $a_i > y_{h-1}$ (so $a_{i+1} = y_h$). But $b_i \geq a_i > y_{h-1}$ so $b_{i+1} \geq y_h = a_{i+1}$.

Thus by induction $b_\ell \geq a_\ell$. Since a_ℓ is greater than or equal to all elements of row ℓ of P' it follows that this is also true of b_ℓ and so $r = l$ contradicting the assumption $r > \ell$. $\qquad\square$

Corollary 2.13. *If the word $w_1 \cdots w_m$ maps to (P, Q). Then $w_i > w_{i+1}$ if and only if $i + 1$ shows up in a row strictly below i in Q.*

Proof. This is combining the lemmas. $\qquad\square$

Remark 2.14. Unfortunately (this fact will cause mild consternation later) it is not true that if $w_1 \cdots w_m$ maps to (P, Q) then $w_i < w_{i+1}$ if and only if $i + 1$ shows up in a column to the right of i in Q. For instance applying Hecke insertion to $\bar{s}_1 \bar{s}_3 \bar{s}_2 \bar{s}_2$, which we write through abuse of notation as 1322 results in:

$$\left\{ \boxed{1} , \boxed{1} \right\}, \left\{ \boxed{1\ 3} , \boxed{1\ 2} \right\}, \left\{ \begin{array}{c} \boxed{1\ 2} \\ \boxed{3} \end{array} , \begin{array}{c} \boxed{1\ 2} \\ \boxed{3} \end{array} \right\}, \left\{ \begin{array}{c} \boxed{1\ 2} \\ \boxed{3} \end{array} , \begin{array}{c} \boxed{1\ 24} \\ \boxed{3} \end{array} \right\}$$

showing that although $4 = 3 + 1$ shows up in a column to the right of 3 in Q, it is not true that $w_3 = 2 < 2 = w_4$. As another example applying Hecke insertion to 1312 results in:

$$\left\{ \boxed{1} , \boxed{1} \right\}, \left\{ \boxed{1\ 3} , \boxed{1\ 2} \right\}, \left\{ \begin{array}{c} \boxed{1\ 3} \\ \boxed{3} \end{array} , \begin{array}{c} \boxed{1\ 2} \\ \boxed{3} \end{array} \right\}, \left\{ \begin{array}{c} \boxed{1\ 2} \\ \boxed{3} \end{array} , \begin{array}{c} \boxed{1\ 2} \\ \boxed{34} \end{array} \right\}$$

showing that although $w_3 = 1 < 2 = w_4$ it is not true that $4 = 3 + 1$ shows up in a column to the right of 3 in Q.

A Hecke word that has been partitioned into $m + 1$ groups of transpositions with decreasing indices is known as a *Hecke factorization*. For instance $(\bar{s}_3 \bar{s}_2)(\bar{s}_3 \bar{s}_2 \bar{s}_1)()(\bar{s}_1)$, which we also may write simply as $(32)(321)()(1)$ is a Hecke factorization with four factors for the permutation $(4, 1, 3, 2) \in S_4$. A Hecke word that has been partitioned into $m + 1$ groups of transpositions with weakly increasing indices is known as a *Hecke grouping*. If $\mathfrak{f}$ is a Hecke factorization or Hecke grouping we denote by $wt(\mathfrak{f})$ the vector whose i^{th} coordinate records the number of entries in the ith factor/group (from left to right) of $\mathfrak{f}$. Denote the set of all Hecke factorizations associated to ω with $m + 1$ factors by $\mathcal{F}_\omega$. Denote the set of all increasing Hecke groupings associated to ω with $m + 1$ factors by $^*\mathcal{F}_\omega$.

Definition 2.15. A (semistandard) *set-valued tableau* or *SVT* of shape λ is a filling of a Young diagram of shape λ with the letters $\{1, \ldots, m, m+1\}$ (repetition allowed) for some integer m such that each box contains a nonempty set of numbers and

- If box b lies left of box b' then $max(b) \leq max(b')$.
- If box b lies above box b' then $max(b) < max(b')$.

Definition 2.16. A (semistandard) *multiset-valued tableau* or *MVT* of shape λ is a filling of a Young diagram of shape λ with the letters $\{1, \ldots, m, m+1\}$ (repetition allowed) for some integer m such that each box contains a nonempty multiset of numbers and

- If box b lies left of box b' then $max(b) \leq max(b')$.
- If box b lies above box b' then $max(b) < max(b')$.

A reverse set-valued tableau or $SVT^{\perp}$ is a tableau P such that its transpose, P^t, is a set-valued tableau. A reverse multiset-valued tableau or $MVT^{\perp}$ is a tableau Q such that its transpose, Q^t, is a multiset-valued tableau. The weight of a tableau T of any four of these types is the vector, $wt(T)$, whose i^{th} coordinate records the number of is in the tableau.

Lemma 2.17 ([BKS$^+$08])**.** *There is a weight preserving bijection between the following sets:*

- *$\mathcal{F}_\omega$ to pairs (P, Q) where $P \in HT_\omega$ and $Q \in SVT^{\perp}$ have the same shape.*
- *$^*\mathcal{F}_\omega$ to pairs (P, Q) where $P \in HT_\omega$ and $Q \in MVT$ have the same shape.*

Proof. A Hecke factorization is just a Hecke word, w, along with a partition of $\{1, 2, \ldots, \ell(w)\}$ into parts such that

- Each part contains consecutive numbers.
- If a and $a + 1$ occur in the same part, $w_a > w_{a+1}$.

But an $SVT^{\perp}$ is just a standard set valued tableau, T, along with a partition of $\{1, 2, \ldots, max(T)\}$ into part such that

- Each part contains consecutive numbers.
- If a and $a + 1$ occur in the same part, $a + 1$ lies strictly lower than a in T.

Corollary 2.10 gives a bijection from Hecke words for ω to pairs (P, Q) of tableaux where $P \in HT_\omega$ and Q is a standard set-valued tableau of the same shape. Corollary 2.13 implies that if $w \to (P, Q)$ then a certain partition of $\{1, 2, \ldots, \ell(w)\}$ turns w into a Hecke factorization if and only if the same partition turns Q into an $SVT^{\perp}$. Thus combining the bijection of corollary 2.10 with the identity on the partitions induces the first weight preserving bijection of the lemma. A Hecke grouping is just a Hecke word, w, along with a partition of $\{1, 2, \ldots, \ell(w)\}$ into parts such that

- Each part contains consecutive numbers.
- If a and $a + 1$ occur in the same part, $w_a \leq w_{a+1}$.

But an MVT is just a standard set valued tableau, T, along with a partition of $\{1, 2, \ldots, max(T)\}$ into part such that

- Each part contains consecutive numbers.
- If $a < a + 1$ occur in the same part, $a + 1$ lies either in the same box as a or strictly right of a in T.

Again corollary 2.13 implies that if $w \to (P, Q)$ under the bijection of Corollary 2.10 then a certain partition of $\{1, 2, \ldots, \ell(w)\}$ turns w into a Hecke grouping if and only if the same partition turns Q into an MVT. Thus combining the bijection of corollary 2.10 with the identity on the partitions induces the second weight preserving bijection of the lemma. $\square$

2.4. **Combinatorial Definition.** A Hecke factorization of an element in S_{n+1} with $n+1$ factors that has only entries with indices of at least i in the ith subdivision is known as a *bounded Hecke factorization*. For instance, $(\bar{s}_3\bar{s}_2\bar{s}_1)(\bar{s}_2)(\bar{s}_3)()$, or $(321)(2)(3)()$ is a bounded Hecke factorization in S_4. (Note that the definition implies the last factor is always empty.)

Definition 2.18. The combinatorial Grothendieck polynomial in the variables $\mathbf{x} = (x_1, \ldots, x_{n+1})$ is defined by:

$$\mathfrak{G}_\omega^c = \sum_{\mathfrak{f} \in \mathfrak{F}_\omega} \mathbf{x}^{wt(\mathfrak{f})}$$

where $\mathfrak{F}_\omega$ is the set of bounded Hecke factorizations for ω.

Theorem 2.19 ([Las90])**.** *We have that* $\mathfrak{G}_\omega^c = \mathfrak{G}_\omega$.

Proof. We prove with induction on inversion number of ω. First, suppose that ω has the maximum inversion number for a permutation in S_{n+1} (i.e, $\omega = \omega_0$). Then $\mathfrak{G}_\omega = \pi_{(\omega_0^{-1}\omega_0)}(x_1^n x_2^{n-1} \cdots x_n^1 x_{n+1}^0) = x_1^n x_2^{n-1} \cdots x_n^1 x_{n+1}^0$. On the other hand it is not difficult to check that the only element of $\mathfrak{F}_{\omega_0}$ is $(\bar{s}_n \cdots \bar{s}_1)(\bar{s}_n \cdots \bar{s}_2) \cdots (\bar{s}_n \bar{s}_{n-1})(\bar{s}_n)()$ which shows that also $\mathfrak{G}_\omega^c = x_1^n x_2^{n-1} \cdots x_n^1 x_{n+1}^0$. Now suppose that ω has fewer inversions than ω_0 and the result is known for all permutations with more inversions than ω. Let s_r be a transposition such that ωs_r has one more inversion than ω. By definition we have $\pi_{s_r}\mathfrak{G}_{\omega s_r} = \mathfrak{G}_\omega$ and by induction we have $\mathfrak{G}_{\omega s_r}^c = \mathfrak{G}_{\omega s_r}$ so that $\pi_{s_r}\mathfrak{G}_{\omega s_r}^c = \mathfrak{G}_\omega$ therefore it suffices to show that $\pi_{s_r}\mathfrak{G}_{\omega s_r}^c = \mathfrak{G}_\omega^c$. In general:

$$\mathfrak{G}_\mu^c = \sum_{\mathfrak{f}\in\mathfrak{F}_\mu} x_1^{|f_1|} \cdots x_{n+1}^{|f_{n+1}|}$$

where f_i refers to the ith factor of the bounded Hecke factorization $\mathfrak{f}$ and $|f_i|$ refers to the length of this factor. It suffices therefore to show that if we fix $\mathfrak{f}^- = (f_1, \ldots, f_{r-1})$ and $\mathfrak{f}^+ = (f_{r+2}, \ldots, f_{n+1})$ then:

$$(2.1) \qquad \pi_{s_r} \sum_{(f_r, f_{r+1})\in\mathfrak{f}_{\omega s_r}^\pm} x_r^{|f_r|}x_{r+1}^{|f_{r+1}|} = \sum_{(f_r, f_{r+1})\in\mathfrak{f}_\omega^\pm} x_r^{|f_r|}x_{r+1}^{|f_{r+1}|}$$

where $\mathfrak{f}_{\omega s_r}^\pm$ (respectively $\mathfrak{f}_\omega^\pm$) is the set of pairs (f_r, f_{r+1}) such that $\mathfrak{f}^- f_r f_{r+1}\mathfrak{f}^+$ is an element of $\mathfrak{F}_{\omega s_r}$ (respectively $\mathfrak{F}_\omega$). Now (f_r, f_{r+1}) maps to some pair of tableaux of the same shape (P, Q) where P is a Hecke tableau for $f_r f_{r+1}$ and Q is an $SVT^\perp$ in the letters $\{1, 2\}$. First we consider the case when $f_r f_{r+1}$ is a Hecke word for a permutation that has no reduced word ending with s_r. This happens if and only if the Hecke insertion tableau for $f_r f_{r+1}$ has the form:

r	$r+1$
$r+1$	$r+2$
$\vdots$	$\vdots$
$r+k-1$	$r+k$
a_0	b_0
a_1	b_1
$\vdots$	$\vdots$
$\vdots$	b_t
$\vdots$	
a_s	

where $r \le r + k \le n$ and $a_0 > r + k$ (unless there are no a_i which we allow as a possibility) and all rows and columns are strictly increasing. We denote this form as the form (0) and denote by $\mathfrak{f}^{\pm}_{\omega s_r}(0)$ the subset of $\mathfrak{f}^{\pm}_{\omega s_r}$ with form (0) (in other words, the subset such that $f_r f_{r+1}$ is a Hecke word for a permutation that has no reduced word ending with s_r). Next we consider the case when $f_r f_{r+1}$ is a Hecke word for a permutation that has a reduced word ending with s_r. This happens if and only if the Hecke insertion tableau for $f_r f_{r+1}$ has the form:

r	$r+1$
$r+1$	$r+2$
$\vdots$	$\vdots$
$\vdots$	$r+k$
$r+k$	b_0
a_1	b_1
$\vdots$	$\vdots$
$\vdots$	b_t
$\vdots$	
a_s	

where $r \le r + k \le n$ and $b_0 > r + k + 1$ (unless there are no b_i, which we allow as a possibility) and all rows and columns are strictly increasing. Among such tableaux there are three possibilities:

(1) $s > t$ and $a_i < b_{i-1}$ for all $i \le t + 1$.
(2) $s > t$ and $a_i \le b_{i-1}$ for all $i \le t + 1$ and $a_i = b_{i-1}$ for some $i \le t + 1$.
(3) $s = t$ or $a_i > b_{i-1}$ for some $i \le t + 1$.

Let $\mathfrak{f}^{\pm}_{\omega s_r}(*)$ denote the subset of $\mathfrak{f}^{\pm}_{\omega s_r}$ such that $f_r f_{r+1}$ is a Hecke word for a permutation that has a reduced word ending with s_r, and the insertion tableau of (f_r, f_{r+1}) is of form $(*)$ above for $* \in \{1, 2, 3\}$.

Claim 2.20. *Let $* = 1$ or $* = 2$. Suppose that applying Hecke insertion to $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(*)$ results in the pair (P, Q). Then Q is the tableau with all 1s in the first column and all 2s in the second column.*

Proof. Otherwise the bottom box of the first column of Q contain a 2. The first step of the inverse of Hecke insertion would be to outsert (with or without box removal) from a box in the first column of P. In light of the inequalities $a_i \le b_{i-1}$ it is not difficult to check that this would result in the outsertion of r from the first row. This would imply that the second factor, f_{r+1} ends in r which is impossible since all elements of f_{r+1} must be greater than or equal to $r+1$ by the boundedness condition. $\qquad\square$

Claim 2.21. *Let $* = 0$ or $* = 3$. Suppose that applying Hecke insertion to $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(*)$ gives the insertion tableau P. Then for any $SVT^{\perp}$, Q of the same shape as P there is exactly one $(f'_r, f'_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(*)$ mapping to (P, Q). The same holds if $\mathfrak{f}^{\pm}_{\omega s_r}(*)$ is replaced by $\mathfrak{f}^{\pm}_{\omega}$.*

Proof. By Lemma 2.17 there is exactly one (unbounded) Hecke factorization of the form (f'_r, f'_{r+1}) mapping to (P, Q). We just need to show that it is bounded, that is, all elements of f'_r are at least r and all elements of f'_{r+1} are at least $r+1$. By the boundedness of (f_r, f_{r+1}) this just means showing there is no r in f'_{r+1}. However, in the cases of $* = 0$ or $\mathfrak{f}^{\pm}_{\omega}$ if f'_{r+1} contains an r it would have to end in r so $f'_r f'_{r+1}$ and $f_r f_{r+1}$ would have to represent a permutation with a reduced word ending in s_r. In the first case this contradicts the definition of $\mathfrak{f}^{\pm}_{\omega}(0)$ and in the second it contradicts the choice of s_r. Moreover, in the case of $* = 3$, the inequality $a_i > b_{i-1}$ for some i implies that the first step of the inverse of Hecke insertion results in a number greater than r eventually being outserted from row 1. Since this must be the smallest number in f'_{r+1} it follows there is no r in f'_{r+1}. $\square$

Claim 2.22. *Suppose $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}$ maps to (P, Q) where Q is the tableau with all 1s in the first row and all 2s in the second row, then the first column of P read bottom to top is f_r and the second column of P read bottom to top is f_{r+1}.*

Proof. Applying inverse Hecke insertion to (P, Q) clearly just pulls out the right and then left column of P. $\square$

Now for some more notation: Recall if two Hecke words represent the same permutation, we denote this by writing a "$\sim$" between them. Moreover, if μ is any permutation, let $\widetilde{\mu}$ denote an arbitrary Hecke word for μ.

We write $\mathfrak{f}^{\pm}_{\omega s_r}(1) = \mathfrak{f}^{\pm}_{\omega s_r}(1a) \cup \mathfrak{f}^{\pm}_{\omega s_r}(1b)$ as follows. Let $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(1)$ and let μ be a permutation such that μs_r is the permutation represented by $f_r f_{r+1}$. Then $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(1a)$ if as concatenated Hecke words $\mathfrak{f}^{-}\widetilde{\mu}\bar{s}_r \not\sim \mathfrak{f}^{-}\widetilde{\mu}$. On the other hand, $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(1b)$ if as concatenated Hecke words $\mathfrak{f}^{-}\widetilde{\mu}\bar{s}_r \sim \mathfrak{f}^{-}\widetilde{\mu}$. (Technically we are abusing notation since $\mathfrak{f}^{-}$ is a Hecke factorization and here we are using it to just represent the Hecke word obtained by erasing the parentheses in this factorization.) All in all, we have so far broken down $\mathfrak{f}^{\pm}_{\omega s_r}$ as the union:

$$\mathfrak{f}^{\pm}_{\omega s_r} = \mathfrak{f}^{\pm}_{\omega s_r}(0) \cup \mathfrak{f}^{\pm}_{\omega s_r}(1a) \cup \mathfrak{f}^{\pm}_{\omega s_r}(1b) \cup \mathfrak{f}^{\pm}_{\omega s_r}(2) \cup \mathfrak{f}^{\pm}_{\omega s_r}(3)$$

For each of $\mathfrak{f}^{\pm}_{\omega}$ and $\mathfrak{f}^{\pm}_{\omega s_r}(0)$ and $\mathfrak{f}^{\pm}_{\omega s_r}(3)$, let $\overline{\mathfrak{f}^{\pm}_{\omega}}$ and $\overline{\mathfrak{f}^{\pm}_{\omega s_r}}(0)$ and $\overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3)$ denote the subset of these sets composed of factors (f_r, f_{r+1}) that map under Hecke insertion to a pair (P, Q) where Q is the tableau with all 1s in the first column and all 2s in the second column.

Claim 2.23. *There is a bijection which sends a pair of factors with lengths (ℓ_1, ℓ_2) to a pair of factors with lengths $(\ell_1 - 1, \ell_2)$ between each of the following pairs of sets:*

(i) $\mathfrak{f}^{\pm}_{\omega s_r}(2) \longrightarrow \overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3)$
(ii) $\mathfrak{f}^{\pm}_{\omega s_r}(1b) \longrightarrow \overline{\mathfrak{f}^{\pm}_{\omega s_r}}(0)$
(iii) $\mathfrak{f}^{\pm}_{\omega s_r}(1a) \longrightarrow \overline{\mathfrak{f}^{\pm}_{\omega}}$

Any factorization into two factors can be represented as a two column array by using the first factor as the first column and the second factor as the second column

where left to right order in factors corresponds to bottom to top order in columns. By claim 2.20 and definitions, we have that:

- $\overline{\mathfrak{f}^{\pm}_{\omega s_r}}(0)$ is exactly the subset of elements of $\mathfrak{f}^{\pm}_{\omega s_r}$ whose corresponding two column array is a valid Hecke tableau of the form 0.
- $\mathfrak{f}^{\pm}_{\omega s_r}(1)$ is exactly the subset of elements of $\mathfrak{f}^{\pm}_{\omega s_r}$ whose corresponding two column array is a valid Hecke tableau of the form 1.
- $\mathfrak{f}^{\pm}_{\omega s_r}(2)$ is exactly the subset of elements of $\mathfrak{f}^{\pm}_{\omega s_r}$ whose corresponding two column array is a valid Hecke tableau of the form 2.
- $\overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3)$ is exactly the subset of elements of $\mathfrak{f}^{\pm}_{\omega s_r}$ whose corresponding two array tableau is a valid Hecke tableau of the form 3.
- $\overline{\mathfrak{f}^{\pm}_{\omega}}$ is exactly the subset of elements of $\mathfrak{f}^{\pm}_{\omega}$ whose corresponding two column array is a valid Hecke tableau.

Example 2.24. Let $\omega = (1, 2, 4, 5, 3, 7, 6) \in S_7$ and suppose $r = 3$. Suppose that $\mathfrak{f}^- = (3)(4)$ and $\mathfrak{f}^+ = ()()()$. Then we have that:

$$\overline{\mathfrak{f}^{\pm}_{\omega s_r}}(0) = \left\{\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 6 & \\ \hline \end{array}\right\}, \quad \mathfrak{f}^{\pm}_{\omega s_r}(1a) = \left\{\begin{array}{|c|c|} \hline 3 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 6 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 \\ \hline 4 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 6 \\ \hline 4 \\ \hline 6 \\ \hline \end{array}\right\}$$

$$\mathfrak{f}^{\pm}_{\omega s_r}(1b) = \left\{\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 \\ \hline 6 \\ \hline \end{array}\right\}, \quad \mathfrak{f}^{\pm}_{\omega s_r}(2) = \left\{\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 & 6 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 6 \\ \hline 6 \\ \hline \end{array}\right\}$$

$$\overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3) = \left\{\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 & 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 6 \\ \hline \end{array}\right\}, \quad \mathfrak{f}^{\pm}_{\omega} = \left\{\begin{array}{|c|c|} \hline 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 6 \\ \hline 6 \\ \hline \end{array}\right\}$$

Proof of (i). We describe the bijection and its inverse on the corresponding arrays. The reader is welcomed to apply the construction given below to the left-hand side of the example below. If the construction is interpreted correctly he will arrive at the right-hand side after applying the bijection. Here $r = 2$ and $k = 1$.

$$(f_r, f_{r+1}) = (876432)(97653) = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline 6 & 7 \\ \hline 7 & 9 \\ \hline 8 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline 6 & 7 \\ \hline 8 & 9 \\ \hline \end{array} = (86432)(97653) = (f'_r, f'_{r+1})$$

Given, $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(2)$ find the maximal j such that the equality $a_j = b_{j-1}$ is satisfied and remove a_j from the first column, sliding all entries below it down one row to form (f'_r, f'_{r+1}) Note that:

- $f_r f_{r+1}$ and $f'_r f'_{r+1}$ represent the same permutation: Find h minimal such that $a_j, a_{j-1}, \ldots, a_{h+1}, a_h$ is a consecutive string of numbers appearing in f_r. We must have $h > 1$ by the condition $b_0 > r + k + 1$ and these numbers must be equal to $b_{j-1}, b_{j-2}, \ldots, b_h, b_{h-1}$ by the condition $a_i \leq b_{i-1}$. Moreover by the maximality of j we must have that either $j - 1 = t$ or else $a_{j+1} < b_j$ whence $a_j < b_j - 1$. This combined with $a_h > a_{h-1}+1$ along with the Hecke relation $a_j a_{j-1} \cdots a_{h+1} a_h a_j a_{j-1} \cdots a_{h+1} a_h \sim a_{j-1} \cdots a_{h+1} a_h a_j a_{j-1} \cdots a_{h+1} a_h$ implies that removing a_j does not change the permutation represented.
- The result is a valid Hecke tableau. First, (f'_r, f_r) is clearly column strict and also row strict, the latter of which is immediate from the maximality of j.
- $(f'_r, f'_{r+1}) \in \overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3)$. Either $s = t + 1$ in which case f'_r and f'_{r+1} have the same length so that the pair is automatically in $\mathfrak{f}^{\pm}_{\omega s_r}(3)$ or else $s \geq t + 2$. In the second case $j + 1 \leq s$ so that when a_j is removed, some a_{j+1} will lie diagonally above b_{j-1} and we have that $a_{j+1} > a_j = b_{j-1}$. Therefore the array of (f'_r, f'_{r+1}) is a valid Hecke tableau of the form (3).

Next we describe the inverse. Let $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3)$. If any j such that the inequality $a_j > b_{j-1}$ is satisfied exists find the minimal such j and insert a copy of b_{j-1} to the first column between a_{j-1} and a_j. If this is not possible then $s = t$ and in this case just add a copy of b_t to the bottom of column one. The result is (f'_r, f'_{r+1}).

- The result is a valid Hecke tableau. Certainly the tableau is still row strict since $b_{j-1} < b_j$ and all the elements below the b_{j-1} that was added to column one have just moved down a row. Moreover the result is also column strict as $a_{j-1} < b_{j-1} < a_j$.
- $(f'_r, f'_{r+1}) \in \mathfrak{f}^{\pm}_{\mu}(2)$ for some μ (it will turn out that $\mu = \omega s_r$). By the minimality of j all elements lying above the added b_{j-1} are less than or equal to the element diagonally above them. By the fact they were moved down a row, all the elements lying below the added b_{j-1} are less than or equal to (in fact strictly less than) the element diagonally above them. Finally the element lying diagonally above the added b_{j-1} is less than or equal to (actually equal to) b_{j-1} so that result is an element of $\mathfrak{f}^{\pm}_{\mu}(2)$ for some μ.

Finally note that

- If $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(2)$ maps to (f'_r, f'_{r+1}) then applying the inverse to (f'_r, f'_{r+1}) gives back (f_r, f_{r+1}): When the element a_j is removed from row j of (f_r, f_{r+1}), a_{j+1} moves into row j and becomes an element greater than the element diagonally above it. Since all elements in the first row of (f_r, f_{r+1}) are less than or equal to the elements diagonally above them this is the minimal j where this occurs in (f'_r, f'_{r+1}). Thus the inverse applied to (f'_r, f'_{r+1}) just adds back $b_{j-1} = a_j$ to column one.
- If $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3)$ maps under the inverse to (f'_r, f'_{r+1}) then applying the inverse to (f'_r, f'_{r+1}) gives back (f_r, f_{r+1}): When the element b_{j-1} is added to row j of (f_r, f_{r+1}), it becomes an element that is equal to the element diagonally above it. Since all elements below it have been moved down a row they are less than the elements diagonally above them. Hence this is

the maximal j where such equality occurs in (f'_r, f'_{r+1}). Thus the forwards map applied to (f'_r, f'_{r+1}) just takes away the b_{j-1} added to column one.

By the second of these we can conclude that the inverse must map $\overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3)$ into $\mathfrak{f}^{\pm}_{\mu}(2)$ for $\mu = \omega s_r$ since the forwards map sends $\mathfrak{f}^{\pm}_{\mu}(2)$ to $\overline{\mathfrak{f}^{\pm}_{\mu}}(3)$ for any μ. Combining all of this shows that the given map is in fact a bijection from $\mathfrak{f}^{\pm}_{\omega s_r}(2)$ to $\overline{\mathfrak{f}^{\pm}_{\omega s_r}}(3)$. $\square$

Proof of (ii) and (iii). We describe the bijection and its inverse on the corresponding two column arrays. The reader is welcomed to apply the construction given below to the left-hand side of the example below. If the construction is interpreted correctly he will arrive at the right-hand side after applying the bijection. Here $r = 2$ and $k = 2$.

$$(f_r, f_{r+1}) = (875432)(8643) = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 4 & 6 \\ \hline 5 & 8 \\ \hline 7 & \\ \hline 8 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline 8 & \\ \hline \end{array} = (87532)(8643) = (f'_r, f'_{r+1})$$

Let $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(1)$. Now simply remove the letter $r + k$ from the first column and move the elements below it up one row to create the two column array corresponding to (f'_r, f'_{r+1}). Note that:

- $\mathfrak{f}^- f'_r f'_{r+1} \mathfrak{f}^+$ represents the correct permutation: Suppose the permutation represented by $f_r f_{r+1}$ is μs_r. The relation $(r+k)\ldots(r)(r+k)\ldots(r+1) \sim (r+k-1)\ldots(r)(r+k)\ldots(r)$ and the fact that $b_0 > r+k+1$ means that $f'_r f'_{r+1}$ represents the permutation μ. Thus if $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(1b)$, then $\mathfrak{f}^- f_r f_{r+1}$ and $\mathfrak{f}^- f'_r f'_{r+1}$ represent the same permutation so $\mathfrak{f}^- f'_r f'_{r+1} \mathfrak{f}^+$ represents ωs_r. On the other hand if $(f_r, f_{r+1}) \in \mathfrak{f}^{\pm}_{\omega s_r}(1a)$, then $\mathfrak{f}^- f_r f_{r+1}$ and $\mathfrak{f}^- f'_r f'_{r+1}$ represent different permutations from which it follows that $\mathfrak{f}^- f'_r f'_{r+1} \mathfrak{f}^+$ must represent ω.
- (f'_r, f'_{r+1}) corresponds to a valid Hecke tableau. The only thing to check is row strictness. But (f_r, f_{r+1}) is of the form (1), thereby satisfying the inequalities $a_i < b_{i-1}$ so moving all the a_i up a one row does not break row strictness.
- The maps map into the proposed images: For bijection (iii) this follows from the first two bullet points above. For bijection (ii), we need to still check the image has form (0). But this is clear since removing $r + k$ from the form (1) always results in the form (0).

Now let $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^{\pm}_{\omega s_r}}(0)$ or $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^{\pm}_{\omega}}$ be represented as a two column tableau of the form (0). (Note that $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^{\pm}_{\omega}}$ cannot have any reduced word for $f_r f_{r+1}$ ending in s_r and therefore automatically has form (0)). Let k be maximal such that $r+1, r+2, \ldots, r+k$ is a consecutive string of elements contained in f_{r+1}. Form $(f'_r f'_{r+1})$ by adding an $r + k$ to column one. Note that:

- $\mathfrak{f}^- f'_r f'_{r+1} \mathfrak{f}^+$ represents the correct permutation: Suppose the permutation represented by $f_r f_{r+1}$ is μ. The relation $(r+k)\ldots(r)(r+k)\ldots(r+1) \sim (r+k-1)\ldots(r)(r+k)\ldots(r)$ and the fact that $b_0 > r+k+1$ and the fact that there is no reduced word for $f_r f_{r+1}$ ending in s_r implies that $f'_r f'_{r+1}$ represents the permutation μs_r. Now if $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^\pm_{\omega s_r}}(0)$, then $\mathfrak{f}^- f_r f_{r+1} \mathfrak{f}^+$ is a Hecke word for ωs_r so that $\mathfrak{f}^- f_r f_{r+1}$ must have a reduced word ending in s_r. Hence $\mathfrak{f}^- f'_r f'_{r+1} \sim \mathfrak{f}^- f_r f_{r+1}$ so $\mathfrak{f}^- f'_r f'_{r+1} \mathfrak{f}^+$ represents ωs_r. On the other hand if $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^\pm_\omega}$, then $\mathfrak{f}^- f_r f_{r+1} \mathfrak{f}^+$ is Hecke word for ω and ω has no reduced word ending in s_r by assumption so $\mathfrak{f}^- f'_r f'_{r+1} \mathfrak{f}^+$ represents ωs_r.
- $(f'_r f'_{r+1})$ is a valid Hecke tableau. This follows from the fact that $r+k$ does not exist in f_r by definition of form (0).
- The inverse maps into the proper sets: Clearly adding $r+k$ to the first column of the form (0) results in the form (1) since the elements moved down a row will be strictly less than the elements diagonally above them. Now, let μ be the permutation representing $f_r f_{r+1}$. If $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^\pm_{\omega s_r}}(0)$, then the first step says $\mathfrak{f}^- \widetilde{\mu} \mathfrak{f}^+$ represents the same permutation as $\mathfrak{f}^- \widetilde{\mu} \bar{s}_r \mathfrak{f}^+$ which in fact means $\mathfrak{f}^- \widetilde{\mu}$ represents the same permutation as $\mathfrak{f}^- \widetilde{\mu} \bar{s}_r$. But μ is a permutation such that μs_r is the permutation represented by $f'_r f'_{r+1}$ so by definition we have that $(f'_r, f'_{r+1}) \in \mathfrak{f}^\pm_{\omega s_r}(1b)$. If $(f_r, f_{r+1}) \in \overline{\mathfrak{f}^\pm_\omega}$, the first step says that $\mathfrak{f}^- f_r f_{r+1} \mathfrak{f}^+$ and $\mathfrak{f}^- f'_r f'_{r+1} \mathfrak{f}^+$ represent different permutations so that we must have $\mathfrak{f}^- \widetilde{\mu} \bar{s}_r \not\sim \mathfrak{f}^- \widetilde{\mu}$. Again μ is a permutation such that μs_r is the permutation represented by $f'_r f'_{r+1}$ so that $(f'_r, f'_{r+1}) \in \mathfrak{f}^\pm_{\omega s_r}(1a)$.

Finally, the given maps are clearly mutual inverses. $\qquad\square$

Claim 2.25. *We have:*
$$\sum_{Q \in SVT^\perp(\ell_1, \ell_2)} x_r^{(\#1)} x_{r+1}^{(\#2)} = \pi_{s_r}(x_r^{\ell_1+1} x_{r+1}^{\ell_2})$$

where the left hand sum is over all $SVT^\perp$ with columns of lengths $\ell_1 \geq \ell_2$ in the letters $\{1,2\}$ and $(\#1)$ and $(\#2)$ is the number of 1s and 2s respectively in Q.

Proof. Counting the elements in the left had sum and doing the required algebra on the right hand side shows both are equal to:
$$x_r^{\ell_1} x_{r+1}^{\ell_2} + x_r^{\ell_1-1} x_{r+1}^{\ell_2+1} + \cdots + x_r^{\ell_2+1} x_{r+1}^{\ell_1-1} + x_r^{\ell_2} x_{r+1}^{\ell_1} +$$
$$x_r^{\ell_1} x_{r+1}^{\ell_2+1} + x_r^{\ell_1-1} x_{r+1}^{\ell_2+2} + \cdots + x_r^{\ell_2+2} x_{r+1}^{\ell_1-1} + x_r^{\ell_2+1} x_{r+1}^{\ell_1}$$
$\qquad\square$

Claim 2.26.
$$\pi_{s_r} \sum_{(f_r, f_{r+1}) \in \mathfrak{f}^\pm_{\omega s_r}(2) \cup \mathfrak{f}^\pm_{\omega s_r}(1b)} x_r^{|f_r|} x_{r+1}^{|f_{r+1}|} = -\pi_{s_r} \sum_{(f_r, f_{r+1}) \in \mathfrak{f}^\pm_{\omega s_r}(3) \cup \mathfrak{f}^\pm_{\omega s_r}(0)} x_r^{|f_r|} x_{r+1}^{|f_{r+1}|}$$

Proof. Because of the bijections of claim 2.23 as well the results of claims 2.20 and 2.21 it suffices to show that if $(f_r, f_{r+1}) \to (f'_r, f'_{r+1})$ under either bijection i or bijection ii then we have:
$$\pi_{s_r}(x_r^{|f_r|} x_{r+1}^{|f_{r+1}|}) = -\pi_{s_r} \sum_{Q \in SVT^\perp(|f'_r|, |f'_{r+1}|)} x_r^{(\#1)} x_{r+1}^{(\#2)}$$

But from claim 2.25 this just says that:

$$\pi_{s_r}(x_r^{|f_r|}x_{r+1}^{|f_{r+1}|}) = -\pi_{s_r}\pi_{s_r}x_r^{|f_r'|+1}x_{r+1}^{|f_{r+1}'|}$$

which follows from part (4) of Lemma 2.2 since $|f_r'| = |f_r| - 1$ and $|f_{r+1'}| = |f_{r+1}|$. $\qquad\square$

It follows from the last claim that the sum on the left hand side of equation 2.1 can be taken over just $\mathfrak{f}_{\omega s_r}^{\pm}(1a)$ rather than all of $\mathfrak{f}_{\omega s_r}^{\pm}$, so we just need to prove that:

Claim 2.27.

$$\pi_{s_r}\sum_{(f_r,f_{r+1})\in\mathfrak{f}_{\omega s_r}^{\pm}(1a)} x_r^{|f_r|}x_{r+1}^{|f_{r+1}|} = \sum_{(f_r,f_{r+1})\in\mathfrak{f}_{\omega}^{\pm}} x_r^{|f_r|}x_{r+1}^{|f_{r+1}|}$$

Proof. Because of the bijection (iii) and the results of claims 2.20 and 2.21 it suffice to show that if $(f_r, f_{r+1}) \to (f_r', f_{r+1}')$ under bijection (iii) then we have:

$$\pi_{s_r}(x_r^{|f_r|}x_{r+1}^{|f_{r+1}|}) = \sum_{Q\in SVT^{\perp}(|f_r'|,|f_{r+1}'|)} x_r^{(\#1)}x_{r+1}^{(\#2)}$$

But this is immediate from 2.25 because $|f_r'| = |f_r| - 1$ and $|f_{r+1'}| = |f_{r+1}|$. $\qquad\square$

$\qquad\square$

3. Double Grothendieck Polynomials

We will give four definitions of the double Grothendieck polynomial, eventually showing that they are all equivalent. To proceed we consider two sets of variables $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$. We extend the action of π_{s_r} linearly over $\mathbb{Z}[y]$ to get an action on $\mathbb{Z}[x, y]$.

A *circled Hecke factorization* is a factorization of a Hecke word into factors, where some of the elements have been circled. Moreover, each factor must be decreasing in the order $①< 1 < ② < \cdots < ⓝ < n$. For instance $(32②)(③21①)()(①)$ is a circled Hecke factorization for the permutation $(4, 1, 3, 2) \in S_4$.

A *bounded circled Hecke factorization* is a circled Hecke factorization with $n + 1$ factors such that all the elements in the ith factor are $\geq ⓘ$. For instance, $(④3②1)(3③)(43③)(4)()$ is a bounded circled Hecke factorization for the permutation $(5, 1, 4, 3, 2) \in S_5$. The *x-weight* of a bounded circled Hecke factorization is the vector whose i^{th} entry records the number of uncircled elements in its ith factor. The *x*-weight of the example above is $(2, 1, 2, 1, 0)$. The *y-weight* of such a factorization is the vector whose i^{th} entry records the number of circled entries that have some value j and appear in some factor k such that $j - k + 1 = i$. The *y*-weight of the example above is $(1, 2, 0, 1, 0)$.

A *double Hecke factorization* is a factorization into an even number of factors where the first half of the factors are increasing in the order $1 < \cdots < n$ and the last half of the factors are decreasing in the order $1 < \cdots < n$. For example $(12)(13)|(21)(32)$ is a double Hecke factorization for $(4, 3, 2, 1) \in S_4$, where we have drawn a "|" between the left half and the right half of the factors for viewing convenience. A *bounded double Hecke factorization* is such a factorization into $2n + 2$ factors where all elements in the ith factor to the right of center are $\geq i$ and all elements in the ith factor to the left of center are $\geq i$. For instance, $()(3)(2)(12)|(31)(32)(3)()$ is a bounded double Hecke factorization for the permutation $(4, 3, 2, 1) \in S_4$. The *x-weight* of a bounded or unbounded double Hecke

factorization is the vector whose i^{th} entry records the number of elements in the ith factor to the right of center. The x-weight of the example above is $(2, 2, 1, 0)$. The y-weight of a bounded or unbounded double Hecke factorization is the vector whose i^{th} entry records the number of entries in the ith factor to the left of center. The y-weight of the example above is $(2, 1, 1, 0)$.

- Let $\mathcal{F}_\omega^\bigcirc$ denote the set of all (unbounded) circled Hecke factorizations of ω into $m + 1$ parts.
- Let $\mathfrak{F}_\omega^\bigcirc$ denote the set of all bounded circled Hecke factorizations of ω into $n + 1$ parts.
- Let $\mathcal{F}_\omega^\square$ denote the set of all (unbounded) double Hecke factorizations of ω into $2m + 2$ parts.
- Let $\mathfrak{F}_\omega^\square$ denote the set of all bounded double Hecke factorizations of ω into $2n + 2$ parts.

If $\mathfrak{f}$ is one of the factorizations above we write $(x, y)^{wt(\mathfrak{f})}$ to mean the monomial $x^{x\text{-}weight(\mathfrak{f})} y^{y\text{-}weight(\mathfrak{f})}$. Finally, if μ is any permutation let X_μ be the set of all pairs of permutations (u, v) such that the concatenation $\widetilde{u}\widetilde{v}$ represents the permutation μ. We can now introduce the following polynomials.

Definition 3.1. The *double Grothendieck polynomial* for ω:

$$\mathfrak{G}_\omega(x, y) = \pi_{(\omega^{-1}\omega_0)} \left(\prod_{i+j \leq n+1} x_i + y_j + x_i y_j \right)$$

Definition 3.2. The *Giambelli double Grothendieck polynomial* for ω:

$$\mathfrak{G}_\omega^G(x, y) = \sum_{(u,v) \in X_\omega} \mathfrak{G}_{u^{-1}}(y) \mathfrak{G}_v(x)$$

Definition 3.3. The *combinatorial double Grothendieck polynomial* for ω:

$$\mathfrak{G}_\omega^C(x, y) = \sum_{\mathfrak{f} \in \mathfrak{F}_\omega^\bigcirc} (x, y)^{wt(\mathfrak{f})}$$

Definition 3.4. The *combinatorial Giambelli double Grothendieck polynomial* for ω:

$$\mathfrak{G}_\omega^{CG}(x, y) = \sum_{\mathfrak{f} \in \mathfrak{F}_\omega^\square} (x, y)^{wt(\mathfrak{f})}$$

Remark 3.5. Definition 3.1 is that given in [Las85]. Definition 3.2 is a generalization of the known Giambelli formula for double Schubert polynomials. Combinatorial definitions other than definitions 3.3 and 3.4 for double Schubert polynomials such as those using RC-graphs as in [BB93] have been given.

Lemma 3.6. *Definitions 3.2 and 3.4 are equivalent.*

Proof. We have:

$$\mathfrak{F}_\omega^\square = \bigcup_{(u,v) \in X_\omega} \mathfrak{F}_{(u,v)}^\square$$

where $\mathfrak{F}_{(u,v)}^\square$ is the subset of $\mathfrak{F}_\omega^\square$ such that the first $n + 1$ factors give a Hecke word for the permutation u and the second $n + 1$ factors give a Hecke word for the permutation v. However, this is precisely the set of all of factorizations such

that reading the first half of the factors from right to left gives a bounded Hecke factorization, $\mathfrak{f}_\ell$, for u^{-1} and reading the second half of the factors from left to right gives a bounded Hecke factorization, $\mathfrak{f}_r$, for v. Moreover, the y-weight of this factorization is the weight of $\mathfrak{f}_\ell$ and the x-weight of this factorization is the weight of $\mathfrak{f}_r$. From this it follows that,

$$\sum_{\mathfrak{f} \in \mathfrak{F}_{(u,v)}^{\square}} (x,y)^{wt(\mathfrak{f})} = \mathfrak{G}_{u^{-1}}(y) \mathfrak{G}_v(x)$$

which completes the proof. $\qquad\square$

To prove the next equivalence of definitions we will need a claim concerning certain pairs of Hecke words: Let μ be a permutation. Denote the set of pairs of Hecke words (a,b) where the concatenation ab is a Hecke word for μ by:

- $^s\vee^s(\mu)$ if a is strictly decreasing and b is strictly increasing.
- $^s\vee^w(\mu)$ if a is strictly decreasing and b is weakly increasing.
- $^w\vee^s(\mu)$ if a is weakly decreasing and b is strictly increasing.
- $^w\vee^w(\mu)$ if a is weakly decreasing and b is weakly increasing.
- $_s\wedge_s(\mu)$ if a is strictly increasing and b is strictly decreasing.
- $_s\wedge_w(\mu)$ if a is strictly increasing and b is weakly decreasing.
- $_w\wedge_s(\mu)$ if a is weakly increasing and b is strictly decreasing.
- $_w\wedge_w(\mu)$ if a is weakly increasing and b is weakly decreasing.

Claim 3.7. *For all $i, j \in \{s, w\}$ there is a bijection from $_i\wedge_j(\mu) \to {}^j\vee^i(\mu)$ such that, denoting word length by $|\cdot|$, if $(b,c) \to (a,d)$ then $|a| = |c|$ and $|b| = |d|$.*

Proof. Denote by $W_k^{ij}(\mu)$ the set of all quadruples of Hecke words (a,b,c,d) such that the concatenation $abcd$ is a Hecke word for μ and such that

- a and c are "j"-ly decreasing.
- b and d are "i"-ly increasing.
- b and c only contain elements from the set $\{1, 2, \ldots, k\}$.
- a and d only contain elements from the set $\{k+1, \ldots, n\}$.

It suffices to find a bijection $W_{k+1}^{ij}(\mu) \to W_k^{ij}(\mu)$, such that if $(a,b,c,d) \to (a',b',c',d')$ then $|a| + |c| = |a'| + |c'|$ and $|b| + |d| = |b'| + |d'|$. We describe the bijection by associating multisets with factors. When referring to the first and third factors a multiset refers to the factor obtained by ordering its elements in a weakly decreasing manner. When referring to the second and fourth factors a multiset refers to the factor obtained by ordering its elements in a weakly increasing manner.

First we suppose that a reduced word for μ has at least one s_k and at least one s_{k+1}. Then, we define a map on $(a,b,c,d) \in W_{k+1}^{ij}(\mu)$ by:

- Remove each k from b. For each k removed from b, add one $k+1$ to d.
- Remove each k from c. For each k removed from c, add one $k+1$ to a.
- Change each $k+1$ in b to k.
- Change each $k+1$ in c to k.

Clearly it has an inverse map on $(a',b',c',d') \in W_k^{ij}(\mu)$ by:

- Change each k in b' to a $k+1$.
- Change each k in c' to $k+1$.
- Remove each $k+1$ from d'. For each $k+1$ removed from d', add a k to b'.
- Remove each $k+1$ from a'. For each $k+1$ removed from a', add a k to c'.

It is not difficult to verify the maps do not change the permutation represented
by the Hecke word formed by the concatenation of the four factors. Nor do they
change the sum of the lengths of the first and third factors, nor the sum of the
lengths of the second and fourth factors. Moreover, if $j = s$, the maps preserve the
strictness of the decrease of the first and third factors, and if $i = s$, the strictness
of the increase of the second and fourth factors. Therefore the above map gives the
desired bijection in the case that a reduced word for μ has at least one s_k and at
least one s_{k+1}.

If a reduced word for μ does contain s_{k+1} but does not contain s_k. Then we use
the following map: Given $(a, b, c, d) \in W_{k+1}^{ij}(\mu)$ we:

- Remove each $k + 1$ from b and add it to d.
- Remove each $k + 1$ from c and add it to a.

Clearly it has an inverse map on $(a', b', c', d') \in W_k^{ij}(\mu)$ by:

- Remove each $k + 1$ from d' and add it to b'.
- Remove each $k + 1$ from a' and add it to c'.

Again the maps do not change the permutation represented by the Hecke word
formed by the concatenation of the four factors, nor do the sum of the lengths of
the first and third factors, nor the sum of the lengths of the second and fourth
factors. And again, if $j = s$, the maps preserve the strict increase of the first and
third factors and if $i = s$ the maps preserve the strict decrease of the second and
fourth factors.

Finally, if no reduced word for μ has s_{k+1} then it is clear that the identity map
gives the needed bijection. This proves the existence of a bijection $W_{k+1}^{ij}(\mu) \to
W_k^{ij}(\mu)$ with the desired properties by explicitly constructing such a map in one of
three possible ways depending on μ. As noted this suffices to prove the claim. $\square$

We denote the map $_i\wedge_j(\mu) \to {}_j\vee_i(\mu)$ by $\downarrow$ and its inverse by $\uparrow$.

Example 3.8. Let $(124)(4331) \in {}_s\wedge_w(\mu)$ for $\mu = (3, 2, 5, 1, 4)$. Then to find
$\downarrow(124)(4331)$ we would compute the following elements of W_k^{sw} for $k = 4, 3, 2, 1, 0$:

$$()(124)(4331)()$$
$$(44)(123)(31)()$$
$$(44)(12)(21)(3)$$
$$(442)(1)(1)(23)$$
$$(4421)()()(123)$$

so that $\downarrow(124)(4331) = (4421)(123) \in {}^w\vee^s(\mu)$.

Lemma 3.9. *Definitions 3.3 and 3.4 are equivalent.*

Proof. We need to find a bijection from $\mathfrak{F}_\omega^\bigcirc$ to $\mathfrak{F}_\omega^\square$ that preserves the x-weight and
the y-weight. The arguments are quite technical and we give an example here that
the reader is welcome to follow along through the proof.

Example 3.10. Let $\omega = (4, 3, 2, 1) \in S_3$. The following sequence of factorizations
would be computed under the bijection of this proof to get from an element of $\mathfrak{F}_\omega^\bigcirc$

to an element of $\mathfrak{F}^{\square}_{\omega}$. The sets $\mathfrak{F}^{jk}_{\omega}$ will be defined inside the proof.

$$(3③②1①)(③2)(3③)() \in \mathfrak{F}^{\circ}_{\omega}$$

$()|$ $\quad (3③②1①)()(③2)(3③)() \in \mathfrak{F}^{13}_{\omega}$

$()|$ $\quad (3)(3②1①)(③2)(3③)() \in \mathfrak{F}^{03}_{\omega}$

$()(3)|$ $\quad (3②1①)(③2)()(3③)() \in \mathfrak{F}^{22}_{\omega}$

$()(3)|$ $\quad (3②1①)(3)(2)(3③)() \in \mathfrak{F}^{12}_{\omega}$

$()(3)|$ $\quad (23)(21①)(2)(3③)() \in \mathfrak{F}^{02}_{\omega}$

$()(3)(23)|$ $\quad (21①)(2)(3③)()() \in \mathfrak{F}^{31}_{\omega}$

$()(3)(23)|$ $\quad (321①)(2)(3)(3)() \in \mathfrak{F}^{21}_{\omega}$

$()(3)(23)|$ $\quad (321①)(2)(3)(3)() \in \mathfrak{F}^{11}_{\omega}$

$()(3)(23)|$ $\quad (13)(321)(3)(3)() \in \mathfrak{F}^{01}_{\omega}$

$()(3)(23)(13)|$ $\quad (321)(3)(3)() \in \mathfrak{F}^{\square}_{\omega}$

For each $k \in \{1, 2, \ldots, n\}$ and $j \in \{0, 1, \ldots, n-k+1\}$ we define the set $\mathfrak{F}^{jk}_{\omega}$ to be the set of factorizations $\mathfrak{f}$ such that:

- $\mathfrak{f}$ contains $n+1-k$ *left* factors which we denote $f_{-(n+1)}, f_{-(n)}, \ldots, f_{-(k+1)}$.
- $\mathfrak{f}$ contains $n+1$ *right* factors which we denote $f_1, f_2, \ldots, f_{n+1}$.
- $\mathfrak{f}$ contains 1 *extra* factor which we denote f_{ex}. (Red in the example.)
- The Hecke word $f_{-(n+1)} \cdots f_{-(k+1)} f_1 \cdots f_j f_{ex} f_{j+1} \cdots f_{n+1}$ represents ω.
- For each i, the left factor $f_{-(i)}$ contains elements from $\{i, \ldots, n\}$.
- For each i, the right factor f_i contains uncircled elements from $\{i, \ldots, n\}$.
- For $i \leq j$, f_i contains circled elements from $\{①, \ldots, ⓢ\} : s = k + i - 1$.
- For $i > j$, f_i contains circled elements from $\{①, \ldots, ⓣ\} : t = k + i - 2$.
- The extra factor f_{ex} contains elements from $\{j + k, \ldots, n\}$
- The left factors and f_{ex} are strictly increasing in the order $1 < \cdots < n$.
- The right factors strictly decrease in the order $① < 1 < \cdots < ⓝ < n$.

If $\mathfrak{f} \in \mathfrak{F}^{jk}_{\omega}$ we define $(x, y)^{wt(\mathfrak{f})}$ to be the monomial such that the power of x_i is the number of uncircled elements in f_i. For $i > k$ the power of y_i is the number of elements in f_{-i}. For $i < k$ the power of y_i is the number of times some $ⓜ$ appears in some factor f_{ℓ} such that $m - \ell + 1 = i$. The power of y_k is the number of times some $ⓜ$ appears in some factor f_{ℓ} such that $m - \ell + 1 = k$ plus the number of elements in f_{ex}. *Essentially what we want to do now is show that we can take a factorization of the form $f_{-(n+1)} \cdots f_{-(k+1)} f_1 \cdots f_j f_{ex} f_{j+1} \cdots f_{n+1}$ and move the extra factor f_{ex} from the right of f_j to its left via some process. If we repeat this process eventually we can pull f_{ex} all the way to the left of $f_1 \cdots f_{n+1}$ and make it f_{-k}. If this in turn can be done for each k it means that if we start with a factorization of the form $f_1 \cdots f_n \in \mathfrak{F}^{\circ}_{\omega}$ we can get one of the form $f_{-(n+1)} \cdots f_{-1} f_1 \cdots f_n$. We will then want to show the latter lies in $\mathfrak{F}^{\square}_{\omega}$.*

We begin by noting that $\mathfrak{F}^{1n}_{\omega} = \mathfrak{F}^{\circ}_{\omega}$ and that $\mathfrak{F}^{01}_{\omega} = \mathfrak{F}^{\square}_{\omega}$. Moreover, we have that $\mathfrak{F}^{0k}_{\omega} = \mathfrak{F}^{(n-k+2)(k-1)}_{\omega}$ for $k > 1$. Hence it suffices to find an x-weight and y-weight preserving bijection from $\mathfrak{F}^{jk}_{\omega}$ to $\mathfrak{F}^{(j-1)k}_{\omega}$ for $j \in \{1, \ldots, n-k+1\}$ and $k \in \{1, \ldots, n\}$. To do the latter it suffices to find a bijection, Ψ_{jk} between pairs (f_j, f_{ex}) such that:

- f_j contains uncircled elements from $\{j, \ldots, n\}$.
- f_j contains circled elements from $\{ⓙ, \ldots, ⓢ\} : s = j + k - 1$.
- f_{ex} contains elements from $\{j + k, \ldots, n\}$
- f_{ex} is strictly increasing in the order $1 < \cdots < n$.
- f_j is strictly decreasing in the order $① < 1 < \cdots < ⓝ < n$.

to pairs (f'_{ex}, f'_j) such that:

- f'_j contains uncircled elements from $\{j, \ldots, n\}$.
- f'_j contains circled elements from $\{ⓙ, \ldots, ⓣ\} : t = j + k - 2$.
- f'_{ex} contains elements from $\{j + k - 1, \ldots, n\}$
- f'_{ex} is strictly increasing in the order $1 < \cdots < n$.
- f'_j is strictly decreasing in the order $① < 1 < \cdots < ⓝ < n$.

with the property that if $(f_j, f_{ex}) \rightarrow (f'_{ex}, f'_j)$ then the Hecke words $f_j f_{ex}$ and $f'_{ex} f'_j$ represent the same permutation and whenever $(f_j)(f_{ex})$ appears in an element of $\mathfrak{F}_\omega^{jk}$ (in the expected position) these two factors make the same contribution to the x-weight and y-weight as the pair $(f'_{ex})(f'_j)$ when it appears in an element of $\mathfrak{F}_\omega^{(j-1)k}$ (in the expected position).

To do this write $f_j = f_j^> f_j^= f_j^<$ where $f_j^>$, $f_j^=$, $f_j^<$ are the parts of f_j composed respectively of elements greater than, equal to, or less than ⓢ (where $s = j + k - 1$) in the order $① < 1 < \cdots < ⓝ < n$. If $f_j^=$ is nonempty then append s to the left of f_{ex} to form f_{ex}^+. Otherwise set $f_{ex}^+ = f_{ex}$. Then let $(g_1, g_2) = \uparrow (f_j^>, f_{ex}^+)$. We define $\Psi_{jk}(f_j, f_{ex}) = (f'_{ex}, f'_j)$ where $f'_{ex} = g_1$ and $f'_j = g_2 f^<$.

On the other hand given a pair (f'_{ex}, f'_j) write $f'_j = f_j'^> f_j'^<$ where $f_j'^>$ and $f_j'^<$ are the parts of f'_j composed respectively of elements greater than or less than ⓢ in the order $① < 1 < \cdots < ⓝ < n$. Next set $(h_1, h_2) = \downarrow (f'_{ex}, f_j'^>)$. Now write $h_2 = h_2^= h_2^>$ where $h_2^=$ and $h_2^>$ are the parts of h_2 composed respectively of elements equal to or greater than s in the order $1 < \cdots < n$. We define $\Psi_{jk}^{-1}(f'_{ex}, f'_j) = (f_j, f_{ex})$ where $f_j = h_1 f_j'^<$ if $h_2^=$ is empty and $f_j = h_1 ⓢ f_j'^<$ otherwise and $f_{ex} = h_2^>$.

The fact that $\uparrow$ and $\downarrow$ preserve the permutation represented along with the commutation of nonadjacent transpositions implies that Ψ and Ψ^{-1} do not change the permutation represented. Moreover the constructions of Ψ and Ψ^{-1} make it clear that they map into the proper images. One can easily check that $\Psi^{-1} \circ \Psi$ is the identity by considering the two cases where either f_j contains a ⓢ or does not. Similarly, one can check that $\Psi \circ \Psi^{-1}$ is the identity by considering the two cases where either the h_2 of the construction of Ψ^{-1} contains an s or does not.

Finally, if $(f_j, f_{ex}) \rightarrow (f'_{ex}, f'_j)$, we need to check these two pairs make the same contributions to the x-weight and y-weight of the factorization they are part of. For the first pair, the contribution to the x-weight is simply to add r to the jth coordinate of the x-weight where r is the number of uncircled elements in f_j. For the second pair, the contribution to the x-weight is simply to add r' to the jth coordinate of the x-weight where r' is the number of uncircled elements in f'_j. Clearly the construction of Ψ implies that $r = r'$. Now the circled elements of f_j and f'_j other than ⓢ (which only affects the kth coordinate of the y-weight since $s - (j - 1) = k$) are the same and f_{ex} and f'_{ex} only affect the kth coordinate of the y-weight. Thus it suffices just to check that $(f_j)(f_{ex})$ and $(f'_{ex})(f'_j)$ make the same contribution to the kth coordinate of the y-weight. If f_j does not contain a ⓢ then f_{ex} and f'_{ex} have the same length ℓ and the contribution to the kth coordinate

of the y-weight is just $\ell + 0$ in either case since neither f_j nor f_j' contain a Ⓢ. If f_j does not contain a Ⓢ then if f_{ex} has length ℓ then f_{ex}' has length $\ell + 1$. The contribution to the kth coordinate of the y-weight from $(f_j)(f_{ex})$ is $(1)+(\ell)$ in since f_j contains one Ⓢ. The contribution to the kth coordinate of the y-weight from $(f_{ex}')(f_j')$ is $(\ell + 1) + (0)$ in since f_j' contains no Ⓢ. □

Example 3.11. Set $n = 9$ and $k = 3$ and $j = 2$. Suppose that $f_j = (9764④③2②)$ and $f_{ex} = (5689)$. Then as in the construction of Ψ_{jk} we set $s = j + k - 1 = 4$ and $f_j^>$ becomes (9764) while $f_j^=$ becomes $(④)$ and $f_j^<$ becomes $(③2②)$. To compute f_{ex}^+ we append a 4 to f_{ex}, and so f_{ex}^+ becomes (45689). Next we set $(g_1, g_2) = \uparrow (9764)(45689)$. To compute this we find the following elements of W_ℓ^{ss} for $\ell = 3, 4, 5, 6, 7, 8, 9$:

$$(9764)()()(45689)$$
$$(976)(4)(4)(5689)$$
$$(976)(45)(5)(689)$$
$$(97)(456)(65)(89)$$
$$(9)(457)(765)(89)$$
$$(9)(4578)(865)(9)$$
$$()(45789)(9865)()$$

and see that $g_1 = (45789)$ and $g_2 = (9865)$. Therefore we get that $f_{ex}' = g_1 = (45789)$ and $f_j' = g_2 f_j^< = (9865③2②)$. All in all, we see that Ψ_{23} sends

$$(9764④③2②)(5689) \rightarrow (45789)(9865③2②)$$

Lemma 3.12. *Definitions 3.1 and 3.2 are equivalent.*

Proof. We must show that $\mathfrak{G}_\omega(x,y) = \mathfrak{G}_\omega^G(x,y)$. We proceed by induction on the number of inversions of $\omega^{-1}\omega_0$. First suppose that this number is 0. That is, $\omega = \omega_0$. First we compute $\mathfrak{G}_{\omega_0}^C(x,y)$: For any $\mathfrak{f} \in \mathfrak{F}_{\omega_0}^\circ$ the boundedness condition implies that for each i, the ith factor of $\mathfrak{f}$ contains a subset of $\{ⓘ, i, \ldots, ⓝ, n\}$. Letting ℓ_i denote the number of distinct numerical values that appear (uncircled, circled, or both) in the ith factor of $\mathfrak{f}$, it is clear that the inversion number of the permutation represented by $\mathfrak{f}$ is bounded by $\sum \ell_i$, which, in turn is bounded by $n + (n-1) + \cdots + 1 + 0 = \binom{n+1}{2}$. But $\binom{n+1}{2}$ actually is the inversion number of $\omega_0 \in S_{n+1}$. Thus $\sum \ell_i = \binom{n+1}{2}$, which means that $\ell_i = n + 1 - i$ for each i or that all the numerical values $\{i, i+1, \ldots, n\}$ show up in the ith factor of $\mathfrak{f}$ (uncircled, circled, or both). In other words, there is a bijection from $\mathfrak{F}_{\omega_0}^\circ$ to the set V of all functions $v : \{(i,j) : i + j \leq n+1\} \rightarrow \{\chi_{10}, \chi_{01}, \chi_{11}\}$. The bijection is given by $\mathfrak{f} \rightarrow v$ where:

- $v(i,j) = \chi_{10}$ if $i + j - 1$ shows up uncircled in factor i of $\mathfrak{f}$.
- $v(i,j) = \chi_{01}$ if $i + j - 1$ shows up circled in factor i of $\mathfrak{f}$.
- $v(i,j) = \chi_{11}$ if $i + j - 1$ shows up uncircled and circled in factor i of $\mathfrak{f}$.

If we take three element set $\{\chi_{10}, \chi_{01}, \chi_{11}\}$ to be the set of the following three functions: $\chi_{10}(a,b) = a$, $\chi_{01}(a,b) = b$, and $\chi_{11}(a,b) = ab$ then we have that if

$\mathfrak{f} \to v$, and we write v_{ij} for $v(i,j)$:

$$(x,y)^{wt(\mathfrak{f})} = \prod_{i+j \leq n+1} v_{ij}(x_i, y_j)$$

It follows that:

$$\sum_{\mathfrak{f} \in \mathfrak{F}_{\omega_0}^{\circ}} (x,y)^{wt(\mathfrak{f})} = \sum_{v \in V} \prod_{i+j \leq n+1} v_{ij}(x_i, y_j) = \prod_{i+j \leq n+1} (x_i + y_j + x_i y_j)$$

This verifies the second equality in the following:

$$\mathfrak{G}_{\omega_0}(x,y) = \prod_{i+j \leq n+1} (x_i + y_j + x_i y_j) = \sum_{\mathfrak{f} \in \mathfrak{F}_{\omega_0}^{\circ}} (x,y)^{wt(\mathfrak{f})} = \mathfrak{G}_{\omega_0}^{C}(x,y) = \mathfrak{G}_{\omega_0}^{G}(x,y)$$

Since the first and third are definitions and the fourth has already been shown this completes the base step of induction. Now suppose that $\omega^{-1}\omega_0$ has at least one inversion. Choose s_r such that ωs_r has more inversions than ω. By definition we have $\pi_{s_r} \mathfrak{G}_{\omega s_r}(x,y) = \mathfrak{G}_{\omega}(x,y)$ and by induction we have $\mathfrak{G}_{\omega s_r}(x,y) = \mathfrak{G}_{\omega s_r}^{G}(x,y)$ so that $\pi_{s_r} \mathfrak{G}_{\omega s_r}^{G}(x,y) = \mathfrak{G}_{\omega}(x,y)$. Therefore it suffices to show that $\pi_{s_r} \mathfrak{G}_{\omega s_r}^{G}(x,y) = \mathfrak{G}_{\omega}^{G}(x,y)$. That is, we must show that:

$$(3.1) \qquad \pi_{s_r} \sum_{(u,v) \in X_{\omega s_r}} \mathfrak{G}_{u^{-1}}(y) \mathfrak{G}_v(x) = \sum_{(u,v) \in X_{\omega}} \mathfrak{G}_{u^{-1}}(y) \mathfrak{G}_v(x)$$

First we write $X_{\omega s_r} = X_{\omega s_r}(1) \cup X_{\omega s_r}(2) \cup X_{\omega s_r}(3)$ where:

(1) $X_{\omega s_r}(1) = \{(u,v) \in X_{\omega s_r} \colon$ no reduced word for v ends in $s_r\}$.
(2) $X_{\omega s_r}(2) = \{(u,v) \in X_{\omega s_r} \colon \exists \nu$ s.t. $\tilde{v} \sim \tilde{\nu}\bar{s}_r, \; \tilde{\nu} \not\sim \tilde{\nu}\bar{s}_r, \; \tilde{u}\tilde{\nu} \sim \tilde{u}\tilde{\nu}\bar{s}_r\}$.
(3) $X_{\omega s_r}(3) = \{(u,v) \in X_{\omega s_r} \colon \exists \nu$ s.t. $\tilde{v} \sim \tilde{\nu}\bar{s}_r, \; \tilde{\nu} \not\sim \tilde{\nu}\bar{s}_r, \; \tilde{u}\tilde{\nu} \not\sim \tilde{u}\tilde{\nu}\bar{s}_r\}$.

Claim 3.13. *There exist bijections*

(i) $X_{\omega s_r}(2) \longrightarrow X_{\omega s_r}(1)$
(ii) $X_{\omega s_r}(3) \longrightarrow X_{\omega}$

Proof. We describe a single map and its inverse that actually gives both the bijections. First let $(u,v) \in X_{\omega s_r}(2)$ or $(u,v) \in X_{\omega s_r}(3)$. In either case we may write $v = \nu s_r$ for a (unique) permutation ν with $\tilde{\nu} \not\sim \tilde{\nu}\bar{s}_r$. We now simply send (u,v) to (u,ν).

If $(u,v) \in X_{\omega s_r}(2)$ then $\tilde{u}\tilde{\nu} \sim \tilde{u}\tilde{\nu}\bar{s}_r$ so that $(u,\nu) \in X_{\omega s_r}$. Moreover, $\tilde{\nu} \not\sim \tilde{\nu}\bar{s}_r$ implies that ν has no reduced word ending in s_r, so $(u,\nu) \in X_{\omega s_r}(1)$. If $(u,v) \in X_{\omega s_r}(3)$ then $\tilde{u}\tilde{\nu} \not\sim \tilde{u}\tilde{\nu}\bar{s}_r$ implies that $(u,\nu) \in X_{\omega}$.

Next let $(u,v) \in X_{\omega s_r}(1)$ or $(u,v) \in X_{\omega s_r}$. Then the inverse map sends (u,v) to $(u, v s_r)$. First, suppose that $(u,v) \in X_{\omega s_r}(1)$. Then the permutation that the Hecke word $\tilde{u}\tilde{v}$ represents is ωs_r which obviously has a reduced word ending in s_r. Thus the permutation represented by $\tilde{u}\tilde{v}\bar{s}_r$ is also ωs_r. Further, in the definition of $X_{\omega s_r}(2)$, replacing v with $v s_r$ and ν with v satisfies the three requirements of the definition, $(\tilde{v}\bar{s}_r) \sim \tilde{v}\bar{s}_r$ (clearly), $\tilde{v} \not\sim \tilde{v}\bar{s}_r$ (because v has no reduced word ending in s_r), and $\tilde{u}\tilde{v} \sim \tilde{u}\tilde{v}\bar{s}_r$ (by the previous sentence). Thus $(u, v s_r) \in X_{\omega s_r}(2)$. On the other hand if $(u,v) \in X_{\omega}$ then the fact that $\tilde{u}\tilde{v}$ represents the permutation ω implies that $\tilde{u}\tilde{v}\bar{s}_r$ represents the permutation ωs_r. Moreover, in the definition of $X_{\omega s_r}(3)$, replacing v with $v s_r$ and ν with v satisfies the three conditions: $(\tilde{v}\bar{s}_r) \sim \tilde{v}\bar{s}_r$ (clearly), $\tilde{v} \not\sim \tilde{v}\bar{s}_r$ (because the fact uv has no reduced word ending in s_r implies the same for v), and $\tilde{u}\tilde{v} \not\sim \tilde{u}\tilde{v}\bar{s}_r$ (because $\tilde{u}\tilde{v} \sim \tilde{\omega}$ and ω has no reduced word ending in s_r). Finally, it is clear the given maps are mutual inverses. $\square$

Claim 3.14.

$$(3.2) \qquad \pi_{s_r} \sum_{(u,v)\in X_{\omega s_r}(2)} \mathfrak{G}_{u^{-1}}(y)\mathfrak{G}_v(x) \quad = \quad -\pi_{s_r} \sum_{(u,v)\in X_{\omega s_r}(1)} \mathfrak{G}_{u^{-1}}(y)\mathfrak{G}_v(x)$$

$$(3.3) \qquad \pi_{s_r} \sum_{(u,v)\in X_{\omega s_r}(3)} \mathfrak{G}_{u^{-1}}(y)\mathfrak{G}_v(x) \quad = \quad \sum_{(u,v)\in X_\omega} \mathfrak{G}_{u^{-1}}(y)\mathfrak{G}_v(x)$$

Proof. To prove 3.2 it suffices to show if $(u, \nu s_r) \to (u, \nu)$ under bijection (i) then,

$$\pi_{s_r}\mathfrak{G}_{\nu s_r}(x) = -\pi_{s_r}\mathfrak{G}_\nu(x)$$

To prove 3.3 it suffices to show if $(u, \nu s_r) \to (u, \nu)$ under bijection (ii) then,

$$\pi_{s_r}\mathfrak{G}_{\nu s_r}(x) = \mathfrak{G}_\nu(x)$$

In either case if $\mu = \mu_1 \cdots \mu_\ell$ is a reduced word for the permutation $(\nu s_r)^{-1}\omega_0$, then $s_r\mu_1 \cdots \mu_\ell$ is a reduced word for $\nu^{-1}\omega_0$. So by the divided difference operator definition, the equations become:

$$\pi_{s_r}(\pi_{\mu_1} \cdots \pi_{\mu_\ell})(x_1^n \cdots x_{n+1}^0) = -\pi_{s_r}(\pi_{s_r}\pi_{\mu_1} \cdots \pi_{\mu_\ell})(x_1^n \cdots x_{n+1}^0)$$

$$\text{and}$$

$$\pi_{s_r}(\pi_{\mu_1} \cdots \pi_{\mu_\ell})(x_1^n \cdots x_{n+1}^0) = (\pi_{s_r}\pi_{\mu_1} \cdots \pi_{\mu_\ell})(x_1^n \cdots x_{n+1}^0)$$

The first follows by part (4) of Lemma 2.2 and the second is immediate. $\square$

Combining equations 3.2 and 3.3 gives equation 3.1 thereby completing the induction step and finishing the proof. $\square$

Remark 3.15. We have now shown all definitions 3.1, 3.2, 3.3, 3.4 are equivalent. The equation resulting from comparing the lowest degree parts the of the two algebraic definitions, 3.1 and 3.2 is known as the Giambelli formula. Being that the arguments of this section are entirely combinatorial we have, in particular, a combinatorial proof of this formula.

4. Stable Grothendieck Polynomials

4.1. **Stable Double Grothendieck Polynomials.** As we will see in this section a certain limit of double Grothendieck polynomials yields a polynomial symmetric with respect to transpositions of the x and y variables amongst themselves. This phenomenon was first studied in the case of single Grothendieck polynomials in [FK94]. In this section we will mostly start by giving combinatorial definitions and then show they are equivalent to the algebraic ones.

Let $\omega \in S_{k+1}$ and choose some $m \geq 0$. Let $\hat\omega \in S_{m+k+1}$ be the permutation of $[1, \ldots, m, (m+1), \ldots, (m+k+1)]$ that fixes the first m entries and applies the permutation ω to the last $k+1$ entries. In other words, $s_{i_1} \cdots s_{i_\ell}$ is a reduced word for ω if and only if $s_{i_1+m} \cdots s_{i_\ell+m}$ is a reduced word for $\hat\omega$. Let $\check{x} = (x_1, \ldots, x_{m+1})$ and $\hat{x} = (x_{m+2}, \ldots, x_{m+k+1})$. Similarly let $\check{y} = (y_1, \ldots, y_{m+1})$ and $\hat{y} = (y_{m+2}, \ldots, y_{m+k+1})$. Write $x = (\check{x}, \hat{x})$ and $y = (\check{y}, \hat{y})$.

Recall that $\mathcal{F}_\omega^\square$ denotes the set of all (unbounded) double Hecke factorizations of ω into $2m + 2$ factors.

Definition 4.1. The stable double Grothendieck polynomial for ω is given by:

$$\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{\mathfrak{f}\in\mathcal{F}_\omega^\square} (\check{x}, \check{y})^{wt(\mathfrak{f})}$$

Proposition 4.2.

$$\mathcal{G}_\omega(\check{x}, \check{y}) = \mathfrak{G}_{\hat{\omega}}(x, y)|_{(\hat{x}=0=\hat{y})}$$

Proof. The combinatorial Giambelli definition of the double Grothendieck function states that:

$$\mathfrak{G}_{\hat{\omega}}(x, y) = \sum_{\mathfrak{f} \in \mathfrak{F}_{\hat{\omega}}^{\square}} (x, y)^{wt(\mathfrak{f})}$$

Therefore we have that

$$\mathfrak{G}_{\hat{\omega}}(x, y)|_{(\hat{x}=0=\hat{y})} = \sum_{\mathfrak{f} \in \mathfrak{F}_{\hat{\omega}}^{\square}(2m+2)} (\check{x}, \check{y})^{wt(\mathfrak{f})}$$

where $\mathfrak{F}_{\hat{\omega}}^{\square}(2m+2)$ is the subset of $\mathfrak{F}_{\hat{\omega}}^{\square}$ where all but the middle $2m+2$ factors are empty. But every Hecke word for $\hat{\omega}$ only contains elements from the set of $\{(m+1), \ldots, (m+k+1)\}$ and the boundedness condition on the central $2m+2$ factors of a factorization of $\mathfrak{F}_{\hat{\omega}}^{\square}(2m+2)$ only requires elements to be greater $\geq i$ for some $i \leq m+1$. Therefore no factorization of $\mathcal{F}_\omega^{\square}$ fails to lie inside of $\mathfrak{F}_{\hat{\omega}}^{\square}(2m+2)$ (after changing each s_i to s_{i+m}), that is, $\mathcal{F}_\omega^{\square} = \mathfrak{F}_{\hat{\omega}}^{\square}(2m+2)$ (after changing each s_i to s_{i+m}). $\square$

A *circled hook Hecke factorization* of ω is a factorization into hook factors. Each hook factor contains a subset of $\{①, 1, ②, 2, \ldots\}$ arranged so that all uncircled factors lie to the left of all circled factors and such that the uncircled elements are decreasing left to right and the circled elements are increasing left to right. Moreover, erasing the circles and parentheses should give a Hecke word for ω. For instance $(32②③)(1②③)(321③)$ is a circled hook Hecke factorization for the permutation $(4, 3, 2, 1) \in S_4$. The *x-weight* of a circled hook Hecke factorization is the vector whose i^{th} entry records the number of uncircled elements in its ith factor. The x-weight of the example above is $(2, 1, 3)$. The *y-weight* of such a factorization is the vector whose i^{th} entry records the number of circled entries in the ith factor. The y-weight of the example above is $(2, 2, 1)$.

Proposition 4.3. *Denote the set of all circled hook Hecke factorizations of ω with $m+1$ factors by $\mathcal{F}_\omega^{\textcircled{\vee}}$. Then:*

$$\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{\mathfrak{f} \in \mathcal{F}_\omega^{\textcircled{\vee}}} (\check{x}, \check{y})^{wt(\mathfrak{f})}$$

Proof. First we show that $\mathcal{G}_\omega(\check{x}, \check{y})$ is symmetric with respect to transposition of y_i and y_{i+1}. This will follow from showing that for any permutation μ and any numbers a and b there are an equal number of pairs of the form:

$(u_1 u_2 \cdots u_a)(v_1 v_2 \cdots v_b): \ u_1 < \cdots < u_a, \ v_1 < \cdots < v_b, \ u_1 \cdots u_a v_1 \cdots v_b \sim \widetilde{\mu}$

and pairs of the form:

$(u_1' u_2' \cdots u_b')(v_1' v_2' \cdots v_a'): \ u_1' < \cdots < u_b', \ v_1' < \cdots < v_a', \ u_1' \cdots u_b' v_1' \cdots v_a' \sim \widetilde{\mu}$

By Lemma 2.17 the number of pairs of the first form is equal to the number of pairs (P, Q) where $P \in HT_{\mu^{-1}}$ and $Q \in SVT^\perp$ have the same shape and Q has b 1s and a 2s. The number of pairs of the second form is equal to the number of pairs (P, Q) where $P \in HT_{\mu^{-1}}$ and $Q \in SVT^\perp$ have the same shape and Q has a 1s and b 2s. But as can be seen in the proof of claim 2.25, these numbers are equal.

Thus the order in which the increasing factors in a factorization of $\mathcal{F}_\omega^\square$ appear amongst themselves is irrelevant. In particular, we may write the first increasing factor (i.e., the one contributing to the exponent of y_1) in the leftmost position, and, in general, the $(i+1)^{st}$ directly to the right of the ith factor for each $i \leq m$.

Further, the existence of the bijection $\downarrow$ implies that the relative order in which the increasing factors and decreasing factors appear in a factorization in $\mathcal{F}_\omega^\square$ is also irrelevant. In particular we can write the first decreasing factor in the leftmost position and then the the first increasing factor to its right, and, in general, the $(i+1)^{st}$ decreasing factor directly to the right of the ith increasing factor and then the $(i+1)^{st}$ increasing factor directly to the right of the $(i+1)^{st}$ decreasing factor. It is clear that this interpretation on $\mathcal{F}_\omega^\square$ is the same as $\mathcal{F}_\omega^{\circledcirc}$ after circling all elements of the increasing factors and then grouping together adjacent pairs of factors into single factors. This proves the proposition. $\square$

An (uncircled) *hook Hecke factorization* of ω is a factorization into factors of the form $p_1 \cdots p_r \cdots p_s$ where $p_1 > \cdots > p_{r-1} > p_r < p_{r+1} < \cdots < p_s$. Moreover, erasing parentheses should give a Hecke word for ω. For instance $(323)(123)(321)$ is a hook Hecke factorization for the permutation $(4,3,2,1) \in S_4$. If the ith factor of a hook Hecke factorization is of the form $p_1 \cdots p_r \cdots p_s$ as above it contributes $r-1$ to the ith coordinate of the x-weight and $s-r$ to the ith coordinate of the y-weight. The example above has x-weight equal to $(1,0,2)$ and y-weight equal to $(1,2,0)$.

Proposition 4.4. *Denote the set of all hook Hecke factorizations of ω with $m+1$ factors by $\mathcal{F}_\omega^\triangledown$. Then:*

$$\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{\mathfrak{f} \in \mathcal{F}_\omega^\triangledown} \left(\prod_{i \in NE(\mathfrak{f})} (x_i + y_i + x_i y_i) \right) (\check{x}, \check{y})^{wt(\mathfrak{f})}$$

where $NE(\mathfrak{f})$ is the set of all i such that the ith factor of $\mathfrak{f}$ is nonempty.

Proof. This follows from proposition 4.3: Every element $\mathfrak{f} \in \mathcal{F}_\omega^\triangledown$ corresponds to $3^{|NE(\mathfrak{f})|}$ distinct elements of $\mathcal{F}_\omega^{\circledcirc}$ by doing the following. For each nonempty factor of the form $p_1 \cdots p_r \cdots p_s$, circling all elements to the right of p_r and then either:

- Circling p_r.
- Not circling p_r.
- Not circling p_r and inserting a new circled p_r immediately to its right.

Moreover, each element of $\mathcal{F}_\omega^{\circledcirc}$ arises this way. Finally, for each $\mathfrak{f} \in \mathcal{F}_\omega^\triangledown$ we have:

$$\sum_{\mathfrak{f}' \in orb(\mathfrak{f})} (\check{x}, \check{y})^{wt(\mathfrak{f}')} = \left(\prod_{i \in NE(\mathfrak{f})} (x_i + y_i + x_i y_i) \right) (\check{x}, \check{y})^{wt(\mathfrak{f})}$$

where $orb(\mathfrak{f})$ is the subset of $\mathcal{F}_\omega^{\circledcirc}$ associated to $\mathfrak{f}$ under the correspondence described above. $\square$

Example 4.5. We use proposition 4.4 to compute $\mathcal{G}_{321}(x_1, x_2, y_1, y_2)$. First, we write down the elements of $\mathcal{F}_{321}^\triangledown$:

$(212)()$, $(212)(2)$, $(212)(1)$, $(212)(21)$, $(212)(12)$, $(212)(212)$,
$(21)(2)$, $(21)(21)$, $(21)(12)$, $(21)(212)$,

$(12)(1)$, $(12)(21)$, $(12)(12)$, $(12)(212)$,
$(1)(21)$, $(1)(212)$,
$(2)(12)$, $(2)(212)$,
$()(212)$.

The factorization $(212)()$ has only its first factor nonempty so the monomial corresponding to its weight, x_1y_1, is multiplied by $(x_1 + y_1 + x_1y_1)$. The factorization $()(212)$ has only its last factor nonempty so the monomial corresponding to its weight, x_2y_2, is multiplied by $(x_2 + y_2 + x_2y_2)$. The remaining 17 factorizations have both parts nonempty so the corresponding monomials are multiplied by $(x_1 + y_1 + x_1y_1)(x_2 + y_2 + x_2y_2)$. The result is:

$$\mathcal{G}_{321}(x_1, x_2, y_1, y_2) = (x_1 + y_1 + x_1y_1) * (x_1y_1)$$
$$+(x_1 + y_1 + x_1y_1)(x_2 + y_2 + x_2y_2) * (x_1 + x_2 + y_1 + y_2)$$
$$+(x_1 + y_1 + x_1y_1)(x_2 + y_2 + x_2y_2) * (x_1x_2 + y_1y_2 + x_1y_2 + x_2y_1 + 2x_1y_1 + 2x_2y_2)$$
$$+(x_1 + y_1 + x_1y_1)(x_2 + y_2 + x_2y_2) * (x_1x_2y_1 + x_1x_2y_2 + x_1y_1y_2 + x_2y_1y_2)$$
$$+(x_1 + y_1 + x_1y_1)(x_2 + y_2 + x_2y_2) * (x_1x_2y_1y_2)$$
$$+(x_2 + y_2 + x_2y_2) * (x_2y_2)$$

Proposition 4.4 is useful in that it streamlines the computation of $\mathcal{G}_\omega(\check{x}, \check{y})$. For instance computing, $\mathcal{G}_{321}(x_1, x_2, y_1, y_2)$ using definition 4.1 or proposition 4.2 requires writing down 159 factorizations whereas doing so with proposition 4.4 only required 19.

4.2. **Single Stable Grothendieck polynomials.** The material of this subsection has been extensively studied. As mentioned, the topic is first introduced in [FK94]. The combinatorics of single stable Grothendieck functions is further studied in places such as [BKS+08], [MPS18], and [HS20]. However, the results are essential to later proofs so we review them here.

Definition 4.6. The stable Grothendieck polynomial for ω is

$$\mathcal{G}_\omega(\check{x}) = \sum_{\mathfrak{f} \in \mathcal{F}_\omega} (\check{x})^{wt(\mathfrak{f})}$$

Definition 4.7. The symmetric Grothendieck polynomial for a partition λ is

$$G_\lambda(\check{x}) = \sum_{T \in SVT^\perp(\lambda)} (\check{x})^{wt(T)}$$

Definition 4.8. The weak stable Grothendieck polynomial for ω is

$$^*\mathcal{G}_\omega(\check{x}) = \sum_{\mathfrak{f} \in {}^*\mathcal{F}_\omega} (\check{x})^{wt(\mathfrak{f})}$$

Definition 4.9. The weak symmetric Grothendieck polynomial for a partition λ is

$$^*G_\lambda(\check{x}) = \sum_{T \in MVT(\lambda)} (\check{x})^{wt(T)}$$

Lemma 4.10. *We have*

$$\mathcal{G}_\omega(\check{x}) = \mathcal{G}_\omega(\check{x}, 0)$$

Proof. This follows from the definition of $\mathcal{G}_\omega(\check{x}, \check{y})$ by plugging in $\check{y} = 0$. $\qquad\square$

Proposition 4.11. *We have*

$$\mathcal{G}_\omega(\check{x}) = \sum_\lambda \sum_{T \in HT_\omega(\lambda)} G_\lambda(\check{x})$$

$$^*\mathcal{G}_\omega(\check{x}) = \sum_\lambda \sum_{T \in HT_\omega(\lambda)} {}^*G_\lambda(\check{x})$$

Proof. These statements follow immediately from lemma 2.17. $\qquad\square$

Definition 4.12. A Grassmannian permutation of $[1, 2, \ldots, k+1]$ is a permutation with at most one descent. In other words a Grassmannian permutation has the form $[a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t]$ where $a_i < a_{i+1}$ and $b_i < b_{i+1}$ but $a_s > b_1$. (There could be no descents if $t = 0$ or $t = k + 1$.)

Lemma 4.13. *There is a bijection from the set P_{k+1} of Grassmannian permutations of $[1, 2, \ldots, k+1]$ which do not fix $k+1$ to the set Λ_{k+1} of partitions whose number of rows plus number of columns is less than or equal to $k + 1$.*

Proof. Let $\omega = [a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t] \in P_{k+1}$. Define $\Gamma(\omega) = \lambda$ where λ is the partition whose r^{th} row has the number of boxes equal to the size of the set $\{i : a_i > b_r\}$. Clearly λ is a partition and has t rows with longest row at most s. Hence the number of rows plus columns is bounded by $s + t = k + 1$. Moreover for any distinct ω and ω' there is a maximal r such that $b_r \neq b_r'$. If $\Gamma(\omega) = \lambda$ and $\Gamma(\omega') = \lambda'$ it follows that $\lambda_r \neq \lambda_r'$. If $\lambda \in \Lambda_{k+1}$ has t rows then let $\omega = [a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t]$ where $b_r = k + 1 - (t - r) - \lambda_r$ for each r (and the a_i and the remaining numbers written in increasing order left to right). Clearly the b_i are increasing left to right and the fact that the number of rows plus columns of λ is less than or equal to $k+1$ implies that all b_i are positive. Since $b_t < k+1$ since $\lambda_t \geq 1$ we see ω does not fix $k+1$. By construction, we have that $\Gamma(\omega) = \lambda$. Thus the map Γ is well defined from P_{k+1} to Λ_{k+1} as well as injective and surjective. $\qquad\square$

Lemma 4.14. *Given $\lambda \in \Lambda_{k+1}$ set $\omega = \Gamma^{-1}(\lambda)$. Then there is exactly one element $T \in HT_\omega$. T has shape λ and if $\omega = [a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t]$ then T has row r, denote T_r, equal to $\bar{s}_{b_r} \bar{s}_{b_r+1} \cdots \bar{s}_{b_r+\lambda_r-1}$.*

Proof. The rows of T read left to right and concatenated from bottom row to top row, i.e., the word $T_u T_{u-1} \cdots T_2 T_1$ can act in two ways to produce the permutation $\omega = [a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t]$ from $[1, 2, \ldots k + 1]$. First, we could read $T_u T_{u-1} \cdots T_2 T_1$ right to left and let each $\bar{s}_i$ in $T_u T_{u-1} \cdots T_2 T_1$ act by interchanging i and $i+1$ if and only if i lies to the left of $i+1$. On the other hand we could read $T_u T_{u-1} \cdots T_2 T_1$ left to right and let each $\bar{s}_i$ in $T_u T_{u-1} \cdots T_2 T_1$ act by interchanging the entries in positions i and $i+1$ if and only if the entry in position i is less than the entry in position $i + 1$. We will use the second interpretation for the proof.

First we show that T must have exactly t rows, that is $u = t$. In order to form $[a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t]$ from $[1, 2, \ldots k+1]$ the element $a_s = k+1$ must move to the left of each element $b_1, b_2, \ldots, b_t$ and can move to the left of no other element. This implies that $T_u T_{u-1} \cdots T_2 T_1$ contains the subsequence $\bar{s}_k \bar{s}_{k-1} \cdots \bar{s}_{k-t+1}$ which in turn implies T has at least t rows. The first column of T read from bottom to top forms a length u decreasing subsequence of $T_u T_{u-1} \cdots T_2 T_1$. When $T_u T_{u-1} \cdots T_2 T_1$

is applied to $[1, 2, \ldots, k+1]$ each $\bar{s}_i$ of this subsequence has nontrivial action and moves i itself one space to the right. Since the subsequence is decreasing each i must be unique. But only $b_1, \ldots, b_t$ can be moved to the right so each such i must be a distinct one of these elements. Thus $u \leq t$.

Now let $j \leq u$ and suppose that for each $r > j$ we know that we have that $T_r = \bar{s}_{b_r} \bar{s}_{b_r+1} \cdots \bar{s}_{b_r+\lambda_r-1}$. Then $T_u T_{u-1} \cdots T_{j+1}$ applied to $[1, 2, \ldots, k+1]$ must give $[1, 2, \ldots, (j+s-\lambda_{j+1}), a_{(s-\lambda_{j+1}+1)}, \ldots, a_{s-1}, a_s, b_{j+1}, \ldots, b_{t-1}, b_t]$. Now consider the entry in the first column in T_j. It must be less than $\bar{s}_{b_{j+1}}$ (this is to say it must be equal to $\bar{s}_i$ for some $i < b_{j+1}$) because that is the entry in the first column of T_{j+1}. It can't lie in the open interval $(\bar{s}_{b_j}, \bar{s}_{b_{j+1}})$ because then it would act by moving a_p to the right for some $p \leq (s-\lambda_{j+1})$. Finally, the first element of T_j cannot be less than $\bar{s}_{b_j}$: The first column of T read from row j up to row 1 gives a length j decreasing subsequence of $T_j T_{j-1} \cdots T_1$. Each $\bar{s}_i$ of this subsequence has nontrivial action and moves i itself one space to the right. Since the subsequence is decreasing each i must be unique. But if the first element of T_j is less than $\bar{s}_{b_j}$ all such i must come from $\bar{s}_{b_1}, \ldots, \bar{s}_{b_{j-1}}$ which is impossible since there should be j of them. Thus the first element of T_j is $\bar{s}_{b_j}$.

Now in order to get $[a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t]$ when $T_j T_{j-1} \cdots T_1$ is applied to $[1, 2, \ldots, (j+s-\lambda_{j+1}), a_{(s-\lambda_{j+1}+1)}, \ldots, a_{s-1}, a_s, b_{j+1}, \ldots, b_{t-1}, b_t]$ it must move a_s to the left of each $b_1, \ldots, b_j$. Since each row is increasing each row by itself can move a_s to the left of at most one of $b_1, \ldots, b_j$. Thus T_j must move a_s to the left of b_j. This is only possible if T_j begins $\bar{s}_{b_j} \bar{s}_{b_j+1} \cdots \bar{s}_{b_j+\lambda_j-1}$. If T_j contained any additional element greater than $\bar{s}_{b_j+\lambda_j-1}$ it would have to act nontrivially and move b_p to the right of b_{p+1} for some $p \geq j$ which is a contradiction. Thus T_j is exactly equal to $\bar{s}_{b_j} \bar{s}_{b_j+1} \cdots \bar{s}_{b_j+\lambda_j-1}$ completing the induction. Thus T must in fact be the tableau described in the lemma. □

Proposition 4.15. *Let $\lambda \in \Lambda_{k+1}$ and set $\omega = \Gamma^{-1}(\lambda)$. Then we have*

$$G_\lambda(\check{x}) = \mathcal{G}_\omega(\check{x})$$

$$and$$

$$^*G_\lambda(\check{x}) = {}^*\mathcal{G}_\omega(\check{x})$$

Proof. This follows from proposition 4.11 because by lemma 4.14 HT_ω is composed of a single tableau of shape λ. □

4.3. **Primed Tableaux.**

Definition 4.16. Consider the alphabet $\{1', 1, 2', 2, \ldots, m', m, (m+1)', (m+1)\}$. Fix some total order $\prec$ on these letters extending the orders $1' < 2' < 3' \cdots$ and $1 < 2 < 3 < \cdots$. Under the order $\prec$ a

(1) *primed set valued tableau* of shape λ, or an element of $PSVT_\prec(\lambda)$,
(2) *primed multiset valued tableau* of shape λ, or an element of $PMVT_\prec(\lambda)$,
(3) *primed set multiset tableau* of shape λ, or an element of $PSMT_\prec(\lambda)$,

is a filling of a Young diagram of shape λ such that

- Each box is nonempty and contains a
  (1) set from $\{1', 1, 2', 2, \ldots\}$.
  (2) multiset from of $\{1', 1, 2', 2, \ldots\}$.
  (3) set from $\{1', 2', \ldots\}$ and a multiset from $\{1, 2, \ldots\}$.
- All of the entries in a box are $\preceq$ all of the entries in the box to its right.

- All of the entries in a box are $\preceq$ all of the entries in the box below it.
- i appears in at most one box in each column.
- i' appears in at most one box in each row.

The x-weight of such a tableau is the vector whose i^{th} coordinate records the number of times i appears in the tableau. The y-weight is the vector whose i^{th} coordinate records the number of times i' appears in the tableau.

Example 4.17. Using the total order $1' \prec 1 \prec 2' \prec 2 \prec \cdots$ the following is a $PSVT$ with x-weight $(2, 1, 2, 3)$ and y-weight $(1, 2, 1, 2)$

$1'$	$12'$	$3'4'$
1	$2'3$	$4'4$
234	4	

Example 4.18. Using the total order $1' \lhd 2' \lhd \cdots \lhd 1 \lhd 2 \lhd \cdots$ the following is a $PSVT$ with x-weight $(3, 3, 3, 2)$ and y-weight $(1, 2, 1, 0)$

$1'2'$	$3'1$	123
$2'1$	2	4
23	34	

Lemma 4.19. *The relative ordering of the primed and unprimed letters in the definitions above is irrelevant. In other words, given any two total orderings $\prec$ and $\lhd$ respecting $1' < 2' < 3' \cdots$ and $1 < 2 < 3 < \cdots$ there is an x-weight and y-weight preserving bijection between $PSVT_{\prec}(\lambda)$ and $PSVT_{\lhd}(\lambda)$ and between $PMVT_{\prec}(\lambda)$ and $PMVT_{\lhd}(\lambda)$ and between $PSMT_{\prec}(\lambda)$ and $PSMT_{\lhd}(\lambda)$.*

Instead of treating each separately, we will simply use PT to mean one of $PSVT$, $PMVT$, or $PSMT$.

Proof. It suffices to suppose that there are some i and j (possibly equal) such that $i \prec j'$ and $j' \lhd i$ and all other pairs of letters have the same relationship in both orders. From this assumption it follows that in both orders there is no other letter between i and j'. If $T \in PT_{\prec}(\lambda)$ or $T \in PT_{\lhd}(\lambda)$, let $\mathcal{B}(T)$ denote all of the boxes of λ that contain at least one i or at least one j'. If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ then label the the upper left to lower right diagonals of λ by $\{D_{-\ell+1}, \ldots, D_{-1}, D_0, D_1, \ldots, D_{\lambda_1 - 1}\}$. It is clear that either $\mathcal{B}(T) \cap D_s$ is a single box or is empty for each s. If p is minimal and q is maximal such that the adjacent diagonals $d_p, d_{p+1}, \ldots, d_{q-1}, d_q$ each have nonempty intersection with $\mathcal{B}(T)$ then we call $B = \{d_p \cap \mathcal{B}(T), \ldots, d_q \cap \mathcal{B}(T)\}$ a connected component of $\mathcal{B}(T)$.

Now suppose $T \in PT_{\prec}(\lambda)$ where $i \prec j'$ and suppose B is a connected component of boxes as described above. Every box from B that has another box from B to its right must contain at least one i and no j'. Every box in B that has another box from B above it must contain at least one j' and no i. By the fact that B has boxes on an adjacent set of diagonals, the only box of B that doesn't fit one of these descriptions is the highest rightmost box. It may have at least one i, at least one j', or at least one of both.

On the other hand suppose $T \in PT_\lhd(\lambda)$ where $j' \lhd i$ and suppose B is a connected component of boxes as described above. Every box from B that has another box from B to its left must contain at least one i and no j'. Every box in B that has another box from B below it must contain at least one j' and no i. By the fact that B has boxes on an adjacent set of diagonals, the only box of B that doesn't fit one of these descriptions is the lowest leftmost box. It may have at least one i, at least one j', or at least one of both. Define a map $\nearrow$ from $PT_\prec(\lambda)$ to $PT_\lhd(\lambda)$ as follows. Suppose that $T \in PT_\prec(\lambda)$. Perform the following to each connected component B of $\mathcal{B}(T)$.

- Remove any is and any j's that appear in the upper rightmost box of B. Record what you have removed.
- Move every remaining i one box to the right.
- Move every remaining j' one box up.
- Fill the lower leftmost box of B with the entries recorded in step one.

After doing this, apart from the entries in the lower leftmost box, every i now has a box in B to its left and every j' has a box in B below it. This holds for all resulting B and it follows that $\nearrow (T) \in PT_\lhd(\lambda)$.

Example 4.20. Suppose that we are given the orderings $1' \prec 2' \prec 1 \prec 2 \prec 3' \prec 3$ and $1' \lhd 2' \lhd 1 \lhd 3' \lhd 2 \lhd 3$. Then under the map $\nearrow$ with $i = 2$ and $j = 3$ the tableau $S \in PSMT_\prec(\lambda)$ on the left below is sent to the tableau $T \in PSMT_\lhd(\lambda)$ on the right below.

$$
S = \begin{array}{|c|c|c|}
\hline
1'2' & 11 & 1223' \\
\hline
2'222 & 2 & 3'3 \\
\hline
3'3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}
\quad \nearrow \quad
\begin{array}{|c|c|c|}
\hline
1'2' & 11 & 13' \\
\hline
2'3' & 222 & 23 \\
\hline
3'223 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array} = T
$$

Next define a map $\swarrow$ from $PT_\lhd(\lambda)$ to $PT_\prec(\lambda)$ as follows. Suppose that $T \in PT_\lhd(\lambda)$. Perform the following to each connected component B of $\mathcal{B}(T)$.

- Remove any is and any j's that appear in the lower leftmost box of B. Record what you have removed.
- Move every remaining i one box to the left.
- Move every remaining j' one box down.
- Fill the upper rightmost box of B with the entries recorded in step one.

After doing this, apart from the entries in the upper rightmost box, every i now has a box in B to its right and every j' has a box in B above it. This holds for all resulting B and it follows that $\swarrow (T) \in PT_\prec(\lambda)$. Clearly the maps $\nearrow$ and $\swarrow$ are inverses and preserve weights. Moreover, all of the arguments in the proof are valid if we substitute any of $PVST$, $PMVT$, or $PSMT$ for PT. □

Define a tableaux, T, to be an element of $PT^\perp(\lambda)$ if T^t is an element of $PT(\lambda)$ where again PT represents any of $PVST$, $PMVT$, or $PSMT$.

Definition 4.21. We define the following polynomials in two sets of $m+1$ variables:

- The symmetric double Grothendieck polynomial:

$$
G_\lambda(\check{x}, \check{y}) = \sum_{T \in PSVT^\perp(\lambda)} (\check{x}, \check{y})^{wt(T)}
$$

- The weak symmetric double Grothendieck polynomial:

$$^*G_\lambda(\check{x}, \check{y}) = \sum_{T \in PMVT(\lambda)} (\check{x}, \check{y})^{wt(T)}$$

- The quasi-weak symmetric double Grothendieck polynomial:

$$^\times G_\lambda(\check{x}, \check{y}) = \sum_{T \in PSMT(\lambda)} (\check{x}, \check{y})^{wt(T)}$$

Definition 4.22. Fix some total order $\prec$ extending the orders $1' < 2' < 3' \cdots$ and $1 < 2 < 3 < \cdots$. A *semistandard primed tableau* of shape λ, or an element of $SSPT_\prec(\lambda)$, is an element of $PSMT_\prec(\lambda)$ with exactly one entry in every box.

Of course, we could replace $PSMT$ above with $PVST$ or $PMST$. Again the choice of $\prec$ is unimportant. The generating function associated to these tableaux is:

$$R_\lambda(\check{x}, \check{y}) = \sum_{T \in SSPT(\lambda)} (\check{x}, \check{y})^{wt(T)}$$

Definition 4.23. Let $\mu \supseteq \lambda$. Label the columns of the inner partition, λ, from the right to left with $\{1, \ldots, \ell\}$ where ℓ is the length of λ. A semistandard flagged tableau of shape μ/λ or element of $SSFT(\mu/\lambda)$ is a semistandard Young tableau of shape μ/λ with entries from $\{1, 2, \ldots, \ell\}$ such that each i must lie strictly to the right of the column labeled i.

Example 4.24. The following is an element of $SSFT((7, 6, 4, 4)/(4, 3, 3, 1))$.

C4	C3	C2	C1			
				1	1	2
			2	3	4	
			3			
	4	4	4			

Proposition 4.25. *There is a bijection from $PSMT(\lambda)$ to pairs (P, Q) where $P \in SSPT(\mu)$ and $Q \in SSFT(\mu/\lambda)$ for some $\mu \supseteq \lambda$. If $T \to (P, Q)$ then the x-weight and y-weight of T and P are the same.*

Proof. We will fix the order $1' \prec 1 \prec 2' \prec 2 \prec \cdots$ for concreteness. We begin with a series of reductions.

- Reduction 1: Let X^k denote the set of pairs (P, Q) where $Q \in SSFT(\mu/\lambda)$ for some μ and such that Q only contains entries from $\{1, \ldots, k\}$ and $P \in PSMT(\mu)$ such that only boxes in the leftmost $\ell - k$ columns of P may contain more than one entry. Then it suffices to find an x-weight and y-weight preserving bijection from X^k to X^{k+1} where the weight of $(P, Q) \in X^k$ or X^{k+1} is defined as the weight of P.
- Reduction 2: Let $P_0 \in PMST(\lambda^{\ell-k-1})$ be arbitrary where $\lambda^{\ell-k-1}$ is the shape formed by the leftmost $\ell - (k+1)$ columns of λ. Now let $\nu \supseteq \lambda$ be arbitrary such that ν/λ contains no box in the leftmost $l - k + 1$ columns of λ. Let $Q_0 \in SSFT(\nu/\lambda)$ be arbitrary such that it only contains entries from $\{1, \ldots, k\}$. Let X_0^k denote the subset of X^k comprised of pairs of the form (P, Q) where $Q = Q_0$ and where P and P_0 agree where they overlap.

Let X_0^{k+1} denote the subset of X^{k+1} comprised of pairs of the form (P', Q') where Q' can be attained by adding a horizontal strip of boxes to Q_0 and filling the boxes of the strip with $k + 1$ and where P' and P_0 agree where they overlap. It now suffices to find an x-weight and y-weight preserving bijection from X_0^k to X_0^{k+1}.

- Reduction 3: Finally, by ignoring the entries of Q_0 and letting ρ correspond to the shape ν after deleting the leftmost $\ell - k - 1$ columns, the above reduces to just finding an x-weight and y-weight preserving bijection for arbitrary ρ:

$$A := PSMT_1(\rho) \to \bigcup_{\tau \supseteq \rho} SSPT(\tau) := B$$

where $PSMT_1(\rho)$ is the subset of $PSMT(\rho)$ with multiple entries per box only permitted in the leftmost column and the union is over all τ such that τ/ρ is a horizontal strip with no box in the leftmost column. This bijection should have the additional property that the smallest entry of each box in the leftmost column of ρ is preserved. This way, any given P_0 can be appended to the left-hand side of an element of A if and only P_0 can be appended to the left-hand side of the image of this element in B, so the bijection from A to B induces a bijection from X_0^k to X_0^{k+1}.

Next we will need the following primed version of RSK insertion:

Definition 4.26. Primed RSK insertion is defined by the following rules:

- Suppose one is given a column C of some $SSPT$. Then an element x is inserted into C as follows:
  (1) If x is an unprimed element then let y be the highest element of C greater than or equal to x. Replace y with x and take the output of the algorithm to be y. If no such y exists, append x to the bottom of C.
  (2) If x is a primed element then let y be the highest element of C greater than x. Replace y with x and take the output of the algorithm to be y. If no such y exists, append x to the bottom of C.
- If one is given a column C of some $SSPT$ and an element x that is greater than $min(C)$ or equal to some unprimed element of C, then x is outserted through C as follows:
  (1) If x is an unprimed element then let y be the lowest element of C less than or equal to x. Replace y with x and take the output of the algorithm to be y.
  (2) If x is a primed element then let y be the lowest element of C less than x. Replace y with x and take the output of the algorithm to be y.
- Given some $T \in SSPT$ an element x is inserted into T, denote $x \hookrightarrow T$, by first inserting it into the first column of T. After that point, every time an output is produced the output is inserted in the column to the right of that from which it came. If T' is the result of inserting x into T, we write $x \hookrightarrow T = T'$.
- Given a corner box of T, b, the element in b is outserted from T, denote $b \hookleftarrow T$, by removing it from its column and outserting it through the column to the left. After that point, every time an output is produced, it is outserted through the column to the left. If $b \hookleftarrow T$ results in the tableau

T' and x is the element eventually outserted from the first column we write $b \leftarrow T = (T', x)$.

It is easy to check that primed RSK insertion has the following properties.

(1) If $x \hookrightarrow T = T'$ then T' is a valid $SSPT$.
(2) If $b \leftarrow T = (T', x)$ then T' is a valid $SSPT$.
(3) If $x \hookrightarrow T = T'$ and $shp(T')/shp(T) = b$ then $b \leftarrow T' = (T, x)$.
(4) If $b \leftarrow T = (T', x)$ then $x \hookrightarrow T' = T$.
(5) Suppose $y < x$ or $y = x$ is an unprimed element. Let $x \hookrightarrow T = T'$ and $y \hookrightarrow T' = T''$. Then the box $shp(T')/shp(T)$ lies in a column strictly to the left of that in which $shp(T'')/shp(T')$ lies.
(6) Suppose $y > x$ or $y = x$ is a primed element. Let $x \hookrightarrow T = T'$ and $y \hookrightarrow T' = T''$. Then the box $shp(T')/shp(T)$ lies in a column weakly to the right of that in which $shp(T'')/shp(T')$ lies.

We now give the need bijection Θ between the sets A and B defined earlier in the proof. Let $P \in A$. Circle (one of) the smallest elements in each box the first column of P. Then in weakly decreasing order insert each remaining element from this column into the tableau formed by all but the first column of P. Now erase the circles and define $\Theta(P)$ to be the result.

Example 4.27.
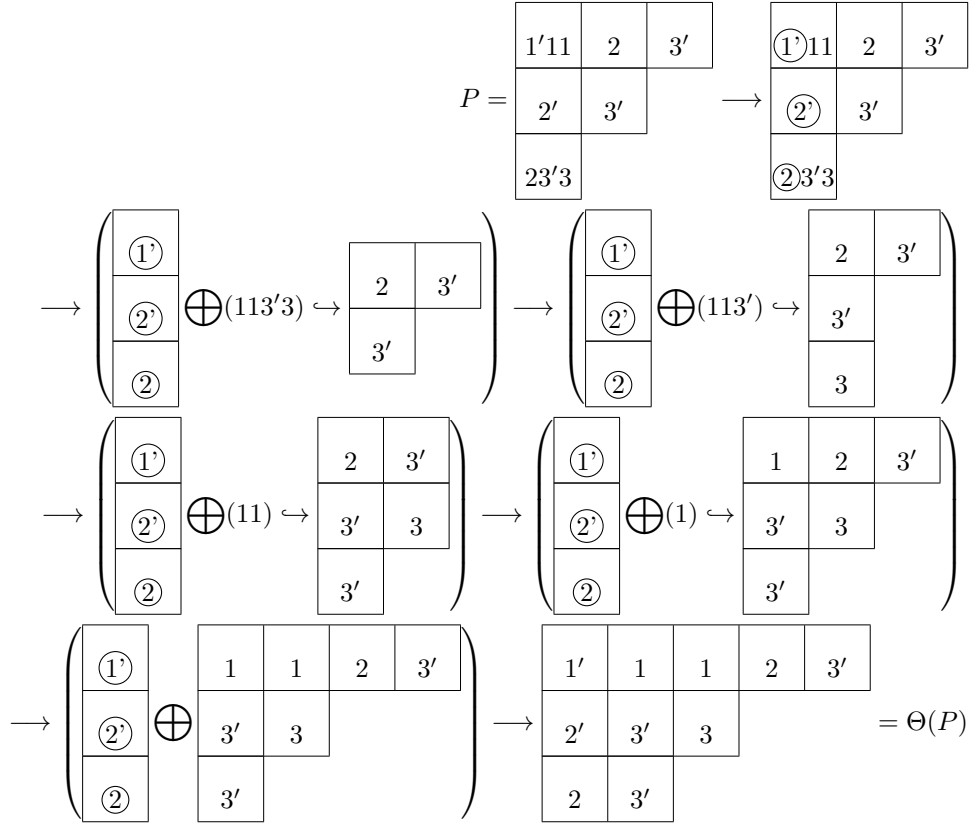
Conversely, given $T \in B$, start by circling all entries in the first column of T. Now consider all the boxes in $shp(T)/\rho$. Starting from the rightmost such box and moving left, outsert the elements in these boxes from the tableau formed by all but the first column of T. Then take all of the elements that were produced in the last step of these outsertions and add them to the first column as follows. Add each element x to the lowest box, b, such that x is greater than the circled element of b or equal to the circled element of b and unprimed. Erase the circles and define $\Theta^{-1}(T)$ to be the result. Note the following:

- If $P \in A$ then $\Theta(P) \in B$. It follows from property (5) that $shp(\Theta(P)/P)$ is a horizontal strip and it clearly does not have a box in column one. It follows from property (1) that the tableau formed by all but the first column of $\Theta(P)$ is a valid $SSPT$ and it is not difficult to verify that the adjoining the first column does not change this.
- If $T \in B$ then $\Theta^{-1}(T) \in A$. By construction $\Theta^{-1}(T)$ has shape ρ and no boxes outside the leftmost column have multiple entries. Property (2) verifies that the tableau formed by all but the leftmost column of $\Theta^{-1}(T)$ is a valid $PSMT$. Property (6) and the invertibility property (4) imply that as we apply outsertion from to boxes from right to left, the elements produced in the last step of outsertion are weakly decreasing and no two consecutive elements are primed and equal. In particular, the set of elements to be added to column one does not contain more than one copy of any primed number. This along with the manner in which elements are added to column one implies that the first column of $\Theta^{-1}(T)$ by itself is a valid $PSMT$. Finally, any element, x, that was outserted from a box b' in the column to the right of the leftmost column will be added to a box b in leftmost column lying weakly below b'. It follows that x is either less than the element, y lying immediately to the right of x (in box b' or some box below b') or equal to y and unprimed. Thus adjoining the leftmost column of $\Theta^{-1}(T)$ to the rest of the tableau does not break any rules required for $\Theta^{-1}(T)$ to be a $PSMT$.
- If $P \in A$ then $\Theta^{-1}(\Theta(P)) = P$. First, the circled elements of $\Theta^{-1}(\Theta(P))$ (before erasing circles) and P (after circling elements) are the same and occur in the same boxes because neither Θ nor Θ^{-1} affects these elements. Next, property (5) implies that the boxes of $shp(\Theta(P))/shp(P)$ appear from left to right when computing $\Theta(P)$. Thus outserting the elements of these boxes from right to left, implies, in light of property (3), that the exact same multiset, M, of elements is added to the leftmost column when computing $\Theta^{-1}(\Theta(P))$ as was removed from the leftmost column when computing $\Theta(P)$ and that P and $\Theta^{-1}(\Theta(P))$ agree on all columns to the right of the leftmost column. Finally, there is only one way in which the elements of the multiset M can possibly be arranged in the boxes of the leftmost column to give a valid $PSMT$ column and maintain the fact that the circled elements are the box minimums. Both P (after circling elements) and $\Theta^{-1}(\Theta(P))$ (before erasing circles) have this property so these two tableaux agree on the noncircled elements of the leftmost column. Thus, in all, these two tableaux agree everywhere, and since making circles and then erasing them does nothing overall, we have $\Theta^{-1}(\Theta(P)) = P$.

- If $T \in B$ then $\Theta(\Theta^{-1}(T)) = T$. Let L be the list of elements outserted from the column to the right of the leftmost column during the computation of $\Theta^{-1}(T)$ (left to right corresponding to first outserted to last outserted). Because of property (6) and invertibility property (4) L is weakly increasing from left to right. The elements of L are the ones added to the leftmost column when computing $\Theta^{-1}(T)$ and it is clear that they are the ones removed from the leftmost column when computing $\Theta(\Theta^{-1}(T))$. Moreover when computing $\Theta(\Theta^{-1}(T))$ they are to be inserted into the the column to the right of the leftmost column in weakly decreasing order. Thus the elements to insert into this column and the order in which to insert them is just the list L read from right to left. It follows now from invertibility property (4) that $\Theta(\Theta^{-1}(T)) = T$.

$\square$

Corollary 4.28. *We have*

$$^{\times}G_\lambda(\check{x}, \check{y}) = \sum_{\mu \supseteq \lambda} \sum_{T \in SSFT(\mu/\lambda)} R_\mu(\check{x}, \check{y})$$

and [MPS18]

$$G_\lambda(\check{x}) = \sum_{\mu \supseteq \lambda} \sum_{T \in SSFT(\mu/\lambda)} s_{\mu^\perp}(\check{x})$$

and [HS20]

$$^*G_\lambda(\check{x}) = \sum_{\mu \supseteq \lambda} \sum_{T \in SSFT(\mu/\lambda)} s_\mu(\check{x})$$

Proof. The first is immediate from Proposition 4.25. The second follows from setting $(\check{x}, \check{y}) = (0, \check{x})$ in the first. The third follows from setting $(\check{x}, \check{y}) = (\check{x}, 0)$. $\square$

Let $\mathbf{x} = (x_1, x_2, \ldots)$ be an infinite list of variables and define:

$$G_\lambda(\mathbf{x}) = \lim_{m \to \infty} G_\lambda(\check{x})$$
$$^*G_\lambda(\mathbf{x}) = \lim_{m \to \infty} {}^*G_\lambda(\check{x})$$
$$\mathcal{G}_\omega(\mathbf{x}) = \lim_{m \to \infty} \mathcal{G}_\omega(\check{x})$$
$$^*\mathcal{G}_\omega(\mathbf{x}) = \lim_{m \to \infty} {}^*\mathcal{G}_\omega(\check{x})$$

Let Ω be the involution on symmetric functions that interchanges the elementary and homogeneous symmetric bases. Equivalently, Ω permutes the base of Schur functions by sending $s_\lambda(\mathbf{x}) \to s_{\lambda^\perp}(\mathbf{x})$.

Corollary 4.29. *We have*

$$\Omega(G_\lambda(\mathbf{x})) = {}^*G_\lambda(\mathbf{x})$$
$$\Omega(\mathcal{G}_\omega(\mathbf{x})) = {}^*\mathcal{G}_\omega(\mathbf{x})$$

Proof. The first follows from proposition 4.28. The second follows from the first along with proposition 4.11 and the linearity of Ω. $\square$

4.4. **Weak Stable Double Grothendieck Polynomials.** A *double Hecke group-ing* is a grouping of a Hecke word into an even number of factors where the first half of the factors are weakly decreasing in the order $1 < \cdots < n$ and the last half of the factors are weakly increasing in the order $1 < \cdots < n$. For example $(3)(22)(31)|(22)()(233)$ is a double Hecke grouping for $(4,3,2,1) \in S_4$, where we have drawn a "|" between the left half and the right half of the factors for view-ing convenience. The *x-weight* of a double Hecke grouping is the vector whose i^{th} entry records the number of elements in the ith factor to the right of center. The x-weight of the example above is $(2,0,3)$. The *y-weight* of a Hecke grouping is the vector whose i^{th} entry records the number of entries in the ith factor to the left of center. The y-weight of the example above is $(2,2,1)$. We denote the set of all double Hecke groupings for ω into $2m+2$ factors by $^*\mathcal{F}_\omega^\square$.

Definition 4.30. The weak stable double Grothendieck polynomial for ω is given by:

$$^*\mathcal{G}_\omega(\check{x},\check{y}) = \sum_{\mathfrak{f} \in {}^*\mathcal{F}_\omega^\square} (\check{x},\check{y})^{wt(\mathfrak{f})}$$

A *circled hook Hecke grouping* of is a grouping of a Hecke word into hook fac-tors. Each hook factor contains a multiset of $\{①, 1, ②, 2, \ldots\}$ arranged so that all uncircled factors lie to the right of all circled factors and such that the uncircled elements are weakly decreasing left to right and the circled elements are weakly increasing left to right. Moreover, erasing the circles and parentheses should give a Hecke word for ω. For instance $(③②223)(②①①23)(③1223)$ is a circled hook Hecke factorization for the permutation $(4,3,2,1) \in S_4$. The *x-weight* of a circled hook Hecke factorization is the vector whose i^{th} entry records the number of uncir-cled elements in its ith factor. The x-weight of the example above is $(3,2,4)$. The *y-weight* of such a factorization is the vector whose i^{th} entry records the number of circled entries in the ith factor. The y-weight of the example above is $(2,3,1)$.

Proposition 4.31. *Denote the set of all circled hook Hecke groupings of ω with $m+1$ factors by $^*\mathcal{F}_\omega^{\circledv}$. Then:*

$$^*\mathcal{G}_\omega(\check{x},\check{y}) = \sum_{\mathfrak{f} \in {}^*\mathcal{F}_\omega^{\circledv}} (\check{x},\check{y})^{wt(\mathfrak{f})}$$

Proof. First we show that $^*\mathcal{G}_\omega(\check{x},\check{y})$ is symmetric with respect to transposition of y_i and y_{i+1}. This will follow from showing that for any permutation μ and any numbers a and b there are an equal number of pairs of the form:

$$(u_1 u_2 \cdots u_a)(v_1 v_2 \cdots v_b): \ u_1 \geq \cdots \geq u_a, \ v_1 \geq \cdots \geq v_b, \ u_1 \cdots u_a v_1 \cdots v_b \sim \tilde{\mu}$$

and pairs of the form:

$$(u'_1 u'_2 \cdots u'_b)(v'_1 v'_2 \cdots v'_a): \ u'_1 \geq \cdots \geq u'_b, \ v'_1 \geq \cdots \geq v'_a, \ u'_1 \cdots u'_b v'_1 \cdots v'_a \sim \tilde{\mu}$$

But this is just the statement that $^*\mathcal{G}_{\mu^{-1}}(y_1, y_2)$ is symmetric which follows from the fact that $^*\mathcal{G}_\omega(\check{y})$ expands in terms of $^*G_\lambda(\check{y})$ and that the latter expands in terms of Schur polynomials. Thus the order in which the decreasing factors in a factorization of $^*\mathcal{F}_\omega^\square$ appear amongst themselves is irrelevant. In particular, we may write the first decreasing factor (i.e., the one contributing to the exponent of y_1) in the leftmost position, and, in general, the $(i+1)^{st}$ directly to the right of

the ith factor for each $i \leq m$. Further, the existence of the bijection $\downarrow$ implies that the relative order in which the increasing factors and decreasing factors appear in a factorization in $^*\mathcal{F}_\omega^\square$ is also irrelevant. In particular we can write the first decreasing factor in the leftmost position and then the first increasing factor to its right, and, in general, the $(i+1)^{st}$ decreasing factor directly to the right of the ith increasing factor and then the $(i+1)^{st}$ increasing factor directly to the right of the $(i+1)^{st}$ decreasing factor. It is clear that this interpretation on $^*\mathcal{F}_\omega^\square$ is the same as $^*\mathcal{F}_\omega^{\circledv}$ after circling all elements of the increasing factors and then grouping together adjacent pairs of factors into single factors. This proves the proposition. $\qquad\square$

Using definition 4.30 or proposition 4.31 to evaluate $^*\mathcal{G}_\omega(\check{x}, \check{y})$ is computationally impractical since the sets $^*\mathcal{F}_\omega^\square$ and $^*\mathcal{F}_\omega^{\circledv}$ are generally infinite. Luckily, there is a way to compute the weak stable double Grothendieck polynomial by applying a substitution of variables to the easier to compute stable double Grothendieck polynomial. Set:

$$^*\check{x} = \left(\frac{x_1}{1 - x_1}, \dots, \frac{x_{k+1}}{1 - x_{k+1}} \right)$$

$$^*\check{y} = \left(\frac{y_1}{1 - y_1}, \dots, \frac{y_{k+1}}{1 - y_{k+1}} \right)$$

Proposition 4.32. *We have:*

$$^*\mathcal{G}_\omega(\check{x}, \check{y}) = \mathcal{G}_\omega(^*\check{y}, ^*\check{x})$$

Proof. There is a surjection from $^*\mathcal{F}_\omega^{\circledv}$ to $\mathcal{F}_\omega^{\circledv}$ given by doing the following to each factor. For each i, remove any instances of $\textcircled{i}$ that are not the leftmost $\textcircled{i}$ to appear in that factor and remove any instances of i that are not the leftmost i to appear in that factor. Then replace every $\textcircled{i}$ with i and every i with $\textcircled{i}$. If $\mathfrak{f}' \in {}^*\mathcal{F}_\omega^{\circledv}$ maps to $\mathfrak{f} \in \mathcal{F}_\omega^{\circledv}$ under this surjection we write $\mathfrak{f}' \in orb(\mathfrak{f})$. Write $wt_{x_i}(\mathfrak{f})$ represents the ith coordinate of the x-weight of $\mathfrak{f}$ and $wt_{y_i}(\mathfrak{f})$ represents the ith coordinate of the y-weight of $\mathfrak{f}$. Then we have

$$^*\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{\mathfrak{f} \in {}^*\mathcal{F}_\omega^{\circledv}} (\check{x}, \check{y})^{wt(\mathfrak{f})} = \sum_{\mathfrak{f} \in \mathcal{F}_\omega^{\circledv}} \left(\sum_{\mathfrak{f}' \in orb(\mathfrak{f})} (\check{x}, \check{y})^{wt(\mathfrak{f}')} \right)$$

$$= \sum_{\mathfrak{f} \in \mathcal{F}_\omega^{\circledv}} \left(\sum_{\mathfrak{f}' \in orb(\mathfrak{f})} \left(\prod_i x_i^{wt_{x_i}(\mathfrak{f}')} y_i^{wt_{y_i}(\mathfrak{f}')} \right) \right)$$

$$= \sum_{\mathfrak{f} \in \mathcal{F}_\omega^{\circledv}} \left(\prod_i \left(\frac{x_i}{1 - x_i} \right)^{wt_{y_i}(\mathfrak{f})} \left(\frac{y_i}{1 - y_i} \right)^{wt_{x_i}(\mathfrak{f})} \right)$$

$$= \sum_{\mathfrak{f} \in \mathcal{F}_\omega^{\circledv}} (^*\check{y}, ^*\check{x})^{wt(\mathfrak{f})} = \mathcal{G}_\omega(^*\check{y}, ^*\check{x})$$

$$\square$$

The proposition above lets us replace the infinite set $^*\mathcal{F}_\omega^{\circledv}$ with the finite one $\mathcal{F}_\omega^{\circledv}$. However, as in proposition 4.4, the latter is still unnecessarily large and can

be replaced with the smaller $\mathcal{F}_\omega^\triangledown$. (Recall the set of all hook Hecke factorizations of ω with $m+1$ factors is denoted by $\mathcal{F}_\omega^\triangledown$ and that for for $\mathfrak{f} \in \mathcal{F}_\omega^\triangledown$ we define $wt_{x_i}(\mathfrak{f})$ to be the number of entries in factor i lying to the left of the minimum entry of factor i and $wt_{y_i}(\mathfrak{f})$ to be the number of entries in factor i lying to the right of the minimum entry of factor i.)

Proposition 4.33. *We have:*

$$
{}^*\mathcal{G}_\omega(\check{x},\check{y}) = \sum_{\mathfrak{f} \in \mathcal{F}_\omega^\triangledown} \left(\prod_{i \in NE(\mathfrak{f})} \frac{x_i + y_i - x_i y_i}{(1-x_i)(1-y_i)} \right) ({}^*\check{y}, {}^*\check{x})^{wt(\mathfrak{f})}
$$

where $NE(\mathfrak{f})$ is the set of all i such that the ith factor of $\mathfrak{f}$ is nonempty.

Proof. By proposition 4.32 we have ${}^*\mathcal{G}_\omega(\check{x},\check{y}) = \mathcal{G}_\omega({}^*\check{y}, {}^*\check{x})$ and applying the formula from proposition 4.4 to the latter gives the right hand expression above after noting that:

$$
\left(\frac{x_i}{1-x_i} \right) + \left(\frac{y_i}{1-y_i} \right) + \left(\frac{x_i}{1-x_i} \right) \left(\frac{y_i}{1-y_i} \right) = \frac{x_i + y_i - x_i y_i}{(1-x_i)(1-y_i)}
$$

$\square$

4.5. Double Stable and Double Symmetric Grothendieck Polynomials.
We are interested now in the relationship between $G_\lambda(\check{x},\check{y})$ and $\mathcal{G}_\omega(\check{x},\check{y})$, as well as the relationship between ${}^*G_\lambda(\check{x},\check{y})$ and ${}^*\mathcal{G}_\omega(\check{x},\check{y})$.

Proposition 4.34. *There is an x-weight and y-weight preserving bijection between the following sets:*

- *$\mathcal{F}_\omega^\square$ to pairs (P,Q) where $P \in HT_\omega$ and $Q \in PSVT^\perp$ have the same shape.*
- *${}^*\mathcal{F}_\omega^\square$ to pairs (P,Q) where $P \in HT_\omega$ and $Q \in PMVT$ have the same shape.*

Proof. We will use the order $1' \lhd 2' \lhd \cdots \lhd 1 \lhd 2 \lhd \cdots$ for the sake of this proof. Let $\mathfrak{f} \in \mathcal{F}_\omega^\square$, respectively $\mathfrak{f} \in {}^*\mathcal{F}_\omega^\square$, and let $\mathfrak{f}_\ell$ represent the leftmost $m+1$ factors of $\mathfrak{f}$ and $\mathfrak{f}_r$ represent the rightmost $m+1$ factors of $\mathfrak{f}$. Suppose that $\mathfrak{f}_\ell$ represents the permutation μ and denote by $\overleftarrow{\mathfrak{f}_\ell}$ the factorization given by reversing the order of the factors of $\mathfrak{f}_\ell$ and reversing the order of the letters within each factor. Note that $\overleftarrow{\mathfrak{f}_\ell}$ is now a Hecke factorization, respectively, Hecke grouping, of μ^{-1}. Apply Hecke insertion to $\overleftarrow{\mathfrak{f}_\ell}$ to obtain a pair (P_ℓ, Q_ℓ) where $P_\ell \in HT_{\mu^{-1}}(\lambda_\ell)$ and $Q_\ell \in SVT^\perp(\lambda_\ell)$, respectively, $Q_\ell \in MVT(\lambda_\ell)$, for some λ_ℓ. Now prime all entries of Q and transpose both tableaux to get a pair $(P_\ell^t, Q_\ell^{t'})$ of shape λ_ℓ^t. Now, proceed with Hecke insertion as if the current insertion tableau were P_ℓ^t and the current recording tableau were $Q_\ell^{t'}$ and exactly the factors of $\mathfrak{f}_r$ remained to be inserted. The only ambiguity to starting in the middle of Hecke insertion like this is not knowing what entry to add to the recording tableau during insertion of the ith factor of $\mathfrak{f}_r$: Use the entry i. Denote the final insertion tableau and recording tableau as P and Q respectively. We can now define the bijection: $\Phi(\mathfrak{f}) = (P, Q)$.

Example 4.35. Let $\mathfrak{f} = (124)(13)|(432)(3) \in \mathcal{F}_{\underset{\omega}{\square}}$. we have $\mathfrak{f}_\ell = (124)(13)$ and $\mathfrak{f}_r = (432)(3)$. First we apply Hecke insertion to $\overleftarrow{\mathfrak{f}_\ell} = (31)(421)$ to find that

$$
P_\ell = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}, \quad
Q_\ell = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}, \quad
P_\ell^t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}, \quad
Q_\ell^{t'} = \begin{array}{|c|c|c|} \hline 1' & 1' & 2' \\ \hline 2' & 2' \\ \cline{1-2} \end{array}
$$

Then we apply Hecke insertion of $\mathfrak{f}_r = (432)(3)$ to the starting pair $(P_\ell^t, Q_\ell^{t'})$

$$
P = P_\ell^t \leftarrow (432)(3) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \leftarrow (432)(3) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}
$$

$$
Q = \begin{array}{|c|c|c|c|} \hline 1' & 1' & 2' & 1 \\ \hline 2' & 2'1 & 2 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array}
$$

There is much to prove:

- P is a Hecke tableau and it represents the permutation ω: Suppose $\mathfrak{f}_\ell$ is a Hecke word for some permutation μ. Now, P_ℓ is formed by applying Hecke insertion to $\overleftarrow{\mathfrak{f}_\ell}$ and so is therefore a Hecke tableau such that its rows read left to right, from bottom row to top row forms a Heck word for μ^{-1}. Since the only requirement for being a Hecke tableau is that the rows and columns are strictly increasing, which is clearly symmetric under transposition, it is also true that P_ℓ^t is a Hecke tableau. Moreover, the columns of P_ℓ^t read from top to bottom from rightmost column to leftmost column give a Hecke word for μ^{-1}. Therefore the columns read from bottom to top from leftmost column to rightmost column give a Hecke word for μ. However:

Claim 4.36. *The column reading word and row reading word of a Hecke tableau, H, represent the same permutation.*

Proof. Let $\omega_k(H)$ be the permutation represented by reading the leftmost k columns of H bottom to top, leftmost column to rightmost column and then, ignoring the first k columns of H, reading rows left to right, bottom row to top row. It suffices to show that $\omega_k(H) = \omega_{k+1}(H)$. Without loss of generality we may assume $k = 0$. Now let $\omega^j(H)$ be the permutation represented by reading the the the lowest j entries of the leftmost column of H from bottom to top and then reading the remaining entries of H by rows, left to right, bottom to top. To show that $\omega_k(H) = \omega_{k+1}(H)$ for $k = 0$ it suffices to show that $\omega^j(H) = \omega^{j+1}(H)$. If a is the entry in the first column of H in $j+1^{st}$ row from the bottom and b is any entry in H in the jth row from the bottom or lower not in the first column of H then $a < b - 1$. Therefore a commutes with all such b which shows that $\omega^j(H) = \omega^{j+1}(H)$. □

Therefore reading the rows of P_ℓ^t left to right, bottom to top also gives a Hecke word for μ. Since $\mathfrak{f}_r$ gives a Hecke word for some permutation ν such that $\widetilde{\mu}\widetilde{\nu} \sim \hat{\omega}$ the properties of Hecke insertion imply that the Hecke word formed by reading the rows of P from left to right, bottom to top also represents ω. All this shows that P is a Hecke tableau and it represents the permutation ω.

- $Q \in PSVT^\perp(\lambda)$, respectively $Q \in PMVT(\lambda)$, where λ is the shape of P: First, $Q_\ell \in SVT^\perp(\lambda_\ell)$, respectively $Q_\ell \in MVT(\lambda_\ell)$, by Lemma 2.17 so it follows that in particular $Q_\ell^{t'} \in PSVT^\perp(\lambda_\ell^t)$, respectively $Q_\ell^{t'} \in PMVT(\lambda_\ell^t)$. On the other hand it also follows from lemma 2.17 that the unprimed entries from Q will give an element of $SVT^\perp(\lambda/\rho)$, respectively $MVT(\lambda/\rho)$, for some $\rho \subseteq \lambda_\ell^t$ such that $\lambda_\ell^t \setminus \rho$ contains no more than one box in any row or column. The fact that the primed and unprimed entries give such tableaux along with the fact that $i' \lhd j$ for any i and j imply that $Q \in PSVT^\perp(\lambda)$, respectively $Q \in PMVT(\lambda)$.

- Φ is injective. Let $\mathfrak{f}, \mathfrak{f}^\times \in \mathcal{F}_\omega^\square$, respectively, $\mathfrak{f}, \mathfrak{f}^\times \in {}^*\mathcal{F}_\omega^\square$. Suppose that $\Phi(\mathfrak{f}) = \Phi(\mathfrak{f}^\times)$ with $\mathfrak{f} \neq \mathfrak{f}^\times$. We use the notation of the construction of Φ. Additionally, let $\mathfrak{p}_\ell$ represent the Hecke factorization given by reading the columns of P_ℓ^t bottom to top from left column to right column. Let $\mathbf{p}_\ell$ represent the Hecke grouping given by reading the rows of P_ℓ^t left to right from bottom row to top row. In all cases use the same notation for corresponding objects associated to $\mathfrak{f}^\times$ but with a $\times$.

  If $\mathfrak{f}_\ell \neq \mathfrak{f}_\ell^\times$ then by lemma 2.17 $(P_\ell, Q_\ell) \neq (P_\ell^\times, Q_\ell^\times)$. But $Q_\ell \neq Q_\ell^\times$ would force $Q \neq Q^\times$ so we must have $P_\ell \neq P_\ell^\times$ and so $P_\ell^t \neq (P_\ell^\times)^t$. Thus either $\mathfrak{f}_\ell \neq \mathfrak{f}_\ell^\times$ in which case $\mathfrak{p}_\ell \neq \mathfrak{p}_\ell^\times$ and $\mathbf{p}_\ell \neq \mathbf{p}_\ell^\times$ or else $\mathfrak{f}_r \neq \mathfrak{f}_r^\times$. Either way, $\mathfrak{p}_\ell \mathfrak{f}_r \neq \mathfrak{p}_\ell^\times \mathfrak{f}_r^\times$ and $\mathbf{p}_\ell \mathfrak{f}_r \neq \mathbf{p}_\ell^\times \mathfrak{f}_r^\times$. But it is easy to see that the insertion tableau of $\mathfrak{p}_\ell$ and $\mathbf{p}_\ell$ is just P_ℓ^t and the insertion tableau of $\mathfrak{p}_\ell^\times$ and $\mathbf{p}_\ell^\times$ is just $P_\ell^{t\times}$. Meanwhile the recording tableaux of $\mathfrak{p}_\ell$ and $\mathfrak{p}_\ell^\times$ are the same and the recording tableaux $\mathbf{p}_\ell$ and $\mathbf{p}_\ell^\times$ are the same. Thus the Hecke factorizations $\mathfrak{p}_\ell \mathfrak{f}_r$ and $\mathfrak{p}_\ell^\times \mathfrak{f}_r^\times$, respectively the Hecke groupings $\mathbf{p}_\ell \mathfrak{f}_r$ and $\mathbf{p}_\ell^\times \mathfrak{f}_r^\times$, would be two distinct elements mapping to the same insertion and recording tableaux under the bijection of lemma 2.17.

- Φ is surjective. Suppose we are given (P, Q) of the same shape λ where $P \in HT_\omega$ and $Q \in PSVT^\perp$, or, respectively $Q \in MVST$. Let Q_{out} denote the skew tableau formed by only taking the unprimed entries of Q. Let Q_{in} denote the tableau formed by taking only the primed entries of Q and then erasing all their prime marks. Take j sufficiently large, (for example more than the number of primed entries in Q) and let Q_{can} be any $SVT^\perp$, respectively MVT such such that erasing all integers less than or equal to j and subtracting j from the rest gives Q_{out}.

  Now use lemma 2.17 to find a Hecke factorization, respectively grouping, $\widetilde{\mathfrak{f}}$ mapping to (P, Q_{can}). Write $\widetilde{\mathfrak{f}} = \widetilde{\mathfrak{f}}_\ell \mathfrak{f}_r$ where $\widetilde{\mathfrak{f}}_\ell$ represents the first j factors of $\widetilde{\mathfrak{f}}$. Suppose the insertion tableau of $\widetilde{\mathfrak{f}}_\ell$ is T. Use lemma 2.17 to find a Hecke factorization, respectively grouping, $\mathfrak{f}_\ell$ mapping to (T^t, Q_{in}^t). Let $\overleftarrow{\mathfrak{f}}_\ell$ represent the result of reversing the order of the factors of $\mathfrak{f}_\ell$ and reversing the order of the entries within each factor. Then we have that

$\Phi(\overleftarrow{\mathfrak{f}_\ell}\mathfrak{f}_r) = (P, Q)$. Now $\overleftarrow{\mathfrak{f}_\ell}\mathfrak{f}_r \in \mathcal{F}^\square_{\omega'}$, respectively $\overleftarrow{\mathfrak{f}_\ell}\mathfrak{f}_r \in {}^*\mathcal{F}^\square_{\omega'}$, for some ω' just by construction. But by the first bullet point we have $\omega' = \omega$.

- Φ preserves the x-weight and the y-weight: Suppose $\Phi(\mathfrak{f}) = (P, Q)$ where $\mathfrak{f} = \mathfrak{f}_\ell\mathfrak{f}_r$. The y-weight of $\mathfrak{f}$ is the vector whose i^{th} coordinate records the number of entries in the ith factor of $\overleftarrow{\mathfrak{f}_\ell}$ which is the number of times i appears in Q_ℓ or equivalently the number of times i' appears in $Q_\ell^{t'}$ or equivalently in Q. This is the definition of the y-weight of Q. The x-weight of $\mathfrak{f}$ is the vector whose i^{th} coordinate records the number of entries in the ith factor of $\mathfrak{f}_r$ which is the number of times i appears in Q. This is the definition of the x-weight of Q.

$\square$

Remark 4.37. If it were not for the unfortunate fact mentioned in Remark 2.14 the whole process of reversing the left side of the factorization and then inserting and then transposing would not be necessary and the proposition could be proved through just inserting the factors directly. We leave it as an open problem to find a way of altering Hecke insertion so it has the additional properties needed for this simpler proof.

Corollary 4.38. *We have*

$$\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_\lambda \sum_{T \in HT_\omega(\lambda)} G_\lambda(\check{x}, \check{y})$$

$$^*\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_\lambda \sum_{T \in HT_\omega(\lambda)} {}^*G_\lambda(\check{x}, \check{y})$$

Proof. This follows from proposition 4.34 $\square$

Proposition 4.39. *Let $\lambda \in \Lambda_{k+1}$ and set $\omega = \Gamma^{-1}(\lambda)$. Then we have*

$$G_\lambda(\check{x}, \check{y}) = \mathcal{G}_\omega(\check{x}, \check{y})$$

$$^*G_\lambda(\check{x}, \check{y}) = {}^*\mathcal{G}_\omega(\check{x}, \check{y})$$

Proof. This follows from corollary 4.38 because by lemma 4.14 HT_ω is composed of a single tableau of shape λ. $\square$

Corollary 4.40. *We have*

$$^*G_\lambda(\check{x}, \check{y}) = G_\lambda({}^*\check{y}, {}^*\check{x})$$

This follows from proposition 4.39 after setting $\omega = \Gamma^{-1}(\lambda)$ and using proposition 4.32.

Lemma 4.41. *Recall that if μ is any permutation then X_μ is the set of all pairs of permutations (u, v) such that the concatenation uv viewed as a Hecke word satisfies $uv \sim \mu$. With this notation we have*

$$\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{(u,v) \in X_\omega} \mathcal{G}_{u^{-1}}(\check{y})\mathcal{G}_v(\check{x})$$

$$^*\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{(u,v) \in X_\omega} {}^*\mathcal{G}_{u^{-1}}(\check{y}) \, {}^*\mathcal{G}_v(\check{x})$$

Proof. We have:

$$\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{\mathfrak{f} \in \mathcal{F}_\omega^\square} (\check{x}, \check{y})^{wt(\mathfrak{f})} =$$

$$\sum_{(u,v) \in X_\omega} \left(\sum_{\mathfrak{f} \in \mathcal{F}_{u^{-1}}} (\check{y})^{wt(\mathfrak{f})} \right) \left(\sum_{\mathfrak{f}' \in \mathcal{F}_v} (\check{x})^{wt(\mathfrak{f}')} \right) = \sum_{(u,v) \in X_\omega} \mathcal{G}_{u^{-1}}(\check{y}) \mathcal{G}_v(\check{x})$$

as well as:

$$^*\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{\mathfrak{f} \in {}^*\mathcal{F}_\omega^\square} (\check{x}, \check{y})^{wt(\mathfrak{f})} =$$

$$\sum_{(u,v) \in X_\omega} \left(\sum_{\mathfrak{f} \in {}^*\mathcal{F}_{u^{-1}}} (\check{y})^{wt(\mathfrak{f})} \right) \left(\sum_{\mathfrak{f}' \in {}^*\mathcal{F}_v} (\check{x})^{wt(\mathfrak{f}')} \right) = \sum_{(u,v) \in X_\omega} {}^*\mathcal{G}_{u^{-1}}(\check{y}) \, {}^*\mathcal{G}_v(\check{x})$$

$\square$

Corollary 4.42. *All of the functions, $\mathcal{G}_\omega(\check{x}, \check{y})$, $^*\mathcal{G}_\omega(\check{x}, \check{y})$, $G_\lambda(\check{x}, \check{y})$, $^*G_\lambda(\check{x}, \check{y})$ are doubly Schur positive.*

Proof. According to lemma 4.41, $\mathcal{G}_\omega(\check{x}, \check{y})$ and $^*\mathcal{G}_\omega(\check{x}, \check{y})$ expand as a sum of a product of stable, respectively weak stable, Grothendieck polynomials in $\check{y}$ and $\check{x}$. By proposition 4.11 each of these in turn expand in terms of symmetric, respectively weak symmetric, Grothendieck polynomials. By corollary 4.28 these polynomials themselves are Schur positive itself. Next, if we set $\omega = \Gamma^{-1}(\lambda)$, then by proposition 4.39 we have $G_\lambda(\check{x}, \check{y}) = \mathcal{G}_\omega(\check{x}, \check{y})$ and $^*G_\lambda(\check{x}, \check{y}) = {}^*\mathcal{G}_\omega(\check{x}, \check{y})$ so $G_\lambda(\check{x}, \check{y})$ and $^*G_\lambda(\check{x}, \check{y})$ are Schur positive by the first half of this proof. $\square$

Let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ be an infinite list of variables and define:

$$G_\lambda(\mathbf{x}, \mathbf{y}) = \lim_{m \to \infty} G_\lambda(\check{x}, \check{y})$$
$$^*G_\lambda(\mathbf{x}, \mathbf{y}) = \lim_{m \to \infty} {}^*G_\lambda(\check{x}, \check{y})$$
$$\mathcal{G}_\omega(\mathbf{x}, \mathbf{y}) = \lim_{m \to \infty} \mathcal{G}_\omega(\check{x}, \check{y})$$
$$^*\mathcal{G}_\omega(\mathbf{x}, \mathbf{y}) = \lim_{m \to \infty} {}^*\mathcal{G}_\omega(\check{x}, \check{y})$$

Let $\Omega_\mathbf{x}$ be the $\mathbb{Z}[\mathbf{y}]$ linear involution on functions symmetric with respect to $\mathbf{x}$ in $\mathbb{Z}[\mathbf{x}, \mathbf{y}]$ which sends $s_\lambda(\mathbf{x}) \to s_{\lambda^\perp}(\mathbf{x})$. Let $\Omega_\mathbf{y}$ be the $\mathbb{Z}[\mathbf{x}]$ linear involution on functions symmetric with respect to $\mathbf{y}$ in $\mathbb{Z}[\mathbf{x}, \mathbf{y}]$ which sends $s_\lambda(\mathbf{y}) \to s_{\lambda^\perp}(\mathbf{y})$.

Corollary 4.43. *We have*

$$\Omega_x \Omega_y(\mathcal{G}_\omega(\mathbf{x}, \mathbf{y})) = {}^*\mathcal{G}_\omega(\mathbf{x}, \mathbf{y})$$
$$\Omega_x \Omega_y(G_\lambda(\mathbf{x}, \mathbf{y})) = {}^*G_\lambda(\mathbf{x}, \mathbf{y})$$

Proof. Using lemma 4.41 (in the first and last step) and proposition 4.29 (in the middle step) we can compute:

$$\Omega_x\Omega_y[\mathcal{G}_\omega(\mathbf{x},\mathbf{y})] =$$

$$\Omega_x\Omega_y\left[\sum_{(u,v)\in X_\omega}\mathcal{G}_{u^{-1}}(\mathbf{y})\mathcal{G}_v(\mathbf{x})\right] =$$

$$\sum_{(u,v)\in X_\omega}{}^*\mathcal{G}_{u^{-1}}(\mathbf{y})\,{}^*\mathcal{G}_v(\mathbf{x}) =$$

$${}^*\mathcal{G}_\omega(\mathbf{x},\mathbf{y})$$

This establishes the first part of the corollary. To see the second part set $\omega = \Gamma^{-1}(\lambda)$. Then by proposition 4.39 and the first part, we have:

$$\Omega_x\Omega_y(G_\lambda(\mathbf{x},\mathbf{y})) =$$
$$\Omega_x\Omega_y(\mathcal{G}_\omega(\mathbf{x},\mathbf{y})) =$$
$${}^*\mathcal{G}_\omega(\mathbf{x},\mathbf{y}) =$$
$${}^*G_\lambda(\mathbf{x},\mathbf{y})$$

Recall that $\mathfrak{G}_\omega(x,y)$ is defined in terms of divided difference operators. Proposition 4.2 says that $\mathcal{G}_\omega(\check{x},\check{y})$ itself can be computed using this definition along with a certain shift and setting some variables to 0. By taking limits, $\mathcal{G}_\omega(\mathbf{x},\mathbf{y})$ can then be computed. Hence, $\mathcal{G}_\omega(\mathbf{x},\mathbf{y})$ can be considered an entirely algebraic object. Moreover, this algebraic object has a rather remarkable algebraic property:

Corollary 4.44. *Let Ω_x^y be defined on functions of in $\mathbf{x}$ and $\mathbf{y}$ by performing the substitutions:*

$$\mathbf{x} \to {}^*\mathbf{y}$$
$$\mathbf{y} \to {}^*\mathbf{x}$$

Then we have:

$$\Omega_x\Omega_y\mathcal{G}_\omega(\mathbf{x},\mathbf{y}) = \Omega_x^y\mathcal{G}_\omega(\mathbf{x},\mathbf{y})$$

Proof. This is just saying that $\Omega_x\Omega_y\mathcal{G}_\omega(\mathbf{x},\mathbf{y}) = \mathcal{G}_\omega({}^*\mathbf{y},{}^*\mathbf{x})$ which follows from combining corollary 4.43 with proposition 4.32.   □

□

5. Quasi-Weak Double Stable Grothendieck polynomials

We originally defined the quasi-weak symmetric Grothendieck polynomial because it was a convenient proof tool. However, it would feel incomplete if we did not quickly consider the corresponding quasi-weak stable Grothendieck polynomial. First, we need yet another type of factorization. A *circled hook Hecke frouping* of ω is a factorization into hook factors. Each hook factor contains a subset of $\{①, ②, \ldots\}$ and a multiset from $\{1, 2, \ldots\}$ arranged so that all circled factors lie to the left of all uncircled factors and such that the circled elements are decreasing left to right and the uncircled elements are weakly increasing left to right. Moreover, erasing the circles and parentheses should give a Hecke word for ω. For instance, $(③②233)(①22)(③②1133)$ is a circled hook Hecke frouping for the permutation $(4, 3, 2, 1) \in S_4$. The *x-weight* of a circled hook Hecke frouping is the vector whose

i^{th} entry records the number of uncircled elements in its ith factor. The x-weight of the example above is $(3, 2, 4)$. The *y-weight* of such a frouping is the vector whose i^{th} entry records the number of circled entries in the ith factor. The y-weight of the example above is $(2, 1, 2)$.

Denote the set of all circled hook Hecke froupings of ω with $m + 1$ factors by $^\times\mathcal{F}_\omega^{\textcircled{\vee}}$. Define the quasi-weak stable Grothendieck polynomial as:

$$^\times\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_{\mathfrak{f}\in{}^\times\mathcal{F}_\omega^{\textcircled{\vee}}} (\check{x}, \check{y})^{wt(\mathfrak{f})}$$

Lemma 5.1. *There is an x-weight and y-weight preserving bijection from $^\times\mathcal{F}_\omega^{\textcircled{\vee}}$ to pairs (P, Q) where $P \in HT_\omega$ and $Q \in PSMT$ have the same shape.*

Proof. A Hecke frouping is just a Hecke word, w, along with a partition of $\{1, 2, \ldots, \ell(w)\}$ into $2m + 2$ parts such that

- Each part contains consecutive numbers.
- If a and $a + 1$ occur in the same one of one of parts number $1, 3, 5, \ldots$ then, $w_a > w_{a+1}$.
- If a and $a + 1$ occur in the same one of one of parts number $2, 4, 6, \ldots$ then, $w_a \leq w_{a+1}$.

But an $PSMT$ is just a standard set valued tableau, T, along with a partition of $\{1, 2, \ldots, max(T)\}$ into groups such that

- Each part contains consecutive numbers.
- If a and $a + 1$ occur in the same one of one of parts number $1, 3, 5, \ldots$ then $a + 1$ lies strictly below a in T.
- If a and $a + 1$ occur in the same one of one of parts number $2, 4, 6, \ldots$ then $a + 1$ lies either in the same box as a or strictly right of a in T.

Corollary 2.13 implies that if $w \to (P, Q)$ under the bijection of Corollary 2.10 then a certain partition of $\{1, 2, \ldots, \ell(w)\}$ turns w into a Hecke frouping if and only if the same partition turns Q into a $PSMT$. Thus combining the bijection of corollary 2.10 with the identity on the partitions induces the weight preserving bijection of the lemma. $\qquad\square$

Corollary 5.2.

$$^\times\mathcal{G}_\omega(\check{x}, \check{y}) = \sum_\lambda \sum_{T\in HT_\omega(\lambda)} {}^\times G_\lambda(\check{x}, \check{y})$$

Proof. This follows from the lemma. $\qquad\square$

Lemma 5.3. *Let $\omega = \Gamma^{-1}(\lambda)$. Then*

$$^\times G_\lambda(\check{x}, \check{y}) = {}^\times\mathcal{G}_\omega(\check{x}, \check{y})$$

Proof. This follows from corollary 5.2 and lemma 4.14. $\qquad\square$

Lemma 5.4. $^\times G_\lambda(\check{x}, \check{y})$ *and* $^\times\mathcal{G}_\omega(\check{x}, \check{y})$ *are doubly Schur positive.*

Proof. By corollary 5.2 it suffices to show the result for $^\times G_\lambda(\check{x}, \check{y})$, which in turn can be shown via corollary 4.28 after we prove that $R_\lambda(\check{x}, \check{y})$ is doubly Schur positive. Recall that:

$$R_\lambda(\check{x}, \check{y}) = \sum_{T\in SSPT(\lambda)} (\check{x}, \check{y})^{wt(T)}$$

Assuming that $SSPT(\lambda)$ is defined using the total order the total order $1' \lhd 2' \lhd \cdots \lhd 1 \lhd 2 \lhd \cdots$ it is clear that:

$$R_\lambda(\check{x}, \check{y}) = \sum_{\mu \subseteq \lambda} s_{\mu^\perp}(\check{y}) s_{\lambda/\mu}(\check{x})$$

where μ represents the shape of formed by the primed elements of a tableau in $SSPT_\lambda$. Since skew Schur polynomials are Schur positive this proves the result. $\square$

Define the quasi-weak stable Grothendieck function as:

$$^\times\mathcal{G}_\omega(\mathbf{x}, \mathbf{y}) = \lim_{m \to \infty} {}^\times\mathcal{G}_\omega(\check{x}, \check{y})$$

Proposition 5.5. *We have*

$$\Omega_x\Omega_y(^\times\mathcal{G}_\omega(\mathbf{x}, \mathbf{y})) = {}^\times\mathcal{G}_\omega(\mathbf{y}, \mathbf{x})$$

Proof. Again if μ is any permutation let X_μ be the set of all pairs of permutations (u, v) such that $\tilde{u}\tilde{v} \sim \tilde{\mu}$. The bijection $\uparrow$ of lemma 3.6 implies that the relative order in which the weakly increasing parts of factors and decreasing parts of factors that appear in a frouping in $^\times\mathcal{F}_\omega^{\circledV}$ is irrelevant. In particular we can go from alternating between decreasing and and weakly increasing parts as in the definition to putting all decreasing parts first followed by all weakly increasing parts. This is what is done to get from (5.2) to (5.3) below. We can also go from putting all weakly increasing parts first followed by all decreasing parts to alternating between decreasing and and weakly increasing parts. This is what is done to get from (5.6) to (5.7) below. Corollary 4.29 has been used between (5.4) and (5.5).

$$(5.1) \qquad \Omega_x\Omega_y(^\times\mathcal{G}_\omega(\mathbf{x}, \mathbf{y})) =$$

$$(5.2) \qquad \Omega_x\Omega_y\left(\lim_{m\to\infty}\left[\sum_{\mathfrak{f}\in{}^\times\mathcal{F}_\omega^{\circledV}}(\check{x}, \check{y})^{wt(\mathfrak{f})}\right]\right) =$$

$$(5.3) \qquad \Omega_x\Omega_y\left(\sum_{(u,v)\in X_\omega}\lim_{m\to\infty}\left[\sum_{\mathfrak{f}\in\mathcal{F}_u}(\check{x})^{wt(\mathfrak{f})}\sum_{\mathfrak{f}'\in{}^*\mathcal{F}_v}(\check{y})^{wt(\mathfrak{f}')}\right]\right) =$$

$$(5.4) \qquad \sum_{(u,v)\in X_\omega}\Omega_x\Omega_y\left(\mathcal{G}_u(\mathbf{x})\,{}^*\mathcal{G}_v(\mathbf{y})\right) =$$

$$(5.5) \qquad \sum_{(u,v)\in X_\omega}{}^*\mathcal{G}_u(\mathbf{x})\mathcal{G}_v(\mathbf{y}) =$$

$$(5.6) \qquad \sum_{(u,v)\in X_\omega}\lim_{m\to\infty}\left[\sum_{\mathfrak{f}\in{}^*\mathcal{F}_u}(\check{x})^{wt(\mathfrak{f})}\sum_{\mathfrak{f}'\in\mathcal{F}_v}(\check{y})^{wt(\mathfrak{f}')}\right] =$$

$$(5.7) \qquad \lim_{m\to\infty}\left[\sum_{\mathfrak{f}\in{}^\times\mathcal{F}_\omega^{\circledV}}(\check{y}, \check{x})^{wt(\mathfrak{f})}\right] =$$

$$(5.8) \qquad {}^\times\mathcal{G}_\omega(\mathbf{y}, \mathbf{x})$$

$\square$

Corollary 5.6. *We have*

$$\Omega_x \Omega_y ({}^\times G_\lambda(\mathbf{x}, \mathbf{y})) = {}^\times G_\lambda(\mathbf{y}, \mathbf{x})$$

Proof. This follows from taking the limit as $m \to \infty$ of lemma 5.3 and applying proposition 5.5. $\qquad\square$

Corollary 5.7. *We have*

$$\Omega_x ({}^\times \mathcal{G}_\lambda(\mathbf{x}, \mathbf{x})) = {}^\times \mathcal{G}_\lambda(\mathbf{x}, \mathbf{x})$$
$$\Omega_x ({}^\times G_\lambda(\mathbf{x}, \mathbf{x})) = {}^\times G_\lambda(\mathbf{x}, \mathbf{x})$$

Proof. Use the lemma above and proposition 5.5. $\qquad\square$

Remark 5.8. ${}^\times \mathcal{G}_\omega(\mathbf{x}, \mathbf{y})$ is not the same as either of the "half-weak" double Grothendieck functions, $\Omega_x(\mathcal{G}_\omega(\mathbf{x}, \mathbf{y}))$ or $\Omega_y(\mathcal{G}_\omega(\mathbf{x}, \mathbf{y}))$. The latter functions are not particularly intereseting as they are in fact just equal to single Grothendieck functions after a certain substitution of variables. In particular:

$$\Omega_x(\mathcal{G}_\omega(\mathbf{x}, \mathbf{y})) = \mathcal{G}_{\omega^{-1}} \left(\frac{x_1}{1 - x_1}, y_1, \frac{x_2}{1 - x_2}, y_2, \cdots \right)$$

$$\Omega_y(\mathcal{G}_\omega(\mathbf{x}, \mathbf{y})) = \mathcal{G}_\omega \left(\frac{y_1}{1 - y_1}, x_1, \frac{y_2}{1 - y_2}, y_2, \cdots \right)$$

Remark 5.9. Since the functions ${}^\times \mathcal{G}_\omega(\mathbf{x}, \mathbf{x})$ and ${}^\times G_\lambda(\mathbf{x}, \mathbf{x})$ are Schur positive and self-dual it is reasonable to ask if they are P-Schur positive. The answer is in fact yes: This is because the function $R_\lambda(\mathbf{x}, \mathbf{x})$ is P-Schur positive. The latter is because an element of $SSPT(\lambda)$ is equivalent to a skew shifted semistandard tableau of shape $(\lambda + \mu)/\mu$ where $\mu = (r, r - 1, \ldots, 2, 1)$ where r is the number of rows of λ. Moreover, skew P-Schur functions are P-Schur positive so this would complete the argument. However, showing P-Schur functions are P-Schur positive is rather irrelevant to this supposedly self-contained paper, so this result is left as a remark.

Example 5.10. We compute the degree 4 part of ${}^\times \mathcal{G}_{321}(x_1, x_2, x_1, x_2)$. Every element of ${}^\times \mathcal{F}_{321}^{\circledcirc}$ contributing a monomial of degree 4 to ${}^\times \mathcal{G}_{321}(x_1, x_2, x_1, x_2)$ can be attained from the one of the factorizations below by circling certain numbers:

$(2122)(), (2112)(),$
$(212)(2), (212)(1), (211)(2), (122)(1), (122)(1),$
$(21)(21), (21)(22), (12)(12), (12)(21), (22)(12), (11)(21), (12)(11), (21)(22),$
$(2)(212), (1)(212), (1)(211), (2)(122), (2)(122),$
$()(2122), ()(2112).$

In particular, for each factorization above with one nonempty part there are 2 possible ways to circle numbers and for each factorization above with two nonempty parts there are 4 possible ways to circle numbers. Moreover, the chosen manner of circling does not affect the monomial contributed by the factorization. Thus we only need consider the 22 factorizations above rather than all 80 elements of ${}^\times \mathcal{F}_{321}^{\circledcirc}$ with 4 numbers. Using this reasoning we can compute that:

$${}^\times \mathcal{G}_{321}(x_1, x_2, x_1, x_2) = 2 * 2x_1^4 + 4 * 5x_1^3 x_2 + 4 * 8x_1^2 x_2^2 + 4 * 5x_1 x_2^3 + 2 * 2x_2^4.$$

Since the P-Schur polynomial $P_{(4,0)}(x_1, x_2) = x_1^4 + 2x_1^3 x^2 + 2x_1^2 x_2^2 + 2x_1 x_2^3 + x_2^4$ and the P-Schur polynomial $P_{(3,1)}(x_1, x_2) = x_1^3 x^2 + 2x_1^2 x_2^2 + x_1 x_2^3$ it follows that we have:

$$^\times \mathcal{G}_{321}(x_1, x_2, x_1, x_2) = 4 * P_{(4,0)}(x_1, x_2) + 12 * P_{(3,1)}(x_1, x_2).$$

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