Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2018 (13)

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MPIM 18-13

CLASSIFICATION OF ONE-CLASS SPINOR GENERA FOR QUATERNARY QUADRATIC FORMS

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Abstract. A quadratic form has a one-class spinor genus if its spinor genus consists of a single equivalence class. In this paper, we determine that there is only one primitive quaternary genus which has a one-class spinor genus but not a one-class genus. In all other cases, the genera of primitive quaternary lattices either have a genus and spinor genus which coincide, or the the genus splits into multiple spinor genera, which in turn split into multiple equivalence classes.

1. Introduction

An integral quadratic form is said to have a one-class (spinor) genus if its (spinor) genus consists of a single class. Recent work of Kirschmer and Lorch [13] completing the determination of all one-class genera of positive definite primitive integral quadratic forms in at least three variables, naturally brings us to revisit the corresponding problem for one-class spinor genera of such forms; that is, the classification of forms whose spinor genera consist of a single class. Throughout this manuscript, all quadratic forms and lattices will be assumed to be positive definite. It was proved by Earnest and Hsia that when such a form has rank at least 5, the notions of one-class genus and one-class spinor genus coincide [7]. However, when the rank is equal to 4, there exist one-class spinor genera which lie in genera containing multiple classes. When the rank is equal to 3, twenty-seven such forms appear in Jagy's list of spinor regular ternary forms of discriminant not exceeding 575,000 that are not regular [12]. In light of the work of the present authors in [3], it is now known that this list is complete; i.e., that it contains representatives from all one-class spinor genera of primitive ternary quadratic forms that do not lie in one-class genera.

To complete the determination of all one-class spinor genera for forms in at least three variables, it thus remains to fully investigate the one-class spinor genera of quaternary quadratic forms. There is one example of a quaternary form which lies in a one-class spinor genus, but not a one-class genus, that has appeared several times in the literature. In his book [21, p. 114], Watson notes that the spinor genus of the quaternary form

$$x^2 + xy + 7y^2 + 3z^2 + 3zw + 3w^2 \tag{1}$$

of discriminant $3^6 = 729$ contains only one class, but its genus contains more than one spinor genus. It can be checked that the genus of this form consists of two spinor genera and a total of three classes.

In the special case of quaternion orders, there are interesting connections between one-class spinor genera and various algebraic properties present in this case. For such lattices over general Dedekind domains in global fields, Nipp [15] gives a characterization of one-class spinor genera of quaternion orders in terms of the ideal theory of the order. In the same paper, Nipp also shows that for ternary lattices the one-class spinor genus property is equivalent to an ideal-theoretic property of an associated quaternion order. In the case of rational quaternion orders, Estes and Nipp [9] give a characterization of the one-class spinor genus property in terms of factorization in the quaternion order, extending investigations of Pall and Williams [18], [23] who characterized the one-class genera of quaternion orders in terms of quaternion factorization and determined the thirty-nine orders having this property. The form (1) appears in both of the papers [15] and [9]. In fact, Parks [19] proved that the lattice corresponding to this form is a representative of the only isomorphism class of definite quaternion orders in rational quaternion algebras that lies in a spinor genus of one class, but a genus consisting of multiple classes.

In order to have a splitting of a genus into multiple spinor genera it is necessary that the discriminant be divisible by a relatively high power of at least one prime. In his book [16] which contains tables of all

2010 *Mathematics Subject Classification.* 11E20, 11E12.

quaternary quadratic forms of discriminant at most 1732, Nipp notes on p. 14 that the only discriminant in this range that could admit multiple spinor genera is 729, and he goes on to show that for all forms of discriminant 729 other than (1) the genus and spinor genus coincide.

Any one-class spinor genera of diagonal quaternaries must appear on the list of ninety-six strictly regular diagonal quaternary forms determined in [4]. To see this, note that an integer primitively represented by any quaternary genus is already represented primitively by every spinor genus in the genus. This is proven, for example, in [21, Theorem 72, p. 114]. A quick check of the sixty-two strictly regular diagonal quaternaries with class number exceeding 2 identifies at most sixteen that have discriminants divisible by a power of at least one prime sufficiently large to admit a splitting into multiple spinor genera. It can be verified using Magma that for all of these forms the genus and spinor genus coincide. Thus there are no new examples of nontrivial one-class quaternary spinor genera to be found among the strictly regular diagonal quaternaries.

In the present paper, we show that in fact the form (1) is a representative of what is essentially the only equivalence class of quaternary quadratic forms that coincides with its spinor genus but not its genus. More precisely we prove:

Theorem 1. *Let f be a primitive integral positive definite quaternary quadratic form for which the spinor genus and class coincide. Then either the genus and class of f coincide or f is equivalent to the form (1).*

To prove this, the general strategy is as follows. We use a transformation defined by Gerstein in [10], similar to Watson's "p-mapping" defined in [22], which simultaneously reduces the class number of a form and the power of p appearing in its discriminant. In this way we are able to reduce one-class spinor genus forms to one-class genus forms and restrict the primes appearing in the discriminant by cross-referencing with the quaternary lattices appearing in the classification by Kirschmer and Lorch in [13]. Once we have a small list of possible prime divisors for discriminants of one-class spinor genus forms, we can systematically eliminate candidate discriminants by using the Watson-type transformation to show that the associated genus does not split into multiple spinor genera, and hence the form is not of interest to us, or by using a version of the Minkowski-Siegel mass formula to show that the spinor genus must contain more than one class. For any cases that don't succumb to these methods, we use the algorithm from [8, Lemma 3] to generate all equivalence classes for forms of a targeted discriminant, and then explicitly compute the structure of the genus and spinor genus in each case.

This strategy makes use of three critical, and interconnected computational components. The first is the online L-Functions and Modular Forms Database [14], of which the lattice database contains the full list of one-class genus forms determined by Kirschmer and Lorch [13]. Downloading the list, the entries can be viewed as objects in the class of quadratic forms in Sagemath [20], enabling quick computations of discriminant and local structure of the forms. From here, it can be easily determined what sort of local splittings and discriminant divisors are admissible by one-class genus forms. In certain cases, we will need to generate a list of possible equivalence classes for a fixed discriminant, this can be done using [8, Lemma 3] and some of the built in functionality for local and global isometry in Magma. Once a list of potential candidates has been computed, local structures can be computed and compared against those already computed in Sagemath. For all remaining candidates, Magma can explicitly compute the structure of the genus and spinor genera.

The paper will be organized as follows. In section 2 we will introduce the relevant notation and terminology for lattices. In section 3 we will introduce the μ_p-transformation and give some explicit local computations. In section 4 we will introduce the Conway-Sloane mass formula and compute bounds on the mass formula for forms with one-class spinor genera. In section 5 we will systematically reduce the list of possible prime divisors of one-class spinor genera, and then using μ_p-transformations, mass formula bounds, and the algorithm from [8, Lemma 3] we will eliminate all possible candidates for one-class spinor genera.

2. Preliminaries and Notation

For the remainder of this paper we will abandon the language of forms, and instead adopt the geometric language of lattices as favored in the preceding literature on this topic, especially [13]; any unexplained notation can be found in [17]. Throughout the paper the term "lattice" will mean a primitive quaternary $\mathbb{Z}$-lattice, L, on a positive definite rational quadratic space (V, Q) with associated bilinear mapping B. The lattice is primitive when the scale ideal $\mathfrak{s}(L)$ equals $\mathbb{Z}$. Let M_L be a Gram matrix for L. Then the form associated to L is the form with coefficient matrix M_L if the norm ideal $\mathfrak{n}(L) = \mathbb{Z}$ (thus giving the classically integral forms), or $\frac{1}{2}M_L$ if $\mathfrak{n}(L) = 2\mathbb{Z}$ (giving the non-classically integral, or integer-valued forms).

The discriminant of L is the determinant of the Gram matrix. We let $\mathcal{D}_s$ denote the set of all possible discriminants of lattices which have one-class spinor genera but not one-class genera, and $\mathcal{P}_s$ the set of all primes dividing the discriminants in $\mathcal{D}_s$. For a prime p, $\mathbb{Z}_p$ will denote the ring of p-adic integers, $\mathbb{Z}_p^\times$ the group of units in $\mathbb{Z}_p$, and for $p \neq 2$, $\triangle$ a non-square unit in $\mathbb{Z}_p^\times$. We denote by L_p the local $\mathbb{Z}_p$-lattice $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. In the dyadic case, $\mathbb{A}$ will be used to denote the anisotropic plane, $A(2,2)$, and $\mathbb{H}$ will denote the hyperbolic plane, $A(0,0)$, as defined in [17, §93B].

Let $h(L)$ and $h_s(L)$ be the class number and spinor class number of L, respectively, and let $g(L)$ be the number of spinor genera in the genus of L. Then L has a one-class genus if $h(L) = 1$ and a one-class spinor genus if $h_s(L) = 1$. A lattice which has $h_s(L) = 1$ and $h(L) > 1$ necessarily has $g(L) > 1$ and therefore has the desired characteristic of a one-class spinor genus but not a one-class genus.

At some points we will need to make use of an algorithm of Pall [18] which is restated in [8, Lemma 3]. For a given discriminant d, a list of equivalence classes of quaternary forms of discriminant d and a prime p, this algorithm returns representatives of all equivalence classes of quaternary forms of discriminant dp^2. In applying this algorithm for the prime $p = 2$, some care must be taken to interpret the results in light of our conventions regarding the correspondence between lattices and their associated forms.

3. The μ_p-transformation

For a lattice L and an integer m, define

$$L^{(m)} = \{x \in L : B(x, L) \subseteq m\mathbb{Z}\}$$

as in [17, §82I], and note that $L^m = mL^\# \cap L$. It is immediate from the definition that

$$(L_p)^{(m)} = \begin{cases} L_p & \text{for } p \nmid m, \text{ and} \\ (L^{(m)})_p & \text{for } p \mid m. \end{cases}$$

for any prime p. Recall that for L with basis $\{e_1, ..., e_n\}$, we have

$$pL = \mathbb{Z}(pe_1) + ... + \mathbb{Z}(pe_n),$$

that is, $\mathfrak{n}(pL) = p^2\mathfrak{n}(L)$ and $\mathfrak{s}(pL) = p^2\mathfrak{s}(L)$. Following the definition given by Gerstein in [10], we define the μ_p-transformation on L by

$$\mu_p L = L + p^{-1}\left(L^{(p^2)}\right).$$

The following Lemma is an adaptation of [10, 3.3] more suitable to our ends.

Lemma 1. *Let p, q be primes. Suppose*

$$L_p \cong L_{(0)} \perp L_{(1)} \perp L_{(2)} \perp ... \perp L_{(t_p)}$$

where each $L_{(i)}$ is p^i-modular or 0 with $L_{(t_p)} \neq 0$. Then,

$$(\mu_p L)_q = \begin{cases} L_q & \textit{for } q \neq p \\ L_{(0)} \perp L_{(1)} \perp p^{-1}\left(L_{(2)} \perp ... \perp L_{(t_p)}\right) & \textit{for } p = q. \end{cases}$$

Proof. If $p \neq q$, then we have

$$(\mu_p L)_q = \left(L + p^{-1}\left(L^{(p^2)}\right)\right)_q = L_q + \left(p^{-1}\left(L^{(p^2)}\right)\right)_q = L_q + \left(L^{(p^2)}\right)_q = L_q.$$

Suppose that $p = q$. We have

$$\mathfrak{s}(L_{(i)}) = p^i\mathbb{Z}_p$$

from the definition of p^i-modularity, and therefore $\left(L_{(i)}\right)^{(p^2)} = pL_{(i)}$ when $i = 0, 1$. From here, we have

$$\begin{aligned} (\mu_p L)_p &= L_p + p^{-1}\left(L_p^{(p^2)}\right) \\ &= L_p + p^{-1}\left(L_{(0)}^{(p^2)} \perp L_{(1)}^{(p^2)} \perp L_{(2)}^{(p^2)} \perp ... \perp L_{(t_p)}^{(p^2)}\right) \\ &= L_p + \left(L_{(0)} \perp L_{(1)} \perp p^{-1}\left(L_{(2)} \perp ... \perp L_{(t_p)}\right)\right) \\ &= L_{(0)} \perp L_{(1)} \perp p^{-1}\left(L_{(2)} \perp ... \perp L_{(t_p)}\right), \end{aligned}$$

which is what we wanted to show. □

From the proof above (and as noted in [10]) it is clear that $\mu_p(L) \neq L$ if and only if L_p has a p^i-modular component for some $i \geq 2$, and hence the sequence

$$\{L, \mu_p(L), {\mu_p}^2(L), ...\}$$

is eventually constant. The constant tail at the end of the sequence will be a lattice which contains only a unimodular and p-modular part. In [10], the global lattice μL is defined as the global lattice whose localization at every p is just the constant term of the corresponding sequence. We define a related lattice $\hat{\mu}L$ to be the global lattice whose localization at every prime p is the last non-unimodular lattice occurring in the above sequence. In this way, we ensure that for any prime p, $p \mid dL$ if and only if $p \mid d\hat{\mu}L$. If the lattice L_p is diagonalizable, this mapping can be described explicitly as follows. For

$$L_p \cong \langle a, p^\beta b, p^\gamma c, p^\delta d\rangle$$

where $a, b, c, d \in \mathbb{Z}_p^\times$ and $1 \leq \beta \leq \gamma \leq \delta$, from Lemma 1 we have

$$(\mu_p(L))_p \cong \begin{cases} \langle a, pb, pc, pd\rangle & \text{if } \beta = \gamma = \delta = 1 \\ \langle a, pb, pc, p^{\delta-2}d\rangle & \text{if } \beta = \gamma = 1, \delta > 1 \\ \langle a, pb, p^{\gamma-2}c, p^{\delta-2}d\rangle & \text{if } \beta = 1, \gamma, \delta > 1 \\ \langle a, p^{\beta-2}b, p^{\gamma-2}c, p^{\delta-2}d\rangle & \text{if } \beta, \gamma, \delta > 1 \end{cases}$$

For a quaternary lattice L and an odd prime p, we define the *p-signature* as the tuple $(0, k, l, m)_p$ which records the power of p appearing in each of the diagonal entries of L_p. There is no harm in setting the first entry in the tuple equal to 0, since we are only dealing with primitive lattices. We've defined $\hat{\mu}L$ so that $(k, l, m, n)_p$ is minimal (but not the all zeros tuple) at each prime $p \mid dL$. Therefore, since $\hat{\mu}L$ is primitive, the possible resulting p-signatures are

$$\begin{matrix} (0,0,0,1)_p & (0,0,1,1)_p & (0,1,1,1)_p \\ (0,0,0,2)_p & (0,0,2,2)_p & (0,2,2,2)_p \end{matrix} \tag{2}$$

for any $p \mid dL$. Hence we have

$$(\hat{\mu}L)_p = \begin{cases} \mu_p^{\frac{\delta-1}{2}}(L_p) \cong \langle a, p^{\beta'}b, p^{\gamma'}c, pd\rangle & \text{if } \delta \text{ is odd} \\ \mu_p^{\frac{\delta}{2}}(L_p) \cong \langle a, p^{\beta'}b, p^{\gamma'}c, d\rangle & \text{if } \delta \text{ is even and } \beta \text{ or } \gamma \equiv 1 \mod 2 \\ \mu_p^{\frac{\delta-2}{2}}(L_p) \cong \langle a, p^{\beta'}b, p^{\gamma'}c, p^2d\rangle & \text{if } \delta \text{ is even and } \beta \equiv \gamma \equiv 0 \mod 2 \end{cases}$$

where $\beta', \gamma' \in \{0, 1\}$ and $\beta'', \gamma'' \in \{0, 2\}$. A consequence of this is that for an odd prime p, $p \mid dL$ if and only if $p \mid d\hat{\mu}L$.

Lemma 2. *For a lattice L, $gen(\hat{\mu}L) = spn(\hat{\mu}L)$.*

Proof. For any odd prime p, the p-signature, $(0, k, l, m)_p$, is one of (2), hence $(\hat{\mu}L)_p$ contains a binary modular component, and consequently $\mathbb{Z}_p^\times \subseteq \theta(O^+((\hat{\mu}L)_p))$. When $p = 2$, if L is split by the scaling of $\mathbb{A}$ or $\mathbb{H}$, then $(\hat{\mu}L)_2$ will be split by some scaling of $\mathbb{A}$ or $\mathbb{H}$, and hence $\mathbb{Z}_2^\times \subseteq \theta(O^+((\hat{\mu}L)_2))$ by [11, Lemma 1]. Therefore L_2 is diagonalizable, with 2-signature $(0, l, m.n)_2$ and hence $\hat{\mu}L_2$ has 2-signature among (2). The 2-signatures which admit a ternary component will give $\mathbb{Z}_2^\times \subseteq \theta(O^+((\hat{\mu}L)_2))$ by [17, 93:20], and the 2-signatures which admit a binary modular components will give the same result by [5, Theorem 3.14]. □

Lemma 3. *Let L and M be global $\mathbb{Z}$-lattices on the same underlying rational quadratic space V. Then*

(1) if $gen(M) = gen(L)$ then $gen(\hat{\mu}M) = gen(\hat{\mu}L)$.
(2) if $M \cong L$ then $\hat{\mu}M \cong \hat{\mu}L$.
(3) if $spn(M) = spn(L)$ then $spn(\hat{\mu}M) = spn(\hat{\mu}L)$.

Proof. Proofs of (1) and (2) follow immediately as in the proof of [10, Lemma 3.5], and the proof of (3) follows similarly to that of (2), except replacing ϕ with $\phi\Sigma$ where $\phi \in O(V)$ and $\Sigma \in O'_{\mathbb{A}}(V)$. □

Lemma 4. *For a lattice L, $h(\hat{\mu}L) \leq h(L)$.*

Proof. See [10, Theorem 3.6]. □

Lemma 5. *For a lattice L, $h_s(\hat{\mu}L) \leq h_s(L)$.*

Proof. This can be shown similarly to [10, Theorem 3.6], as stated in [7]. □

Theorem 2. *If $h_s(L) = 1$ then $h(\hat{\mu}L) = 1$.*

Proof. Suppose that $\mathrm{spn}(L)$ is a one-class spinor genus, that is, $h_s(L) = 1$. It follows from Lemma 5 that $h_s(\hat{\mu}L) = 1$. Moreover, from Lemma 2 we know that $\mathrm{spn}(\hat{\mu}L) = \mathrm{gen}(\hat{\mu}L)$, and consequently $h(\hat{\mu}L) = 1$. Therefore $\hat{\mu}L$ has class number one. □

From an examination of the discriminants of the lattices appearing in [13], it can be see that

$$\mathcal{P} = \{2, 3, 5, 7, 11, 13, 17, 23\}$$

is the set of all prime divisors of the discriminants of quaternary lattices with class number one.

Corollary 6. *If a prime p divides d for some $d \in \mathcal{D}_s$, then $p \in \mathcal{P}$.*

Based on this, we know that

$$\mathcal{P}_s \subseteq \{2, 3, 5, 7, 11, 13, 17, 23\}.$$

but we will see shortly that we can quickly eliminate several of these candidates. Suppose that L is a quaternary lattices with $h_s(L) = 1$. Then from Corollary 6 we know that

$$dL = \prod_{p \in \mathcal{P}} p^{a_p}.$$

where $0 \leq a_p$. One consequence of this is that for any prime $p \notin \mathcal{P}$, L_p is unimodular, and hence $\mathbb{Z}_p^\times \subseteq \theta(O^+(L_p))$.

4. The Conway-Sloane mass formula

In this section, we will compute the mass as presented by Conway and Sloane in [2]. In general, then mass of lattice L is given by

$$m(L) = \sum_{i=1}^{h(L)} \frac{1}{\mid O(L_i) \mid}$$

and summing instead over the representatives of the spinor genus, we obtain the spinor mass

$$m_s(L) = \sum_{i=1}^{h_s(L)} \frac{1}{\mid O(L_i) \mid}.$$

As noted in [7] and [16], the mass is distributed evenly over the spinor genera of the genus of any given L, so mass satisfies

$$m_s(L) = \frac{m(L)}{g(L)}.$$

At any prime p, we consider the local splitting of L into its Jordan components,

$$L_p = L_{(-1)} \perp L_{(0)} \perp L_{(1)} \perp L_{(2)} \perp \ldots$$

where each $L_{(i)}$ is p^i-modular (and possibly 0-dimensional, making this a finite sum). When $p = 2$ a component is called *type I* if $\mathfrak{n}(L) = \mathfrak{s}(L)$ and type *type II* if $\mathfrak{n}(L) = 2\mathfrak{s}(L)$. Using the mass formula of Conway and Sloane, formula (2) from [2], specialized to the quaternary case, we have

$$m(L) = 2\pi^{-5} \cdot \prod_{j=1}^{4} \Gamma\left(\frac{j}{2}\right) \cdot \prod_p 2m_p(L) = \pi^{-4} \cdot \prod_p 2m_p(L),$$

where $m_p(L)$ is the local p-mass given by

$$m_p(L) = \prod_{-1 \leq i} M_p\left(L_{(p^i)}\right) \cdot \prod_{-1 \leq i < j} \left(p^{i-j}\right)^{\frac{1}{2}n(i)n(j)} \cdot 2^{n(I,I) - n(II)}.$$

where $n(i)$ is the dimension of $L_{(i)}$. We refer to the left-hand product in $m_p(L)$ as the *diagonal product* and the other product as the *cross product.* The value $n(I, I)$ counts the number of adjacent pairs, $L_{(i)}$ and

$L_{(i+1)}$, that are both of type I, and $n(II)$ is the sum of all dimensions of type II components in the Jordan decomposition (the $n(I)$ and $n(II)$ values are only relevant in the case when $p = 2$).

Lemma 7. *If L is a primitive quaternary lattice with $dL = q^n$ for some $n > 0$ and q an odd prime, and L_q has 1-dimensional modular components, $L_{(i)}$, for $i \in \{0, k, l, m\}$ with $0 < k < l < m$ and 0-dimensional components for all other i, then*

$$m(L) \geq \frac{q^{\frac{3m+l-k}{2}}}{2^9 \cdot 3 \cdot 5} \cdot (1 - q^{-2})^2$$

when n is even, and

$$m(L) > \frac{q^{\frac{3m+l-k}{2}}}{2^8 \cdot 3^2 \cdot 5} \cdot (1 - q^{-4})$$

when n is odd.

Proof. Suppose that L is a primitive quaternary lattice with $dL = q^n$ for some $n > 0$ and q prime. We will bound the mass of L by first computing the local p-mass at each prime p. When $p = 2$, then we have the 2-adic splitting

$$L_2 = L_{(-1)} \perp L_{(0)} \perp L_{(1)}$$

where both $L_{(-1)}$ and $L_{(1)}$ are 0-dimensional, and

$$L_{(0)} \cong \begin{cases} \langle \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \rangle, \\ \mathbb{A} \perp \mathbb{A} \cong \mathbb{H} \perp \mathbb{H}, \text{ or,} \\ \mathbb{A} \perp \mathbb{H} \end{cases} \tag{3}$$

where $\epsilon_i \in \mathbb{Z}_2^\times$. In the first case of (3), $L_{(0)}$ is free of type I. Consequently both $L_{(-1)}$ and $L_{(1)}$ are 0-dimensional bound forms, and therefore each contribute 1/2 to the diagonal product. If $\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3 \cdot \epsilon_4 \equiv \pm 1 \mod 8$, then $L_{(0)}$ has species 2+, otherwise $L_{(0)}$ has species 2−. For the cross product, we have

$$\prod_{-1 \leq i < j} \left(2^{i-j}\right)^{\frac{1}{2}n(i)n(j)} \cdot 2^{n(I,I)-n(II)} = 1$$

since there is only one component of non-zero dimension. Therefore, we obtain

$$m_2(L) = \begin{cases} \frac{1}{4} & \text{if } L_{(0)} \text{ has species } 2+ \\ \frac{1}{12} & \text{if } L_{(0)} \text{ has species } 2-. \end{cases} \tag{4}$$

In the second two cases of (3), $L_{(0)}$ is free of type II, and therefore $L_{(-1)}$ and $L_{(1)}$ are 0-dimensional free forms which only contribute 1 to the diagonal product. When $dL_{(0)} \equiv 1 \mod 8$ then $L_{(0)}$ has species 4+, and when $dL_{(0)} \equiv -3 \mod 8$ then $L_{(0)}$ has species 4−. For the cross product, we get

$$\prod_{-1 \leq i < j} \left(2^{i-j}\right)^{\frac{1}{2}n(i)n(j)} \cdot 2^{n(I,I)-n(II)} = 2^{-4}$$

since there is a single component which has non-zero dimension, and $n(II) = 4$. Therefore, we have

$$m_2(L) = \begin{cases} \frac{1}{18} & \text{if } L_{(0)} \text{ has species } 4+ \\ \frac{1}{30} & \text{if } L_{(0)} \text{ has species } 4-. \end{cases} \tag{5}$$

We have exhausted all possibilities for local structure at 2, therefore combining equations (4) and (5), we can begin to bound $m(L)$ by

$$m(L) \geq \pi^{-4} \cdot \frac{1}{3 \cdot 5} \cdot \prod_{p \neq 2} 2m_p(L). \tag{6}$$

When $p = q$, then we suppose $L_{(i)}$ is 1-dimensional for $i \in \{k, l, m\}$ where $0 < k < l < m$, and all other components are 0-dimensional. The 1-dimensional terms each have species 1 and therefore each contribute 1/2 to the diagonal product, and the 0-dimensional components all contribute 1. For the cross product, we have

$$\prod_{-1 \leq i < j} \left(q^{i-j}\right)^{\frac{1}{2}n(i)n(j)} = \left[\frac{q^k}{q^0} \cdot \frac{q^l}{q^0} \cdot \frac{q^l}{q^k} \cdot \frac{q^m}{q^0} \cdot \frac{q^m}{q^k} \cdot \frac{q^m}{q^l}\right]^{\frac{1}{2}} = q^{\frac{3m+l-k}{2}}.$$

Therefore, combining the diagonal product and the cross product, we obtain

$$m_q(L) = \frac{q^{\frac{3m+l-k}{2}}}{2^4} \tag{7}$$

and with this we can further improve upon (6), obtaining

$$m(L) \geq \pi^{-4} \cdot \frac{q^{\frac{3m+l-k}{2}}}{2^3 \cdot 3 \cdot 5} \cdot \prod_{p \neq 2,q} 2m_p(L). \tag{8}$$

If $p \neq 2, q$, then L_p is unimodular and therefore $L_{(0)}$ is 4-dimensional, while all other components of L_p are 0-dimensional. Because of this, the cross product equals 1 for $m_p(L)$ whenever $p \neq 2, q$. When n is even, then dL is always a quadratic residue modulo p, and therefore $L_{(0)}$ has genus 4+, and

$$m_p(L) = \frac{1}{2(1-p^{-2})^2}.$$

In this case we can refine the bound in (8) to obtain

$$m(L) \quad \geq \quad \pi^{-4} \cdot \frac{q^{\frac{3m+l-k}{2}}}{2^3 \cdot 3 \cdot 5} \cdot \prod_{p \neq 2,q} \frac{1}{(1-p^{-2})^2}$$

and hence

$$m(L) \geq \pi^{-4} \cdot \frac{q^{\frac{3m+l-k}{2}}}{2^3 \cdot 3 \cdot 5} \cdot (1-2^{-2})^2 \cdot (1-q^{-2})^2 \cdot \zeta(2)^2 = \frac{q^{\frac{3m+l-k}{2}}}{2^9 \cdot 3 \cdot 5} \cdot (1-q^{-2})^2$$

which is the inequality we wanted to reach.

On the other hand, when n is odd, then the local p-mass depends on the square class of dL. If q is a quadratic residue modulo p then $L_{(0)}$ has species 4+, otherwise it has species 4−, and thus

$$m_p(L) = \begin{cases} \frac{1}{2(1-p^{-2})^2} & \text{when } L_{(0)} \text{ has species } 4+ \\ \frac{1}{2(1-p^{-4})} & \text{when } L_{(0)} \text{ has species } 4-. \end{cases}$$

With the additional observation that

$$\frac{1}{(1-p^{-2})^2} > \frac{1}{(1-p^{-4})}$$

we can further improve our bound (8) on $m(L)$ by

$$m(L) \quad > \quad \pi^{-4} \cdot \frac{q^{\frac{3m+l-k}{2}}}{2^3 \cdot 3 \cdot 5} \cdot \prod_{p \neq 2,q} \frac{1}{(1-p^{-4})}$$

and thus

$$m(L) > \pi^{-4} \cdot \frac{q^{\frac{3m+l-k}{2}}}{2^3 \cdot 3 \cdot 5} \cdot (1-2^{-4}) \cdot (1-q^{-4}) \cdot \zeta(4) = \frac{q^{\frac{3m+l-k}{2}}}{2^8 \cdot 3^2 \cdot 5} \cdot (1-q^{-4})$$

which is the desired inequality. □

Using this bound on $m(L)$ we can begin to bound the powers of certain primes appearing in the discriminant of a lattice L having a one-class spinor genus.

Lemma 8. *For a lattice, L, with $dL = q^n$ for some $n > 0$ and q an odd prime. If $h_s(L) = 1$ and $h(L) > 1$ then L_q has 1-dimensional modular components, $L_{(i)}$, for $i \in \{0, k, l, m\}$ with $0 < k < l < m$ and 0-dimensional components for all other i, and*

$$3m + l - k \leq \begin{cases} 16 & \textit{for } q = 3 \textit{ and } n \textit{ even}, \\ 17 & \textit{for } q = 3 \textit{ and } n \textit{ odd}, \\ 11 & \textit{for } q = 5, \\ 9 & \textit{for } q = 7. \end{cases}$$

Proof. Suppose that L is a primitive quaternary lattice with $dL = q^n$ for some $n > 0$ and q an odd prime. Suppose also that $h_s(L) = 1$ while $h(L) > 1$. Then, since L_p is unimodular at every prime $p \neq q$, it follows that L_p does not contain any binary modular components, or else $\mathbb{Z}_p^\times \subseteq \theta(O^+(L_p))$ at every prime p, which would mean $\mathrm{gen}(L) = \mathrm{spn}(L)$ and hence $h(L) = 1$. Since $h(L) > 1$, we may conclude that $\mathrm{gen}(L)$ splits into multiple spinor genera, and since p is odd, we may say more precisely that $g(L) = 2$. Since $\mid O(L) \mid \geq 2$, if we can show that $m(L) > 1$, then we will have shown that $m_s(L) > 1/2$, and consequently the sum

$$m_s(L) = \sum_{i=1}^{h_s(L)} \frac{1}{\mid O(L_i)}$$

must be taken over more than one class. In other words, if $m(L) > 1$ then $h_s(L) > 1$.

From Lemma 7, in order to show that $m(L) > 1$, it suffices to show that

$$\frac{q^{\frac{3m+l-k}{2}}}{2^9 \cdot 3 \cdot 5} \cdot (1 - q^{-2})^2 \geq 1 \tag{9}$$

for n even, and

$$\frac{q^{\frac{3m+l-k}{2}}}{2^8 \cdot 3^2 \cdot 5} \cdot (1 - q^{-4}) > 1 \tag{10}$$

for n odd. □

5. Reducing the list of possible prime divisors

Lemma 9. $\mathcal{P}_s \subseteq \{2, 3, 5, 7\}$

Proof. To prove this claim, we need to show that $11, 13, 17$ and 23 all fail to appear in $\mathcal{P}_s$. The general strategy will be as follows. For a given $p \in \{11, 13, 17, 23\}$ we will suppose that L is a primitive quaternary lattice for which $h_s(L) = 1$, $h(L) > 1$ and $p \mid dL$. We know from Theorem 2 that $h(\hat{\mu}L) = 1$ and therefore $\hat{\mu}L$ appears in the table of 481 lattices in [13]. Moreover, from the definition of $\hat{\mu}L$, we know that $p \mid d\hat{\mu}L$, so we can narrow down the possible candidates for $\hat{\mu}L$. For each candidate we will consider the associated p-signatures, $(k, l, m, n)_p$, for $\hat{\mu}L$. From here, we will define L' to be a lattice which descends to $\hat{\mu}L$ by one iteration of the μ_p-transformation, that is, $\mu_p(L') = \hat{\mu}L$. Next, we will examine the p-signatures of such L', taking note that $h_s(L') = 1$ and $L'_q \cong \hat{\mu}L_q$ for every $q \neq p$. In most cases, we will see that the p-signature of L' forces $h(L') = 1$ and hence $\mathrm{gen}(L') = \mathrm{spn}(L')$, meaning L' is among the lattices in [13], which will lead to a contradiction.

Suppose that $p = 11$, then $\hat{\mu}L$ corresponds to one of 9 lattices in [13] having a discriminant divisible by 11. These lattices all have discriminants of the form $2^r \cdot 3^s \cdot 11^t$ for non-negative integers r, s, t, and 11-signatures from among

$$(0,0,0,1)_{11} \quad (0,0,1,1)_{11} \quad (0,1,1,1)_{11}.$$

Letting L' be a lattice for which $\mu_{11}(L') = \hat{\mu}L$, then when $\hat{\mu}L$ has 11-signature $(0,0,0,1)_{11}$ then L' has 11-signature from among

$$(0,0,1,2)_{11} \quad (0,0,0,3)_{11} \quad (0,1,2,2)_{11} \quad (0,0,2,3)_{11} \quad (0,2,2,3)_{11},$$

and when $\hat{\mu}L$ has 11-signature $(0,1,1,1)_{11}$ then L' has 11-signature from among

$$(0,1,1,3)_{11} \quad (0,1,3,3)_{11} \quad (0,3,3,3)_{11}.$$

In all of these cases, L'_{11} contains a binary modular component, so we know that $\mathbb{Z}_{11}^\times \subseteq \theta(O^+(L'_{11}))$, and since $L'_p \cong \hat{\mu}L'_p$ for every prime $p \neq 11$, we still have $\mathbb{Z}_p^\times \subseteq \theta(O^+(L'_p))$ for every prime p. Consequently $\mathrm{gen}(L') = \mathrm{spn}(L')$ and $h_s(L') = 1$, implying that $h(L') = 1$. However, this is impossible since it can be checked that the list of lattices in [13] doesn't contain any lattice admitting such an 11-signature. On the other hand, when $\hat{\mu}L$ has 11-signature $(0,0,1,1)_{11}$, then there is only one lattice in [13] with such an 11-signature, and it has the local structure

$$(\hat{\mu}L)_{11} \cong \langle 1, \triangle \rangle \perp 11\langle 1, \triangle \rangle$$

where $\triangle$ denotes a non-square unit in $\mathbb{Z}_{11}^\times$. This implies that L'_{11} will either contain a binary modular component, or a sublattice of the form

$$\langle 1 \rangle \perp 2^{11^2}\langle \triangle \rangle \text{ or } \langle \triangle \rangle \perp 2^{11^2}\langle 1 \rangle.$$

and so in any case $\mathbb{Z}_{11}^\times \subseteq \theta(O^+(L'_{11}))$. Therefore, again we have $h(L') = 1$ and $h_s(L') = 1$, but L' must have 11-signature from among,

$$(0,0,3,3)_{11} \quad (0,0,1,3)_{11} \quad (0,1,2,3)_{11} \quad (0,1,1,2)_{11} \quad (0,2,3,3)_{11},$$

and no such 11-signatures appears in [13]. Thus, we may conclude that $11 \notin \mathcal{P}_s$.

When $p = 13$ the argument proceeds similarly. In this case μL must correspond to one of only 3 lattices among those in [13] which have discriminant divisible by 13. These have discriminants $13, 13^2$ and 13^3, and 13-signatures

$$(0,0,0,1)_{13} \quad (0,0,1,1)_{13} \quad (0,1,1,1)_{13},$$

respectively. Moreover, when $\hat{\mu}L$ has 13-signature $(0,0,1,1)_{13}$, we have

$$(\hat{\mu}L)_{13} \cong \langle 1, \triangle\rangle \perp 13\langle 1, \triangle\rangle.$$

From here the argument proceeds precisely as above, and we conclude that $13 \notin \mathcal{P}_s$.

When $p = 17$ the argument is further simplified, since here $d\hat{\mu}L$ is either 17 or 17^3 with respective 17-signatures,

$$(0,0,0,1)_{17} \quad (0,1,1,1)_{17},$$

from which we can deduce that $17 \notin \mathcal{P}_s$.

When $p = 23$ then $d\hat{\mu}L$ is one of $3 \cdot 23, 3^3 \cdot 23, 3 \cdot 23^3$ or $3^3 \cdot 23^3$ with respective 23-signatures

$$(0,0,0,1)_{23} \quad (0,1,1,1)_{23}.$$

From here we can use precisely the argument as used above to conclude that $23 \notin \mathcal{P}_s$. □

Lemma 10. $7 \notin \mathcal{P}_s$

Proof. Suppose that $7 \in \mathcal{P}_s$, and suppose that L is a primitive quaternary lattice which has $7 \mid dL$, $h_s(L) = 1$ and $h(L) > 1$. Then $\hat{\mu}L$ must be among the lattices appearing in [13] which can have either odd or even discriminants. We will consider these cases separately. The proof will proceed similarly to the proof of Lemma 9, where we define L' to be the lattice which descends to $\hat{\mu}L$ by one iteration of the μ_7-transformation, and thus $\mu_7(L') = \hat{\mu}L$ and $L'_p \cong \hat{\mu}L_p$ for every $p \neq 7$.

Suppose that $d\hat{\mu}L$ is odd. Then $d\hat{\mu}L = 3^k \cdot 7^m$ for non-negative integers k and m, and possible local 7-signatures

$$(0,0,0,1)_7 \quad (0,0,1,1)_7 \quad (0,0,0,1)_7,$$

where, in particular, $\hat{\mu}L$ corresponding to the signature $(0,0,1,1)_7$ has discriminant 7^2. When $\hat{\mu}L$ has 7-signature $(0,0,0,1)_7$ then L' has 7-signature from among

$$(0,0,1,2)_7 \quad (0,0,0,3)_7 \quad (0,1,2,2)_7 \quad (0,0,2,3)_7 \quad (0,2,2,3)_7,$$

and when $\hat{\mu}L$ has 7-signature $(0,1,1,1)_7$ then L' has 7-signature from among

$$(0,1,1,3)_7 \quad (0,1,3,3)_7 \quad (0,3,3,3)_7.$$

But in all of these cases L'_7 contains a binary modular component, which means that $h(L') = 1$ and $h_s(L') = 1$, leading to a contradiction, since no lattices among the list in [13] admit such 7-signatures. On the other hand, when $\hat{\mu}L$ has 7-signature $(0,0,1,1)_7$, then L' has 7-signature from among

$$(0,0,3,3)_7 \quad (0,0,1,3)_7 \quad (0,2,3,3)_7 \quad (0,1,2,3)_7 \quad (0,1,1,2)_7.$$

The first three of these can be immediately ruled out since they imply $h(L') = 1$, but no lattices among those in [13] admit such 7-signatures. When L' has 7-signature $(0,1,2,3)_7$ we know that $d(\mu_7(L')) = 7^2$ and hence $dL' = 7^6$, therefore we may conclude from Lemma 8 that $h_s(L') > 1$, since $3(3) + 2 - 1 > 9$. But $h_s(L') \leq h_s(L) = 1$, so this leads to a contradiction. When L' has 7-signature $(0,1,1,2)_7$, then $h(L') = 1$, and L' corresponds to the unique lattice in [13] with such a 7-signature, which has $dL' = 7^4$. In this case we define L'' to be the lattice which descends to L' by one iteration of the μ_7-transformation. Then dL'' must be a power of 7, and L'' must have a 7-signature from among

$$(0,1,1,4)_7 \quad (0,3,3,4)_7 \quad (0,1,3,4)_7.$$

The first two cases can be immediately ruled out since they contain a binary modular component but don't appear in [13], and the third case can be ruled out by Lemma 8 since $3(4) + 3 - 1 > 9$.

Suppose that $d\hat{\mu}L$ is even. Then $d\hat{\mu}L = 2^k \cdot 7^m$ for non-negative integers k and m, and possible 7-signatures

$$(0,0,0,1)_7 \quad (0,0,1,1)_7 \quad (0,1,1,1)_7.$$

By the same argument used in the odd case, we can immediately rule out 7-signatures $(0,0,0,1)_7$ and $(0,1,1,1)_7$. Suppose that $\hat{\mu}L$ has 7-signature $(0,0,1,1)_7$. There is a unique lattice among the lattice in [13] with 7-signature $(0,0,1,1)_7$ and even discriminant, this lattice has 2-adic structure

$$(\hat{\mu}L)_2 \cong \mathbb{A} \perp 2^2\mathbb{A}.$$

Now L' must have 7-signature from among

$$(0,1,1,2)_7 \quad (0,0,1,3)_7 \quad (0,0,3,3)_7 \quad (0,2,3,3)_7 \quad (0,1,2,3)_7$$

the first four of which can be immediately ruled out since [13] does not contain any lattices with even discriminant divisible by 7, admitting such a 7-signature. On the other hand, when L' has 7-signature $(0,1,2,3)_7$, we will use the Conway-Sloane mass formula from [2] to show that $h_s(L') > 1$. Since $L'_2 \cong (\hat{\mu}L)_2$ we have

$$m_2(L') = m_2(\hat{\mu}L) = \frac{1}{3} \cdot \frac{1}{3} \cdot \left(\frac{2^2}{2^0}\right)^{\frac{1}{2}\cdot 2\cdot 2} \cdot 2^{-4} = \frac{1}{3^2}$$

and since $dL' = 2^4 \cdot 7^6$, we have

$$m(L') = \pi^{-4} \cdot 2 \cdot m_2(L') \cdot 2 \cdot m_7(L') \cdot (1-2^{-2})^2 \cdot (1-7^{-2})^2 \cdot \zeta(2)^2 = 7 > 1.$$

Since $m(L') > 1$ it follows that $m_s(L') > 1/2$ and hence $h_s(L') > 1$, leading to a contradiction. □

Lemma 11. $5 \notin \mathcal{P}_s$.

Proof. Suppose that L is a primitive quaternary lattice with $5 \mid dL$, $h_s(L) = 1$ and $h(L) > 1$. Then $5 \mid d\hat{\mu}L$ and $\hat{\mu}L$ must be from among the lattices in [13]. But among the lattice in [13], all discriminants divisible by 5 are of the form $2^r \cdot 3^s \cdot 5^t$ where r, s are non-negative integers, and $t > 0$. Moreover, we know that the 5-signature of $\hat{\mu}L$ must be from among

$$(0,0,0,1)_5 \quad (0,0,1,1)_5 \quad (0,1,1,1)_5.$$

When $\hat{\mu}L$ has 5-signature $(0,0,0,1)_5$ then L' has 5-signature from among

$$(0,0,1,2)_5 \quad (0,0,0,3)_5 \quad (0,1,2,2)_5 \quad (0,0,2,3)_5 \quad (0,2,2,3)_5,$$

and when $\hat{\mu}L$ has 5-signature $(0,1,1,1)_5$ then L' has 5-signature from among

$$(0,1,1,3)_5 \quad (0,1,3,3)_5 \quad (0,3,3,3)_5.$$

We can immediately rule out all possible 5-signatures for L' except for $(0,0,1,2)_5$ and $(0,1,2,2)_5$ since they correspond to L' with $h(L') = 1$, but no lattices in [13] admit such 5-signatures. If L' has 5-signature $(0,0,1,2)_5$, then $h(L') = 1$ and hence L' must be one of two lattices in [13] with this signature, both of which have discriminant $dL' = 5^3$. If we define L'' to be the lattice that descends to L' by one iteration of the μ_5-transformation, then $\mu_5(L'') = L'$ and $L''_p \cong L'_p \cong (\hat{\mu}L)_p$ for every prime $p \neq 5$. Consequently, dL'' is a power of 5, and L'' has a 5-signature from among

$$(0,0,1,4)_5 \quad (0,0,3,4)_5 \quad (0,1,2,4)_5 \quad (0,2,3,4)_5$$

and hence by Lemma 8 in all of these cases $h_s(L'') > 1$. Similarly, when L' has 5-signature $(0,1,2,2)_5$ then we know that $h(L') = 1$ and again L' must be one of two lattices [13] with this signature, both of which have $dL' = 5^5$. Hence, dL'' is a power of 5, and L'' has a 5-signature from among

$$(0,1,4,4)_5 \quad (0,3,4,4)_5$$

so again it follows from Lemma 8 and $h_s(L'') > 1$.

Suppose that $\hat{\mu}L$ has 5-signature $(0,0,1,1)_5$. Then $d\hat{\mu}L = 5^2$ or $2^2 \cdot 5^2$, and $\hat{\mu}L$ is one of 4 possible lattices in [13] which have $m_2(\hat{\mu}L)$ equal to $1/4, 1/8$ or $1/36$ (computed using Sagemath [20]). If we define L'' as in the previous paragraph, then L'' has a 5-signature from among

$$(0,0,3,3)_5 \quad (0,0,1,3)_5 \quad (0,2,3,3)_5 \quad (0,1,2,3)_5 \quad (0,1,1,2)_5.$$

and in the usual way the first three of these 5-signatures can be eliminated. When L' has 5-signature $(0,1,2,3)_5$, then

$$m_2(L') = m_2(\hat{\mu}L)$$

and

$$m_5(L') = \frac{5^{\frac{3(3)+2-1}{2}}}{2^4} = \frac{5^5}{2^4}$$

and hence

$$m(L') = \pi^{-4} \cdot 2 \cdot m_2(L') \cdot 2 \cdot m_5(L') \cdot (1-2^{-2})^2 \cdot (1-5^{-2})^2 \cdot \zeta(2)^2 = m_2(\hat{\mu}L) \cdot \frac{3^2 \cdot 5}{2^2}.$$

But for any possible choice of $m_2(\hat{\mu}L)$, it follows that $m_s(L') = m(L')/g(L')$ is not of the form $1/\mid O(L') \mid$, and hence $h_s(L') > 1$. When L' has 5-signature $(0,1,1,2)_5$, then $h(L') = 1$, so we define L'' to be the lattice which descends to L' by one iteration of the μ_5-transformation. Then L'' has 5-signature from among

$$(0,1,1,4)_5 \quad (0,3,3,4)_5 \quad (0,1,3,4)_5$$

and again we can immediately rule out the first two 5-signatures by the usual method. When L'' has 5-signature $(0,1,3,4)_5$, then $m_2(L'') = m_2(L') = m_2(\hat{\mu}L) \geq 1/36$, and

$$m_5(L'') = \frac{5^{\frac{3(4)+3-1}{2}}}{2^4} = \frac{5^7}{2^4}$$

and hence

$$m(L'') = \pi^{-4} \cdot 2 \cdot m_2(L'') \cdot 2 \cdot m_5(L'') \cdot (1-2^{-2})^2 \cdot (1-5^{-2})^2 \cdot \zeta(2)^2 \geq \frac{5^3}{2^4}$$

and since $m(L'') > 1$ we may conclude that $m_s(L'') > 1/2$ and hence $h_s(L'') > 1$. □

In the proof of the following lemma we will make use of results from [5], and we remind the reader that in the terminology of that paper, a lattice M is said to have *even order* if $Q(P(M)) \subseteq \mathbb{Z}_2^\times {\mathbb{Q}_2^\times}^2$ and *odd order* if $Q(P(M)) \subseteq 2\mathbb{Z}_2^\times {\mathbb{Q}_2^\times}^2$, where $P(M)$ denotes the set of all primitive anisotropic vectors whose associated symmetries are in $O(M)$. A unary modular component, $2^m\langle\epsilon\rangle$ where $\epsilon \in \mathbb{Z}_2^\times$, has odd or even order according to the parity of m. Recall also that from [5, Proposition 3.2], a binary lattice M has even order if and only if $M \cong A(1,0)$ or $A(1,4\epsilon)$ and M has odd order if and only if $M \cong A(0,0)$ or $A(2,2\epsilon)$ where $\epsilon \in \mathbb{Z}_2^\times$ and

$$A(a,b) = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$$

as defined in [17, §93B]. It follows hence that

$$\langle 1,1\rangle, \langle 3,3\rangle, \langle 3,7\rangle \text{ or } \langle 1,5\rangle, \tag{11}$$

are precisely the binary unimodular $\mathbb{Z}_2$-lattices which are neither odd nor even. We also recall the definition of *Type E* given in [6, p. 531], particularly noting that when a $\mathbb{Z}_2$-lattice is of type E, then its spinor norm group contains to full group of units.

Lemma 12. *There are no powers of 2 appearing in $\mathcal{D}_s$.*

Proof. Suppose that L is a primitive quaternary lattice with $dL = 2^n$ for some $n > 0$, and suppose that $h_s(L) = 1$ which $h(L) > 1$. Since L_p is unimodular at every odd prime, we may conclude that L_2 does not contain any improper modular components or modular components of dimension 3 or 4, or else $\text{gen}(L) = \text{spn}(L)$ and hence $h(L) = 1$. Therefore, L must have the 2-adic splitting

$$L_2 \cong \langle\epsilon_1\rangle \perp 2^k\langle\epsilon_2\rangle \perp 2^l\langle\epsilon_3\rangle \perp 2^m\langle\epsilon_4\rangle$$

where $\epsilon_i \in \mathbb{Z}_2^\times$, and $0 \leq k \leq l \leq m$.

Suppose that $k = 0$, and $l = m$, and hence L_2 has two binary modular components,

$$N \cong \langle\epsilon_1, \epsilon_2\rangle \text{ and } M \cong 2^m\langle\epsilon_3, \epsilon_4\rangle,$$

where $m > 0$. According to [5, Theorem 3.14], if M and N both have odd order, both have even order, or one of each, then $\mathbb{Z}_2^\times \subseteq \theta(O^+(L_2))$ by [5, Theorem 3.14 (i) and (ii)], and hence $\text{gen}(L) = \text{spn}(L)$, implying $h(L) = 1$. Therefore we may suppose that one of M or N must be from among the lattices in (11), and if the other lattice has odd or even order, then $\mathbb{Z}_2^\times \subseteq \theta(O^+(L_2))$ by [5, Theorem 3.14 (iii)]. Therefore we may suppose that both M and N have neither odd nor even order. If $M \not\cong N$ or if $m < 4$, then $\mathbb{Z}_2^\times \subseteq \theta(O^+(L_2))$ by [5, Theorem 3.14 (iv)]. Thus, in order to simultaneously have $h_s(L) = 1$ and $h(L) > 1$, we may assume that $M \cong N$ is from among the binary forms in (11) and $m \geq 4$. Now we will compute the Conway-Sloane mass formula as given in [2], namely,

$$m(L) = \pi^{-4} \cdot 2 \cdot m_2(L) \cdot \prod_{p \neq 2} 2m_p(L).$$

Since

$$L_2 \cong L_{(-1)} \perp L_{(0)} \perp_{(1)} \perp ... \perp L_{(m-1)} \perp L_{(m)} \perp L_{(m+1)},$$

we have $L_{(i)}$ contributing $1/2$ to the diagonal product for $i \in \{-1, 1, m-1, m+1\}$. Moreover, since $L_{(0)}$ and $L_{(m)}$ are both 2-dimensional free type I forms with octane value ± 2, they have species 1 and therefore each contribute $1/2$ to the diagonal product. Therefore,

$$m_2(L) = \left(\frac{1}{2}\right)^6 \cdot \left(\frac{2^m}{2^0}\right)^{\frac{1}{2}\cdot 2\cdot 2} \cdot 2^{0-0} = 2^{2m-6}$$

and since $dL_2 = 2^{2m} \in {\mathbb{Q}^{\times}}^2$ we have

$$m(L) = \pi^{-4} \cdot 2^{2m-5} \cdot (1-2^{-2})^2 \cdot \zeta(2)^2 = 2^{2m-11}$$

Since $g(L) = 2, 4$, it follows that $m_s(L) = m(L)/g(L) > 1/2$ for any $m \geq 7$, and thus $m_s(L)$ not of the form $1/ \mid O(L) \mid$, implying that $h_s(L) > 1$. On the other hand, when $m = 4, 5, 6$ we will use the algorithm from [8, Lemma 3] to determine all possible genera for lattices with 2-signature $(0, 0, m, m)_2$. When $m = 4$, then the algorithm produces 4 genera with 2-signature $(0, 0, 4, 4)_2$ and 2-adic structure $M \perp 2^4 M$, with representative lattices

$$L_1 = \begin{bmatrix} 2 & 0 & 1 & -2 \\ 0 & 2 & 1 & -2 \\ 1 & 1 & 5 & -2 \\ -2 & -2 & -2 & 20 \end{bmatrix} L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 4 \\ 0 & 2 & 5 & 2 \\ 0 & 4 & 2 & 20 \end{bmatrix} L_3 = \begin{bmatrix} 8 & 0 & 2 & 4 \\ 0 & 2 & -1 & 0 \\ 2 & -1 & 3 & 1 \\ 4 & 0 & 1 & 10 \end{bmatrix} L_4 = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 12 & -2 & 6 \\ 0 & -2 & 3 & -1 \\ -1 & 6 & -1 & 6 \end{bmatrix}.$$

Checking the structure of each $\text{gen}(L_i)$ in Magma we see that only $\text{gen}(L_1)$ splits into multiple spinor genera, both containing multiple classes, and for the remaining cases $\text{gen}(L_i) = \text{spn}(L_i)$. When $m = 5$, then there are only 3 possible genera with signature $(0, 0, 5, 5)_2$ and 2-adic structure $M \perp 2^5 M$, and they have representative lattices

$$L_1 = \begin{bmatrix} 3 & 1 & -1 & -1 \\ 1 & 4 & -2 & -2 \\ -1 & -2 & 4 & 4 \\ -1 & -2 & 4 & 36 \end{bmatrix} L_2 = \begin{bmatrix} 2 & 0 & -1 & -2 \\ 0 & 2 & 1 & -2 \\ -1 & 1 & 9 & 0 \\ -2 & -2 & 0 & 36 \end{bmatrix} L_3 = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 12 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 11 \end{bmatrix}.$$

Of these genera, $\text{gen}(L_1) = \text{spn}(L_1)$, while $\text{gen}(L_2)$ and $\text{gen}(L_3)$ split into multiple spinor genera, each containing several classes. When $m = 6$, then there are 4 possible genera with 2-signature $(0, 0, 6, 6)_2$ and 2-adic structure $M \perp 2^6 M$, with representative lattices

$$L_1 = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 5 & -2 & 0 \\ 0 & -2 & 5 & 0 \\ 0 & 0 & 0 & 64 \end{bmatrix} L_2 = \begin{bmatrix} 2 & 0 & 1 & -2 \\ 0 & 8 & 2 & 4 \\ 1 & 2 & 9 & 0 \\ -2 & 4 & 0 & 36 \end{bmatrix} L_3 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 6 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 64 \end{bmatrix} L_4 = \begin{bmatrix} 6 & 0 & 1 & 6 \\ 0 & 6 & 3 & 2 \\ 1 & 3 & 7 & 2 \\ 6 & 2 & 2 & 28 \end{bmatrix}$$

Again, a check of these genera in Magma reveals that $\text{gen}(L_1) = \text{spn}(L_1)$ and $\text{gen}(L_3) = \text{spn}(L_3)$, while the remaining genera split into two spinor genera, each containing several classes.

Next, suppose that $k = 0$ and $0 < l < m$ and hence

$$L_2 \cong \langle \epsilon_1, \epsilon_2 \rangle \perp 2^l \langle \epsilon_3 \rangle \perp 2^m \langle \epsilon_4 \rangle.$$

The unary components are either odd or even according to the parity of l and m. If the binary component is either odd or even, then in any case $\mathbb{Z}_2^{\times} \subseteq \theta(O^+(L_2))$ by [5, Theorem 3.14 (i) and (ii)]. Therefore we may suppose that the binary component is neither odd nor even, and hence is one of the lattices in (11). If $l < 4$ or if $m - l < 4$ then $\mathbb{Z}_2^{\times} \subseteq \theta(O^+(L_2))$ by [5, Theorem 3.14 (iv)], therefore we may assume that $l \geq 4$ and $k \geq 8$. Now we will compute the mass formula, $m(L)$. To compute the diagonal product, we have a decomposition

$$L_2 \cong L_{(-1)} \perp L_{(0)} \perp_{(1)} \perp ... \perp L_{(l-1)} \perp L_{(l)} \perp L_{(l+1)} \perp ... \perp L_{(m-1)} \perp L_{(m)} \perp L_{(m+1)},$$

each of the 0-dimensional forms is bound since it is adjacent to a form of type I, and therefore $L_{(i)}$ contributes $1/2$ to the diagonal product for $i \in \{-1, 1, l-1, l+1, m-1, m+1\}$. Moreover, the binary part is free of type I with octane value ± 2, and therefore has species 1, and the two unary parts of free of type 1 with octane value ± 1 and therefore have species 0+. Therefore, computing the local mass we have

$$m_2(L) = \left(\frac{1}{2}\right)^7 \cdot (2^{m-l})^{\frac{1}{2}\cdot 1\cdot 1} \cdot (2^m)^{\frac{1}{2}\cdot 2\cdot 1} \cdot (2^l)^{\frac{1}{2}\cdot 2\cdot 1} = 2^{\frac{3m+2l-14}{2}} \geq 2^9,$$

and hence

$$m(L) \quad > \quad \pi^{-4} \cdot 2^{10} \cdot (1 - 2^{-4}) \cdot \zeta(4) = \frac{2^5}{3}.$$

But since $g(L) = 2, 4$, this implies that $m_s(L) = h(L)/g(L) > 1/2$, and therefore we may conclude that $h_s(L) > 1$.

Next, suppose that $0 < k$ and either $k = l$ or $l = m$, and hence

$$L_2 \cong \begin{cases} \langle \epsilon_1 \rangle \perp 2^k \langle \epsilon_2, \epsilon_3 \rangle \perp 2^m \langle \epsilon_4 \rangle & \text{when } k = l \\ \langle \epsilon_1 \rangle \perp 2^k \langle \epsilon_2 \rangle \perp 2^m \langle \epsilon_3, \epsilon_4 \rangle & \text{when } l = m. \end{cases}$$

In either case, by the argument in the preceding paragraph, we may suppose that the binary component is neither odd nor even, and we may further assume that $k \leq 4$ and $m \leq 8$. Therefore,

$$m_2(L) = \begin{cases} 2^{\frac{3m-14}{2}} & \text{when } k = l \\ 2^{\frac{4m+k-14}{2}} & \text{when } l = m \end{cases}$$

and

$$m(L) > \pi^{-4} \cdot 2 \cdot m_2(L) \cdot (1 - 2^{-4}) \cdot \zeta(4) = \frac{m_2(L)}{2^4 \cdot 3}$$

and hence $m(L) > 2$ in all but the exceptional case when $k = l = 4$ and $m = 8$. In this exceptional case, we have

$$L_2 \cong \langle \epsilon_1 \rangle \perp 2^4 \langle \epsilon_2, \epsilon_3 \rangle \perp 2^8 \langle \epsilon_4 \rangle.$$

This means

$$\mu_2\left(L\right)_2 \cong \langle \epsilon_1 \rangle \perp 2^2 \langle \epsilon_2, \epsilon_3 \rangle \perp 2^6 \langle \epsilon_4 \rangle.$$

has class number one by [5, Theorem 3.14], and consequently must correspond to a lattice in [13] with 2-signature $(0, 2, 2, 6)_2$. There is only one such lattice in [13] and it has the local 2-adic splitting

$$\langle 3 \rangle \perp 2^2 \langle 3, 7 \rangle \perp 2^6 \langle 7 \rangle,$$

and hence

$$L_2 \cong \langle 3 \rangle \perp 2^4 \langle 3, 7 \rangle \perp 2^8 \langle 7 \rangle.$$

From here we may conclude from [5, Theorem 3.14] that

$$\theta(O^+(L_2)) = \{c \in \mathbb{Q}_2^\times : (c, -5) = 1\} = \{1, 5, 6, 14\}\mathbb{Q}_2^{\times 2}.$$

Now we can use the formula given in [17, 102:7] to count the number of proper spinor genera in the genus of L_2, namely,

$$g^+(L) = \left[J_{\mathbb{Q}} : \mathbb{Q}^\times \prod_p \theta(O^+(L_p)) \right],$$

where $J_{\mathbb{Q}}$ denotes the group of rational ideles. For an arbitrary $\mathbf{x} = (x_2, x_3, x_5, ...) \in J_{\mathbb{Q}}$, we will show that $\mathbf{x}$ is in $\mathbb{Q}^\times \prod_p \theta(O^+(L_p))$. Multiplying $\mathbf{x}$ by a suitably chosen scalar, a, we know that ax_p is a unit at every prime p. If ax_2 is either 1 or 5, then $a\mathbf{x} \in \mathbb{Q}^\times \prod_p \theta(O^+(L_p))$. On the other hand, if ax_2 is either 3 or 7, then $2a\mathbf{x} \in \mathbb{Q}^\times \prod_p \theta(O^+(L_p))$. Therefore, we may conclude that there is only one proper spinor genus in the genus of L, and since $g(L) \leq g^+(L) \leq 2g(L)$ (cf. [17, 102:2]), we may conclude that $g(L) = 1$, and consequently $h_s(L) = 1$ implies $h(L) = 1$ for such a form.

Now it only remains to deal with the case when $0 < k < l < m$, and

$$L_2 \cong \langle \epsilon_1 \rangle \perp 2^k \langle \epsilon_2 \rangle \perp 2^l \langle \epsilon_3 \rangle \perp 2^m \langle \epsilon_4 \rangle$$

where $\epsilon_i \in \mathbb{Z}_2^\times$; we will consider the cases when k is odd or even separately.

First, we suppose that k is even, so $k = 2k'$ for some natural number k', and define $L' = \mu_2^{k'}(L)$. Then,

$$L'_2 \cong \langle \epsilon_1, \epsilon_2 \rangle \perp 2^{l-k}\langle \epsilon_3 \rangle \perp 2^{m-k}\langle \epsilon_4 \rangle,$$

where $\langle \epsilon_1, \epsilon_2 \rangle$ is a proper binary modular component. If $l - k = 1$, then L' has 2-signature $(0, 0, 1, m - k)_2$, hence $\mathbb{Z}_2^\times \subseteq \theta(O^+(L_2))$ by [5, Theorem 3.14], implying that $h(L') = 1$. Therefore L' is among the lattices in [13], and must have a signature from among

$$(0, 0, 1, 2)_2 \quad (0, 0, 1, 3)_2 \quad (0, 0, 1, 4)_2,$$

all of which in turn would make L of type E, meaning that $h(L) = 1$. On the other hand, if $l - k = 2$, then then L' has 2-signature $(0, 0, 2, m - k)_2$, and again $h(L') = 1$ by [5, Theorem 3.14]. Therefore L' is among the lattices in [13], and must have a signature from among

$$(0, 0, 2, 3)_2 \quad (0, 0, 2, 5)_2 \quad (0, 0, 2, 4)_2,$$

and for the first two 2-signatures this once again forces L to be of type E, implying $h(L) = 1$. On the other hand, when L' has 2-signature $(0, 0, 2, 4)_2$, it is possible that L' lifts either to a lattice with 2-signature $(0, 2, 4, 6)_2$ or to a lattice with 2-signature $(0, 0, 4, 6)_2$. But using the algorithm from [8, Lemma 3] we generate all possible genera bearing such 2-signatures, and a check in Magma reveals that all of the associated spinor genera split into multiple classes. Therefore in this case, we are assured that $h_s(L') > 1$. Cases where $l - k \geq 3$ can be reduced to one of these two cases above by repeated applications of μ_2 to L'.

Suppose that k is odd, so $k = 2k' + 1$ for some natural number k', and as above, define $L' = \mu_2^{k'}(L)$. Then,

$$L'_2 \cong \langle \epsilon_1 \rangle \perp 2\langle \epsilon_2 \rangle \perp 2^{l-k+1}\langle \epsilon_3 \rangle \perp 2^{m-k+1}\langle \epsilon_4 \rangle.$$

Now we will consider the possibility that $l - k$ is either odd or even. If $l - k$ is odd, then $l - k + 1 = 2\ell'$ for some natural number ℓ', and letting $L'' = \mu_2^{\ell'-1}(L')$, we have

$$L''_2 \cong \langle \epsilon_1 \rangle \perp 2\langle \epsilon_2 \rangle \perp 2^2\langle \epsilon_3 \rangle \perp 2^{m-l+2}\langle \epsilon_4 \rangle.$$

But then clearly L'' is of Type E and hence $h(L') = 1$ and has 2-signature $(0, 1, 2, m - l + 2)_2$ where $m - l + 2 > 2$, but no such signature exists among the lattices in [13], so this case cannot occur. On the other hand, suppose that $l - k$ is even, so $l - k = 2\ell'$ for some natural number ℓ', and define $L'' = \mu_2^{\ell'-1}(L')$. Then,

$$L'_2 \cong \langle \epsilon_1 \rangle \perp 2\langle \epsilon_2 \rangle \perp 2^{2\ell'+1}\langle \epsilon_3 \rangle \perp 2^{m-k}\langle \epsilon_4 \rangle.$$

and hence

$$L''_2 \cong \langle \epsilon_1 \rangle \perp 2\langle \epsilon_2 \rangle \perp 2^3\langle \epsilon_3 \rangle \perp 2^{m-l+3}\langle \epsilon_4 \rangle.$$

which is always of Type E, and hence has class number 1 and 2-signature $(0, 1, 3, m-l+3)_2$ where $m-l+3 > 3$, but no such lattice exists in [13]. □

Lemma 13. *Only* $3^6 \in \mathcal{D}_s$.

Proof. Suppose that L is a lattice with $h_s(L) = 1$ and $h(L) >$ for which $3 \mid dL$. Then in view of the previous lemmas we know that $dL = 2^k \cdot 3^m$ where k is a non-negative integer and m is a positive integer. By Theorem 2 we know that $h(\hat{\mu}L) = 1$, and consequently $\hat{\mu}L$ appears in among the lattices in [13], and must have one of the following 3-signatures types,

$$(0, 0, 0, 1)_3 \quad (0, 0, 1, 1)_3 \quad (0, 1, 1, 1)_3 \quad (0, 0, 0, 2)_3 \quad (0, 0, 2, 2)_3 \quad (0, 2, 2, 2)_3.$$

In each of these cases, it is immediately obvious that $\mathbb{Z}_p^\times \subseteq \theta(O^+(L_p))$ at every odd prime p, and it follows from [5, Theorem 3.14] that $\mathbb{Z}_2^\times \subseteq \theta(O^+(L_2))$. Define L' to the the lattice for which $\mu_3(L') = \hat{\mu}L$. Then $L'_p \cong \hat{\mu}L_p$ at every prime $p \neq 3$, and L' must have 3-signature from among

$$\begin{array}{llllll}
(0,0,0,3)_3 & *(0,0,1,3)_3 & (0,1,1,3)_3 & (0,0,0,4)_3 & (0,0,4,4)_3 & (0,4,4,4)_3 \\
*(0,0,1,2)_3 & *(0,1,1,2)_3 & (0,1,3,3)_3 & (0,0,2,4)_3 & (0,2,4,4)_3 & \\
*(0,1,2,2)_3 & *(0,1,2,3)_3 & (0,3,3,3)_3 & (0,2,2,4)_3 & & \\
(0,0,2,3)_3 & *(0,2,3,3)_3 & & & & \\
(0,2,2,3)_3 & (0,0,3,3)_3. & & & &
\end{array}$$

We can immediately rule out all but the starred cases, since these signatures would have to correspond to an L' with $h(L') = 1$, but no such signatures appear among the lattices in [13]. Suppose that L' has one of the starred 3-signatures above, then L' has $h(L') = 1$ (except in certain exceptional cases corresponding to

$(0,1,2,3)_3)$ and $h_s(L') = 1$. In these cases, we define L'' to be the lattice for which $\mu_3(L'') = L'$. Here we observe again that for every prime $p \neq 3$ we have $L''_p \cong L'_p \cong \hat{\mu} L_p$. Then L'' has a 3-signature coming from among

$$\begin{array}{cccccc} (0,0,1,4)_3 & (0,1,4,4)_3 & (0,0,1,5)_3 & (0,1,1,4)_3 & *(0,1,4,5)_3 & (0,4,5,5)_3 \\ (0,0,3,4)_3 & (0,3,4,4)_3 & (0,0,3,5)_3 & *(0,1,3,4)_3 & *(0,3,4,5)_3 & \\ *(0,2,3,4)_3 & & *(0,1,2,5)_3 & (0,3,3,4)_3 & & \\ *(0,2,1,4)_3 & & *(0,2,3,5)_3. & & & \end{array}$$

Once again, all but the starred cases correspond to signatures that would force $h(L'') = 1$, but no such 3-signatures appear in [13], so these can be immediately eliminated. The remaining signatures (including $(0,1,2,3)_3$) will be dealt with case by case.

First, suppose that L'' has 3-signature $(0,2,3,4)_3$ or $(0,1,2,4)_3$. In this case $\mu_3(L'') = L'$ has 3-signature $(0,0,1,2)_3$, and therefore L' is in [13]. By searching among the lattices in [13] with 3-signature $(0,0,1,2)_3$ we find, using Sagemath, that any such lattice has $m_2(L') = 1/6$. Consequently, $m_2(L'') = m_2(L) = 1/6$. From here, we can compute upper and lower bounds for the Conway-Sloane mass for L'' using the formula give in [2]. Since

$$\frac{1}{(1-p^{-4})} < \frac{1}{(1-p^{-2})^2}$$

and

$$m_3(L'') = \frac{3^{\frac{13}{2}}}{2^4}$$

we can underestimate $m(L'')$ by

$$m^-(L'') = \pi^{-4} \cdot 2 \cdot \frac{1}{6} \cdot 2 \cdot \frac{3^{\frac{13}{2}}}{2^4} \cdot (1-2^{-4}) \cdot (1-3^{-4}) \cdot \zeta(4) = \frac{3^{1/2} \cdot 5}{2^4} \approx 0.5412$$

and overestimate $m(L'')$ by

$$m^+(L'') = \pi^{-4} \cdot 2 \cdot \frac{1}{6} \cdot 2 \cdot \frac{3^{\frac{13}{2}}}{2^4} \cdot (1-2^{-2})^2 \cdot (1-3^{-2})^2 \cdot \zeta(2)^2 = \frac{3^{3/2}}{2^3} \approx 0.6495$$

where $m^-(L'') < m(L'') < m^+(L'')$. But since $g(L'') = 2$, and $m_s(L'') = m(L'')/g(L'')$ this means that

$$0.2707 < m_s(L'') < 0.3248.$$

Consequently, $m_s(L'')$ is not of the form $1/\mid O(L'') \mid$, and therefore $h_s(L'') > 1$.

Next, suppose that L'' has 3-signature $(0,1,2,5)_3$ or $(0,2,3,5)_3$. Again, we know that $h(L'') > 1$ since no such signatures appear in [13]. On other other hand, we know that $\mu_3(L'') = L'$ does appear in [13], and so by searching among that lattice in [13], and using Sagemath, we determine that $m_2(L'') = m_2(L) = 1/6$ or $1/18$. Since

$$m_3(L'') = \frac{3^8}{2^4}$$

for either signature, we obtain

$$m(L'') = \pi^{-4} \cdot 2 \cdot m_2(L'') \cdot 2 \cdot \frac{3^8}{2^4} \cdot (1-2^{-2})^2 \cdot (1-3^{-2})^2 \cdot \zeta(2)^2 = m_2(L'') \cdot \frac{3^4}{2^4}.$$

Since $m_2(L') = m_2(L'')$, this implies

$$m_s(L'') = \frac{m(L'')}{g(L'')} = m_2(L') \cdot \frac{3^4}{2^5},$$

but this will always have at least one power of 3 in the numerator, and hence is not of the form $1/\mid O(L'') \mid$. Therefore, we may conclude that $h_s(L'') > 1$. The cases when L'' has 3-signatures $(0,1,3,4)_3, (0,1,4,5)_3$ and $(0,3,4,5)_3$ follow similarly, except in these case

$$m(L'') = m_2(L') \cdot \begin{cases} \frac{3^3}{4} & \text{for 3-signature } (0,1,3,4)_3 \\ \frac{3^5}{4} & \text{for 3-signature } (0,1,4,5)_3 \\ \frac{3^4}{4} & \text{for 3-signature } (0,3,4,5)_3, \end{cases}$$

where the possibilities for $m_2(L') = m_2(L'')$ are

$$\frac{1}{2^2\cdot 3^2},\ \frac{1}{2\cdot 3^2},\ \frac{1}{2^2\cdot 3},\ \frac{1}{3^2},\ \frac{1}{2^2},\ \frac{1}{3} \text{ or } 1.$$

But again, in every case $m_s(L'')$ is left with a 3 in the numerator, and hence is not of the form $1/\mid O(L'')\mid$.

Finally, we deal with the case where L' has 3-signature $(0,1,2,3)_3$. From [16], we know that there are 33 isometry classes of lattices with discriminant 3^6, and of these, only 6 have 3-signature $(0,1,2,3)_3$, namely,

$$L_1 = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 6 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 1 & 0 & 0 & 14 \end{bmatrix} L_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 9 & 18 \end{bmatrix} L_3 = \begin{bmatrix} 6 & 3 & 3 & 3 \\ 3 & 6 & 0 & 3 \\ 3 & 0 & 8 & 4 \\ 3 & 3 & 4 & 8 \end{bmatrix}$$

where L_1, L_2 and L_3 are the representative classes for a single genus, and

$$M_1 = \begin{bmatrix} 4 & 1 & 1 & 2 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 4 & -1 \\ 2 & 2 & -1 & 16 \end{bmatrix} M_2 = \begin{bmatrix} 4 & 2 & 1 & 1 \\ 2 & 4 & -1 & 2 \\ 1 & -1 & 10 & 4 \\ 1 & 2 & 4 & 10 \end{bmatrix} M_3 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 8 & -1 & 2 \\ 1 & -1 & 8 & 2 \\ 1 & 2 & 2 & 8 \end{bmatrix},$$

where M_1, M_2 and M_3 are representatives for three distinct genera, each with class number 1. Here $\operatorname{gen}(L_1)$ is the example mentioned in the introduction, first observed by Watson in [21], and later confirmed by Nipp in [16], for which the genus splits into two spinor genera, namely $\operatorname{spn}(L_1)$ and $\operatorname{spn}(L_2) = \operatorname{spn}(L_3)$, and $h_s(L_1) = 1$ while $h_s(L_2) = h_s(L_3) = 2$. Consequently, any lattice L which descends to L_2 or L_3 by a series of μ_p-transformation will already have $h_s(L) > 1$. On the other hand, it is still possible to have lattice, L, descend to L_1, M_1, M_2 or M_3 by a series of μ_p-transformations, which has spinor class number 1. If L descends by μ_3, then this would imply that there is a lattice with spinor class number 1 and 3-signature

$$(0,3,3,4)_3 \quad (0,1,4,5)_3 \quad (0,3,4,5)_3$$

all of these would lead to a contradiction, since $(0,3,3,4)_3$ would have class number 1 but doesn't appear in [13], and the $(0,1,3,5)_3$ and $(0,3,4,5)_3$ have already be ruled out in the preceding paragraphs using the mass formula. Therefore, the only possibility is that L descends to one of L_1, M_1, M_2 or M_3 by a series of μ_2-transformations. If we can show that that there are no lattice with spinor class number 1 and discriminant $2^k\cdot 3^6$ for $k = 2,4,6$, then we are done. Using this list of 33 isometry classes with discriminant 3^6 to seed the algorithm in [8, Lemma 3], we can generate all possible isometry classes of lattices with discriminants $2^k\cdot 3^6$ for $k = 2,4,6$. Generating this list in Magma, we obtain 18 genera, 63 genera, and 135 genera corresponding to discriminants $2^2\cdot 3^6$, $2^4\cdot 3^6$ and $2^6\cdot 3^6$, respectively. Narrowing this list down to only the genera which admit 3-signature $(0,1,2,3)_3$, there are 8 genera, 28 genera, and 60 genera corresponding to discriminants $2^2\cdot 3^6$, $2^4\cdot 3^6$ and $2^6\cdot 3^6$, respectively. Among these, there is only genus which is has class number 1(note, magma does return a second imprimitive isometry class) and is therefore in [13], namely,

$$K_1 = \begin{bmatrix} 4 & 2 & -1 & 0 \\ 2 & 10 & 4 & 0 \\ -1 & 4 & 10 & 3 \\ 0 & 0 & 3 & 12 \end{bmatrix}.$$

which has local 2-adic structure $\mathbb{H} \perp 2\langle 1,7\rangle$. Consequently any lattice which descends to K_1 by a μ_2-transformation must have 2-signature $(0,0,3,3)_2$, but we already know from the algorithm that all lattice of discriminant $2^6\cdot 3^6$ have spinor class number greater than 1. □

The proof of Theorem 1 now follows by combining the above results and the fact that Nipp's tables [16] explicitly cover the discriminant 729.

6. Acknowledgments

The second author wishes to thank the Max Planck Institute for Mathematics for the hospitality and support provided throughout her visit during which much of this research took place.

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