

EMBEDDINGS OF \mathbb{C}^* -SURFACES INTO WEIGHTED PROJECTIVE SPACES

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ABSTRACT. Let V be a normal affine surface which admits a \mathbb{C}^* - and a \mathbb{C}_+ -action. Such surfaces were classified e.g., in [FlZa₁, FlZa₂], see also the references therein. In this note we show that in many cases V can be embedded as a principal Zariski open subset into a hypersurface of a weighted projective space. In particular, we recover a result of D. Daigle and P. Russell, see Theorem A in [DR].

1. INTRODUCTION

If $V = \text{Spec } A$ is a normal affine surface equipped with an effective \mathbb{C}^* -action, then its coordinate ring A carries a natural structure of a \mathbb{Z} -graded ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$. As was shown in [FlZa₁], such a \mathbb{C}^* -action on V has a hyperbolic fixed point if and only if $C = \text{Spec } A_0$ is a smooth affine curve and $A_{\pm 1} \neq 0$. In this case the structure of the graded ring A can be elegantly described in terms of a pair (D_+, D_-) of \mathbb{Q} -divisors on C with $D_+ + D_- \leq 0$. More precisely, A is the graded subring

$$A = A_0[D_+, D_-] \subseteq K_0[u, u^{-1}], \quad K_0 := \text{Frac } A_0,$$

where for $i \geq 0$

$$(1) \quad A_i = \{f \in K_0 \mid \text{div } f + iD_+ \geq 0\} u^i \quad \text{and} \quad A_{-i} = \{f \in K_0 \mid \text{div } f + iD_- \geq 0\} u^{-i}.$$

This presentation of A (or V) is called in [FlZa₁] the *DPD-presentation*. Furthermore two pairs (D_+, D_-) and (D'_+, D'_-) define equivariantly isomorphic surfaces over C if and only if they are equivalent that is,

$$D_+ = D'_+ + \text{div } f \quad \text{and} \quad D_- = D'_- - \text{div } f \quad \text{for some } f \in K_0^\times.$$

In this note we show that if such a surface V admits also a \mathbb{C}_+ -action then it can be \mathbb{C}^* -equivariantly embedded (up to normalization) into a weighted projective space as a hypersurface minus a hyperplane; see Theorem 2.3 and Corollary 2.5 below. In particular we recover the following result of Daigle and Russell [DR].

Theorem 1.1. *Let V be a normal Gizatullin surface¹ with a finite divisor class group. Then V can be embedded into a weighted projective plane $\mathbb{P}(a, b, c)$ minus a hypersurface. More precisely:*

- (a) *If $V = V_{d,e}$ is toric² then V is equivariantly isomorphic to the open part³ $\mathbb{D}_+(z)$ of the weighted projective plane $\mathbb{P}(1, e, d)$ equipped with homogeneous coordinates $(x : y : z)$ and with the 2-torus action $(\lambda_1, \lambda_2).(x : y : z) = (\lambda_1 x : \lambda_2 y : z)$.*

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¹That is, V admits a completion by a linear chain of smooth rational curves; see Section 3 below.

²See 3.1(a) below.

³We use the standard notation $\mathbb{V}_+(f) = \{f = 0\}$ and $\mathbb{D}_+(f) = \{f \neq 0\}$.

(b) If V is non-toric then $V \cong \mathbb{D}_+(xy - z^m) \subseteq \mathbb{P}(a, b, c)$ for some positive integers a, b, c satisfying $a + b = cm$ and $\gcd(a, b) = 1$.

2. EMBEDDINGS OF \mathbb{C}^* -SURFACES INTO WEIGHTED PROJECTIVE SPACES

According to Proposition 4.8 in [FlZa₁] every normal affine \mathbb{C}^* -surface V is equivariantly isomorphic to the normalization of a weighted homogeneous surface V' in \mathbb{A}^4 . In some cases (described in *loc.cit.*) V' can be chosen to be a hypersurface in \mathbb{A}^3 . Cf. also [Du] for affine embeddings of some other classes of surfaces.

In Theorem 2.3 below we show that any normal \mathbb{C}^* -surface V with a \mathbb{C}_+ -action is the normalization of a principal Zariski open subset of some weighted projective hypersurface.

In the proofs we use the following observation from [Fl].

Proposition 2.1. *Let $R = \bigoplus_{i \geq 0} R_i$ be a graded R_0 -algebra of finite type containing the field of rational numbers \mathbb{Q} . If $z \in R_d$, $d > 0$, is an element of positive degree then the group of d th roots of unity $E_d \cong \mathbb{Z}_d$ acts on R and then also on $R/(z - 1)$ via*

$$\zeta \cdot a = \zeta^i \cdot a \quad \text{for } a \in R_i, \zeta \in E_d,$$

with ring of invariants $(R/(z - 1))^{E_d} \cong (R[1/z])_0$. Consequently

$$(\text{Spec } R/(z - 1))/E_d \cong \mathbb{D}_+(z)$$

is isomorphic to the complement of the hyperplane $\{z = 0\}$ in $\text{Proj}(R)$.

Let us fix the notations.

2.2. Let $V = \text{Spec } A$ be a normal \mathbb{C}^* -surface with DPD-presentation

$$A = \mathbb{C}[t][D_+, D_-] \subseteq \mathbb{C}(t)[u, u^{-1}].$$

If V carries a \mathbb{C}_+ -action then according to [FlZa₂], after interchanging (D_+, D_-) and passing to an equivalent pair, if necessary, we may assume that

$$(2) \quad \begin{aligned} D_+ &= -\frac{e_+}{d}[0] \quad \text{with } 0 < e_+ \leq d, \\ D_- &= -\frac{e_-}{d}[0] - \frac{1}{k}D_0 \end{aligned}$$

with an integral divisor D_0 , where $D_0(0) = 0$. We choose a polynomial $Q \in \mathbb{C}[t]$ with $D_0 = \text{div}(Q)$; so $Q(0) \neq 0$.

Theorem 2.3. *Let F be the polynomial*

$$(3) \quad F = x^k y - s^{k(e_+ + e_-)} Q(s^d/z) z^{\deg Q} \in \mathbb{C}[x, y, z, s],$$

which is weighted homogeneous of degree⁴ $k(e_+ + e_-) + d \deg Q$ with respect to the weights

$$(4) \quad \deg x = e_+, \quad \deg y = ke_- + d \deg Q, \quad \deg z = d, \quad \deg s = 1.$$

Then the surface V as in 2.2 above is equivariantly isomorphic to the normalization of the principal Zariski open subset $\mathbb{D}_+(z)$ of the hypersurface $\mathbb{V}_+(F)$ in the weighted projective 3-space

$$(5) \quad \mathbb{P} = \mathbb{P}(e_+, ke_- + d \deg Q, d, 1).$$

⁴We note that $e_+ + e_- = d(-D_+(0) - D_-(0)) \geq 0$.

Proof. With $s = \sqrt[d]{t}$ the field $L = \text{Frac}(A)[s]$ is a cyclic extension of $K = \text{Frac}(A)$. Its Galois group is the group of d th roots of unity E_d acting on L via the identity on K and by $\zeta \cdot s = \zeta \cdot s$ if $\zeta \in E_d$. Let A' be the normalization of A in L . According to Proposition 4.12 in [FlZa₁]

$$A' = \mathbb{C}[s][D'_+, D'_-] \subseteq \mathbb{C}(s)[u, u^{-1}]$$

with $D'_\pm = \pi_d^*(D_\pm)$, where $\pi_d : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the covering $s \mapsto s^d$. Thus

$$(D'_+, D'_-) = \left(-e_+[0], -e_-[0] - \frac{1}{k}\pi_d^*(D_0) \right) = \left(-e_+[0], -e_-[0] - \frac{1}{k}\text{div}(Q(s^d)) \right).$$

The element $x = s^{e_+}u \in A'_1$ is a generator of A'_1 as a $\mathbb{C}[s]$ -module. According to Example 4.10 in [FlZa₁] the graded algebra A' is isomorphic to the normalization of

$$(6) \quad B = \mathbb{C}[x, y, s]/(x^k y - s^{k(e_++e_-)}Q(s^d)).$$

The cyclic group E_d acts on A' via

$$\zeta \cdot x = \zeta^{e_+}x, \quad \zeta \cdot y = \zeta^{ke_-}y, \quad \zeta \cdot s = \zeta s$$

with invariant ring A . Clearly this action stabilizes the subring B . Assigning to x, y, z, s the degrees as in (4), F as in (3) is indeed weighted homogeneous. Since $F(x, y, 1, s) = x^k y - s^{k(e_++e_-)}Q(s^d)$, the graded algebra

$$R = \mathbb{C}[x, y, z, s]/(F)$$

satisfies $R/(z-1) \cong B$. Applying Proposition 2.1 $V = \text{Spec } A$ is isomorphic to the normalization of $\mathbb{D}_+(z) \cap \mathbb{V}_+(F)$ in the weighted projective space \mathbb{P} . \square

Remark 2.4. In general not all weights of the weighted projective space \mathbb{P} in (5) are positive. Indeed it can happen that $ke_- + d \deg Q \leq 0$. In this case we can choose $\alpha \in \mathbb{N}$ with $ke_- + d(\deg Q + \alpha) > 0$ and consider instead of F the polynomial

$$(7) \quad \tilde{F} = x^k y - s^{k(e_++e_-)}Q(s^d/z)z^{\deg Q + \alpha} \in \mathbb{C}[x, y, z, s],$$

which is now weighted homogeneous of degree $k(e_+ + e_-) + d(\deg Q + \alpha)$ with respect to the *positive* weights

$$(8) \quad \deg x = e_+, \quad \deg y = ke_- + d(\deg Q + \alpha), \quad \deg z = d, \quad \deg s = 1.$$

As before $V = \text{Spec } A$ is isomorphic to the normalization of the principal open subset $\mathbb{D}_+(z)$ of the hypersurface $\mathbb{V}_+(F)$ in the weighted projective space

$$\mathbb{P} = \mathbb{P}(e_+, ke_- + d(\deg Q + \alpha), d, 1).$$

In certain cases it is unnecessary in Theorem 2.3 to pass to normalization.

Corollary 2.5. *Assume that in (2) one of the following conditions is satisfied.*

- (i) $k = 1$;
- (ii) $e_+ + e_- = 0$, and D_0 is a reduced divisor.

Then $V = \text{Spec } A$ is equivariantly isomorphic to the principal open subset $\mathbb{D}_+(z)$ of the weighted projective hypersurface $\mathbb{V}_+(F)$ as in (3) in the weighted projective space \mathbb{P} from (5).

Proof. In case (i) the hypersurface in \mathbb{A}^3 with equation

$$F(x, y, 1, s) = xy - s^{e_+ + e_-} Q(s^d) = 0$$

is normal. In other words, the quotient $R/(z-1)$ of the graded ring $R = \mathbb{C}[x, y, z, s]/(F)$ is normal and so is its ring of invariants $(R/(z-1))^{E_d}$. Comparing with Theorem 2.3 the result follows.

Similarly, in case (ii)

$$F(x, y, 1, s) = x^k y - Q(s^d).$$

Since the divisor D_0 is supposed to be reduced and $D_0(0) = 0$, the polynomials $Q(t)$ and then also $Q(s^d)$ both have simple roots. Hence the hypersurface $F(x, y, 1, s) = 0$ in \mathbb{A}^3 is again normal, and the result follows as before. \square

Remark 2.6. The surface V as in 2.2 is smooth if and only if the divisor D_0 is reduced and $-m_+ m_- (D_+(0) + D_-(0)) = 1$, where $m_{\pm} > 0$ is the denominator in the irreducible representation of $D_{\pm}(0)$, see Proposition 4.15 in [FlZa₁]. It can happen, however, that V is smooth but the surface $\mathbb{V}_+(F) \cap \mathbb{D}_+(z) \subseteq \mathbb{P}$ has non-isolated singularities. For instance, if in 2.2 $D_0 = 0$ (and so $Q = 1$), then V is an affine toric surface⁵. In fact, every affine toric surface different from $(\mathbb{A}_*^1)^2$ or $\mathbb{A}^1 \times \mathbb{A}_*^1$ appears in this way, see Lemma 4.2(b) in [FKZ₁].

In this case the integer $k > 0$ can be chosen arbitrarily. For any $k > 1$, the affine hypersurface $V_+(F) \cap \mathbb{D}_+(z) \subseteq \mathbb{P}$ with equation $x^k y - s^{k(e_+ + e_-)} = 0$ has non-isolated singularities and hence is non-normal. Its normalization $V = \text{Spec } A$ can be given as the Zariski open part $\mathbb{D}_+(z)$ of the hypersurface $V_+(xy' - s^{e_+ + e_-})$ in $\mathbb{P}' = \mathbb{P}(e_+, e_-, d, 1)$ (which corresponds to the choice $k = 1$). Indeed, the element $y' = s^{e_+ + e_-} / x \in K$ with $y'^k = y$ is integral over A . However cf. Theorem 1.1(a).

Example 2.7. (*Danilov-Gizatullin surfaces*) We recall that a Danilov-Gizatullin surface $V(n)$ of index n is the complement to a section S in a Hirzebruch surface Σ_d , where $S^2 = n > d$. By a remarkable result of Danilov and Gizatullin up to an isomorphism such a surface only depends on n and neither on d nor on the choice of the section S , see e.g., [DaGi, CNR, FKZ₃] for a proof.

According to [FKZ₁, §5], up to conjugation $V(n)$ carries exactly $(n-1)$ different \mathbb{C}^* -actions. They admit DPD-presentations

$$(D_+, D_-) = \left(-\frac{1}{d}[0], -\frac{1}{n-d}[1] \right), \quad \text{where } d = 1, \dots, n-1.$$

Applying Theorem 2.3 with $e_+ = 1$, $e_- = 0$, and $k = n-d$, the \mathbb{C}^* -surface $V(n)$ is the normalization of the principal open subset $\mathbb{D}_+(z)$ of the hypersurface $\mathbb{V}_+(F_{n,d}) \subseteq \mathbb{P}(1, d, d, 1)$ of degree n , where

$$F_{n,d}(x, y, z, s) = x^{n-d} y - s^{n-d} (s^d - z).$$

Taking here $d = 1$ it follows that $V(n)$ is isomorphic to the normalization of the hypersurface $x^{n-1} y - (s-1)s^{n-1} = 0$ in \mathbb{A}^3 .

As our next example, let us consider yet another remarkable class of surfaces. These were studied from different viewpoints in [MM, Theorem 1.1], [FlZa₃, Theorem 1.1(iii)], [GMMR, 3.8-3.9], [KK, Theorem 1.1. and Example 1], [Za, Theorem 1(b) and Lemma

⁵See 3.1(a) below.

7]. Collecting results from *loc.cit.* and from this section, we obtain the following equivalent characterizations.

Theorem 2.8. *For a smooth affine surface V , the following conditions are equivalent.*

- (i) V is not Gizatullin and admits an effective \mathbb{C}^* -action and an \mathbb{A}^1 -fibration $V \rightarrow \mathbb{A}^1$ with exactly one degenerate fiber, which is irreducible⁶.
- (ii) V is \mathbb{Q} -acyclic, $\bar{k}(V) = -\infty$ ⁷ and V carries a curve $\Gamma \cong \mathbb{A}^1$ with $\bar{k}(V \setminus \Gamma) \geq 0$.
- (iii) V is \mathbb{Q} -acyclic and admits an effective \mathbb{C}^* - and \mathbb{C}_+ -actions. Furthermore, the \mathbb{C}^* -action possesses an orbit closure $\Gamma \cong \mathbb{A}^1$ with $\bar{k}(V \setminus \Gamma) \geq 0$.
- (iv) The universal cover $\tilde{V} \rightarrow V$ is isomorphic to a surface $x^k y - (s^d - 1) = 0$ in \mathbb{A}^3 , with the Galois group $\pi_1(V) \cong E_d$ acting via $\zeta.(x, y, s) = (\zeta x, \zeta^{-k} y, \zeta^e s)$, where $k > 1$ and $\gcd(e, d) = 1$.
- (v) V is isomorphic to the \mathbb{C}^* -surface with DPD presentation $\text{Spec } \mathbb{C}[t][D_+, D_-]$, where

$$(D_+, D_-) = \left(-\frac{e}{d}[0], \frac{e}{d}[0] - \frac{1}{k}[1] \right) \quad \text{with } 0 < e \leq d \quad \text{and } k > 1.$$

- (vi) V is isomorphic to the Zariski open subset

$$\mathbb{D}_+(x^k y - s^d) \subseteq \mathbb{P}(e, d - ke, 1), \quad \text{where } 0 < e \leq d \quad \text{and } k > 1.$$

Proof. In view of the references cited above it remains to show that the surfaces in (v) and (vi) are isomorphic. By Corollary 2.5(ii) with $e_+ = -e_- = e$, the surface V as in (v) is isomorphic to the principal open subset $\mathbb{D}_+(z)$ in the weighted projective hypersurface

$$V_+(x^k y - (s^d - z)) \subseteq \mathbb{P}(e, d - ke, d, 1).$$

Eliminating z from the equation $x^k y - (s^d - z) = 0$ yields (vi). \square

These surfaces admit as well a constructive description in terms of a blowup process starting from a Hirzebruch surface, see [GMMR, 3.8] and [KK, Example 1].

An affine line $\Gamma \cong \mathbb{A}^1$ on V as in (ii) is distinguished because it cannot be a fiber of any \mathbb{A}^1 -fibration of V . In fact, there exists a family of such affine lines on V , see [Za].

Some of the surfaces as in Theorem 2.8 can be properly embedded in \mathbb{A}^3 as *Bertin surfaces* $x^e y - x - s^d = 0$, see [FlZa₂, Example 5.5] or [Za, Example 1].

3. GIZATULLIN SURFACES WITH A FINITE DIVISOR CLASS GROUP

A *Gizatullin surface* is a normal affine surface completed by a zigzag i.e., a linear chain of smooth rational curves. By a theorem of Gizatullin [Gi] such surfaces are characterized by the property that they admit two \mathbb{C}_+ -actions with different general orbits.

In this section we give an alternative proof of the Daigle-Russell Theorem 1.1 cited in the Introduction. It will be deduced from the following result proven in [FKZ₂, Corollary 5.16].

Proposition 3.1. *Every normal Gizatullin surface with a finite divisor class group is isomorphic to one of the following surfaces.*

⁶Since V is not Gizatullin there is actually a unique \mathbb{A}^1 -fibration $V \rightarrow \mathbb{A}^1$. A surface V as in (i) is necessarily a \mathbb{Q} -homology plane (or \mathbb{Q} -acyclic) that is, all higher Betti numbers of V vanish.

⁷As usual, \bar{k} stands for the logarithmic Kodaira dimension.

(a) The toric surfaces $V_{d,e} = \mathbb{A}^2/E_d$, where the group $E_d \cong \mathbb{Z}_d$ of d -th roots of unity acts on \mathbb{A}^2 via

$$\zeta.(x, y) = (\zeta x, \zeta^e y).$$

(b) The non-toric \mathbb{C}^* -surfaces $V = \text{Spec } \mathbb{C}[t][D_+, D_-]$, where

$$(9) \quad (D_+, D_-) = \left(-\frac{e}{m}[p], \frac{e}{m}[p] - c[q] \right) \quad \text{with } c \geq 1, \quad p, q \in \mathbb{A}^1, \quad p \neq q,$$

and with coprime integers e, m such that $1 \leq e < m$.

Conversely, any normal affine \mathbb{C}^* -surface V as in (a) or (b) is a Gizatullin surface with a finite divisor class group.

Let us now deduce Theorem 1.1.

Proof of Theorem 1.1. To prove (a), we note that according to 2.1 the cyclic group E_d acts on the ring $\mathbb{C}[x, y, z]/(z-1) \cong \mathbb{C}[x, y]$ via $\zeta.x = \zeta x$, $\zeta.y = \zeta^e y$, and $\zeta.z = z$, where

$$\deg x = 1, \quad \deg y = e, \quad \text{and} \quad \deg z = d.$$

Hence $\mathbb{D}_+(z) = \text{Spec } \mathbb{C}[x, y]^{E_d} = V_{d,e}$, as required in (a).

To show (b) we consider $V = \text{Spec } A$ as in 3.1(b), where

$$A = \mathbb{C}[t][D_+, D_-] \subseteq \mathbb{C}(t)[u, u^{-1}].$$

By definition (1) the homogeneous pieces $A_{\pm 1}$ of A are generated as $\mathbb{C}[t]$ -modules by the elements

$$u_+ = tu \quad \text{and} \quad u_- = (t-1)^c u^{-1},$$

and similarly $A_{\pm m}$ by

$$v_+ = t^e u^m \quad \text{and} \quad v_- = t^{-e} (t-1)^{cm} u^{-m}.$$

Thus

$$u_+^m = t^{m-e} v_+, \quad u_-^m = t^e v_-, \quad \text{and} \quad u_+ u_- = t(t-1)^c.$$

The algebra A is the integral closure of the subalgebra generated by u_{\pm}, v_{\pm} and t .

Consider now the normalization A' of A in the field $L = \text{Frac}(A)[u'_+]$, where

$$(10) \quad u'_+ = \sqrt[d]{v_+} \quad \text{with} \quad d = cm.$$

Clearly the elements $\sqrt[m]{v_+} = t^{\frac{e-m}{m}} u_+$ and then also $t^{\frac{e-m}{m}}$ both belong to L . Since e and m are coprime we can choose $\alpha, \beta \in \mathbb{Z}$ with $\alpha(e-m) + \beta m = 1$. It follows that the element $\tau := t^{\frac{1}{m}} = t^{\alpha \frac{e-m}{m}} t^{\beta}$ is as well in L whence being integral over A we have $\tau \in A'$.

The element u'_+ as in (10) also belongs to A' and as well $u'_- = \sqrt[d]{v_-} \in A'$. Now $v_+ v_- = (t-1)^{cm}$, so taking d th roots we get for a suitable choice of the root u'_- ,

$$(11) \quad u'_+ u'_- = \tau^m - 1.$$

We note that u_{\pm}, v_{\pm} and t are contained in the subalgebra $B = \mathbb{C}[u'_+, u'_-, \tau] \subseteq A'$. The equation (11) defines a smooth surface in \mathbb{A}^3 . Hence B is normal and so

$$A' = B \cong \mathbb{C}[u'_+, u'_-, \tau]/(u'_+ u'_- - (\tau^m - 1)).$$

By Lemma 3.2 below, for a suitable $\gamma \in \mathbb{Z}$ the integers $a = e - \gamma m$ and d are coprime. We may assume as well that $1 \leq a < d$. We let E_d act on A' via $\zeta.u'_+ = \zeta^a u'_+$ and

$\zeta|A = \text{id}_A$. Since $\gcd(a, d) = 1$, A is the invariant ring of this action. We claim that the action of E_d on (u'_+, u'_-, τ) is given by

$$(12) \quad \zeta.u'_+ = \zeta^a u'_+, \quad \zeta.u'_- = \zeta^{-a} u'_- = \zeta^b u'_- \quad \text{and} \quad \zeta.\tau = \zeta^c \tau,$$

where $b = d - a$. Indeed, the equality $u'_+{}^c = t^{\frac{e-m}{m}} u_+ = \tau^{e-m} u_+$ implies that $\zeta.\tau^{e-m} = \zeta^{ac} \tau^{e-m}$. Since $\tau = \tau^{\alpha(e-m)} t^\beta$ the element $\zeta \in E_d$ acts on τ via $\zeta.\tau = \zeta^{\alpha a} \tau$. In view of the congruence $\alpha a \equiv 1 \pmod{m}$ the last expression equals $\zeta^c \tau$. Now the last equality in (12) follows. In the equation $u'_+ u'_- = \tau^m - 1$ the term on the right is invariant under E_d . Hence also the term on the left is. This provides the second equality in (12).

The algebra $B = \mathbb{C}[u'_+, u'_-, \tau]$ is naturally graded via

$$\deg u'_+ = a, \quad \deg u'_- = b, \quad \text{and} \quad \deg \tau = c.$$

According to Proposition 2.1 $\text{Spec } A = \text{Spec } A^{E_d}$ is the complement of the hypersurface $\mathbb{V}_+(f)$ of degree $d = a + b$ in the weighted projective plane

$$\text{Proj}(B) = \mathbb{P}(a, b, c), \quad \text{where} \quad f = u'_+ u'_- - \tau^m,$$

proving (b). □

To complete the proof we still have to show the following elementary lemma.

Lemma 3.2. *Assume that $e, m \in \mathbb{Z}$ are coprime. Then for every $c \geq 2$ there exists $\gamma \in \mathbb{Z}$ such that $\gamma m - e$ and c are coprime.*

Proof. Write $c = c'\gamma$ such that c' and m have no common factor and every prime factor of γ occurs in m . Then for every $\gamma \in \mathbb{Z}$ the integers $\gamma m - e$ and γ have no common prime factor. Indeed, such a prime must divide m and then also $e = \gamma m - (\gamma m - e)$. Hence it is enough to establish the existence of $\gamma \in \mathbb{Z}$ such that $\gamma m - e$ and c' are coprime. However, the latter is evident since the residue classes of γm , $\gamma \in \mathbb{Z}$, in $\mathbb{Z}_{c'}$ cover this group. □

Remark 3.3. 1. Two triples $(1, e, d)$ and $(1, e', d)$ as in Theorem 1.1(a) define the same affine toric surface if and only if $ee' \equiv 1 \pmod{d}$, see [FlZa₁, Remark 2.5].

2. As follows from Theorem 0.2 in [FKZ₂], the integers c, m in Theorem 1.1(b) are invariants of the isomorphism type of V . Indeed, the fractional parts of both divisors D_\pm as in (9) being nonzero and concentrated at the same point, there is a unique DPD presentation for V up to interchanging D_+ and D_- , passing to an equivalent pair and applying an automorphism of the affine line $\mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$.

Furthermore, from the proof of Theorem 1.1 one can easily derive that

$$a \equiv e \pmod{m} \quad \text{and} \quad b = mc - a \equiv -e \pmod{m}.$$

Therefore also the pair (a, b) is uniquely determined by the isomorphism type of V up to a transposition and up to replacing (a, b) by $(a', b') = (a - sm, b + sm)$, while keeping $\gcd(a', b') = 1$.

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