On the Integrability of the n–Centre Problem

Andreas Knauf* Iskander A. Taimanov†

Abstract

It is known that for $n \geq 3$ centres and positive energies the n-centre problem of celestial mechanics leads to a flow with a strange repellor and positive topological entropy.

Here we consider the energies above some threshold and show: Whereas for arbitrary $g > 1$ independent integrals of Gevrey class g exist, no real-analytic (that is, Gevrey class 1) independent integral exists.

1 Introduction

In [BT] the existence of a smoothly integrable geodesic flow on a compact manifold with positive topological entropy was established. Positivity of topological entropy is seen as an indication of complex dynamics, whereas integrability of a Hamiltonian flow is a metaphor for its simplicity. So coexistence of these two properties may not have been expected. In fact, as we show here, such a coexistence takes place in natural physical problems.

To be more specific, in this note we consider the n–centre problem of celestial mechanics in $d = 2$ and 3 dimensions. We denote by $\vec{s}_k \in \mathbb{R}^d$, $Z_k \in \mathbb{R} \setminus \{0\}$ the location resp. strength of the k–th centre, assuming $\vec{s}_k \neq \vec{s}_l$ for $k \neq l$. Then the Hamiltonian function

$$\hat{H} : T^*\hat{M} \to \mathbb{R} \quad , \quad \hat{H}(\vec{p}, \vec{q}) = \tfrac{1}{2}\vec{p}^{\,2} + V(\vec{q}),$$

*Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstr. $1\frac{1}{2}$, D–91054 Erlangen, Germany. e-mail: knauf@mi.uni-erlangen.de

†Institute of Mathematics, 630090 Novosibirsk, Russia. e-mail: taimanov@math.nsc.ru

with potential

$$V : \hat{M} \to \mathbb{R} \quad , \quad V(\vec{q}) = -\sum_{k=1}^{n} \frac{Z_k}{\|\vec{q} - \vec{s}_k\|}, \tag{1}$$

on the cotangent bundle $T^*\hat{M}$ of configuration space

$$\hat{M} := \mathbb{R}^d \setminus \{\vec{s}_1, \ldots, \vec{s}_n\}$$

generates a – in general incomplete – flow.

Denoting by $\hat{\omega} := \sum_{i=1}^{d} dq_i \wedge dp_i \restriction_{T^*\hat{M}}$ the restricted canonical symplectic form, there exists a unique smooth extension

$$(P, \omega, H) \quad \text{of the hamiltonian system} \quad (T^*\hat{M}, \hat{\omega}, \hat{H}) \tag{2}$$

such that the flow $\Phi : \mathbb{R} \times P \to P$ of H is complete (see [Kn], Thm. 5.1).

Concerning integrability of the flow, the following is known:

- For $n = 1$ this Hamiltonian system is integrable, with angular momentum

  $$\vec{L} : P \to \mathbb{R}^3 \quad , \quad \vec{L}(x) = \begin{cases} (\vec{q} - \vec{s}_1) \times \vec{p} & , \quad x = (\vec{p}, \vec{q}),\ \vec{q} \neq \vec{s}_1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

  for dimension $d = 3$ being a real analytic constant of motion. For $Z_1 > 0$ this is called the Kepler problem.

- For $n = 2$ centres one introduces elliptic prolate coordinates to analytically integrate the flow Φ, see e.g. [Ar] or [Th].

- For $n \geq 3$ centres and $d = 2$ Bolotin showed in [Bo] the nonexistence of an analytic integral of the motion which is non–constant on an energy shell $H^{-1}(E)$, $E > 0$, see also the discussion in Fomenko [Fo].

- For $d = 3$ and a collinear configuration of centres the angular momentum w.r.t. that axis is an additional constant of the motion, independent of the number n of centres. However, for $d = 3$ it was proved by the first author [Kn] for sufficiently large energies $E > E_{\text{th}}$ and by Bolotin and Negrini [BN, BN2] for nonnegative energies $E \geq 0$ that the topological entropy of the flow, restricted to the set b_E of bounded orbits on $H^{-1}(E)$ is positive (and $h_{\rm top}(E) = 0$ for $b_E = \emptyset$). Furthermore $h_{\rm top}(E)$ is zero for $n = 1$ and 2, and $h_{\rm top}(E) > 0$ if $n \geq 3$ and all centres being attracting or not more than two $\vec{s}_k$ being on a line (for collinear configurations with $Z_1, \ldots, Z_n < 0$ one has $h_{\rm top}(E) = 0$ for $E > 0$).

In the present paper it is proved that

- *for $d = 2$, attracting centres ($Z_k > 0$) and $E > 0$,*
- *for $d = 3$, arbitrary $Z_k \neq 0$, non-collinear configurations of centres and $E > E_{\rm th}$ where the threshold energy level $E_{\rm th}$ is determined by the data Z_k, $\vec{s}_k$, $k = 1, \ldots, n$,*

the n-centre problem restricted onto the energy level E admits $d-1$ independent integrals of motion which are smooth (and moreover are of the Gevrey class g for any $g > 1$) (see **Thm. 1***). Therefore the restricted problem is smoothly integrable.*

On the other hand if the affine span of the (non-collinear) centres equals $\mathbb{R}^3$ then the restricted problem is not integrable in the real-analytic sense, that is Gevrey class 1 (see **Thm. 2***).*

The article is based on the analysis of the n-centre problem given in the paper by the first author [Kn] where, in particular, it is shown that for high energies there are maps which relate scattering orbits to their asymptotics given by scattering orbits of the Kepler problem which is integrable (the necessary facts extracted from [Kn] are exposed in sections 2–3 of the present paper). Such a relation of the n-centre problem with an integrable problem leads to the smooth integrability of the n-centre problem which is proved here. Here we use a trick for constructing smooth first integrals from discontinuous preserved quantities similar to others used in [Bu, BT]

We would like to notice that in [Bo] the nonexistence of an additional analytic integral of motion for the two-dimensional problem was derived by Bolotin from Kozlov's theorem [Ko] which reads that the geodesic flow of (real)-analytic metric on a closed oriented surface of genus $g > 1$ does not admit an additional analytic first integral.

Although one can expect that the integrability of a problem by analytic integrals of motion implies vanishing of the topological entropy it is not proved until now. Therefore the results from [BN, Kn] do not imply the nonexistence of a complete family of analytic first integrals. We prove that by an analysis of the set formed by bounded orbits which supports the restricted flow with positive topological entropy.

2 Known Smoothness Results

There are three basic types of motion: bounded, scattering and trapped, corresponding to the disjoint subsets $b, s, t \subset P$ with

$$b^\pm := \{x \in P \mid \vec{q}\,(\pm\mathbb{R}^+, x) \text{ is bounded } \} \quad , \quad b := b^+ \cap b^-,$$
$$s^\pm := \{x \in P \mid x \notin b^\pm \text{ and } H(x) > 0\} \quad , \quad s := s^+ \cap s^-$$

and $t := s^+ \triangle s^-$. The orbits in s go to spatial infinity in both time directions, but because of the long range character of the effective potential of strength $Z_\infty := \sum_{k=1}^n Z_k$, one describes these limits by comparison with (regularized) Kepler flow $\Phi_\infty^t : P_\infty \to P_\infty$ generated by the extension of

$$\hat{H}_\infty : T^*(\mathbb{R}^d \setminus \{0\}) \to \mathbb{R} \quad , \quad \hat{H}_\infty(\vec{p}, \vec{q}) := \tfrac{1}{2}\vec{p}^{\,2} - \frac{Z_\infty}{\|\vec{q}\|} \tag{3}$$

Identifying the two phase spaces P and P_∞ outside a region projecting to a ball in configuration space which contains all singularities, the *Møller transformations*

$$\Omega^\pm : P_{\infty,+} := \{x \in P_\infty \mid H_\infty(x) > 0\} \to s^\pm \ , \ \Omega^\pm := \lim_{t\to\pm\infty} \Phi^{-t} \circ \mathrm{Id} \circ \Phi_\infty^t \tag{4}$$

exist as pointwise limits, and are measure-preserving diffeomorphisms (Thm. 6,3 and 6.5 of [Kn]). Similarly the asymptotic limits of the momentum

$$\vec{p}^{\,\pm} : s^\pm \to \mathbb{R}^d \quad , \quad \vec{p}^{\,\pm}(x_0) := \lim_{t\to\pm\infty} \vec{p} \circ \Phi^t(x_0) \tag{5}$$

are smooth. Finally we define *time delay* $\tau : s \to \mathbb{R}$ of a scattering state $x \in s$ by

$$\tau(x) := \lim_{R\to\infty} \int_{\mathbb{R}} \Big(\sigma(R) \circ \Phi^t(x) - \tag{6}$$
$$\tfrac{1}{2}(\sigma_\infty(R) \circ \Phi_\infty^t \circ \Omega_*^+(x) + \sigma_\infty(R) \circ \Phi_\infty^t \circ \Omega_*^-(x))\Big)\, dt,$$

where $\sigma(R) : P \to \{0,1\}$ and $\sigma_\infty(R) : P_\infty \to \{0,1\}$ are the characteristic functions $\sigma(R)(x) := \theta(R - \|\vec{q}(x)\|)$ and similarly for $\sigma_\infty(R)$. That asymptotic difference between the time spent by the orbit and its Kepler limits inside a ball of large radius diverges near $b \cup t$. However τ is smooth, as the Møller transformations are.

3 Analyticity Properties

Proposition 1 *For a potential V of the form (1) the following maps are real-analytic:*

1. *the flow $\Phi : \mathbb{R} \times P \to P$*
2. *the Møller transformations $\Omega^{\pm} : P_{\infty,+} \to s^{\pm}$ and asymptotic momenta $\vec{p}^{\pm} : s^{\pm} \to \mathbb{R}^d$*
3. *the time delay $\tau : s \to \mathbb{R}$.*

Proof.
1) We first indicate the definition of phase space P in order to show that for the potential (1) the smooth extension (2) actually works in the real-analytic category (in [Kn] more general non-analytic potentials V were considered). We assume $d = 3$, the case of $d = 2$ dimensions following by restriction.

For small $\varepsilon > 0$ in the phase space neighbourhood

$$\hat{U}_l^{\varepsilon} := \left\{ (\vec{p}, \vec{q}) \in T^*\hat{M} \;\middle|\; \|\vec{q} - \vec{s}_l\| < \varepsilon,\ |\vec{p}|^2 > \frac{3}{2} \frac{Z_l}{\|\vec{q} - \vec{s}_l\|} \right\}. \tag{7}$$

of the lth centre the following real-analytic coordinates are used:

- The restriction of the Hamiltonian function, which splits into

$$\hat{H}(\vec{p}, \vec{q}) = \hat{H}_l(\vec{p}, \vec{q}) + W_l(\vec{q}) \quad \text{with} \quad \hat{H}_l(\vec{p}, \vec{q}) := \tfrac{1}{2}\vec{p}^2 - \frac{Z_l}{\|\vec{q} - \vec{s}_l\|} \tag{8}$$

  and

$$W_l(\vec{q}) = \sum_{i \neq l} \frac{-Z_i}{\|\vec{q} - \vec{s}_i\|}.$$

- The angular momentum

$$\hat{L}_l : \hat{U}_l^{\varepsilon} \to \mathbb{R}^3 \quad , \quad \hat{L}_l(\vec{p}, \vec{q}) := (\vec{q} - \vec{s}_l) \times \vec{p}. \tag{9}$$

  relative to the position $\vec{s}_l$.

- The time $\hat{T}_l : \hat{U}_l^\varepsilon \to \mathbb{R}$ after which the Kepler orbit generated by $\hat{H}_l$ is in its pericentre w.r.t. $\vec{s}_l$

$$\hat{T}_l(\vec{p},\vec{q}) := \int_{r_{\min}(\vec{p},\vec{q})}^{\|\vec{q}-\vec{s}_l\|} \frac{r\,dr}{\sqrt{2r^2\hat{H}_l(\vec{p},\vec{q}) + 2Z_l r - \hat{L}_l^2(\vec{p},\vec{q})}} \cdot \operatorname{sign}((\vec{q}-\vec{s}_l)\cdot\vec{p}). \tag{10}$$

The neighbourhood $\hat{U}_l^\varepsilon \subset T^*\hat{M}$ is defined in a way which makes the pericentre unique. Its distance from $\vec{s}_l$ equals

$$r_{\min}(\vec{p},\vec{q}) := \begin{cases} \frac{-Z_l+\sqrt{Z_l^2+2\hat{H}_l(\vec{p},\vec{q})\hat{L}_l^2(\vec{p},\vec{q})}}{2\hat{H}_l(\vec{p},\vec{q})} & ,\hat{H}_l \neq 0 \\ \hat{L}_l^2(\vec{p},\vec{q})/2Z_l & ,\hat{H}_l = 0 \end{cases}. \tag{11}$$

- The Runge-Lenz vector relative to $\vec{s}_l$ is given by

$$\vec{F}_l : \hat{U}_l^\varepsilon \to \mathbb{R}^3 \quad , \quad \vec{F}_l(\vec{p},\vec{q}) := \vec{p} \times \hat{L}_l(\vec{p},\vec{q}) - Z_l \frac{\vec{q}-\vec{s}_l}{\|\vec{q}-\vec{s}_l\|}.$$

Since $|\vec{F}_l|^2 > Z_l^2/4 > 0$, the *pericentral direction*

$$\hat{F}_l : \hat{U}_l^\varepsilon \to S^2 \quad , \quad \hat{F}_l := \vec{F}_l/|\vec{F}_l| \tag{12}$$

is well-defined.

As $\hat{L}_l \cdot \hat{F}_l = 0$, we get a real-analytic diffeomorphism onto the image

$$\hat{\mathcal{Y}} : \hat{U}_l^\varepsilon \to T^*(\mathbb{R}\times S^2)\setminus \bar{0}, \quad (\vec{p},\vec{q}) \mapsto (\hat{T}_l, \hat{L}_l; \hat{H}_l, \hat{F}_l)$$

onto a punctured neighbourhood of the zero section $\bar{0}$ of the cotangent bundle. The flow generated by $\hat{H}_l$ is linearized in these coordinates, and extended on $U_l^\varepsilon := \hat{U}_l^\varepsilon \cup (\mathbb{R}\times S^2)$ by

$$\mathcal{Y} := (T_l, \mathbf{L}_l; H_l, \mathbf{F}_l) : U_l^\varepsilon \to T^*(\mathbb{R}\times S^2)\ ,\ \mathcal{Y}(x) := \begin{cases} (0,0;h,\mathbf{f}) & (h,\mathbf{f}) \in \mathbb{R}\times S^2 \\ \hat{\mathcal{Y}}(x) & \text{otherwise} \end{cases}$$

to a full neighbourhood of the zero section.

In a final step the hamiltonian flow $\Phi : \mathbb{R}\times P \to P$ generated the continuous extension $H : P \to \mathbb{R}$ of the Hamiltonian function $\hat{H} : T^*\hat{M} \to \mathbb{R}$ is linearized by slightly changing the coordinates $\mathcal{Y}$ using the (real-analytic but incomplete) flow generated by $\hat{H}$. See [Kn] for details.

In summary, we obtain a real-analytic extension (2) of the Hamiltonian system $(T^*\hat{M}\,, \hat{\omega}, \hat{H}\,)$. Thus the Hamiltonian vector field X_H (defined by $i_{X_H}\omega = dH$) is real-analytic, too, and thus the flow $\Phi : \mathbb{R} \times P \to P$ is known to be real-analytic (see, e.g. [Ho]), proving assertion 1.

2) This, however is insufficient to show real-analyticity of the Møller transformations and asymptotic momenta. It is known that even for smooth potentials these maps may be very non-smooth, see [Si].

There exists an energy-dependent *virial radius* $R_{\mathrm{vir}} > 0$, with

$$\frac{d}{dt}\langle \vec{q}, \vec{p}\rangle > \frac{E}{2} > 0 \qquad \text{if } \|\vec{q}\| \geq R_{\mathrm{vir}}(E) \text{ and } E := H(\vec{p}, \vec{q}). \tag{13}$$

If we assume $q_0 := \|\vec{q}_0\| \geq R_{\mathrm{vir}}(E)$ and $\langle \vec{q}_0, \vec{p}_0\rangle \geq 0$, then

$$\|\vec{q}(t)\| \geq q_0 \cdot \langle t\rangle_\lambda \quad \text{for all } t \geq 0, \text{ with } \quad \lambda := \sqrt{E/2}/q_0 \tag{14}$$

and

$$\langle t\rangle_\lambda := \sqrt{1 + (\lambda t)^2} \quad , \quad \langle t\rangle := \langle t\rangle_1 .$$

In particular a trajectory leaving the ball of radius $R_{\mathrm{vir}}(E)$ cannot reenter this ball in the future but must go to spatial infinity.

Analyticity estimates for the trajectory are then derived from the integral equation with initial conditions $x_0 = (\vec{p}_0, \vec{q}_0)$

$$\vec{q}(t, x_0) = \vec{q}_0 + t\vec{p}_0 - \int_0^t \int_0^s \nabla V(\vec{q}(\tau, x_0))\, d\tau\, ds,$$

using the decay property of the potential

$$\left|\partial^\beta V(\vec{q})\right| \leq |\beta|! \left(\frac{C}{\|\vec{q}\|}\right)^{|\beta|+1} \qquad (\beta \in \mathbb{N}_0^3)$$

outside the virial radius, and (14).

Setting

$$\|\vec{w}\|_\lambda := \sup_{t \geq 0} \frac{\|\vec{w}(t)\|}{\langle t\rangle_\lambda}$$

we obtain for $\gamma := (\alpha, \beta)$, $\partial_{x_0}^\gamma := \partial_{p_0}^\alpha \partial_{q_0}^\beta$

$$\|\partial_{x_0}^\gamma \vec{q}(\cdot, x_0)\|_\lambda \leq \alpha!\, q_0^{-|\beta| + \delta_{|\gamma|,1}} E^{-\frac{1}{2}|\alpha| - 1 + \delta_{|\gamma|,1}}. \tag{15}$$

This allows us to perform the (locally uniform w.r.t. x_0) time limit in

$$\partial_{x_0}^{\gamma}(\vec{p}(t,x_0)-\vec{p}_0) = - \tag{16}$$

$$\sum_{N=1}^{g} \sum_{\substack{\gamma^{(1)}+\ldots+\gamma^{(N)}=\gamma \\ |\gamma^{(i)}|>0}} \int_0^t D^N \nabla V(\vec{q}(\tau,x_0)) \left(\partial_{x_0}^{\gamma^{(1)}}\vec{q}(\tau,x_0),\ldots,\partial_{x_0}^{\gamma^{(N)}}\vec{q}(\tau,x_0)\right) d\tau.$$

with $g := |\gamma| \geq 1$, and to conclude that $\vec{p}^+$ is real-analytic at x_0. We can substitute the assumption $x_0 \in s^+$ for the stronger assumptions $q_0 := \|q_0\| \geq R_{\mathrm{vir}}(E)$, $\langle \vec{q}_0, \vec{p}_0 \rangle \geq 0$, as initial conditions meeting the first lead to data meeting the second one after some t. The same holds for $\vec{p}^-$ by reversibility ($\vec{p}^-(\vec{p}_0,\vec{q}_0) = -\vec{p}^+(-\vec{p}_0,\vec{q}_0)$).

The proof of real-analyticity of the Møller transforms is based on the integral equation

$$\partial_{x_0}^{\gamma}\vec{r}(t) = \int_t^\infty \int_s^\infty \partial_{x_0}^{\gamma} \nabla \left(\frac{-Z_\infty}{\left\|\vec{Q}(\tau;x_0)\right\|} - V(\vec{q}(\tau,x_0)) \right) d\tau\, ds$$

for $\vec{r}(t;x_0) := \vec{q}(t,x_0) - \vec{Q}(t;x_0)$ with Kepler hyperbola $\left(\vec{P}(t;x_0),\vec{Q}(t;x_0)\right) = \Phi_\infty^t(X_0)$. Inspecting the proof of smoothness for the Møller transforms in [Kn], one sees that the estimates for $\partial_{x_0}^{\gamma}\vec{r}$ can be dominated by $|\gamma|! C^{|\gamma|}$.

3) For a scattering state $x \in s$ time delay equals

$$\begin{aligned} \tau(x) &= \tfrac{1}{2}\lim_{R\to\infty} \int_{\mathbb{R}} \sigma^+ \circ \Phi^t(x) \cdot \left(\sigma_\infty(R)\circ\Omega_*^+ - \sigma_\infty(R)\circ\Omega_*^-\right)(\Phi^t(x))dt \\ &+ \tfrac{1}{2}\lim_{R\to\infty} \int_{\mathbb{R}} \sigma^- \circ \Phi^t(x) \cdot \left(\sigma_\infty(R)\circ\Omega_*^- - \sigma_\infty(R)\circ\Omega_*^+\right)(\Phi^t(x))dt, \end{aligned}$$

with $\sigma^\pm(\vec{p},\vec{q}) := \theta(\pm\vec{q}\cdot\vec{p})$, see [KK] and $\Omega_*^\pm := (\Omega^\pm)^{-1}$. Using the intertwining property $\Omega_*^\pm \circ \Phi^t = \Phi_\infty^t \circ \Omega_*^\pm$ this can be written entirely in terms of Møller transformations and the Kepler flow:

$$\begin{aligned} \tau(x) = &\tfrac{1}{2}\lim_{R\to\infty} \int_{\mathbb{R}} \sigma^+ \circ \Phi_\infty^t(x) \cdot \left(\sigma_\infty(R)\circ\Phi_\infty^t\circ\Omega_*^+(x) - \sigma_\infty(R)\circ\Phi_\infty^t\circ\Omega_*^-(x)\right) dt \\ &- \tfrac{1}{2}\lim_{R\to\infty} \int_{\mathbb{R}} \sigma^- \circ \Phi_\infty^t(x) \cdot \left(\sigma_\infty(R)\circ\Phi_\infty^t\circ\Omega_*^+(x) - \sigma_\infty(R)\circ\Phi_\infty^t\circ\Omega_*^-(x)\right) dt. \end{aligned}$$

With Assertion 2 and the analog of formula (10) this implies analyticity of τ. □

4 Gevrey Integrals of Motion

We now show the existence of independent constants of motion for all energies $E > E_{\rm th}$. In [Kn] many estimates are shown to hold true above a threshold energy $E_{\rm th} \geq 0$:

In [Kn] many estimates are shown to hold true above a threshold energy $E_{\rm th} \geq 0$:

- For $d = 2$ and attracting centres ($Z_k > 0$) $E_{\rm th} = 0$.
- For $d = 3$, arbitrary $Z_k \neq 0$ and non-collinear configurations of the $\vec{s}_k \in \mathbb{R}^3$ the existence of such a threshold is proven.

In particular the set b_E of bounded orbits of energy $E > E_{\rm th}$ is shown to be of measure zero (and has a Cantor set structure for $n \geq 3$).

According to the standard definition (see, e.g. [AM], Def. 5.2.20) functions $f_0, \ldots, f_k : \tilde{P} \to \mathbb{R}$ on a symplectic manifold $(\tilde{P}, \tilde{\omega})$ are called *independent* if the set of singular points of $F := f_0 \times \ldots \times f_k : \tilde{P} \to \mathbb{R}^k$ has measure zero.

To simplify discussion, we set

$$\tilde{P} := H^{-1}([E_1, E_2])$$

for an arbitrary energy interval, $E_{\rm th} \leq E_1 \leq E_2 < \infty$, and $f_0^g \equiv f_0 := H|_{\tilde{P}}$. Then for a parameter $g > 1$ we define for $k = 1, 2$

$$f_k^g : \tilde{P} \to \mathbb{R} \quad , \quad f_k^g(x) := \begin{cases} p_k^+(x) \exp\left(-e^{C(g)\langle\tau(x)\rangle}\right) & , \quad x \in s \\ 0 & , \quad x \notin s, \end{cases} \tag{17}$$

where $C(g) = \frac{C_2}{g-1}$ with C_2 to be defined below. Putting things together we get a map $F^g : \tilde{P} \to \mathbb{R}^3$.

The notation is chosen so that F^g belongs to the Gevrey class of index g.

Now we collect some information about Gevrey functions, which were introduced in [Ge].

Definition 1 For $g \geq 1$ and an open set $\Omega \subset \mathbb{R}^n$ a function $f \in C^\infty(\Omega)$ is called of *Gevrey class* g if for every compact $K \subset \Omega$ there exist $A_K, C_K > 0$ with

$$\max_{x \in K} |\partial^\alpha f(x)| \leq A_K \, C_K^{|\alpha|} \, (\alpha!)^g \qquad (\alpha \in \mathbb{N}_0^n).$$

Here $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$ if $\alpha = (\alpha_1, \ldots, \alpha_n)$.
The vector space of these functions is denoted by $G_g(\Omega)$.

Then $G_1(\Omega)$ is the space of real-analytic functions, and $G_{g'}(\Omega) \supset G_g(\Omega)$ for $g' > g$. $G_g(\Omega)$ is stable w.r.t. partial derivatives, compositions, and the implicit function theorem holds within the class. For $g > 1$ Gevrey partitions of unity exist.

Example 1 For $g > 1$

$$f^g(x) := \begin{cases} \exp(-x^{-1/(g-1)}) & , \quad x > 0 \\ 0 & , \quad x \leq 0, \end{cases}$$

is a function in $G_g(\mathbb{R})$ but not in $G_{g'}(\mathbb{R})$ for any $g' < g$, see e.g. [Ju].

For a real-analytic manifold we can define the spaces of Gevrey functions, as the defining family of bounds is preserved by coordinate changes.

Theorem 1 *For all $g > 1$ the functions f_0^g, f_1^g, f_2^g are independent, in involution and of Gevrey class g.*

Proof.
In Thm. 12.8 of [Kn] it was shown that the set b_E of bounded orbits of energy E is of Liouville measure zero for all $E > E_{\mathrm{th}}$. As the set $t_E := \{x \in H^{-1}(E) \mid x \in T\}$ of trapped orbits is always of Liouville measure zero, on $\tilde{P}$ the set $\tilde{P} \setminus s$ is of measure zero. On $\tilde{P} \cap s$ the functions f_1^g and f_2^g are real-analytic, using Proposition 1, whereas they are zero on $\tilde{P} \setminus s$.

We study their decay near $\tilde{P} \setminus s$ in order to prove that they are in $G^g(\tilde{P})$.

All orbits in $\tilde{P} \setminus s$ enter the interaction zone, so we need only orbits in $\tilde{P} \cap s$ entering the interaction zone. W.l.o.g. we assume R_{vir} to be constant on $[E_1, E_2]$. Then the region

$$\tilde{I} := \{x \in \tilde{P} \mid \|q(x)\| \leq R_{\mathrm{vir}}\}$$

projecting to the interaction zone is compact. The restriction of the real-analytic flow $\Phi : \mathbb{R} \times P \to \mathbb{R}$ to a domain of the form $[-\varepsilon, \varepsilon] \times \tilde{I}$ thus has partial derivatives $|\partial^\alpha \Phi(t, x)| \leq \tilde{C}_1 \tilde{C}_2^{|\alpha|} \alpha!$. We conclude, using Corollary 1 that for *arbitrary* $t \in \mathbb{R}$ such that $\Phi(t, x) \in \tilde{I}$, too,

$$|\partial^\alpha \Phi(t, x)| \leq C_1 \exp(C_2 |\alpha| \ \langle t \rangle) \ \alpha!. \tag{18}$$

Next we analyse time delay (6) for orbits entering $\tilde{I}$. By compactness of $\tilde{I}$ there exists a $\tilde{R} \geq R_{\mathrm{vir}}$ such that all Kepler orbits $\{\Phi_\infty^t \circ \Omega_*^\pm(x) \mid t \in \mathbb{R}\}$

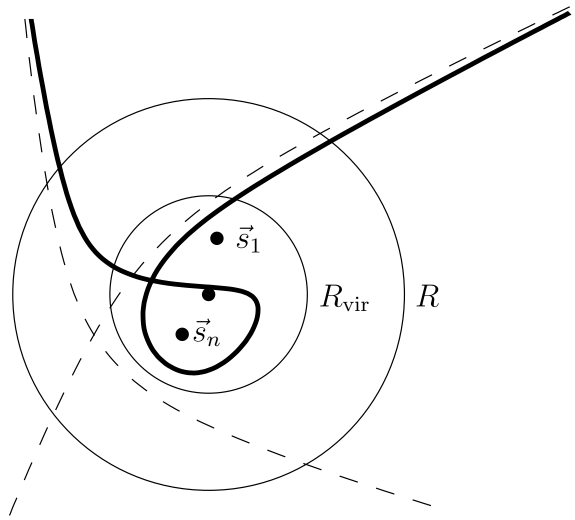

Figure 1: Scattering trajectory in configuration space (solid line), with associated Kepler hyperbolae (broken lines)

through points $x \in \tilde{I}$ enter a configuration space region of radius $\tilde{R}$. In (6) we only consider radii $R \geq \tilde{R}$ and, denoting by $\tau_R(x)$ the integral (6), we split this quantity in

$$\tau_R(x) = \tau_R^+(x) + \tau_R^-(x) + \tau^0(x).$$

Here

$$\begin{aligned} \tau_R^{\pm}(x) \quad &:= \quad \int_{\mathbb{R}} \big(\sigma^+ \circ \Phi_t(x) \cdot \sigma(R)(1 - \sigma(R_{\mathrm{vir}})) \circ \Phi^t(x) \\ &\qquad -\tfrac{1}{2}\sigma_\infty(R) \circ \Phi_\infty^t \circ \Omega_*^{\pm}(x)\big)\, dt \end{aligned} \tag{19}$$

and

$$\tau^0(x) := \int_{\mathbb{R}} \sigma(R_{\mathrm{vir}}) \circ \Phi^t(x)\, dt. \tag{20}$$

Whereas the limits $\tau^{\pm}(x) := \lim_{R\to\infty} \tau_R^{\pm}(x)$ of (19) meet Gevrey class 1 estimates uniform on $\tilde{I} \cap s$, $\tau^0(x)$ defined in (20), the time spent by the orbit in the interaction zone, is not uniformly bounded on $\tilde{I} \cap s$ for $n \geq 2$ centres, see Figure 1.

We need only consider scattering states in a slightly smaller domain $\tilde{I}_\varepsilon := \{x \in \tilde{P} \mid \|q(x)\| \leq R_{\mathrm{vir}} - \varepsilon\}$. By (13) the configuration space trajectories with initial conditions in $\tilde{I}_\varepsilon \cap s$ intersect the boundary of the interaction zone with angles uniformly bounded away from zero. Therefore (18) shows that for $x \in \tilde{I}_\varepsilon \cap s$

$$|\partial^\alpha \tau^0(x)| \leq C_1 \exp(C_2|\alpha|\,\langle \tau(x)\rangle)\ \alpha!,$$

and $\tau = \tau^+ + \tau^- + \tau^0$ satisfies the same kind of estimate (with enlarged constants).

A similar kind of reasoning applies to the asymptotic momenta $\vec{p}^{\pm}$:

$$|\partial^\alpha p_k^+(x)| \leq C_1 \exp(C_2|\alpha|\,\langle \tau(x)\rangle)\ \alpha!$$

for $x \in \tilde{I}_\varepsilon \cap s$.

¿From this one can conclude that f_1^g and f_2^g are of Gevrey class g for $C(g)$ large enough, see Proposition 3.

The functions $\vec{p}^{\pm}$ and τ are Φ_t–invariant. Therefore f_1^g and f_2^g are Φ_t–invariant, too. As $f_0^g = H \setminus \tilde{P}$ generates the flow Φ_t on $\tilde{P}$, the Poisson brackets $\{f_0^g, f_1^g\}$ and $\{f_0^g, f_2^g\}$ vanish.

By definition $\vec{p}^{\pm}(x) = \lim_{t\to\pm\infty} p \circ \Phi_t(x)$, and the decay estimates (16) imply that

$$\left\{\lim_{t\to\infty} p_1 \circ \Phi_t(x), \lim_{t\to\infty} p_2 \circ \Phi_t(x)\right\} = \lim_{t\to\infty}\{p_1, p_2\} \circ \Phi_t(x) = 0.$$

Thus the Poisson bracket $\{f_1^g, f_2^g\}$ vanishes, too.

To see that the functions f_k^g are independent, it suffices to show that the (analytic) restriction

$$F^g|_{\tilde{P}\cap s}$$

has a measure zero set of singular points. As the energy range $[E_1, E_2]$ under consideration consists of regular values of H, we may consider instead the independence of f_1^g and f_2^g, restricted to $H^{-1}(E) \cap s$, for $E \in [E_1, E_2]$.

Independence of f_1^g and f_2^g, restricted to an energy surface $H^{-1}(E)$ with $E \in [E_1, E_2]$ follows from their real-analyticity on $H^{-1}(E) \cap s$ and the independence of p_1^+ and p_2^+, remarking that for $\langle \vec{q}, \vec{p}\rangle \to \infty$, $\vec{p}^+((\vec{p}, \vec{q})) \to \vec{p}$. $\square$

Proposition 2 *Consider real-analytic maps*

$$f : \Omega_f \to \mathbb{R}^d \quad , \quad g : \Omega_g \to \Omega_f$$

with open sets $\Omega_f, \Omega_d \subset \mathbb{R}^d$*, meeting*

$$|\partial^\alpha f_i| \leq M_f\, C_f^{|\alpha|}\, \alpha! \quad , \quad |\partial^\alpha g_i| \leq M_g\, C_g^{|\alpha|}\, \alpha! \qquad (i = 1\,,\,\ldots, d, \alpha \in \mathbb{N}_0^d).$$

Then

$$|\partial^\alpha (f\circ g)_i| \le M_f\, C_{fg}^{|\alpha|}\,\alpha! \quad \textit{with} \quad C_{fg} := C_g(1+C_f d M_g)d.$$

Proof. The chain rule has the form

$$\partial^\alpha f\circ g = \sum_{k=1}^{n} \sum_{A_1,\ldots,A_k} D^k f(\partial^{\alpha_{A_1}} g,\ldots,\partial^{\alpha_{A_k}} g)$$

with $n=|\alpha|$, $(A_1,\ldots,A_k)$ running through the k–partitions of $\{1,\ldots,n\}$ and $\alpha_{A_l}\in\mathbb{N}_0^d$ is the multiindex of size $|\alpha_{A_l}|=|A_l|$ corresponding to $A_l\subset\{1,\ldots,n\}$ in a, say lexicographic, ordering of the n partial derivatives. Therefore with $\lambda := C_f d M_g$

$$\begin{aligned}
|\partial^\alpha (f\circ g)| &\le \sum_{k=1}^{n} M_f C_f^k d^k k! \sum_{A_1,\ldots,A_k}\prod_{l=1}^{k} M_g C_g^{|A_l|}\alpha_{A_l}! \\
&= M_f C_g^n \sum_{k=1}^{n} (C_f d M_g)^k k! \sum_{A_1,\ldots,A_k}\prod_{l=1}^{k} \alpha_{A_l}! \\
&\le M_f C_g^n \sum_{k=1}^{n} \lambda^k k! \sum_{A_1,\ldots,A_k}\prod_{l=1}^{k} |A_l|! \\
&= M_f C_g^n n! \sum_{k=1}^{n} \lambda^k \tbinom{n-1}{k-1} \\
&= M_f C_g^n \lambda(1+\lambda)^{n-1} n!,
\end{aligned}$$

since $k!\sum_{A_1,\ldots,A_k}\prod_{l=1}^{k}|A_l|! = n!\binom{n-1}{k-1}$. As in d dimensions

$$|\alpha|! \le d^{|\alpha|}\,\alpha! \qquad (\alpha\in\mathbb{N}_0^d),$$

the estimate follows. □

This estimate is iterated in order to estimate the flow for long times:

Corollary 1 *Assume that for* $k=1,\ldots,t$ *the maps*

$$f^{(k)} : \Omega^{(k)} \to f^{(k)}\big(\Omega^{(k)}\big) \subset \Omega^{(k+1)} \quad \textit{on} \quad \Omega^{(k)}\subset\mathbb{R}^d$$

admit the uniform estimates $\left|\partial^\alpha f_i^{(k)}\right| \le M\tilde{C}^{|\alpha|}\,\alpha!$ *with* $M\ge 1$.

Then their iterates $T^{(k)} := f^{(k)} \circ T^{(k-1)}$ *with* $T^{(0)} := \mathrm{Id}_{\Omega_1}$ *are estimated by*

$$\left|\partial^\alpha T_i^{(k)}\right| \leq M \exp(C\,|\alpha|\,k)\,\alpha! \qquad (\alpha \in \mathbb{N}_0^d, k = 1, \ldots, d) \tag{21}$$

with $C := \ln(d) + Md\tilde{C}$.

Proof. (21) holds for $k = 1$, and is assumed to hold for some $k < t$. Then by Proposition 2

$$\begin{aligned} |\partial^\alpha T_i^{(k+1)}| &\leq M(\exp(Ck)(1 + Md\tilde{C})d)^{|\alpha|}\,\alpha! \\ &\leq M \exp(C(k+1))^{|\alpha|}\,\alpha!, \end{aligned}$$

proving (21). $\square$

Proposition 3 *For the choice* $C(g) := \frac{C_2}{g-1}$ *in (17) the functions* f_k^g *are of Gevrey class* g.

Proof. As $|\partial^\alpha \langle\tau(x)\rangle| \leq C_1 \exp(C_2|\alpha| \langle\tau(x)\rangle)\,\alpha!$,

$$\begin{aligned} |\partial^\alpha \exp(C(g)\langle\tau(x)\rangle)| &\leq \exp\big(C(g)\langle\tau(x)\rangle\big) \sum_{k=1}^{|\alpha|} C(g)^k \sum_{A_1,\ldots,A_k} \prod_{l=1}^{k} \partial^{\alpha_{A_l}} \langle\tau\rangle \\ &\leq \exp\big((C(g) + C_2 n)\langle\tau(x)\rangle\big) \left(\sum_{k=1}^{n} C(g)^k C_1^k \tbinom{n-1}{k-1}\right) \alpha! \\ &= \exp\big((C(g) + C_2 n)\langle\tau(x)\rangle\big)(1 + C(g)C_1)^n\,\alpha! \end{aligned}$$

with $n := |\alpha|$.

The $\langle\tau\rangle$ dependent part of $|\partial^\alpha f_k^g|$ is of the form

$$\exp\left(-e^{C(g)\langle\tau\rangle}\right) e^{C_2 n\langle\tau\rangle} \leq x^x,$$

as it has its maximum for $\langle\tau\rangle$ with

$$x := e^{C(g)\langle\tau\rangle} = \tfrac{C_2 n}{C(g)}.$$

Thus the choice $C(g) := \frac{C_2}{g-1}$ leads to the proof. $\square$

5 Nonexistence of Analytic Integrals of Motion

We start with the following simple observation for a hamiltonian system (P, ω, H):

Proposition 4 *Let $\gamma \subset \Sigma_E := H^{-1}(E)$ be a periodic orbit of the Hamiltonian flow Φ_t generated by the Hamiltonian function H which is isolated on the energy surface Σ_E. Then there is no additional integral of motion which is functionally independent of H on γ.*

Proof. Let us assume that there is an additional integral of motion $J : P \to \mathbb{R}$. By localizing J around γ if necessary, the Hamiltonian flow Ψ_s generated by J exists for all times $s \in \mathbb{R}$. Since these integrals are in involution, their flows commute:

$$\Phi_t \Psi_s = \Psi_s \Phi_t \qquad (t, s \in \mathbb{R}).$$

Let T be the period of γ and $x \in \gamma$. We have

$$\Phi_T \Psi_s(x) = \Psi_s \Phi_T(x) = \Psi_s(x) \qquad (s \in \mathbb{R}).$$

This implies that Ψ_s maps (within a given energy surface Σ_E) periodic orbits of the flow Φ_t into periodic orbits of this flow. But $\gamma \subset \Sigma_E$ is an isolated periodic orbit and therefore $\Psi_s(x) \in \gamma$ for all $s \in \mathbb{R}$. This implies that H and J are functionally dependent on γ. $\square$

Remark 1 Note that *single* isolated periodic orbits need *not* form an obstruction to the existence of an additional analytic integral of motion $J : P \to \mathbb{R}$, independent of H in the sense defined in Sect. 4. If the periodic orbit γ is hyperbolic, then J must be constant on its stable and unstable manifold, but still on a neighbourhood of $\gamma \subset \Sigma_E$ the singular set of $J|_{\Sigma_E}$ may only consist of γ which is of measure zero.

Easy examples for this are given by motion in a smooth potential $V : \mathbb{R}^2 \to \mathbb{R}$ which are rotationally symmetric. Then angular momentum is an independent analytic integral of the motion, but circular hyperbolic orbits may exist for some energies.

In the present context the energy surface Σ_E ($E > E_{\text{th}}$) for the two–centre problem contains exactly one bounded orbit γ. γ is periodic and collides with the two centres. As commented in the Introduction, the two–centre problem is analytically integrable.

Let us return to the n-centre problem. By Thm. 12.8 from [Kn] the set of bounded orbits b_E on the energy level $E > E_{\rm th}$ consists in hyperbolic trajectories, and for $n \geq 3$ is locally homeomorphic to a product of a Cantor set and the interval. Moreover its Liouville measure vanishes. We now show that b_E *can* form an obstruction to the existence of independent analytic integrals:

Theorem 2 *If the affine span of the (non-collinear) centres $\vec{s}_1, \ldots, \vec{s}_n$ equals $\mathbb{R}^3$, then for $E > E' \geq E_{\rm th}$ the n–centre problem does not admit a pair of independent analytic integrals $I_1, I_2 : \Sigma_E \to \mathbb{R}$ of motion on the energy surface $\Sigma_E := H^{-1}(E)$.*

Proof. Assume that $I_1, I_2 : \Sigma_E \to \mathbb{R}$ are analytic integrals of motion. Then on the five–dimensional manifold Σ_E we consider the *singular set*

$$S := \{x \in \Sigma_E \mid \mathrm{rank}(DI(x)) < 2\}$$

with $I := (I_1, I_2)$.

By Prop. 4 the periodic orbits within b_E belong to the (closed) set S (as in Prop. 4 we consider functions $J : P \to \mathbb{R}$, we use appropriate smooth extensions J of $I_k : \Sigma_E \to \mathbb{R}$).

In [Kn] it was shown that for energies $E > E_{\rm th}$ the periodic orbits are dense within b_E. Thus $b_E \subset S$, too.

According to [vdD] (see also [Ta] where a sketch for more general subanalytic sets appears) S admits an analytic simplicial decomposition which is locally finite and whose simplices are semianalytic. As $b_E \subset S$ is compact, there exist (disjoint) simplices $\Delta_1, \ldots, \Delta_m$ with

$$b_E \subset \bigcup_{i=1}^{m} \Delta_i. \tag{22}$$

Now we assume I_1, I_2 to be indendent integrals, contrary to the statement of the theorem. Then $K := \max_i \dim(\Delta_i) < \dim(\Sigma_E) = 5$. As the set S is Φ_t–invariant, we consider the transversal intersections

$$\tilde{\Delta}_i := \Delta_i \cap \mathcal{H}_E \tag{23}$$

of the simplices with the Poincaré surface $\mathcal{H}_E \subset \Sigma_E$ (defined in Sect. 10 of [Kn]). We have $\tilde{K} := \max_i \dim(\tilde{\Delta}_i) = K - 1 < \dim(\mathcal{H}_E) = 4$. The intersection $\Lambda_E := b_E \cap \mathcal{H}_E$ of the bounded orbits with the Poincaré surface has the form

$$\Lambda_E = \Lambda_E^+ \cap \Lambda_E^-,$$

$\Lambda_E^\pm \subset \mathcal{H}_E$ being the (un–) stable manifolds, consisting of two–dimensional leaves which intersect transversally.

It is known from Hasselblatt [Ha] that the Hölder regularity of (the distributions of) $\Lambda_E^\pm$ can be controlled by the so-called bunching constant. For the case of the n-centre problem Prop. 11.2 of [Kn] controls the expansion resp. contraction rates of the Poincaré map on $\Lambda_E^\pm$, which differ from a constant times E by $\mathcal{O}(E^0)$. Thus applying [Ha] we have $C^{2-\varepsilon}$ regularity of $\Lambda_E^\pm$ for all large energies $E > E_{\mathrm{th}}$.

Now as $\tilde{K} < 4$, for each $\tilde{\Delta}_i$ in (23) at least one of the intersections

$$\Delta_i^\pm := \tilde{\Delta}_i \cap \Lambda_E^\pm$$

must be of dimension ≤ 1. By reversibility of the flow we assume w.l.o.g. that $\dim(\Delta_1^-) \leq 1$ and derive a contradiction.

As Λ_E is a Cantor set (see Thm. 12.8 of [Kn]), we assume w.l.o.g. that Δ_1^- contains a sequence $(x^{(i)})_{i\in\mathbb{N}}$ of points $x^{(i)} \in \Lambda_E$ converging to $x \in \Delta_1^-$. By going to a subsequence, if necessary, we assume that the unit vectors $v^{(i)} \in T_x\mathcal{H}_E$ with $\exp_x(t^{(i)}v^{(i)}) = x^{(i)}$ have a limit $v := \lim_{i\to\infty} v^{(i)}$ (which is in fact independent of the choice of Riemannian metric used to define the exponential map).

In [Kn] starting from the alphabet $\mathcal{S} := \{1, \ldots, n\}$ the space of symbol sequences

$$\mathbf{X} := \left\{\underline{k} \in \mathcal{S}^{\mathbb{Z}} \mid \forall\, i \in \mathbb{Z} : k_{i+1} \neq k_i\right\}$$

is equipped with the metric

$$d(\underline{k}, \underline{l}) := \sum_{i\in\mathbb{Z}} 2^{-|i|} \cdot (1 - \delta_{k_i,l_i}), \qquad (\underline{k}, \underline{l} \in \mathbf{X}).$$

Then denoting the *shift* by

$$\sigma : \mathbf{X} \to \mathbf{X} \quad , \quad \sigma(\underline{k})_i := k_{i+1} \quad (i \in \mathbb{Z}),$$

there exists a Hölder homeomorphism

$$\mathcal{F}_E : \mathbf{X} \to \Lambda_E \tag{24}$$

conjugating σ and the restriction of the Poincaré map to Λ_E, see Lemma 12.2 of [Kn].

The Poincaré surface is the disjoint union

$$\mathcal{H}_E = \bigcup_{\substack{k,l=1\\k\neq l}}^{n} \mathcal{H}_E^{k,l},$$

the $\mathcal{H}_E^{k,l}$ being open regions in the intersection of a five-dimensional affine space and Σ_E. Starting with

$$V_E(k_0,k_1) := W_E(k_0,k_1) := \mathcal{H}_E^{k_0,k_1},$$

in [Kn] for $(k_{-m},\ldots,k_0)$ admissible (that is $k_l \neq k_{l+1}$) the nested subsets

$$W_E(k_{-m},\ldots,k_0) := W_E(k_{-1},k_0) \cap \mathcal{P}_E(W_E(k_{-m},\ldots,k_{-1})), \qquad (25)$$

resp. for $(k_0,\ldots,k_m)$ admissible

$$V_E(k_0,\ldots,k_m) := V_E(k_0,k_1) \cap \mathcal{P}_E^{-1}(V_E(k_1,\ldots,k_m)), \qquad (26)$$

were defined, using the Poincaré map $\mathcal{P}_E$.

By going to subsequences we can assume that the points $x^{(i)} \in \Lambda_E^-$ correspond to symbol sequences $\underline{k}^{(i)} \in \mathbf{X}$ which are related to the symbol sequence $\underline{k} := \mathcal{F}_E^{-1}(x) \in \mathbf{X}$ of x by

$$k_j^{(i)} = k_j \qquad (i \in \mathbb{N}; j > \chi(i))$$

but $k_{\chi(i)}^{(i)} \neq k_{\chi(i)}$, where $\chi(i) \to -\infty$.

By construction the vector v is tangent to the one-dimensional manifold Δ_1^- at x.

We now show the existence of a second sequence $(y^{(i)})_{i\in\mathbb{N}}$ of points $y^{(i)} \in \Lambda_E$ converging to x, but with the following property: There exists an $\alpha \in (0,1)$ such that writing the points in the form

$$y^{(i)} = \exp_x(s^{(i)} w^{(i)})$$

with units vectors $w^{(i)} \in T_x\mathcal{H}_E$,

$$d(w^{(i)}, \mathrm{span}(v)) \geq |s^{(i)}|^{\alpha+1}. \qquad (27)$$

Namely as the number n of centres is ≥ 4 (which follows from our assumption on the positions $\vec{s}_1, \ldots, \vec{s}_n$) we find symbol sequences $\underline{l}^{(i)} \in \mathbf{X}$ with

$$l_j^{(i)} = k_j \qquad (i \in \mathbb{N}; j < \chi(i))$$

but $l_{\chi(i)}^{(i)} \neq k_{\chi(i)}$, and

$$\text{affine span}\left(\vec{s}_{k_{\chi(i)-1}}, \vec{s}_{k_{\chi(i)}}, \vec{s}_{k^{(i)}_{\chi(i)}}, \vec{s}_{l^{(i)}_{\chi(i)}}\right) = \mathbb{R}^3. \tag{28}$$

Then the $y^{(i)} := \mathcal{F}_E(\underline{l}^{(i)})$ converge to

$$\lim_{i\to\infty} y^{(i)} = \mathcal{F}_E\left(\lim_{i\to\infty} \underline{l}^{(i)}\right) = \mathcal{F}_E(\underline{k}) = x.$$

Next we consider the geometric situation at the (early) time $\chi(i)$. More precisely we set

$$\underline{\tilde{k}} := \sigma^{\chi(i)}(\underline{k}) \quad , \quad \underline{\tilde{k}}^{(i)} := \sigma^{\chi(i)}(\underline{k}^{(i)}) \quad \text{and} \quad \underline{\tilde{l}}^{(i)} := \sigma^{\chi(i)}(\underline{l}^{(i)}).$$

Thus

$$(\tilde{k}_1, \tilde{k}_2) = (\tilde{k}_1^{(i)}, \tilde{k}_2^{(i)}) = (\tilde{l}_1^{(i)}, \tilde{l}_2^{(i)}) = \left(k_{\chi(i)+1}, k_{\chi(i)+2}\right)$$

but $\tilde{k}_0 \neq \tilde{k}_0^{(i)} \neq \tilde{l}_0^{(i)} \neq \tilde{k}_0$.

By the conjugacy property of (24) the identities $\tilde{x} := \mathcal{F}_E(\underline{\tilde{k}}) = \mathcal{P}_E^{\chi(i)}(x)$, $\tilde{x}^{(i)} := \mathcal{F}_E\left(\underline{\tilde{k}}^{(i)}\right) = \mathcal{P}_E^{\chi(i)}(x^{(i)})$ and $\tilde{y} := \mathcal{F}_E\left(\underline{\tilde{l}}^{(i)}\right) = \mathcal{P}_E^{\chi(i)}\left(y^{(i)}\right)$ are true.

All of these points are contained in the local stable manifold

$$V_E(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \ldots) \subset V_E(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) \subset \mathcal{H}_E^{\tilde{k}_1, \tilde{k}_2},$$

but at the same time in the following disjoint sets

$$\tilde{x} \in W_E(\tilde{k}_0, \tilde{k}_1, \tilde{k}_2) \quad , \quad \tilde{x}^{(i)} \in W_E(\tilde{k}_0^{(i)}, \tilde{k}_1, \tilde{k}_2) \quad , \quad \tilde{y}^{(i)} \in W_E(\tilde{l}_0^{(i)}, \tilde{k}_1, \tilde{k}_2).$$

In $\mathcal{H}_E^{\tilde{k}_1, \tilde{k}_2} \subset \mathcal{H}_E$ the minimal angle between vectors from (points in)

$$W_E(\tilde{k}_0, \tilde{k}_1, \tilde{k}_2) \cap V_E(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) \quad \text{to} \quad W_E(\tilde{k}_0^{(i)}, \tilde{k}_1, \tilde{k}_2) \cap V_E(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$$

and vectors from

$$W_E(\tilde{k}_0, \tilde{k}_1, \tilde{k}_2) \cap V_E(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) \quad \text{to} \quad W_E(\tilde{l}_0^{(i)}, \tilde{k}_1, \tilde{k}_2) \cap V_E(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$$

is bounded away from zero by some $\Delta > 0$ (see (26) and (25)). This follows from (28). As there are only finitely many (at most n, to be more precise) choices for each of the indices $\tilde{k}_0, \tilde{k}_1, \tilde{k}_2, \tilde{k}_0^{(i)}$ and $\tilde{l}_0^{(i)}$, this bound is uniform.

So the angle between the unit vectors $\tilde{v}^{(i)}$ and $\tilde{w}^{(i)}$ defined by

$$\tilde{x}^{(i)} = \exp_{\tilde{x}}(\tilde{s}^{(i)}\tilde{v}^{(i)}) \quad , \quad \tilde{y}^{(i)} = \exp_{\tilde{x}}(\tilde{t}^{(i)}\tilde{w}^{(i)})$$

is bounded below by Δ. But by Prop. 11.2 of [Kn] the contraction rates of these vectors w.r.t. the Poincaré $\mathcal{P}_E$ differ at most by the order $\mathcal{O}(1/E)$.

By estimate (11.2) of [Kn] we conclude that after iterating $-\chi(i) > 0$ times these vectors still have an angle bounded away from zero by

$$\Delta\left(1 - \frac{C}{E}\right)^{|\chi(i)|}$$

whereas their length is reduced at least by a factor $(cE)^{|\chi(i)|}$.

Thus by enlarging the energy E, we can choose an arbitrarily small $\alpha > 0$ in (27).

But (27) is incompatible with $y^{(i)}$ belonging to the one–dimensional submanifold Δ_1^-.

As we can apply the same argument to all $x \in \Lambda_E$, we have derived a contradiction to the finite covering assumption (22). $\square$

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