On reduction of Hilbert-Blumenthal varieties

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§1. Introduction

Let F be a totally real number field of degree g and O_F be its ring of integers. A Hilbert-Blumenthal variety parameterizes the isomorphism classes of O_F-abelian varieties of dimension g with a certain condition (and with certain additional structure). The purpose of the condition is to exclude some bad points in characteristic p such that the integral model becomes flat. In [R], Rapoport used the condition that the Lie algebra of the O_F-abelian scheme over a base scheme S is a locally free $O_F \otimes S$-module. We will call it the Rapoport condition and the Rapoport locus for the defined moduli space. The condition was modified later by Deligne and Pappas [DP] in order to confirm the properness of the compactification constructed in [R] in the case of bad reduction. We will call the modified condition the Deligne-Pappas condition, and the Deligne-Pappas space for corresponding moduli space, see (2.4) and (5.1) for the precise definitions. Over $\mathbb{Z}_{(p)}$ the Deligne-Pappas condition is equivalent to that the O_F-abelian scheme admits a prime-to-p polarization.

This paper is based on the important work of Rapoport [R], and that of Deligne-Pappas [DP]. It serves several purposes. All of them go to understand the geometry of reduction of Hilbert-Blumenthal varieties. In this introduction we assume that there is only one prime v of O_F lying over p. Let $\mathcal{O} := O_F \otimes \mathbb{Z}_p$, $k(v)$ be the residue field of O_F at v, and $\pi \in O_F$ be a uniformizer of $\mathcal{O}$. Let k be an algebraically closed field of characteristic p.

One purpose of this paper is to study various conditions and invariants of O_F-abelian varieties using Dieudonné modules. Three invariants–the Lie types (see (2.3)), the slopes (Newton polygons), and a-numbers (see (2.10))–are discussed and three conditions–the Rapoport condition, the Deligne-Pappas condition, and the Kottwitz determinant condition–are also discussed. In [DP], Deligne and Pappas have defined a stratification on the reduction $\mathcal{M}_v^{\mathrm{DP}}$ of the Deligne-Pappas space by the Lie types. In this paper we define a stratification on the reduction $\mathcal{M}$ of the Rapoport locus by the a-numbers. We did not discuss the Ekedahl-Oort stratification as it was the content of the work by F. Andreatta and E. Goren. The readers interested in the Ekedahl-Oort stratification are referred to their work.

Along this direction we obtain the following results

1. The Deligne-Pappas condition is equivalent to the Kottwitz determinant condition for O_F-abelian varieties of characteristic p.

2. The Kottwitz determinant condition is an open and close condition in the moduli space of polarized O_F-abelian varieties of characteristic p.

3. The Rapoport locus is the smooth locus of the Deligne-Pappas space.

4. The set slope(A) of slopes of an O_F-abelian variety A is either $\{\frac{i}{g},\dots,\frac{i}{g},\frac{g-i}{g},\dots,\frac{g-i}{g}\}$ for some $0\le i\le g/2$, or $\{\frac{1}{2},\dots,\frac{1}{2}\}$.

5. Every isogeny class of polarized O_F-abelian varieties over k contains one which satisfies the Rapoport condition.

The next problem is whether these invariants arising from the Dieudonné modules in question can be realized by abelian varieties. We solve this problem by establishing the following result (Theorem 7.5)

Theorem *Any quasi-polarized $\mathcal{O}$-p-divisible group (H,λ,ι) over k is isomorphic to the p-divisible attached to a polarized O_F-abelian variety.*

This is the integral analogue of the Manin's problem. The result implies that the strata defined by these invariants are all not empty. As an application we construct a polarized O_F-abelian surface of characteristic p in the complement of the Rapoport locus which is specialized by a point of characteristic 0. To the author's knowledge, this is the first example in the literature that shows invalidity of the valuation criterion for the Rapoport locus at good reduction, even though the statement has been known, due to Deligne and Pappas [DP, Introduction], for almost ten years.

The proof of the algebraization theorem goes as follows. Using the weak Grothendieck conjecture, we algebrise the p-divisible groups in the Rapoport locus. Then we algebrise the rest of p-divisible groups by classifying the formal isogeny classes explicitly. The classification uses a result of Rapoport-Zink [RZ] and of Rapoport-Richartz [RR].

Another purpose of this paper is to generalize the main results of E. Goren and F. Oort [GO] on the a-number stratification, the slope stratification and their relation, removing the assumption of p. To describe our results, we introduce some notations. Let $S(g)$ denote the set $\{i\in\mathbb{Z}; 0\le i\le \frac{g}{2}\}\cup\{\frac{g}{2}\}$ with the order induced from $\mathbb{Q}$. For each $i\in S(g)$, we denote by $s(i)$ the set of slopes $\{\frac{i}{g},\dots,\frac{i}{g},\frac{g-i}{g},\dots,\frac{g-i}{g}\}$. Let $\lceil m\rceil$ denote the smallest integer not less than the real number m. For each O_F-abelian variety A over k, the reduced (usual) a-number $t(A)$ of A is defined to be the dimension of the k-vector space $\mathrm{Hom}(\alpha_p, A[\pi])$. In this direction we establish the following results

1. For each a-number $\underline{a}$, we define a closed subscheme $\mathcal{M}_{\ge\underline{a}}$ of $\mathcal{M}$. It has the property that $x\in\mathcal{M}_{\ge\underline{a}}(k)$ if and only if the a-number of $x\ge\underline{a}$. We show (Theorem 5.4) that the closed subscheme $\mathcal{M}_{\ge\underline{a}}$ is smooth over $\mathrm{Spec}k(v)$ of pure dimension $g-|\,\underline{a}\,|$.

2. An a-number $\underline{a}=(a^i)$ is called *spaced* if $a^ia^{i+1}=0,\ \forall\, i\in\mathbb{Z}/f\mathbb{Z}$ (cf. [GO, p.112]). If $\underline{a}$ is spaced, then the points in $\mathcal{M}_{\ge\underline{a}}$ of slope $s(|\underline{a}|)$ are dense in $\mathcal{M}_{\ge\underline{a}}$ (Theorem 6.8).

3. The generic point of each irreducible component of $\mathcal{M}_{\geq \underline{a}}$ has slope $\geq s(\lambda(\underline{a}))$, where $\lambda(\underline{a}) := \max\{|\underline{b}|; \underline{b} \subset \underline{a}, \underline{b}$ is spaced$\}$ (cf. [GO, p.112]) (Corollary 6.9).

4. Let $x : \operatorname{Spec} k \to \mathcal{M}$ be a geometric point of reduced a-number one. Then for each $m \in S(g), s(m) \leq \operatorname{slope}(x)$, the reduced closed subscheme $\mathcal{M}_x^{\geq s(m)}$ of slope $\geq s(m)$ in $\mathcal{M}_x$ is formally smooth of codimension $\lceil m \rceil$ and its generic point has slope $s(m)$ (Theorem 6.12).

5. The weak Grothendieck conjecture (see [GO. Sec. 5.1]) for $\mathcal{M}$ holds (Corollary 6.15).

6. The strong Grothendieck conjecture (see [GO. Sec. 5.1]) for $\mathcal{M}_v^{\mathrm{DP}}$ holds when p is totally ramified in F (Theorem 6.19).

The methods are based on the previous works of F. Oort [O2], E. Goren and F. Oort [GO], and the author [Y]. Using the explicit deformation method developed by Norman [N], Norman-Oort [NO], T. Zink [Z2], we construct the universal deformation of the Dieudonné module in the Rapoport locus. Then we study systematically the a-number stratification and the slope stratification on the formal neighborhood around the point. The results above are extracted from the systematic iteration of Frobenius map. It is possible to extract finer information beyond the reduced a-number one from the result of our computation. We leave this possible generalization in the future when the finer information becomes useful. We follow the approach of [O2] and [Y], which is different from that of Goren and Oort [GO]. Therefore, we give a new proof of several results of theirs: Thm. 2.3.4, Cor. 2.3.11–13, Thm. 5.2.1, Thm. 5.4.4, and the key cases of Thm. 5.4.11 of [GO]. Andreatta and Goren earlier obtained similar results in the case when p is totally ramified. Our work is independent from their results.

In [C], C.-L. Chai studied the combinatoric properties of Newton polygons for connected reductive quasi-split algebraic groups, inspired by earlier works of R. Kottwitz [K1], K.-Z. Li and F. Oort [LO], M. Rapoport and M. Richartz [RR]. He gave a group-theoretic conjectural description of the dimensions of Newton strata of good reduction of Shimura varieties. The main motivation of this work is to examine whether his group-theoretic description of Newton strata for quasi-split groups applies for the simplest case of bad reduction. The results of this work support his description even when the reductive group G in question is no longer unramified. One may expect that Chai's description applies for the bad reduction case as well, under the assumption of the existence of good integral models of Shimura varieties for certain special subgroups $K_p \subset G(\mathbb{Q}_p)$. We confirm (Corollary 6.20) a question of Chai [C, Question 7.6, p.984] on the dimensions of Newton strata in the Hilbert-Blumenthal cases when p is totally ramified.

The results of this work reveal an important feature: the stratifications on the smooth locus of bad reduction of Hilbert-Blumenthal varieties behave very similarly as those on the good reduction. The reader can compare the results 1–6 above and those of [GO] for the good reduction case. A fundamental question is whether this feature holds for more general PEL-type Shimura varieties. We have no idea but expect that the cases of type C still hold.

The following is the structure of this paper. In Section 2 we describe the structure of Dieudonné module of general polarized O_F-abelian varieties of dimension g. In Section 3 we classify the formal isogeny classes explicitly. In Section 4 We provide the normal forms of the Dieudonné modules in the Rapoport locus. In Section 5 we give the natural generalization the a-number stratification on the Rapoport locus and study its properties. In Section 6 we study systematically the a-number stratification and the slope stratification on formal neighborhoods in the Rapoport locus, by the methods of [N], [NO], [Z], and [O2] as explained before. In Section 7 we establish the theorem of algebraization concerning the p-divisible groups in question. In Section 8 we construct the desired example as explained before. In the last section we do a computation of the Hecke correspondence. Using this, we describes the singularities of the supersingular locus near the superspecial point constructed in Section 8.

The problems addressed in this paper are all local, hence we use the convenient language of algebraic stacks. The reader is free to replace the word "algebraic stack(s)" by "scheme(s)" in this paper by adding an auxiliary level structure.

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§2. Dieudonné modules

(2.1) Let F be a totally real field of degree g and O_F be its ring of integers. To simplify notation, we will assume that *there is only one prime of O_F lying over p in Section 2–6.* Let e be the ramification index and f be the residue degree of this prime v, thus $g = ef$. Denote by F_v the completion of F at v and $\mathcal{O}$ the ring of integers in F_v. Let $\mathcal{O}^{\mathrm{ur}}$ denote the ring of integers in the maximal étale extension of $\mathbb{Z}_p$ in $\mathcal{O}$. The ring $\mathcal{O}^{\mathrm{ur}}$ is isomorphic to $W(\mathbb{F}_{p^f})$. Let π be a uniformizer of $\mathcal{O}$, chosen from O_F. Let $P(T)$ be the monic irreducible polynomial of π over $\mathcal{O}^{\mathrm{ur}}$.

(2.2) Let (A, λ, ι) be a polarized O_F-abelian variety of dimension g over a perfect field k containing $\mathbb{F}_{p^f}$, and let M be its covariant Dieudonné module. The Dieudonné module M is a free $\mathcal{O} \otimes_{\mathbb{Z}_p} W(k)$-module of rank two equipped with a non-degenerate alternating pairing

$$M \times M \to W(k)$$

such that $\langle ax, y\rangle = \langle x, ay\rangle$ for all $x, y \in M$ and $a \in \mathcal{O}$ and on which the Frobenius F and Verschiebung V commute with the action of $\mathcal{O}$. We call it briefly a *quasi-polarized*

$\mathcal{O}$*-Dieudonné module*. The ring $\mathcal{O} \otimes_{\mathbb{Z}_p} W(k)$ is isomorphic to

$$\bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} W(k)[T]/(\sigma_i(P(T)),$$

where $\sigma_i, i \in \mathbb{Z}/f\mathbb{Z}$, are embeddings of $\mathcal{O}^{\mathrm{ur}}$ into $W(k)$, arranged in a way that $\sigma\sigma_i = \sigma_{i+1}$. Set $W^i := W(k)[T]/(\sigma_i(P(T))$ and denote again by π the image of T in W^i. The action of the Frobenius map σ on $\mathcal{O} \otimes_{\mathbb{Z}_p} W(k)$ through the right factor gives a map $\sigma : W^i \to W^{i+1}$ which sends $a \mapsto a^\sigma$ for $a \in W(k)$ and $\sigma(\pi) = \pi$. We also have $W^i \otimes_W k = k[\pi]/(\pi^e)$ and $\mathcal{O} \otimes_{\mathbb{Z}_p} k = \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} k[\pi]/(\pi^e)$. Let

$$M^i := \{x \in M \mid ax = \sigma_i(a)x, \ \forall\ a \in \mathcal{O}^{\mathrm{ur}}\}$$

be the σ_i-component of M, which is a free W^i-module of rank two. We have the decomposition

$$M = M^0 \oplus M^1 \oplus \cdots \oplus M^{f-1}$$

in which $F : M^i \to M^{i+1}$, $V : M^{i+1} \to M^i$. The summands M^i, M^j are orthogonal with respect to the pair $\langle\, ,\rangle$ for $i \neq j$. Conversely, a Dieudonné module together with such a decomposition and these properties is a quasi-polarized $\mathcal{O}$-Dieudonné module.

(2.3) The Lie algebra $\mathrm{Lie}(A)$ of A also admits a decomposition

$$\mathrm{Lie}(A) = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \mathrm{Lie}(A)^i$$

and $\mathrm{Lie}(A)^i = M^i/VM^{i+1}$ is a $k[\pi]/(\pi^e)$-module generated by at most two elements. Let

$$\mathrm{Lie}(A)^i = k[\pi]/(\pi^e) \cdot x_1^i \oplus k[\pi]/(\pi^e) \cdot x_2^i \simeq k[\pi]/(\pi^{e_1^i}) \oplus k[\pi]/(\pi^{e_2^i}).$$

Hence we obtain an invariant $(\{e_1^i, e_2^i\})_i$, which we call the Lie type of A and denote it by $\mathrm{lie}(A)$.

(2.4) We say an O_F-abelian variety satisfies *the Deligne-Pappas condition* if it admits a separable O_F-linear polarization. It is easy to see that an $\mathcal{O}$-Dieudonné module can be separably quasi-polarized if and only if $e_1^i + e_2^i = e$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. The latter is equivalent to the condition that the k-dimensions of $\mathrm{Lie}(A)^i$ are all the same. This is equivalent to the Kottwitz determinant condition [K2, Sec. 5]. It follows that the Deligne-Pappas condition implies the Kottwitz determinant condition. We will see (3.10) that any O_F-abelian variety admits an O_F-linear polarization λ. The Kottwitz determinant condition then implies that the polarization λ is a power of π times a separable polarization. Therefore, these two conditions are equivalent.

(2.5) The invariant $\mathrm{lie}(A) = (\{e_1^i, e_2^i\})_i$ has the property that $e_j^i \le e, i \in \mathbb{Z}/f\mathbb{Z}, j = 1, 2$, $\sum_i e_1^i + e_2^i = g$, and the invariant $e_1^i + e_2^i$ is a locally constant function in a family. The latter follows from the fact that the σ_i-component $\mathrm{Lie}(A)^i$ of $\mathrm{Lie}(A)$ is a locally free sheaf. One can see from this that the Kottwitz determinant condition is an open condition for the reduction of Hilbert-Blumenthal varieties.

In [DP], Deligne and Pappas showed that the stratum of each Lie type $(\{e_1^i, e_2^i\})_i$, in the Deligne-Pappas space, is a regular locally closed subset, and has dimension $g - 2\sum_i \min\{e_1^i, e_2^i\}$ provided the stratum is non-empty. However, non-emptiness follows directly from the algebraization theorem (7.5).

(2.6) Let $\mathrm{Def}[A, \iota]$ denote the deformation functor of the O_F-abelian variety (A, ι) over k. It follows from crystalline theory that $\mathrm{Def}[A, \iota](k[\epsilon]) = \mathrm{Hom}_{k\otimes O_F}(VM/pM, M/VM)$. From $\dim_k \mathrm{Hom}_{k[\pi]/\pi^e}(k[\pi]/(\pi^a), k[\pi]/(\pi^b)) = \min\{a, b\}$, we compute that

$$\dim_k \mathrm{Def}[A, \iota](k[\epsilon]) = \sum_{i\in\mathbb{Z}/f\mathbb{Z}} \sum_{1\le j,k,\le 2} \min\{e_j^i, e - e_k^i\}. \tag{1}$$

If (A, ι) satisfies the Deligne-Pappas condition, then

$$\dim_k \mathrm{Def}[A, \iota](k[\epsilon]) = ef + 2 \sum_{i\in\mathbb{Z}/f\mathbb{Z}} \min\{e_1^i, e_2^i\}. \tag{2}$$

(2.7) Assume that (A, λ, ι) is a *separably* polarized O_F-abelian variety, that is, it lies in the Deligne-Pappas space. Let $\mathrm{Def}[A, \lambda, \iota]$ denote the deformation functor of (A, λ, ι). We can choose a $k[\pi]/(\pi^e)$-basis $\{x_1^i, x_2^i\}$ of $(M/pM)^i$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that $(VM/pM)^i$ is generated by $\{\pi^{e_1^i} x_1^i, \pi^{e_2^i} x_2^i\}$ and $\langle \pi^{e-1} x_1^i, x_2^i\rangle = 1$, $\forall\, i \in \mathbb{Z}/f\mathbb{Z}$. The first order universal deformation (over the deformation ring R) of the O_F-abelian variety (A, ι) is given by the following data:

$$\widetilde{Fil} \subset H_1^{\mathrm{cris}}(A/R),\ \widetilde{Fil} = \oplus \widetilde{Fil}^i, \quad \widetilde{Fil}^i =< X_1^i, X_2^i >_{R[\pi]/(\pi^e)},$$

where

$$X_1^i = \pi^{e_1^i} x_1^i + \sum_{j=0}^{e_1^i-1} a_j^i \pi^j x_1^i + \sum_{j=0}^{e_2^i-1} b_j^i \pi^j x_2^i, \quad X_2^i = \pi^{e_2^i} x_2^i + \sum_{j=0}^{e_1^i-1} c_j^i \pi^j x_1^i + \sum_{j=e_2^i-e_1^i}^{e_2^i-1} d_j^i \pi^j x_2^i.$$

Here we assume that $e_1^i \le e_2^i$ for simplicity. From $\langle X_1^i, X_2^i\rangle = 0$, the vector space $\mathrm{Def}[A, \lambda, \iota](k[\epsilon])$ is cut down by the equations $d_{e_2^i-1}^i + a_{e_1^i-1}^i - b_{e_1^i-1}^i = 0$, which is empty if $\min\{e_1^i, e_2^i\} = 0$. Hence we conclude that

$$\dim_k \mathrm{Def}[A, \lambda, \iota](k[\epsilon]) = ef + \sum_{i\in\mathbb{Z}/f\mathbb{Z}} \left(2\min\{e_1^i, e_2^i\} - \min\{e_1^i, e_2^i, 1\}\right). \tag{3}$$

This shows that $\dim_k \mathrm{Def}[A, \lambda, \iota](k[\epsilon]) = g$ if and only if (A, λ, ι) satisfies the Rapoport condition.

(2.8) Proposition *The Rapoport locus is the smooth locus in the Deligne-Pappas space.*

PROOF. It follows from the dimension statement of [DP, Thm. 2.2] and (3). ∎

(2.9) Remark It is well known that the forgetful morphism $\mathrm{Def}[A, \lambda, \iota] \to \mathrm{Def}[A, \iota]$ is formally étale if (A, λ, ι) satisfies the Rapoport condition, and that the Rapoport locus is smooth. In [DP, Thm 2.2], Deligne and Pappas concluded that the singular locus had codimension two. However, they actually showed that the complement of the Rapoport locus had codimension two. Proposition 2.8 fills the harmless gap of their assertion on the dimension of the singular locus. From (2) and (3), one can see that the forgetful morphism $\mathrm{Def}[A, \lambda, \iota] \to \mathrm{Def}[A, \iota]$ is not formally étale any more when (A, λ, ι) does not satisfy the Rapoport condition.

(2.10) Let $\mathbf{a}(A)$ denote the *a-module* of A, which is defined to be the cokernel of the Frobenius map F on $\mathrm{Lie}(A) = M/VM$:

$$M/VM \xrightarrow{F} M/VM \to \mathbf{a}(A) \to 0.$$

Note that in the covariant theory, the Frobenius map F is induced from the Verschiebung morphism $V : A^{(p)} \to A$ via the covariant functor.

Let each σ_i-component of $\mathbf{a}(A)$ be

$$\mathbf{a}(A)^i \simeq k[\pi]/(\pi^{a_1^i}) \oplus k[\pi]/(\pi^{a_2^i})$$

for some $\{a_1^i, a_2^i\}$. We define the *a-number* $a(A)$ *of* A to be the invariant $a(A) = (\{a_1^i, a_2^i\})_i$. Let $|a(A)|$ denote the usual a-number of A, which is the dimension of the k-vector space $\mathrm{Hom}_k(\alpha_p, A^t)$. If the O_F-abelian variety (A, ι) satisfies the Rapoport condition, we write $a(A) = (a^i)_i$ for $a(A) = (\{0, a^i\})_i$.

(2.11) Lemma *The a-module* $\mathbf{a}(A)$ *is canonically isomorphic to the k-linear dual* $\mathrm{Hom}_k(\alpha_p, A^t)^*$ *of* $\mathrm{Hom}_k(\alpha_p, A^t)$ *as* $k \otimes_{\mathbb{Z}} O_F$*-modules.*

PROOF. Let M^* denote the *contravariant* Dieudonné functor. Then we have

$$\begin{aligned}\mathrm{Hom}_k(\alpha_p, A^t) &= \mathrm{Hom}_{W[F,V]}(M^*(A^t), k) \simeq \mathrm{Hom}_{W[F,V]}(M(A), k) \\ &= \mathrm{Hom}_k(M(A)/(F, V)M(A), k) = \mathbf{a}(A)^*. \ \blacksquare\end{aligned}$$

(2.12) Note that A is ordinary if and only if $|a(A)| = 0$. It is easy to see that the latter condition implies that $\{e_1^i, e_2^i\} = \{0, e\}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. Therefore, ordinary points satisfy the Rapoport condition automatically. As non-ordinary points do not specialize to the boundary of Deligne-Pappas space, the complement of the Rapoport locus is in the interior of the space. Therefore, the compactification constructed by Rapoport provides that of the Deligne-Pappas space [DP, Introduction].

(2.13) Let $a(A) = (\{a_1^i, a_2^i\})_i, a_1^i \leq a_2^i$, be the a-number of A and $\mathrm{lie}(A) = (\{e_1^i, e_2^i\})_i$ be the Lie type. It follows from elementary divisors that there are two W^i-bases $\{x_1^i, x_2^i\}, \{y_1^i, y_2^i\}$ of M^i such that

$$FM^{i-1} =< \pi^{e-e_1^{i-1}} x_1^i, \pi^{e-e_2^{i-1}} x_2^i >, \text{ and } VM^{i+1} =< \pi^{e_1^i} y_1^i, \pi^{e_2^i} y_2^i > .$$

If $e_1^i = \min\{e_1^i, e_2^i, e - e_1^{i-1}, e - e_2^{i-1}\}$, write $y_1^i = ax_1^i + bx_2^i, y_2^i = cx_1^i + dx_2^i$, then

$$FM^{i-1} + VM^{i+1} =< \pi^{e_1^i} x_1^i, \pi^{e_2^i} x_2^i, \pi^{e-e_1^{i-1}} bx_2^i, \pi^{\ e^{i-1} i_2} dx_2^i > .$$

We obtain a bound for a-numbers: $a_1^i = e_1^i$ and $\min\{e_2^i, e - e_1^{i-1}, e - e_2^{i-1}\} \leq a_2^i \leq e_2^i$ when $e_1^i = \min\{e_1^i, e_2^i, e - e_1^{i-1}, e - e_2^{i-1}\}$.

(2.14) Lemma *Let* $a(A) = (\{a_1^i, a_2^i\})_i$ *and* $\mathrm{lie}(A) = (\{e_1^i, e_2^i\})_i$. *Then* $\mathrm{lie}(A^t) = (\{e - e_1^i, e - e_2^i\})_i$ *and the a-number* $(\{b_1^i, b_2^i\})_i$ *of the dual abelian variety* A^t *is given as follows:* $b_1^i = \min\{e - e_1^i, e - e_2^i, e_1^{i-1}, e_2^{i-1}\}$ *and* $b_1^i + b_2^i = (a_1^i + a_2^i) + (e_1^{i-1} + e_2^{i-1}) - (e_1^i + e_2^i)$.

PROOF. Write $\overline{M} = M/pM$ and $\overline{M^t} = M^t/pM^t$, where M^t is the Dieudonné module of A^t. From the definition of $\mathrm{lie}(A) = (\{e_1^i, e_2^i\})_i$, one concludes that $V\overline{M^{i+1}}$ is isomorphic to $k[\pi]/(\pi^{e-e_1^i}) \oplus k[\pi]/(\pi^{e-e_2^i})$. The perfect pairing $\overline{M} \times \overline{M^t} \to k$ induces a perfect pairing

$$V\overline{M^{i+1}} \times \overline{M^{t,i}}/V\overline{M^{t,i+1}} \to k,$$

thus $\mathrm{lie}(A^t) = (\{e - e_1^i, e - e_2^i\})_i$.

Let $W^i := F\overline{M^{i-1}} \cap V\overline{M^{i+1}}$ in $\overline{M^i}$. We have $\dim_k W^i = \dim_k F\overline{M^{i-1}} + \dim_k V\overline{M^{i+1}} - \dim_k(F\overline{M^{i-1}} + V\overline{M^{i+1}}) = (a_1^i + a_2^i) + (e_1^{i-1} + e_2^{i-1}) - (e_1^i + e_2^i)$. The perfect pairing $\overline{M} \times \overline{M^t} \to k$ induces a perfect pairing

$$W^i \times \left(M^t/(F,V)M^t\right)^i \to k,$$

thus $b_1^i + b_2^i = (a_1^i + a_2^i) + (e_1^{i-1} + e_2^{i-1}) - (e_1^i + e_2^i)$. The assertion $b_1^i = \min\{e - e_1^i, e - e_2^i, e_1^{i-1}, e_2^{i-1}\}$ is obtained from (2.13) and $\mathrm{lie}(A^t) = (\{e - e_1^i, e - e_2^i\})_i$. ∎

(2.15) Let $\mathcal{D}^{-1} = (\pi^{-d})$ be the inverse of the different of $\mathcal{O}$ over $\mathbb{Z}_p$. There is a unique $W \otimes \mathcal{O}$-bilinear pair $(\, ,) : M \times M \to W \otimes \mathcal{D}^{-1}$ such that $\langle x, y \rangle = \mathrm{Tr}_{W \otimes \mathcal{O}/W}(x, y)$. For each W^i-basis x_1^i, x_2^i of M^i, the π-adic valuation $\mathrm{ord}_\pi(x_1^i, x_2^i)$ is independent of the choice of basis, and the degree of the quasi-polarization is p^R, where

$$R = 2 \sum_{i \in \mathbb{Z}/f\mathbb{Z}} \mathrm{ord}_p \, \mathrm{Norm}_{W^i/W}(\pi^d(x_1^i, x_2^i))).$$

We can choose two W^i-bases $\{x_1^i, x_2^i\}, \{y_1^i, y_2^i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$Vy_1^{i+1} = \pi^{e_1^i} x_1^i, \quad Vy_2^{i+1} = \pi^{e_2^i} x_2^i.$$

It follows from $(Vx, Vy) = p(x, y)^{\sigma^{-1}}$ that we get

$$\mathrm{ord}_\pi(y_1^{i+1}, y_2^{i+1}) = \mathrm{ord}_\pi(x_1^i, x_2^i) + (e_1^i + e_2^i - e).$$

If the $\min_{i \in \mathbb{Z}/f\mathbb{Z}} \mathrm{ord}_\pi(x_1^i, x_2^i) = -d$, say that $i = 0$ achieves the minimum, then

$$\mathrm{ord}_\pi(x_1^i, x_2^i) = -d + \sum_{k=0}^{i-1} (e_1^k + e_2^k - e)$$

and

$$R = 2 \sum_{i=1}^{f-1} \sum_{k=0}^{i-1} (e_1^k + e_2^k - e).$$

We can divide by a power of π such that $\min_{i \in \mathbb{Z}/f\mathbb{Z}} \mathrm{ord}_\pi(x_1^i, x_2^i) = -d$.

(2.16) Proposition *There exists a number N, depending on g, such that* $\mathrm{ord}_p(\deg \lambda) \le N$ *for any polarized O_F-abelian variety (A, λ, ι) of dimension g such that the polarization λ is not an integral multiple of another polarization.*

Proof. Take any number greater than R as above, e.g. $N = ef(f-1)$. ∎

§3. Formal isogeny classes

In the rest of this paper we will assume that k *is an algebraically closed field of characteristic* $p > 0$.

(3.1) In this section we address the following question: For every isogeny class of polarized O_F-abelian varieties over k, is there one which satisfies the Rapoport condition? Two polarized O_F-abelian varieties $(A_i, \lambda_i, \iota_i), i = 1, 2$, are isogenous if there exists an O_F-linear isogeny $\varphi : A_1 \to A_2$ such that $\varphi^* \lambda_2 = a\lambda_1$ for some $a \in \mathbb{Z}$. We answer this question in Corollary 3.9.

(3.2) Lemma *Let M be a quasi-polarized $\mathcal{O}$-Dieudonné module. Then the set* $\mathrm{slope}(M)$ *of slopes of M is either $\{\frac{i}{g}, \ldots, \frac{i}{g}, \frac{g-i}{g}, \ldots, \frac{g-i}{g}\}$ for some $0 \le i \le g/2$, or $\{\frac{1}{2}, \ldots, \frac{1}{2}\}$.*

Proof. Suppose that M is not supersingular. The F-isocrystal $M \otimes B(k)$ contains $M_{a,b}^r \oplus M_{b,a}^r$, where $M_{a,b}$ is the simple F-isocrystal of slope $a/(a+b)$. We want to show that $2(a+b)r = 2g$. We first have an embedding $F_v \to \mathrm{End}(M_{a,b}^r) = M_r(D_{a/(a+b)})$ and any maximal commutative subalgebra of the latter has $\mathbb{Q}_p$-dimension $r(a+b)$, so $g | r(a+b)$. On the other hand, we have $2g \ge 2r(a+b)$, from $M \otimes B(k) \supset M_{a,b}^r \oplus M_{b,a}^r$. Hence $M \otimes B(k) = M_{a,b}^r \oplus M_{b,a}^r$. This completes the proof. ∎

(3.3) Let $S(g)$ denote the subset of $\mathbb{Q}$ which parameterizes possible slopes arising from O_F-abelian varieties

$$S(g) := \{i \in \mathbb{Z}; 0 \le i \le \frac{g}{2}\} \cup \{\frac{g}{2}\}.$$

For each $i \in S(g)$, we denote by $s(i)$ the set of slopes $\{\frac{i}{g}, \dots, \frac{i}{g}, \frac{g-i}{g}, \dots, \frac{g-i}{g}\}$. The order on $S(g)$ induced from $\mathbb{Q}$ is compatible with the Grothendieck's specialization theorem.

(3.4) Example. Let $M = M^0 \oplus \dots \oplus M^{f-1}$ be an $\mathcal{O}$-Dieudonné module, where M^i is a free W^i-module of rank two generated by x_1^i, x_2^i with $x_2^i \in VM^{i+1}$. Let

$$\begin{aligned} Fx_1^i &= a_{11}^{i+1} x_1^{i+1} + a_{12}^{i+1} x_2^{i+1} \\ Fx_2^i &= pa_{21}^{i+1} x_1^{i+1} + pa_{22}^{i+1} x_2^{i+1} \end{aligned} \tag{4}$$

be the action of the Frobenius F. Then

$$\begin{aligned} F^f x_1^0 &= \alpha x_1^0 + \beta x_2^0 \\ F^f x_2^0 &= \gamma x_1^0 + \delta x_2^0, \end{aligned} \tag{5}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A_1^{(f-1)} \cdots A_{f-1}^{(1)} A_f, \quad A_i := \begin{pmatrix} a_{11}^i & a_{12}^i \\ pa_{21}^i & pa_{22}^i \end{pmatrix}, \ A_f = A_0.$$

Here we write $A^{(n)}$ for A^{σ^n}. Note that M satisfies the Rapoport condition.

Let $a, b \in \mathbb{Z}, a + b = g, 0 \le a \le b$. Write $a = de + r, 0 \le r < e$. Take

$$A_i = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \text{ for } 1 \le i \le 2d, \quad A_i = \begin{pmatrix} 1 & 1 \\ -p & 0 \end{pmatrix} \text{ for } 2d < i < f, \text{ and } A_f = \begin{pmatrix} \pi^r & 1 \\ -p & 0 \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = p^d \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}, \quad \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \prod_{i=2d+1}^{f} A_i \equiv \begin{pmatrix} \pi^r & 1 \\ 0 & 0 \end{pmatrix} \mod p$$

Note that $A_i^\sigma = A_i$ for all $i \in \mathbb{Z}/\mathbb{Z}/f\mathbb{Z}$. The characteristic polynomial of F^f is $X^2 - (\alpha + \beta)X + p^f$ and $\mathrm{ord}_p(\alpha + \beta) = \frac{a}{e}$. Therefore slope$(M) = \{\frac{a}{g}, \dots, \frac{a}{g}, \frac{b}{g}, \dots, \frac{b}{g}\}$.

(3.5) Remark The example constructed above is non-trivial, due to the following constrain. Let $M = M_{a,b} \oplus M_{b,a}$, $a + b = g$, $a \ne b$ be an $\mathcal{O}$-Dieudonné module which satisfies the Rapoport condition, where $M_{a,b}$ (resp. $M_{b,a}$) is a Dieudonné submodule of slope $\frac{b}{a+b}$ (resp. $\frac{a}{a+b}$). Then a is a multiple of e. Therefore, one can not construct a special Dieudonné module, in the sense of Manin, with an $\mathcal{O}$-action which satisfies the Rapoport condition and has arbitrary possible slopes in Lemma 3.2. The proof of this fact is as follows.

Let $M_{a,b} = M^0 \oplus \dots \oplus M^{f-1}$ as in (2.2), where M^i is a free W^i-module of rank one. We can choose a basis x_i for M^i such that $Vx_{i+1} = \pi^{n_i} x_i$ for $1 \le i \le f - 1$ and $Vx_1 = u_0 \pi^{n_0} x_0, u_0 \in (W^0)^\times$, where $n_i = 0$ or e as it satisfies the Rapoport condition. As the slope of $M_{a,b}$ is $\frac{b}{a+b}$, we have $V^f x_0 = \pi^a u x_0, u \in (W^0)^\times$, hence $a = n_0 + \dots + n_{f-1}$.

(3.6) Let V be a 2-dimensional F_v-vector space with a non-degenerate alternating form ψ on V with values in $\mathbb{Q}_p$ such that $\psi(ax,y)=\psi(x,ay)$ for $a\in F_v,\ x,y\in V$. Let G be the algebraic group of F_v-linear similitudes over $\mathbb{Q}_p$ and let G_1 be the derived group of G. The algebraic group G_1 is simply-connected. Denote by c the multiplier map $G\to\mathbb{G}_{\mathrm{m}}$.

Let M be a quasi-polarized Dieudonné module. We choose an isomorphism between $M_0\otimes_W B(k)$ and $V\otimes_{\mathbb{Q}_p} B(k)$ for skew-symmetric F_v-modules over $B(k)$. Let $b\in G(B(k))$ be the element obtained by the transport of structure of the Frobenius F of M, and let ν be associated slope homomorphism from the pro-torus $\mathbf{D}$ into G, [K1, Sec. 4]. Let J_b be the algebraic group over $\mathbb{Q}_p$ which represents the group functor ([RZ, Prop. 1.12])

$$R\mapsto\{g\in G(R\otimes_{\mathbb{Q}_p} B(k)); g(b\sigma)=(b\sigma)g\}.$$

(3.7) Lemma $H^1(\mathbb{Q}_p,J_b)=0.$

Proof. As k algebraically closed, we may assume that b satisfies the *decent equation* ([RZ, 1.8]): $(b\sigma)^s=s\nu(p)\sigma^s$ for some $s\in\mathbb{N}$. If M is supersingular, then J_b is an inner form of G [RZ, Remark 1.15]. Therefore, $H^1(\mathbb{Q}_p,J_b)=H^1(\mathbb{Q}_p,G)=0$. If M is not supersingular, then J_b is an inner form of the centralizer $G_{s\nu(p)}$, which is $\mathrm{Res}_{Q_{p^s}/\mathbb{Q}_p}(F_v\otimes\mathbb{Q}_{p^s})^\times\times\mathbb{Q}_p^\times$. Therefore, $H^1(\mathbb{Q}_p,J_b)=0$. ∎

(3.8) Corollary *Let M_1 and M_2 are two quasi-polarized $\mathcal{O}$-Dieudonné modules. If* $\mathrm{slope}(M_1)=\mathrm{slope}(M_2)$*, then $M_1\otimes_W B(k)$ and $M_2\otimes_W B(k)$ are isomorphic as quasi-polarized F-isocrystals with the $\mathcal{O}$-action.*

Proof. This follows from [RR, Prop. 1.17] and Lemma 3.7. ∎

(3.9) Corollary *Any polarized O_F-abelian variety over k is isogenous to one which satisfies the Rapoport condition.*

Proof. It follows from (3.4), Corollary 3.8, and surjectivity of the multiplier map $c: G(\mathbb{Q}_p)\to\mathbb{Q}_p^\times$ that the statement holds for the associated p-divisible group. Then the assertion follows from a theorem of Tate. ∎

(3.10) Corollary *Any O_F-abelian variety over k admits an O_F-linear polarization.*

Proof. This follows from Corollary 3.9 and [R, Prop. 1.10].

§4. Normal forms

(4.1) Let M be a non-ordinary separably quasi-polarized $\mathcal{O}$-Dieudonné module over k which satisfies the Rapoport condition. Let $a(M)=(a^i)_i$ be the a-number of M. Denote by $\tau(M)$ the *a-index* of M, which is defined to be the subset of $\mathbb{Z}/f\mathbb{Z}$:

$$\{i\in\mathbb{Z}/f\mathbb{Z}; a^i\neq 0\}.$$

Write τ for $\tau(M)$.

Denote by $\overline{M}$ the quotient of M module πM. The module $\overline{M}$ is a $2f$-dimensional vector space over k together with a k-linear action by $\mathcal{O}/\pi = \mathcal{O}^{\mathrm{ur}}/p$ that commutes with the action of F and V. The perfect pairing $\langle\,,\rangle$ on M induces an perfect pairing $\langle\,,\rangle$ on $\overline{M}$ that satisfies $\langle Fx, y\rangle = \langle x, Vy\rangle^p$ and $\langle ax, y\rangle = \langle x, ay\rangle$ for all $x, y \in \overline{M}, a \in \mathcal{O}^{\mathrm{ur}}/p$. The decomposition of $\overline{M}$ (2.2) into σ_i-eigenspaces

$$\overline{M} = \oplus_{i\in\mathbb{Z}/f\mathbb{Z}} \overline{M}^i$$

respects the perfect pairing as before (2.2).

As M satisfies the Rapoport condition, $V\overline{M}$ and $F\overline{M}$ are free $k \otimes \mathcal{O}^{\mathrm{ur}}$-module of rank one, and we have that $F\overline{M} = \ker V$ and $V\overline{M} = \ker F$. It follows that $\overline{M}$ is isomorphic to N/pN for a separably quasi-polarized $\mathcal{O}^{\mathrm{ur}}$-Dieudonné module N over k of rank $2f$ with the same a-index as M.

(4.2) Proposition *There exists a k-basis $\{x_i, y_i\}$ of $\overline{M}^i$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

- $\langle x_i, y_i\rangle = 1$ *and* $y_i \in V\overline{M}$ *for all* $i \in \mathbb{Z}/f\mathbb{Z}$,
- $F\,x_i = \begin{cases} x_{i+1} & \text{if } i+1 \notin \tau \\ -y_{i+1} & \text{if } i+1 \in \tau \end{cases} \quad V\,y_i = \begin{cases} y_{i-1} & \text{if } i \notin \tau \\ 0 & \text{if } i \in \tau \end{cases} \quad V\,x_i = \begin{cases} y_{i-1} & \text{if } i \in \tau \\ 0 & \text{if } i \notin \tau \end{cases}$

PROOF. It follows from [Y, Prop. 4.1] or [GO, Thm. 3.2.8]. ∎

Note that this proposition gives the classification of the π-torsion subgroup scheme $A[\pi]$ of separably polarized O_F-abelian varieties over k, classified by the a-indices.

The following results generalize [Y, Prop. 4.1, Prop 4.2, Lemma 4.3], the proofs are the same.

(4.3) Proposition *There exists a W^i-basis $\{X_i, Y_i\}$ of M for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

(i) $\langle X_i, Y_i\rangle = 1$ *and* $Y_i \in (VM)^i$, $\forall\, i \in \mathbb{Z}/f\mathbb{Z}$,

(ii) $F\,X_i = \begin{cases} X_{i+1} & \text{if } i+1 \notin \tau \\ -Y_{i+1} + c_{i+1}\pi X_{i+1} & \text{if } i+1 \in \tau \end{cases} \qquad F\,Y_i = \begin{cases} pY_{i+1} & \text{if } i+1 \notin \tau \\ pX_{i+1} & \text{if } i+1 \in \tau \end{cases}$
for some $c_{i+1} \in W^{i+1}$.

(4.4) Proposition *There exists a W^i-basis $\{X_i, Y_i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

(i) $\langle X_i, Y_i\rangle \in (W^i)^\times$ *and* $Y_i \in (VM)^i$, $\forall\, i \in \mathbb{Z}/f\mathbb{Z}$,

(ii) $F\,X_i = \begin{cases} X_{i+1} & \text{if } i+1 \notin \tau \\ Y_{i+1} + c_{i+1}\pi X_{i+1} & \text{if } i+1 \in \tau \end{cases} \quad F\,Y_i = \begin{cases} \pi^e Y_{i+1} & \text{if } i+1 \notin \tau \\ \pi^e X_{i+1} & \text{if } i+1 \in \tau, \end{cases}$
for some $c_{i+1} \in W^{i+1}$

Note that the i-th component a^i of $a(M)$ is $\min\{e, \mathrm{ord}_\pi(c_i\pi)\}$ for $i \in \tau$.

(4.5) Lemma *If M is superspecial, i.e. $|a(M)| = g$. Then*

(1) There exists a W^i-basis $\{X_i, Y_i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that

- $Y_i \in (VM)^i$ *and* $\langle X_i, Y_i\rangle = 1$,
- $FX_i = -Y_{i+1},\ \ FY_i = pX_{i+1}$,

for all $i \in \mathbb{Z}/f\mathbb{Z}$.

(2) There exists a W^i-basis $\{X_i, Y_i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that

- $Y_i \in (VM)^i$ *and* $\langle X_i, Y_i\rangle = u_i \in (W^i)^\times$ *with* $u_i^{\sigma^r} = u_i$, *where* $r = \mathrm{lcm}(2, f)$,
- $FX_i = Y_{i+1},\ \ FY_i = pX_{i+1}$,

for all $i \in \mathbb{Z}/f\mathbb{Z}$.

(4.6) Let M be a non-ordinary separably quasi-polarized $\mathcal{O}$-Dieudonné module satisfying the Rapoport condition. Let $a(M) = (a^i)$ be the a-number of M and $\tau = \tau(M) = \{n_1, n_2, \dots n_t\}$ be the a-index of M, where $t = |a(M/\pi M)|$, the *reduced usual a-number of M*. We assume that $0 \le n_1 < n_2 < \dots < n_t < f$ and let $n_0 := n_t - f$ and $n_{t+1} = n_1$. Denote by $A(e, f)$ the set of a-numbers

$$A(e, f) := \{(a^i); i \in \mathbb{Z}/f\mathbb{Z}, a^i \in \mathbb{Z}, 0 \le a^i \le e\},$$

with the partial order that $(a^i) \le (b^i)$ if $a^i \le b^i$, $\forall\, i \in \mathbb{Z}/f\mathbb{Z}$. An a-number $(a^i) \in A(e, f)$ is called *spaced* if $a^i a^{i+1} = 0$, $\forall i \in \mathbb{Z}/f\mathbb{Z}$, cf. [GO, Sec. 1, p.112].

By Proposition 4.4, we can choose a W^i-basis $\{X_i, Y_i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$\begin{pmatrix} FX_{i-1} \\ FY_{i-1} \end{pmatrix} = A_i \begin{pmatrix} X_i \\ Y_i \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 1 & 0 \\ 0 & \pi^e \end{pmatrix} \text{ if } i \notin \tau; \quad A_i = \begin{pmatrix} \pi^{a^i} c_i & 1 \\ \pi^e & 0 \end{pmatrix} \text{ if } i \in \tau,$$

for some $c_i \in W^i$ for $i \in \tau$ and c_i is a unit if $a^i < e$.

Let $\ell_i := n_i - n_{i-1}$ for $1 \le i \le t$. We have

$$\begin{pmatrix} F^{\ell_i} X_{n_{i-1}} \\ F^{\ell_i} Y_{n_{i-1}} \end{pmatrix} = \begin{pmatrix} \pi^{a^i} c_{n_i} & 1 \\ \pi^{e\ell_i} & 0 \end{pmatrix} \begin{pmatrix} X_{n_i} \\ Y_{n_i} \end{pmatrix}, \quad \forall 1 \le i \le t.$$

For $1 \le s \le t$, suppose that

$$\begin{pmatrix} F^{\ell_s+\ell_{s+1}+\cdots+\ell_t}X_{n_{s-1}} \\ F^{\ell_s+\ell_{s+1}+\cdots+\ell_t}Y_{n_{s-1}} \end{pmatrix} = \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix} \begin{pmatrix} X_{n_t} \\ Y_{n_t} \end{pmatrix},$$

for some coefficients $\alpha_s, \beta_s, \gamma_s, \delta_s$. It follows from

$$\begin{aligned} \begin{pmatrix} F^{\ell_{s-1}+\ell_s+\cdots+\ell_t}X_{n_{s-2}} \\ F^{\ell_{s-1}+\ell_s+\cdots+\ell_t}Y_{n_{s-2}} \end{pmatrix} &= \begin{pmatrix} F^{\ell_s+\ell_{s+1}+\cdots+\ell_t}(F^{\ell_{s-1}}X_{n_{s-2}}) \\ F^{\ell_s+\ell_{s+1}+\cdots+\ell_t}(F^{\ell_{s-1}}Y_{n_{s-2}}) \end{pmatrix} \\ &= \begin{pmatrix} \pi^{a^{n_{s-1}}} c_{n_{s-1}}^{(\ell_s+\ell_{s+1}+\cdots+\ell_t)} & 1 \\ \pi^{e\ell_{s-1}} & 0 \end{pmatrix} \begin{pmatrix} F^{\ell_s+\ell_{s+1}+\cdots+\ell_t}X_{n_{s-1}} \\ F^{\ell_s+\ell_{s+1}+\cdots+\ell_t}Y_{n_{s-1}} \end{pmatrix} \end{aligned} \tag{6}$$

that

$$\begin{pmatrix} \alpha_{s-1} & \beta_{s-1} \\ \gamma_{s-1} & \delta_{s-1} \end{pmatrix} = \begin{pmatrix} u_{s-1} & 1 \\ \pi^{e\ell_{s-1}} & 0 \end{pmatrix} \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix}, \quad \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix} = \begin{pmatrix} \pi^{a^{n_t}} c_{n_t} & 1 \\ \pi^{e\ell_t} & 0 \end{pmatrix}, \tag{7}$$

where $u_{s-1} = \pi^{a^{n_{s-1}}} c_{n_{s-1}}^{(\ell_s+\ell_{s+1}+\cdots+\ell_t)}$. Recall that we write $a^{(n)}$ for a^{σ^n} (3.4). Therefore,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \begin{pmatrix} u_1 & 1 \\ \pi^{e\ell_1} & 0 \end{pmatrix} \begin{pmatrix} u_2 & 1 \\ \pi^{e\ell_1} & 0 \end{pmatrix} \cdots \begin{pmatrix} u_t & 1 \\ \pi^{e\ell_t} & 0 \end{pmatrix}. \tag{8}$$

(4.7) When $c_i = 0,\ \forall\, i \in \tau$, if t is even, $t = 2d$, we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \pi^{e(\ell_2+\ell_4+\cdots+\ell_{2d})} & 0 \\ 0 & \pi^{e(\ell_1+\ell_3+\cdots+\ell_{2d-1})} \end{pmatrix}$$

and $\mathrm{slope}(M) = s(i)$ (3.3), where $i = \min\{e(\sum_{1\le j\le d} \ell_{2j}), e(\sum_{1\le j\le d} \ell_{2j-1})\}$. If t is odd, $t = 2d+1$, we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^2 = \begin{pmatrix} \pi^g & 0 \\ 0 & \pi^g \end{pmatrix}$$

and M is supersingular.

(4.8) Proposition *Let M be a separably quasi-polarized $\mathcal{O}$-Dieudonné module over k. If $a(M)$ is spaced, i.e. $\ell_i \ge 2\ \forall\, 1 \le i \le t$, then* $\mathrm{slope}(M) \ge s(|a(M)|)$ *(3.3).*

PROOF. We may assume that M is non-ordinary. It follows from (8) that

$$\begin{pmatrix} F^f\pi^e X_{n_t} \\ F^f Y_{n_t} \end{pmatrix} = A \begin{pmatrix} \pi^e X_{n_t} \\ Y_{n_t} \end{pmatrix}, \quad A = \prod_{i=1}^{t} A_i = \pi^{|a(M)|} \prod_{i=1}^{t} \left(\pi^{-a^{n_i}} A_i\right), \quad A_i = \begin{pmatrix} u_t & \pi^e \\ \pi^{e(\ell_t-1)} & 0 \end{pmatrix}.$$

Let N be the $\mathcal{O}$-Dieudonné submodule of M generated by $\pi^e X_{n_t}, Y_{n_t}$. As $F^f(N) \subset \pi^{|a(M)|}N$, we get $\mathrm{slope}(M) \ge s(|a(M)|)$. ∎

(4.9) When $t=|\tau|=1$, say $\tau=\{0\}$, we have

$$\begin{pmatrix} F^f X_0 \\ F^f Y_0 \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ \pi^g & 0 \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}.$$

for some $c_0 \in \pi^{a^0} W^0$. It is easy to see that M satisfies a Cayley-Hamilton equation ([O2],[Y]) $F^{2f}X_0 - c_0^{(f)} F^f X_0 - \pi^g X_0 = 0$, thus $\mathrm{slope}(M) = s(i)$, where $i = \min\{\frac{g}{2}, \mathrm{ord}_\pi(c_0)\}$.

(4.10) When $t=|\tau|=2$, say $\tau=\{0,\ell_2\}$ and $\ell_2 \le \ell_1$, we have

$$\begin{pmatrix} F^f X_{\ell_1} \\ F^f Y_{\ell_1} \end{pmatrix} = \begin{pmatrix} u_1 & 1 \\ \pi^{e\ell_1} & 0 \end{pmatrix} \begin{pmatrix} u_2 & 1 \\ \pi^{e\ell_2} & 0 \end{pmatrix} \begin{pmatrix} X_{\ell_1} \\ Y_{\ell_1} \end{pmatrix},$$

for some coefficients u_1, u_2 ($u_1 = c_0^{(\ell_2)}, u_2 = c_{\ell_2}$). Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} u_1 u_2 + \pi^{e\ell_2} & u_1 \\ \pi^{e\ell_2} u_2 & \pi^{e\ell_2} \end{pmatrix} \tag{9}$$

If $u_1 = 0$, then the matrix is $\begin{pmatrix} \pi^{e\ell_2} & 0 \\ \pi^{e\ell_1} u_2 & \pi^{e\ell_1} \end{pmatrix}$, hence $\mathrm{slope}(M) = s(e\ell_2)$. If $u_1 \neq 0$, then we have the Cayley-Hamilton equation

$$\beta F^{2f} X_{\ell_1} - (\beta^{(f)}\delta + \beta\alpha^{(f)}) F^f X_{\ell_1} + \beta^{(f)}(\alpha\delta - \beta\delta) X_{\ell_1} = 0.$$

As $\mathrm{ord}_\pi(\delta) = e\ell_1 \ge \frac{g}{2}$, we see that $\mathrm{slope}(M) = s(i)$, where $i = \min\{\frac{g}{2}, \mathrm{ord}_\pi(a^{(f)}\}$. In both cases we have that $\mathrm{slope}(M) = s(i)$, where $i = \min\{\frac{g}{2}, \mathrm{ord}_\pi(u_1 u_2 + \pi^{e\ell_2})\}$.

§5. a-number stratification

(5.1) Let p be a fixed prime number. Let $\mathcal{M}^{\mathrm{DP}}$ denote the moduli stack over $\mathrm{Spec}\mathbb{Z}_{(p)}$ of polarized O_F-abelian varieties (A, λ, ι) of dimension $g = [F:\mathbb{Q}]$ with the polarization λ of prime-to-p degree. It is a separated Deligne-Mumford algebraic stack over $\mathrm{Spec}\mathbb{Z}_{(p)}$ locally of finite type. In [DP], Deligne and Pappas showed that the algebraic stack $\mathcal{M}^{\mathrm{DP}}$ is flat and a locally complete intersection over $\mathrm{Spec}\mathbb{Z}_{(p)}$ of relative dimension g, and the closed fibre $\mathcal{M}^{\mathrm{DP}} \otimes \mathbb{F}_p$ is geometrically normal and has singularities of codimension at least two. It follows from Deligne-Pappas' results that the irreducible components of geometric special fibre $\mathcal{M}^{\mathrm{DP}} \otimes \overline{\mathbb{F}_p}$ are in bijection correspondence with those of geometric generic fibre $\mathcal{M}^{\mathrm{DP}} \otimes \overline{\mathbb{Q}}$. Those are parameterized by the isomorphism classes of non-degenerate skew-symmetric O_F-modules $H_1(A(\mathbb{C}), \mathbb{Z})$ for all $(A, \lambda, \iota) \in \mathcal{M}^{\mathrm{DP}}(\mathbb{C})$.

Let $\mathcal{M}^R$ denote the Rapoport locus of $\mathcal{M}^{\mathrm{DP}}$. It is the smooth locus of $\mathcal{M}^{\mathrm{DP}}$ (2.8). Let $\mathcal{M}$ denote the reduction $\mathcal{M}^R \otimes_{\mathbb{Z}} k(v)$ of $\mathcal{M}^R$ modulo v, where $k(v)$ is the residue field of O_F at v. We will define the stratification on $\mathcal{M}$ by a-numbers scheme-theoretically, (cf. [Y, Sec. 3]).

(5.2) Let S be a locally noetherian scheme over $\mathrm{Spec} k(v)$ and $\pi_A : (A, \lambda, \iota) \to S \in \mathcal{M}(S)$ a polarized O_F-abelian scheme over S. The sheaf $R^1(\pi_A)_*(\mathcal{O}_A)$ is a locally free rank one $O_F \otimes \mathcal{O}_S$-module. It admits a decomposition (2.2)

$$R^1(\pi_A)_*(\mathcal{O}_A) = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} R^1(\pi_A)_*(\mathcal{O}_A)^i$$

with respect to the action by O_F. Each component $R^1(\pi_A)_*(\mathcal{O}_A)^i$ is a locally free $O_S[\pi]/(\pi^e)$-module of rank one and it admits a filtration of locally free O_S-modules

$$R^1(\pi_A)_*(\mathcal{O}_A)^i = \mathcal{F}^{i,0} \supset \mathcal{F}^{i,1} \supset \cdots \supset \mathcal{F}^{i,e} = 0, \quad \mathcal{F}^{i,j} := \ker \pi^{e-j}.$$

Let $F_{A/S} : A \to A^{(p)}$ be relative Frobenius morphism over S, where $A^{(p)} := A \times_{S, F_{\mathrm{abs}}} S$ and F_{abs} is the absolute Frobenius morphism on S. It induces an $\mathcal{O}_S \otimes O_F$-linear morphism $F^*_{A/S} : R^1(\pi_A)_*(\mathcal{O}_A)^{(p)} \to R^1(\pi_A)_*(\mathcal{O}_A)$.

Let $\underline{a} = (a^i) \in A(e, f)$ (4.6) be an a-number. Let $\mathcal{M}_{\geq \underline{a}}$ denote the substack of $\mathcal{M}$ whose objects $\pi_A : (A, \lambda, \iota) \to S$ satisfying the following condition

$$F^*_{A/S}\left(R^1(\pi_A)_*(\mathcal{O}_A)^{(p),i}\right) \subset \mathcal{F}^{i,a^i}, \quad \forall\, i \in \mathbb{Z}/f\mathbb{Z}. \tag{10}$$

It is clear that the condition (10) is closed and locally for Zariski topology $\mathcal{M}_{\geq \underline{a}}$ is defined by finitely many equations. Therefore $\mathcal{M}_{\geq \underline{a}}$ is a closed algebraic substack of $\mathcal{M}$.

It is clear that $\mathcal{M}_{\geq \underline{a}}$ is non-empty as it contains $E \otimes O_F$ with a prime-to-p polarization, where E is a supersingular elliptic curve.

(5.3) Lemma *Let $(A, \lambda, \iota) \in \mathcal{M}(k)$ and $a(A) = (a^i)$ be the a-number of A (2.10). Then* $\operatorname{coker} F^*_{A/k} = \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} k[\pi]/(\pi^{a^i})$.

PROOF. Let M^* denote the contravariant Dieudonné module of A. We identify $M^*/pM^* = H^1_{DP}(A/k)$ and have $H^1(A, \mathcal{O}_A) = M^*/VM^*$ and $\operatorname{coker} F^*_{A/k} = M^*/(F, V)M^*$. On the other hand, the quasi-polarization gives an isomorphism $M \simeq M^*$ of $\mathcal{O}$-Dieudonné modules, which induces an isomorphism $M/(F, V)M \simeq M^*/(F, V)M^*$ of $O_F \otimes k$-modules. ∎

(5.4) Theorem *The algebraic stack $\mathcal{M}_{\geq \underline{a}}$ is smooth over* $\mathrm{Spec} k(v)$ *of pure dimension* $g - |\,\underline{a}\,|$.

PROOF. Let $x : \mathrm{Spec} k \to \mathcal{M}_{\geq \underline{a}}$ be a geometric point. Then there is an affine open neighborhood U of x and a polarized O_F-abelian scheme $(A, \lambda, \iota) \in \mathcal{M}(U)$ such that $R^1(\pi_A)_*(\mathcal{O}_A)^{(p)}$ and $R^1(\pi_A)_*(\mathcal{O}_A)$ are free $\mathcal{O}_U \otimes O_F$-modules. Let $x_i^{(p)}$ and x_i be $O_U[\pi]/(\pi^e)$-bases of $R^1(\pi_A)_*(\mathcal{O}_A)^{(p),i}$ and $R^1(\pi_A)_*(\mathcal{O}_A)^i$ respectively for each $i \in \mathbb{Z}/f\mathbb{Z}$. Let

$$F^*_{A/S}(x_i^{(p)}) = \sum_{0 \leq j < e} f_{i,j} \pi^j x_i, \quad f_{i,j} \in \mathcal{O}_U.$$

Then locally $\mathcal{M}_{\geq \underline{a}}$ is defined by the equations $f_{i,j}$, where $(i, j) \in I = \{(i, j); i \in \mathbb{Z}/f\mathbb{Z}, 0 \leq j < a^i\}$. Therefore, $\dim_x \mathcal{M}_{\geq \underline{a}} \geq g - |I| = g - |\,\underline{a}\,|$.

Let $\overline{M} := H^1_{cris}(A_x/k), \overline{M}^{(p)} := H^1_{cris}(A_x^{(p)}/k) = \overline{M} \otimes_{k,\sigma} k$. We have the Hodge filtrations:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Fil^{(p)} & \longrightarrow & \overline{M}^{(p)} & \longrightarrow & Q^{(p)} := H^1(A_x, \mathcal{O}_{A_x})^{(p)} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \\
0 & \longrightarrow & Fil & \longrightarrow & \overline{M} & \longrightarrow & Q := H^1(A_x, \mathcal{O}_{A_x}) & \longrightarrow & 0
\end{array}$$

By Proposition 4.3 and Lemma 5.3, we can choose a $k[\pi]/(\pi^e)$-basis $\{x_i, y_i\}$ be of $(\overline{M})^i$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that $y_i \in Fil^i$ and $F^*(x_i \otimes 1) = -y_i + c_i x_i$ for some $c_i \in \pi^{a^i} k[\pi]/(\pi^e)$. For any PD-extension R of k, the differential forms x_i, y_i have unique horizontal liftings in $\widetilde{M} := H^1_{cris}(A/R)$ with respect to the Gauss-Manin connection, which we will denote the liftings by x_i, y_i again.

Let $R := k[[\underline{t}]]/(\underline{t})^2$ be the first order universal deformation ring of $(A_x, \lambda_x, \iota_x)$, where $\underline{t} = (t_{i,j}), i \in \mathbb{Z}/f\mathbb{Z}, 0 \le j < e$. The first order universal deformation $(\tilde{A}, \tilde{\lambda}, \tilde{\iota})$ gives rise to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{Fil}^{(p)} & \longrightarrow & \widetilde{M}^{(p)} & \longrightarrow & \widetilde{Q}^{(p)} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \\
0 & \longrightarrow & \widetilde{Fil} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{Q} & \longrightarrow & 0
\end{array}$$

where $\widetilde{Fil} = \mathrm{span}\langle y_i + \sum_{0 \le j < e} t_{i,j} \pi^j x_i \rangle$, $\widetilde{Fil}^{(p)} = \mathrm{span}\langle (y_i + \sum_{0 \le j < e} t_{i,j} \pi^j x_i) \otimes 1 \rangle = \mathrm{span}\langle y_i \otimes 1 \rangle$, $\widetilde{Q} = \mathrm{span}\langle [x_i] \rangle$, and $\widetilde{Q}^{(p)} = \mathrm{span}\langle [x_i \otimes 1] \rangle$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. It follows from

$$F^*([x_i \otimes 1]) = [-y_i] = \sum_{0 \le j < e} t_{i,j} [\pi^j x_i]$$

that $R/(t_{i,j})_{(i,j) \in I}$ is the first order deformation ring of $\mathcal{M}_{\ge \underline{a}}$ at x and the tangent space has dimension $g - |\,\underline{a}\,| \le \dim_x \mathcal{M}_{\ge \underline{a}}$. The assertion follows. ∎

(5.5) Corollary *The ordinary points are dense in* $\mathcal{M}^{\mathrm{DP}} \otimes \mathbb{F}_p$.

PROOF. It follows form Theorem 5.4 and density of the Rapoport locus. ∎

(5.6) Let $\mathcal{M}_{\underline{a}}$ denote the subset of $\mathcal{M}$ whose points have a-number $\underline{a}$. It is a locally closed subset of $\mathcal{M}$, hence as a locally closed algebraic substack of $\mathcal{M}$ with the reduced induced structure. It follows from Lemma 5.3 that $(A, \lambda, \iota) \in \mathcal{M}_{\ge \underline{a}}(k)$ if and only if $a(A) \ge \underline{a}$. It follows from Theorem 5.4 that $\mathcal{M}_{\underline{a}}$ is an dense open substack of $\mathcal{M}_{\ge \underline{a}}$ and $\mathcal{M}_{\ge \underline{a}}$ is the scheme-theoretic closure of $\mathcal{M}_{\underline{a}}$ in $\mathcal{M}$. This justifies our notation.

§6. Deformations of Dieudonné modules

(6.1) We follow the convenient setting of [N, Sec. 0] and [CN, Sec. 2, p. 1011]. As we will only deal with smooth functors, the deformation theory developed by P. Norman [N] and

Norman-Oort [NO] is enough for our purpose. We refer the reader to [Z2] for the generalized theory of displays over more general base ring.

Let R be a commutative ring of characteristic p. Let $W(R)$ denote the ring of Witt vectors over R, equipped with the Verschiebung τ and Frobenius σ:

$$(a_0, a_1, \dots)^\tau = (0, a_0, a_1, \dots)$$
$$(a_0, a_1, \dots)^\sigma = (a_0^p, a_1^p, \dots).$$

Let $\mathrm{Cart}_p(R)$ denote the Cartier ring $W(R)[F][\![V]\!]$ modulo the relations

- $FV = p$ and $VaF = a^\tau$,
- $Fa = a^\sigma F$ and $Va^\tau = aV$, $\forall\, a \in W(R)$.

A left $\mathrm{Cart}_p(R)$-module is *uniform* if it is complete and separated in the V-adic topology. A uniform $\mathrm{Cart}_p(R)$-module M is *reduced* if V is injective on M and M/VM is a free R-module. A *Dieudonné module over* R is a finitely generated reduced uniform $\mathrm{Cart}_p(R)$-module.

There is an equivalence of categories between the category of finite dimensional commutative formal group over R and the category of Dieudonné module over R. We denote this functor by $\mathbf{D}_*$. The tangent space of a formal group G is canonically isomorphic to $\mathbf{D}_*(G)/V\mathbf{D}_*(G)$.

(6.2) Let $\underline{a} = (a^i)$ be an a-number and $(A_0, \lambda_0, \iota_0) \in \mathcal{M}_{\geq \underline{a}}(k)$ be a *non-ordinary* polarized O_F-abelian variety. By Lemma 3.2, the associated p-divisible group $G_0 = A_0[p^\infty]$ is connected, hence is a smooth formal group. As the forgetful functor $\mathrm{Def}[A_0, \lambda_0, \iota_0] \to \mathrm{Def}[A_0, \iota_0]$ induces an equivalence of deformation functors, we will consider deformations of O_F-abelian varieties and their associated formal groups.

Let I be the set $\{(i,j); i \in \mathbb{Z}/f\mathbb{Z}, a^i \leq j < e-1\}$ and let $R := k[\![t_{i,j}]\!]_{(i,j)\in I}$. We have

$$\mathcal{O} \otimes_{\mathbb{Z}_p} W(R) = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} W(R)[T]/(\sigma_i(P(T))).$$

Set $W_R^i := W(R)[T]/(\sigma_i(P(T)))$ and denote again by π the image of T in W_R^i.

Let M_0 be the covariant Dieudonné module of A_0. Let $\underline{b} = (b^i)$ be the a-number of A_0 and τ be the a-index of M_0. By Proposition 4.4, we can choose a W^i-basis of M_0^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$FX_{i-1} = \begin{cases} X_i & \text{if } i \notin \tau \\ c_i X_i + Y_i & \text{if } i \in \tau \end{cases} \qquad FY_{i-1} = \begin{cases} \pi^e Y_i & \text{if } i \notin \tau \\ \pi^e X_i & \text{if } i \in \tau, \end{cases} \tag{11}$$

for some $c_i \in \pi^{b^i} W^i$.

By [NO, Lemma 0.2], we construct a Dieudonné module M_R over R with a $\mathrm{Cart}_p(R)$-linear action by $\mathcal{O}$ as follows. It is the $\mathrm{Cart}_p(R)$-module generated by $\{\pi^j X_i, \pi^j Y_i\}_{i\in\mathbb{Z}/f\mathbb{Z}, 0\le j<e}$ with the relations

$$F X_{i-1} = \begin{cases} X_i & \text{if } i \notin \tau \\ c_i X_i + (Y_i + \sum_{a^i \le j < e} T_{i,j}\pi^j X_i) & \text{if } i \in \tau \end{cases} \tag{12}$$

$$Y_{i-1} = \begin{cases} Vu(Y_i + \sum_{a^i \le j < e} T_{i,j}\pi^j X_i) & \text{if } i \notin \tau \\ VuX_i & \text{if } i \in \tau, \end{cases} \tag{13}$$

where $T_{i,j}$ is the Teichmüller lift of $t_{i,j}$, $\forall\,(i,j) \in I$, $u = p^{-1}\pi^e$. We impose the natural W_R^i-module structure on the free $W(R)$-submodule generated by $\{\pi^j X_i, \pi^j Y_i\}_{0\le j<e}$, and impose the trivial W_R^k-module structure on this submodule for $k \neq i$. The action of $\mathcal{O}$ on M_R comes from the natural embedding $\mathcal{O} \hookrightarrow \mathcal{O} \otimes_{\mathbb{Z}_p} W(R)$.

(6.3) Lemma *The $\mathcal{O}$-Dieudonné module M_R over R is isomorphic to the universal deformation of M_0 for $\mathcal{M}_{\ge \underline{a}}$.*

PROOF. It is clear that $\mathrm{Cart}_p(k) \otimes_{\mathrm{Cart}_p(R)} M_R = M_0$. On M_R/VM_R, we have

$$F[X_{i-1}] = c_i[X_i] + \sum_{a^i \le j < e} t_{i,j}\pi^j[X_i], \quad \forall\, i \in \tau.$$

By the Serre-Tate theorem, we obtain a morphism $\mathrm{Spf}\, R \to \mathcal{M}^\wedge_{\ge \underline{a}, x}$. By [N, Thm. 1], this construction induces an injection of tangent spaces. By Theorem 5.4, $\mathcal{M}_{\ge \underline{a}}$ is smooth and $\dim \mathcal{M}_{\ge \underline{a}} = \dim R$. Hence the morphism is an isomorphism. ∎

(6.4) Remark Another construction of M_R is using tensor products modulo relations. Let P_R is a free $\mathcal{O} \otimes_{\mathbb{Z}_p} W(R)$-module of rank two, with a W_R^i-basis $\{X_i, Y_i\}$ for each component P_R^i. Then we construct M_R to be the quotient of $\mathrm{Cart}_p(R) \otimes_{W(R)} P_R$ modulo the relations (12) and (13).

(6.5) Let M_R be as in (6.2). Let

$$T_i := \begin{cases} \sum_{a^i \le j < e} T_{i,j}\pi^j & \text{if } i \notin \tau \\ c_i + \sum_{a^i \le j < e} T_{i,j}\pi^j & \text{if } i \in \tau. \end{cases} \tag{14}$$

Then we have

$$\begin{pmatrix} FX_{i-1} \\ FY_{i-1} \end{pmatrix} = A_i \begin{pmatrix} X_i \\ Y_i \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 1 & 0 \\ T_i\pi^e & \pi^e \end{pmatrix} \text{ if } i \notin \tau; \quad A_i = \begin{pmatrix} T_i & 1 \\ \pi^e & 0 \end{pmatrix} \text{ if } i \in \tau.$$

(6.6) Lemma *The non-ordinary locus of* $\mathcal{M}_x^\wedge$ *is defined by* $\prod_{i\in\tau} t_{i,0}=0$.

PROOF. Take $\underline{a}=(0,0,\dots,0)$. On $M_R/(VM_R+\pi M_R)$, we have

$$F[X_{i-1}]=[X_i] \quad \text{if } i\notin\tau; \quad F[X_{i-1}]=t_{i,0}[X_i] \quad \text{if } i\in\tau.$$

As each slope stratum is reduced, it induces a closed reduced subscheme, if not empty, of $\mathcal{M}_x^\wedge$. A point $\mathfrak{p}\in\operatorname{Spec}R$ is in the non-ordinary locus if and only if any of $t_{i,0}, i\in\tau$, vanishes on $R/\mathfrak{p}$. Therefore the defining equation is $\prod_{i\in\tau} t_{i,0}=0$. ∎

(6.7) We continue with (6.5). Let $t=|a(M_0/\pi M_0)|$ and $\tau=\{n_1,n_2,\dots n_t\}$. We assume that $0\le n_1<n_2<\dots<n_t<f$ and let $n_0:=n_t-f$ and $n_{t+1}=n_1$. Let $\ell_i:=n_i-n_{i-1}$ for $1\le i\le t$. (4.6)

From (6.5), we have

$$\begin{aligned}\begin{pmatrix}F^{\ell_i}X_{n_{i-1}}\\ F^{\ell_i}Y_{n_{i-1}}\end{pmatrix}&=\begin{pmatrix}1&0\\ T^{(\ell_i-1)}_{n_{i-1}+1}\pi^e&\pi^e\end{pmatrix}\cdots\begin{pmatrix}1&0\\ T^{(1)}_{n_{i-1}+\ell_i-1}\pi^e&\pi^e\end{pmatrix}\begin{pmatrix}T_{n_i}&1\\ \pi^e&0\end{pmatrix}\begin{pmatrix}X_{n_i}\\ Y_{n_i}\end{pmatrix}\\ &=\begin{pmatrix}1&0\\ V_i'&\pi^{e(\ell_i-1)}\end{pmatrix}\begin{pmatrix}U_i'&1\\ \pi^e&0\end{pmatrix}\begin{pmatrix}X_{n_i}\\ Y_{n_i}\end{pmatrix}\\ &=\begin{pmatrix}U_i'&1\\ U_i'V_i'+\pi^{e\ell_i}&V_i'\end{pmatrix}\begin{pmatrix}X_{n_i}\\ Y_{n_i}\end{pmatrix}\end{aligned}\tag{15}$$

where

$$U_i':=T_{n_i}, \quad \text{and} \quad V_i':=\sum_{j=1}^{\ell_i-1}T^{(\ell_i-j)}_{n_{i-1}+j}\pi^{ej}.$$

Recall that we write $T^{(n)}$ for T^{σ^n} (3.4) (4.6). For $1\le s\le t$, suppose that

$$\begin{pmatrix}F^{\ell_s+\ell_{s+1}+\dots+\ell_t}X_{n_{s-1}}\\ F^{\ell_s+\ell_{s+1}+\dots+\ell_t}Y_{n_{s-1}}\end{pmatrix}=\begin{pmatrix}\alpha_s&\beta_s\\ \gamma_s&\delta_s\end{pmatrix}\begin{pmatrix}X_{n_t}\\ Y_{n_t}\end{pmatrix},$$

for some coefficients $\alpha_s,\beta_s,\gamma_s,\delta_s$ in $W_R^{n_t}$. It follows from the same computation (6) of (4.6) that

$$\begin{pmatrix}\alpha_{s-1}&\beta_{s-1}\\ \gamma_{s-1}&\delta_{s-1}\end{pmatrix}=\begin{pmatrix}U_{s-1}&1\\ U_{s-1}V_{s-1}+\pi^{e\ell_{s-1}}&V_{s-1}\end{pmatrix}\begin{pmatrix}\alpha_s&\beta_s\\ \gamma_s&\delta_s\end{pmatrix},\tag{16}$$

$$\begin{pmatrix}\alpha_t&\beta_t\\ \gamma_t&\delta_t\end{pmatrix}=\begin{pmatrix}U_t&1\\ U_tV_t+\pi^{e\ell_t}&V_t\end{pmatrix},\tag{17}$$

where

$$U_s := T_{n_s}^{(\ell_{s+1}+\ell_{s+2}+\cdots+\ell_t)} \quad \text{and} \quad V_s := \sum_{i=1}^{\ell_s-1} T_{n_{s-1}+i}^{(\ell_s+\cdots+\ell_t-i)} \pi^{ei}.$$

Therefore,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \prod_{i=1}^{t} \begin{pmatrix} U_i & 1 \\ U_iV_i + \pi^{e\ell_i} & V_i \end{pmatrix} \tag{18}$$

(6.8) Theorem *If $\underline{a} = (a^i)$ is spaced, then the points in $\mathcal{M}_{\geq \underline{a}}$ of slope $s(|\underline{a}|)$ are dense in $\mathcal{M}_{\geq \underline{a}}$.*

PROOF. We will show that for each point $x = (A_0, \lambda_0, \iota_0) \in \mathcal{M}_{\underline{a}}(k)$, there is a deformation in $\mathcal{M}_{\geq \underline{a}}$ whose generic point has slope $s(|\underline{a}|)$. Let M_0 the Dieudonné module of A_0. Let $R = k[\![t_{i,a^i}]\!], i \in \tau(M_0)$. We construct a deformation M_R by (6.2) with $T_{i,j} = 0$ except for $i \in \tau(M_0)$ and $j = a^i$. Note that $V_i = 0$ in (18). We have (cf. 4.8)

$$\begin{pmatrix} F^f \pi^e X_{n_t} \\ F^f Y_{n_t} \end{pmatrix} = \pi^{|\underline{a}|} A \begin{pmatrix} \pi^e X_{n_t} \\ Y_{n_t} \end{pmatrix}, \quad A = \prod_{i=1}^{t} \begin{pmatrix} U_i & \pi^{e-a^{n_i}} \\ \pi^{e(\ell_i-1)-a^{n_i}} & 0 \end{pmatrix},$$

where $U_i = (c_{n_i} + T_{n_i,a^{n_i}})^{(\ell_{i+1}+\ell_{i+2}+\cdots+\ell_t)}$ for some c_{n_i} in W^{n_i}. We may assume that $e(\ell_i - 1) > a^{n_i}$ for some i, otherwise $|\underline{a}| = \frac{g}{2}$ and there is nothing to prove. Assume that $e(\ell_1 - 1) > a^{n_1}$, we have

$$\begin{aligned} A &\equiv \begin{pmatrix} U_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_2 & * \\ * & 0 \end{pmatrix} \cdots \begin{pmatrix} U_t & * \\ * & 0 \end{pmatrix} \mod \pi \\ &\equiv \begin{pmatrix} P & * \\ 0 & 0 \end{pmatrix} \mod \pi \end{aligned} \tag{19}$$

where P is a nonzero polynomial in U_i's. Hence P is a unit in $W(K)[\pi]$, where K is the perfection of $\mathrm{Frac}(R)$. Therefore $M_K = \mathrm{Cart}_p(K) \otimes_{\mathrm{Cart}_p(R)} M_R$ has slope $s(|\underline{a}|)$. ∎

(6.9) Corollary *The generic point of each irreducible component of $\mathcal{M}_{\geq \underline{a}}$ has slope $\geq s(\lambda(\underline{a}))$, where $\lambda(\underline{a}) := \max\{|\underline{b}|; \underline{b} \subset \underline{a}, \underline{b}$ is spaced$\}$ (cf. [GO, p.112]).*

PROOF. This follows from Proposition 4.8 and Grothendieck's specialization theorem. ∎

(6.10) Remark Goren and Oort showed [GO, Thm. 5.4.11] that when p is inert in F, the inequality $\geq$ in Corollary 6.9 can be replaced by equation $=$. However, the equality does not hold in general. I do not have the sharp formula of the slopes of the generic points of each a-number stratum. When the a-numbers are spaced, the equality can be achieved as it stands in Theorem 6.8. Those are the key cases of [GO, Thm. 5.4.11]. Our proof is different from theirs.

(6.11) Let $x = (A_0, \lambda_0, \iota_0)$ and M_0 be as in (6.2). Suppose that *the reduced (usual) a-number* $t := |\tau|$ *of* x *is one*, where τ is the a-index of M_0. We assume that $\tau = \{0\}$ for simplicity. By Proposition 4.4, we choose a basis for M_0 as in (11) of (6.2).

Let $\ell = \mathrm{ord}_\pi(c_0)$. We have $a^0 = \min\{e, \ell\}$ and $\mathrm{slope}(M_0) = s(\min\{\frac{g}{2}, \ell\})$ by (4.9).

Take $\underline{a} = (0, 0, \dots 0)$ and construct the universal deformation M_R of M_0 for $\mathcal{M}$ by (6.2). It is the Dieudonné module of the formal group attached to the universal formal deformation $(\widetilde{A}, \widetilde{\lambda}, \widetilde{\iota})$ over $\mathcal{M}^\wedge_x = \mathrm{Spf}\, R$. For each $m \in S(g)$, $m \le \ell$, let $\mathcal{M}^{\ge s(m)}_x$ denote the reduced closed subscheme (in $\mathcal{M}^\wedge_x$) of slope $\ge s(m)$. We will find the defining equations of the subscheme $\mathcal{M}^{\ge s(m)}_x$ and show that the generic point of $\mathcal{M}^{\ge s(m)}_x$ has slope $s(m)$.

From (18), we have

$$\begin{pmatrix} F^f X_0 \\ F^f Y_0 \end{pmatrix} = \begin{pmatrix} U_1 & 1 \\ U_1 V_1 + \pi^g & V_1 \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix},$$

where

$$U_1 = T_0 \quad \text{and} \quad V_1 = \sum_{i=1}^{f-1} T_i^{(f-i)} \pi^{ei}.$$

Therefore we have the Cayley-Hamilton equation $F^{2f} X_0 - (U_1^{(f)} + V_1) F^f X_0 - \pi^g = 0$. The subscheme $\mathcal{M}^{\ge s(m)}_x$ is defined by the equations obtained from "$\mathrm{ord}_\pi(U_1^{(f)} + V_1) \ge m$".

From (14) of (6.5), write

$$\begin{aligned} U_1^{(f)} + V_1 &= c_0^{(f)} + \sum_{i=0}^{f-1} \sum_{j=0}^{e-1} T_{i,j}^{(f-i)} \pi^{ei+j} \\ &= c_0^{(f)} + \sum_{i=0}^{g-1} T_k \pi^k \end{aligned} \tag{20}$$

where $T_k := T_{i,j}^{(f-i)}, t_k = t_{i,j}$ for $k = ei + j, 0 \le j \le e-1$. We can see that the defining equations for $\mathcal{M}^{\ge s(m)}_x$ are $t_0 = t_1 = \dots t_{m-1} = 0$. Let K_m be the perfection of the generic residue field of $\mathcal{M}^{\ge s(m)}_x$. The element T_m is a unit in $W(K_m)$, therefore the generic point of $\mathcal{M}^{\ge s(m)}_x$ has slope $s(m)$.

(6.12) Theorem *Let* $x : \mathrm{Spec} k \to \mathcal{M}$ *be a geometric point of reduced a-number one. Then each closed subscheme* $\mathcal{M}^{\ge s(m)}_x$*, for* $m \in S(g)$*,* $s(m) \le \mathrm{slope}(x)$ *(3.3), is formally smooth of codimension* $\lceil m \rceil$ *and its generic point has slope* $s(m)$*. Here* $\lceil m \rceil$ *denotes the smallest integer not less than* m*.*

(6.13) Corollary *The strong Grothendieck conjecture (see [GO. Sec. 5.1]) holds on the open subset of* $\mathcal{M}$ *of reduced a-number* ≤ 1*.*

(6.14) Corollary *The strong Grothendieck conjecture for $\mathcal{M}$ holds when p is totally ramified in F.*

(6.15) Corollary *The weak Grothendieck conjecture (see [GO. Sec. 5.1]) for $\mathcal{M}$ holds.*

PROOF. It follows from Lemma 7.3 that there is a supersingular point of reduced a-number one. Then the assertion follows from Theorem 6.12. ∎

(6.16) *In the rest of this section we assume that p is totally ramified in F.* Denote by $\mathcal{M}_v^{\rm DP}$ the reduction $\mathcal{M}^{\rm DP} \otimes_{\mathbb{Z}} k(v)$ of $\mathcal{M}^{\rm DP}$ modulo v. Let $x = (A_0, \lambda_0, \iota_0) \in \mathcal{M}_v^{\rm DP}(k)$ be a geometric point and let $\mathrm{lie}(A_0) = \{e_1, e_2\}$ and $a(A_0) = \{a_1, a_2\}$. We assume that $e_1 \le e_2$ and $a_1 \le a_2$. Let M_0 be the covariant Dieudonné module of A_0. We can choose two $W[\pi] = \mathcal{O} \otimes W$-bases $\{X_1, X_2\}$, $\{Y_1, Y_2\}$ of M_0 such that

$$VuY_1 = \pi^{e_2} X_1 \quad \text{and} \quad VuY_2 = \pi^{e_1} X_2,$$

where $pu = \pi^e$. We have

$$FX_1 = \pi^{e_1} Y_1 \quad \text{and} \quad FX_2 = \pi^{e_2} Y_2.$$

Write $Y_1 = \alpha X_1 + \beta X_2$ and $Y_1 = \gamma X_1 + \delta X_2$. Then we have

$$\begin{pmatrix} FX_1 \\ FX_2 \end{pmatrix} = \begin{pmatrix} \pi^{e_1}\alpha & \pi^{e_1}\beta \\ \pi^{e_2}\gamma & \pi^{e_2}\delta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{and} \quad (F, V)M = < \pi^{e_1} X_2, \pi^{e_2} X_1, \pi^{e_1}\alpha X_1 >_{W[\pi]} .$$

It follows that $a_1 = e_1$ and $a_2 = \min\{e_2, e_1 + \mathrm{ord}_\pi(\alpha)\}$. As $e_2 \ge \frac{e}{2}$, by the argument in (4.10) we have $\mathrm{slope}(A_0) = s(i)$, where $i = \min\{\frac{e}{2}, e_1 + \mathrm{ord}_\pi(\alpha)\}$. Note that the a-number $a(A_0)$ determines the invariant $\mathrm{lie}(A_0)$ and $\mathrm{slope}(A_0)$: $e_1 = a_1$ and $\mathrm{slope}(A_0) = s(\min\{\frac{e}{2}, a_2\})$.

(6.17) In [DP], Deligne and Pappas have defined a closed algebraic substack $\mathcal{N}_i$ of $\mathcal{M}_v^{\rm DP}$ which classifies the objects of Lie type $\{e_1, e_2\}$ with $e_1 \ge i$. They have shown that the complement $\mathcal{N}_i^o$ of $\mathcal{N}_{i+1}$ in $\mathcal{N}_i$ is a smooth algebraic stack of dimension $e - 2i$ if it is non-empty. It follows from their results and Theorem 7.5 that points of Lie type $\{e_1, e_2\}$ are dense in $\mathcal{N}_{e_1}$. It follows from (6.16) that any point in $\mathcal{N}_{e_1}$ has slope $\ge s(e_1)$. The following lemma confirms the density of points of slope $s(e_1)$ in $\mathcal{N}_{e_1}$.

(6.18) Lemma *If $a_2 > a_1$, then there is a deformation $(\widetilde{A}, \widetilde{\lambda}, \widetilde{\iota})$ over $k[\![t]\!]$ of $(A_0, \lambda_0, \iota_0)$ whose generic point has a-number $(a_1, a_2 - 1)$.*

PROOF. As the forgetful map $\mathrm{Def}[A_0, \lambda_0, \iota_0] \to \mathrm{Def}[A_0, \iota_0]$ induces an equivalence of deformation functors in $\mathcal{N}_{e_1}$, we will construct a deformation of O_F-abelian varieties in $\mathcal{N}_{e_1}$. By the reduction step in [DP, 4.3] and the construction of (6.2), we can construct an $\mathcal{O}$-Dieudonné module M_R over $R := k[\![t]\!]$ of M_0 such that

$$\begin{aligned} FX_1 &= \pi^{e_1}\alpha X_1 + \pi^{e_1}\beta(X_2 + T\pi^{a_2 - e_1 - 1} X_1) \\ FX_2 &= \pi^{e_2}\gamma X_1 + \pi^{e_2}\delta(X_2 + T\pi^{a_2 - e_1 - 1} X_1), \end{aligned} \tag{21}$$

where T is the Teichmülller lift of t. Note that β is a unit in $W[\pi]$: if $\beta \equiv 0 \mod \pi$, then α is a unit and $a_2 = a_1$. By base change to $K := k((t))^{\text{perf}}$, we have

$$(F, V)M_K =< \pi^{e_1} X_2, \pi^{e_2} X_1, (\pi^{e_1}\alpha + \pi^{a_2-1}\beta T)X_1 >_{W(K)[\pi]},$$

thus $a(M_K) = (a_1, a_2 - 1)$. This completes the proof. ∎

(6.19) Theorem *The strong Grothendieck conjecture for $\mathcal{M}_v^{\text{DP}}$ holds when p is totally ramified in F.*

Proof. Let $x \in \mathcal{M}_v^{\text{DP}}(k)$ be a geometric point with a-number (a_1, a_2) and slope $s(i)$, where $i = \min\{\frac{e}{2}, a_2\}$. It suffices to deform the point x to a point of slope $s(i-1)$. If $a_2 > a_1$, then by Lemma 6.18 we can deform to a point of slope $s(i-1)$. Suppose that $a_2 = a_1 = e_1$, we have slope$(x) = s(e_1)$. By (6.17), we can deform to a point of Lie type $\{e_1 - 1, e_2 + 1\}$. As points of slope $s(e_1 - 1)$ are dense in $\mathcal{N}_{e_1-1}$, we can further deform to a point of slope $s(e_1 - 1)$. ∎

(6.20) Corollary *Each Newton stratum of $\mathcal{M}_v^{\text{DP}}$ of slope $s(m)$, $m \in S(g)$ (3.3), has pure dimension $g - \lceil m \rceil$ when p is totally ramified in F.*

Proof. This follows from the purity of Newton strata [dJO] and Theorem 6.19.

(6.21) Remark Corollary 6.20 confirms a question of Chai [C, Question 7.6, p.984] in the Hilbert-Blumenthal cases when p is totally ramified.

§7. An algebraization theorem

(7.1) In this section we do not assume that there is only one prime of the totally real field F lying over p. Let $v_1, \dots, v_s$ be primes of F over p, $F_p := F \otimes \mathbb{Q}_p = F_{v_1} \oplus \cdots \oplus F_{v_s}$, and $\mathcal{O} := O_F \otimes \mathbb{Z}_p = \mathcal{O}_{v_1} \oplus \cdots \oplus \mathcal{O}_{v_s}$. The possible slopes are parameterized by $S(g_1) \times \cdots \times S(g_s)$ (3.3), where $g_i = [F_{v_i} : \mathbb{Q}_p]$. For each O_F-abelian variety A, the associated p-divisible group $A(p) := A[p^\infty]$ has a decomposition $A(p) = A(p)_{v_1} \oplus \cdots \oplus A(p)_{v_s}$.

Let $k(p)$ denote the smallest finite field containing all residue fields $k(v_i)$. Denote by $\mathcal{M}_p^{\text{DP}}$ the reduction $\mathcal{M}^{\text{DP}} \otimes_Z k(p)$ of $\mathcal{M}^{\text{DP}}$ and $\mathcal{M}$ the reduction $\mathcal{M}^R \otimes_Z k(p)$ of the Rapoport locus $\mathcal{M}^R$.

(7.2) A quasi-polarized $\mathcal{O}$-p-divisible group over k is a tuple (H, λ, ι), where H is a p-divisible group, $\lambda : H \to H^t$ an isogeny such that $\lambda^t = -\lambda$, and an algebra homomorphism $\iota : \mathcal{O} \hookrightarrow \text{End}_k(H)$ such that $\lambda\iota(a) = \iota(a)^t\lambda$ for all $a \in \mathcal{O}$. Here H^t stands for the Serre dual of H, equipped with a natural action by $\mathcal{O}$. We identify $(H^t)^t$ with H by the natural isomorphism $H \simeq (H^t)^t$. We call such a tuple (H, λ, ι) over k *algebrisable* if it is attached to a polarized O_F-abelian variety (A, λ_A, ι_A) over k.

(7.3) Lemma *Any supersingular quasi-polarized $\mathcal{O}$-p-divisible group (H, λ, ι) over k is algebrisable.*

PROOF. Let E be a supersingular elliptic curve over k, and $A' := E \otimes O_F$. It is clear that A' satisfies the Rapoport condition. By [R, Prop. 1.10], there exists a separable O_F-linear polarization λ' on A'. Let $(H_1, \lambda_1, \iota_1)$ be the p-divisible group attached to (A', λ', ι'). It follows from Corollary 3.8 that $(H_1, \lambda_1, \iota_1)$ is isogenous to (H, λ, ι). By a theorem of Tate, there is a polarized O_F-abelian variety (A, λ_A, ι_A) whose p-divisible group is isomorphic to (H, λ, ι). ∎

(7.4) Theorem *(1) The weak Grothendieck conjecture for $\mathcal{M}$ holds.*

(2) The strong Grothendieck conjecture for $\mathcal{M}_p^{\mathrm{DP}}$ holds when all residue degrees f_i are one.

PROOF. (1) It follows from Lemma 7.3 that there is a supersingular polarized O_F-abelian variety A such that each component $A(p)_{v_i}$ of the associated p-divisible group $A(p)$ has reduced a-number one. Then the assertion follows from the theorem of Serre-Tate and Theorem 6.12.

(2) This follows from the theorem of Serre-Tate and Theorem 6.19. ∎

(7.5) Theorem *Any quasi-polarized $\mathcal{O}$-p-divisible group (H, λ, ι) over k is algebrisable.*

PROOF. It follows from Theorem 7.4 (1) that any slope in $S(g_1) \times \cdots \times S(g_s)$ can be realized by a point in $\mathcal{M}$. Then the theorem follows from Corollary 3.8. ∎

§8. An example

In this section we construct a separably polarized O_F-abelian scheme A over a complete DVR R whose close fibre A_k does not satisfy the Rapoport condition.

Let $g = 2$ and $p > 3$ be a ramified prime in the totally quadratic real field F. We have $O_F \otimes W(k) = W(k)[\pi], \pi^2 = p$. Let M be a free $W(k)[\pi]$-module generated by e_1, e_2 with the Verschiebung action

$$Ve_1 = \pi e_2, Ve_2 = \pi e_1.$$

and with the alternating form determined by

$$\langle e_1, e_2' \rangle = \langle e_1', e_2 \rangle = 1$$

and other pairing are 0 for the W-basis e_1, e_1', e_2, e_2', where $e_1' := \pi e_1, e_2' := \pi e_2$. By the algebraization theorem, there is a polarized O_F-abelian variety A_0 over k with the prescribed Dieudonné module M.

We have the Hodge filtration of $M/pM = H^1_{DP}(A_0/k)^*$

$$0 \to VM/pM = k\bar{e}'_1 \oplus k\bar{e}'_2 \to M/pM \to M/VM = k\bar{e}_1 \oplus \bar{e}_2 \to 0.$$

Clearly, A_0 does not satisfy the Rapoport condition, as $\pi = 0$ on M/VM.

Let $R := W(k)[\sqrt{p}]$. By Grothendieck-Messing's theory, Serre-Tate's Theorem and Grothendieck's Existence Theorem, we can lift the abelian variety A_0 with the additional structure over R by lifting the Hodge filtration with respect to the addition structure. Let N be the R-submodule of $M \otimes_W R$ generated by $e'_1 + \sqrt{p}e_1$ and $e'_2 - \sqrt{p}e_2$. It is easy to check that N is stable by O_F-action, $N \otimes_R k = VM/pM$ and $\langle N, N \rangle = 0$. Thus, we get a desired polarized O_F-abelian scheme over R.

§9. A computation of the Hecke correspondence

(9.1) Let M_0 be the quasi-polarized $\mathcal{O}$-Dieudonné module of the polarized abelian variety A_0 in the previous section. Let N be the quasi-polarized Dieudonné module containing M_0 with $VN = M$ and $\langle\, , \rangle_N = \frac{1}{p}\langle\, , \rangle$. We have

$$\begin{array}{ll} Ve_1 = \pi e_2 = e'_2 & Fe'_2 = pe_1 \\ Ve_2 = \pi e_1 = e'_1 & Fe'_1 = pe_2 \\ Ve'_1 = \pi V e_1 = pe_2 & Fe_2 = e'_1 \\ Ve'_2 = \pi V e_2 = pe_1 & Fe_1 = e'_2 \end{array}$$

and $\langle e_1, e'_2 \rangle = 1, \langle e'_1, e_2 \rangle = 1$. From $N = \frac{1}{p}FM$, we have

$$N =< e_1, e_2, \frac{1}{p}e'_1, \frac{1}{p}e'_2 >_W .$$

Write $X_1 = e_1, X_2 = e_2, X'_1 = \frac{1}{p}e'_1, X'_2 = \frac{1}{p}e'_2$, we have

$$M_0 =< X_1, X_2, pX'_1, pX'_2 >_W .$$

and $\langle X_1, X'_2 \rangle_N = \langle X'_1, X'_2 \rangle_N = 1$. Denote by $\bar{N}$ the quotient of N modulo pN and $\bar{M}_0$ the image of M_0 in $\bar{N}$. Write x_i, x'_i the image of X_i, X'_i in $\bar{N}$.

(9.2) Let $\mathcal{X}$ be space of π-invariant maximal isotropic "Dieudonné " subspaces of $\bar{N}$ over k

$$\mathcal{X}(k) = \{0 \subset \bar{M} \subset \bar{N}; \dim \bar{M} = 2,\ \pi\bar{M} \subset \bar{M}, F\bar{M} \subset \bar{M}, V\bar{M} \subset \bar{M}, \langle \bar{M}, \bar{M} \rangle_N = 0\}$$

We regard $\mathcal{X}$ as a reduced subscheme over k. It is a closed subscheme of the Lagrangian Grassmanian $\mathrm{L}G(2,4)$, hence a projective variety.

There is a finite morphism $\mathrm{pr} : \mathcal{X} \to \mathcal{M}^{\mathrm{DP}}_p$ (cf. [Y. Sec. 14]) sending $\bar{M}_0 \mapsto (A_0, \lambda_0, \iota_o)$ and the morphism pr factors through the supersingular locus $\mathcal{S}^{\mathrm{DP}}$ of $\mathcal{M}^{\mathrm{DP}}_p$.

We have $\bar{M}_0 =< x_1, x_2 >_k \in \mathcal{X}(k)$. Let $\bar{M}_t \in \mathcal{X}$ be the k-subspace of $\bar{N}$ generated by

$$\tilde{x}_1 = x_1 + t_{11}x'_1 + t_{12}x'_2, \quad \tilde{x}_2 = x_2 + t_{21}x'_1 + t_{22}x'_2.$$

The points $\bar{M}_t$ form a Zariski open neighborhood of $\bar{M}_0$ in $\mathcal{X}$, which we denote by $\mathcal{U}$. We will show that $\mathcal{U} \simeq \mathrm{Spec}k[t_1, t_2, t_3]/(t_1^{p+1} - t_2^{p+1}, t_1^2 + t_2t_3)$.

(9.3) From $\langle \bar{M}_t, \bar{M}_t \rangle_N = 0$, we get $t_{11} + t_{22} = 0$. From $X_i' = \frac{1}{p} e_i' = \frac{1}{\pi} e_i = \frac{1}{\pi} X_i$, we have

$$\pi x_i' = x_i, \ \pi x_i = 0, i = 1, 2.$$

One computes

$$\pi \tilde{x}_1 = t_{11} x_1 + t_{12} x_2 = t_{11} \tilde{x}_1 + t_{12} \tilde{x}_2 - (t_{11}^2 + t_{12} t_{21}) x_1' - (t_{11} t_{12} + t_{12} t_{22}) x_2'$$

and concludes from $\pi \tilde{x}_1 \in \bar{M}_t$ that

$$t_{11}^2 + t_{12} t_{21} = 0, \text{ and } t_{12}(t_{11} + t_{22}) = 0. \tag{22}$$

Similarly from $\pi \tilde{x}_2 \in \bar{M}_t$, one gets

$$t_{22}^2 + t_{12} t_{21} = 0, \text{ and } t_{21}(t_{11} + t_{22}) = 0. \tag{23}$$

For the stability of $\bar{M}_t$ by F and V, one computes

$$F \tilde{x}_1 = t_{11}^p x_2 + t_{12}^p x_1 = t_{11}^p \tilde{x}_2 + t_{12}^p \tilde{x}_1 - (t_{11}^p t_{21} + t_{12}^p t_{11}) x_1' - (t_{11}^p t_{22} + t_{12}^{p+1}) x_2'.$$

and obtains

$$t_{11}^p t_{21} + t_{12}^p t_{11} = 0 \tag{24}$$

$$t_{11}^p t_{22} + t_{12}^{p+1} = 0 \tag{25}$$

Similarly from $F \tilde{x}_2 \in \bar{M}_t$, one obtains

$$t_{21}^{p+1} + t_{22}^p t_{11} = 0 \tag{26}$$

$$t_{21}^p t_{22} + t_{22}^p t_{12} = 0 \tag{27}$$

Applying $V \bar{M}_t \subset \bar{M}_t$, one does not get new equations but (24)–(27). From (25), one has $t_{12} = \alpha t_{11}, \alpha^{p+1} = 1$. From (26), one has $t_{21} = \beta t_{22} = -\beta t_{11}, \beta^{p+1} = 1$. From (22), one has $1 = \alpha\beta$. These parameters satisfy the the equations (24) and (27).

We computed that $\mathcal{U} = \{(t_{11}, t_{12}, t_{21}, t_{22}) = (t, \alpha t, -\frac{1}{\alpha} t, -t); \alpha^{p+1} = 1\}$, hence $\mathcal{U} \simeq \operatorname{Spec} k[t_1, t_2, t_3]/(t_1^{p+1} - t_2^{p+1}, t_1^2 + t_2 t_3)$. Compared with a result of [BG, p. 476, 3.], the morphism pr maps $\mathcal{X}$ *onto* the $(p+1)$ irreducible components of $\mathcal{S}^{\mathrm{DP}}$ containing $(A_0, \lambda_0, \iota_0)$.

(9.4) Proposition *There is a Zariski open neighborhood $\mathcal{V}$ of 0 in $\mathcal{U}$ such that* $\mathrm{pr} : \mathcal{V} \to \mathcal{S}^{\mathrm{DP}}$ *is an étale neighborhood of $\mathcal{S}^{\mathrm{DP}}$ at $(A_0, \lambda_0, \iota_0)$.*

PROOF. Choose a finite étale cover $\mathcal{M}_{p,n}^{\mathrm{DP}} \to \mathcal{M}_p^{\mathrm{DP}}$ by adding a prime-to-p level structure with $n \geq 3$. Choose a lift $(A_0, \lambda_0, \iota_0, \eta_0)$ of $(A_0, \lambda_0, \iota_0)$. Then there is a lift $\widetilde{\mathrm{pr}} : \mathcal{X} \to \mathcal{M}_{p,n}^{\mathrm{DP}}$ which sends $\bar{M}_0$ to $(A_0, \lambda_0, \iota_0, \eta_0)$. The morphism $\widetilde{\mathrm{pr}}$ becomes a closed immersion as the automorphisms invloved are trivial. It again factors through the supersingular locus $\mathcal{S}_n^{\mathrm{DP}}$. As $\mathcal{X}$ and $\mathcal{S}_n^{\mathrm{DP}}$ are reduced schemes, $\mathcal{X}$ is ismorphic to its imagae. The imgage is the union of the $p+1$ irreducible components of $\mathcal{S}^{\mathrm{DP}}$ containing $(A_0, \lambda_0, \iota_0, \eta_0)$. Then there is a Zariski open $\mathcal{V}$ of $\mathcal{U}$ such that the $\mathcal{V}$ is an étale neighborhood of $\mathcal{S}^{\mathrm{DP}}$ at $(A_0, \lambda_0, \iota_0)$. ∎

§10. Superspecial points

(10.1) Shimura's exact mass formula. Let B be a quaternion division algebra over a totally real number field F, and let σ denothe standard involution of B. Let V be a left B-module of rank m, and let φ be a quaternion Hermitian form on V. That is,

$$\varphi : V \times V \to B$$

such that $\varphi(x,y) = \varphi(y,x)^\sigma$ and $\varphi(ax,by) = a\varphi(x,y)b^\sigma$ for all $x,y \in V$ and $a,b \in B$. Let G^φ denote the unitary group of φ. It is a reductive group over F and we will regard it as a reductive algebraic group over $\mathbb{Q}$ via the Weil restriction of scalars from F to $\mathbb{Q}$. Assume that $G^\varphi(\mathbb{R})$ is compact. For any open compact subgroup K of $G^\varphi(\mathbb{A}_f)$, the mass of K is defined as follows. Let $\{c_1, c_2, \dots, c_h\}$ be a set of (complete) representatives of the double coset space $G^\varphi(\mathbb{Q})\backslash G^\varphi(\mathbb{A}_f)/K$, and let $\Gamma_i := G^\varphi(\mathbb{Q}) \cap c_i K c_i^{-1}$. Note that each Γ_i is finite by the assumption of $G^\varphi(\mathbb{R})$. The mass of K is defined to be

$$\mathrm{mass}(K) := \sum_{i=1}^{h} \frac{1}{\#\Gamma_i}.$$

For the general defintion of the mass of K (without the compactness assumption), we refer to [S, Introduction, p. 67]. It follows easily from the definition that for two open compact subgroups $K_1 \subset K_2$, one has $\mathrm{mass}(K_1) = [K_2 : K_1]\mathrm{mass}(K_2)$.

Choose a maximal order O_B of B. Let L be an O_B-lattice in V which is maximal along the lattices on which φ takes its value in O_B. Let K_0 be the maximal compact subgroup of $G^\varphi(\mathbb{A}_f)$ which stabilizes the adele lattice $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. In [S, Introduction, p. 68] Shimura gives an explicit formula

$$\mathrm{mass}(K_0) = |D_F|^{m^2} \cdot \prod_{k=1}^{m} \left\{ D_F^{1/2} \left[(2k-1)!(2\pi)^{-2k}\right]^d \zeta_F(2k) \right\} \prod_{i=1}^{s} \prod_{k=1}^{m} \left\{ (N(\mathfrak{p}_i)^k + (-1)^k \right\}, \tag{28}$$

where D_F is the discriminant of F over $\mathbb{Q}$, d is the degree of F, and $\mathfrak{p}_i$ are the primes of O_F at which B is ramified.

Note that the maximality of L is a local property. That is, L is maximal if and only if each $L_\mathfrak{p}$ is maximal for all primes $\mathfrak{p}$ of F.

(10.2) One can express the exact formula in terms of special values of zeta functions at negative integers. Let F be a totally real field of degree d. Set

$$\Lambda_F(s) := A^s \Gamma(s/2)^d \zeta_F(s), \quad A := D_F^{1/2} \pi^{-d/2}.$$

Then one has the functional equation for $\zeta_F(s)$:

$$\Lambda_F(1-s) = \Lambda_F(s).$$

It gives

$$\zeta_F(s) = D_F^{1/2} \cdot D_F^{-s} \pi^{-\frac{d}{2}+ds} \frac{\Gamma(\frac{1-s}{2})^d}{\Gamma(\frac{s}{2})^d} \cdot \zeta_F(1-s)$$

Let $s = 2k$, one gets

$$\zeta_F(2k) = D_F^{1/2} \cdot D_F^{-2k} \pi^{-\frac{d}{2}+2dk} \frac{\Gamma(\frac{1-2k}{2})^d}{\Gamma(k)^d} \cdot \zeta_F(1-2k).$$

On the other hand, one has

$$\frac{\Gamma(\frac{1-2k}{2})}{\Gamma(k)} = \frac{(-1)^k \cdot \frac{2^{2k-1}(k-1)!}{(2k-1)!} \cdot \Gamma(1/2)}{(k-1)!} = \frac{(-1)^k \cdot 2^{2k-1} \cdot \sqrt{\pi}}{(2k-1)!}.$$

The term in (28) can be expressed

$$\begin{aligned} & D_F^{1/2} \left[(2k-1)!(2\pi)^{-2k}\right]^d \zeta_F(2k) \\ =& D_F^{1-2k} \left[(2k-1)!(2\pi)^{-2k}\right]^d \cdot \left[\frac{(-1)^k \cdot 2^{2k-1} \cdot \sqrt{\pi}}{(2k-1)!}\right]^d \cdot \pi^{-\frac{d}{2}+2dk} \cdot \zeta_F(1-2k) \\ =& D_F^{1-2k} \left[\frac{(-1)^k}{2}\right]^d \cdot \zeta_F(1-2k). \end{aligned} \tag{29}$$

Hence we get

$$\mathrm{mass}(K_0) = \prod_{k=1}^{m} \left\{ \left[\frac{(-1)^k}{2}\right]^d \cdot \zeta_F(1-2k) \right\} \cdot \prod_{i=1}^{s} \prod_{k=1}^{m} \{(N(\mathfrak{p}_i)^k + (-1)^k\} \tag{30}$$

(10.3) If $F = \mathbb{Q}$ and B is the quaternion algbera which is ramified at $\{\infty, p\}$, then we have

$$\mathrm{mass}(K_0) = \frac{(-1)^{m(m+1)/2}}{2^m} \left\{ \prod_{k=1}^{m} \zeta(1-2k) \right\} \cdot \prod_{k=1}^{m} \{(p^k + (-1)^k\} \tag{31}$$

If $m = 1$ and B is ramified only at the primes over p and infinite places, then we have

$$\mathrm{mass}(K_0) = \left[\frac{-1}{2}\right]^d \cdot \zeta_F(-1) \cdot \prod_{i=1}^{s} \{(N(\mathfrak{p}_i) + (-1)\} \tag{32}$$

(10.4) Let $x = (A_0, \lambda_0, \iota_0)$ be a supersingular polarized O_F-abelian variety over k of dimension $g = md$. Let G_x denote the group scheme over $\mathrm{Spec}\mathbb{Z}$ whose group of R-points, for each commutative ring R, is

$$G_x(R) = \{\phi \in (\mathrm{End}_{O_F}(A_0) \otimes R)^\times ; \phi'\phi = 1\},$$

where the map $\phi \mapsto \phi'$ is the Rasoti involution induced by λ_0.

Let Λ_x denote the set of the isomorphism classes of supersingular polarized O_F-abelian varieties (A, λ, ι) of dimension g over k such that

(i) the Dieudonné module $M(A)$ is isomorphic to $M(A_0)$ as quasi-polarized $\mathcal{O}$-Dieudonné modules, and

(ii) the Tate module $T_\ell(A)$ is isomorphic to $T_\ell(A_0)$ as non-degenerate alternating $O_F \otimes \mathbb{Z}_\ell$-modules for all $\ell \neq p$.

Let Λ'_x be the subset of Λ_x consists of objects (A, λ, ι) such that

(iii) there exists an element $\phi \in \mathrm{Hom}_{O_F}(A_0, A) \otimes \mathbb{Q}$ such that $\phi^*\lambda = \lambda_0$.

(10.5) Theorem *There is a natural bijection of pointed sets between Λ'_x and $G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}})$.*

PROOF. For any $(A, \lambda, \iota) \in \Lambda'_x$, there is an $\mathbb{Q}$-polarized O_F-linear isogeny $\phi : (A_0, \lambda_0, \iota_0) \to (A, \lambda, \iota)$, by definition. Then there is an $\phi_\ell \in G_x(\mathbb{Q}_\ell)$ for all primes ℓ such that $\phi\phi_\ell \in \mathrm{Isom}(A_0(\ell), A(\ell))$. The element ϕ_ℓ lies in $G_x(\mathbb{Z}_\ell)$ for almost all ℓ and it is determined by ϕ up to a right translation of $G_x(\mathbb{Z}_\ell)$. Therefore, ϕ gives an element $(\phi_\ell) \in G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}})$. Let ϕ' be another $\mathbb{Q}$-polarized O_F-linear isogeny from A_0 to A. Then we have $\phi' g = \phi$ for some $g \in G_x(\mathbb{Q})$ and $(\phi'_\ell)G_x(\hat{\mathbb{Z}}) = (g\phi_\ell)G_x(\hat{\mathbb{Z}})$. Therefore, we get a well-defined map σ from Λ'_x to $G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}})$.

Proof of injectivity. Let $\alpha : (A_0, \lambda_0, \iota_0) \to (A, \lambda_A, \iota_A)$ and $\beta : (A_0, \lambda_0, \iota_0) \to (B, \lambda_B, \iota_B)$ be two $\mathbb{Q}$-polarized O_F-linear isogenies such that $\sigma(\alpha) = \sigma(\beta)$. Then there exist $\phi \in G_x(\mathbb{Q})$ and $(\phi_\ell) \in G_x(\mathbb{A}_f)$ such that $\alpha\phi_\ell \in \mathrm{Isom}(A_0(\ell), A(\ell))$ and $\beta\phi\phi_\ell \in \mathrm{Isom}(A_0(\ell), B(\ell))$ for all ℓ. The polarized O_F-quasi-isogeny $\beta\phi\alpha^{-1} = \beta\phi\phi_\ell(\alpha\phi_\ell)^{-1} : A \to B$ is an isomorphism at every place, hence it is an isomorphism.

Proof of surjectivity. Given an element $[(\phi_\ell)] \in G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}})$, then there is a positive integer N such that $N\phi_\ell : A_0(\ell) \to A_0(\ell)$ is an *isogeny* for all ℓ. Let H be the product of all finite group $\ker(N\phi_\ell)$. Since each $N\phi_\ell$ is O_F-linear, H is stable under the O_F-action. Take $A := A_0/H$, then there is a natural O_F-action, ι, on A such that the projection $\pi : A_0 \to A$ is an O_F-linear morphism. The degree of π is N^{2g}. The polarization $N\lambda_0$ descends to a fractional polarization λ on A. Note that we have the induced isomorphism $A(\ell) \to A_0(\ell)$ of quasi-polarized ℓ-divisible groups. Therefore the polarization λ is ℓ-adic integral for all ℓ, hence it is a polarization on A. Therefore, $N^{-1}\pi : (A_0, \lambda_0, \iota_0) \to (A, \lambda, \iota)$ gives an element in Λ'_x and it induces the element $[(\phi_\ell)]$. ∎

(10.6) Proposition (J.-K. Yu) *Let $(A_i, \lambda_i, \iota_i), i = 1, 2$ be two supersingular polarized O_F-abelian varieties over k. Then there exists $\varphi \in \mathrm{Hom}_{O_F}(A_1, A_2) \otimes \mathbb{Q}$ such that $\varphi^*\lambda_2 = \lambda_1$. That is, the condition (iii) of (10.4) is satisfied automatically.*

PROOF. Choose an isogeny $\varphi : A_1 \to A_2$. By Noether-Skolem's thoerem, φ can be chosen to be O_F-linear. Consider the O_F-linear polarizations λ_1 and $\lambda'_1 := \varphi^*\lambda_2$ on A_1. Those induce two positive involutions $*_1$ and $*'_1$ on the semi-simple algebra $B := \mathrm{End}(A_1) \otimes \mathbb{Q}$. By [Z3, Satz 1.1], there exists a positive element $c \in B^\times$, $c = c^{*_1}$, such $x^{*'_1} = c^{-1}x^{*_1}c, \forall x \in B$. Let $x \in F$ then we get $c \in C := \mathrm{End}_{O_F}(A_1) \otimes \mathbb{Q}$. Consider the algebraic variety P over $\mathbb{Q}$ defined by $\{X \in C; X^{*_1}X = c\}$. The algebraic group $G_1 := \{g \in C^\times; g^{*_1}g = 1\}$ over $\mathbb{Q}$ acts on P on the left. The variety P is a G_1-torsor, hence associates an element ξ in $H^1(\mathbb{Q}, G_1)$. By [K2, Lemma 2.11], $P(\mathbb{R}) \neq \emptyset$. Since G_1 is semi-simple and simply-connected, $H^1(\mathbb{Q}_p, G_1) = 0$. Therefore ξ is locally trivial everywhere. By Hasse principle for G_1, ξ is trivial. Then there exists $g \in G_1(\mathbb{Q})$ such that $g^{*_1}g = c$. Replacing φ by φg, we have $\lambda_1' = q\lambda_1$ for some $q \in \mathbb{Q}^\times$. Note that q is positive. It follows from the fact that the polarizations lie in the positive cone of the Néron-Severi group $\mathrm{NS}(A_1) \otimes \mathbb{R}$. Therefore q is a norm and we we can find a $\varphi \in \mathrm{Hom}_{O_F}(A_1, A_2) \otimes \mathbb{Q}$ such that $\varphi^*\lambda_2 = \lambda_1$. This completes the proof.

(10.7) Corollary $\Lambda_x' = \Lambda_x$.

(10.8) Lemma *Let* $(A, \lambda, \iota) \in \Lambda_x$ *and* $[c]$ *be the correspoding double coset. Then* $\mathrm{Aut}(A, \lambda, \iota) \simeq \Gamma_c$*, where* $\Gamma_c := G_x(\mathbb{Q}) \cap cG_x(\hat{\mathbb{Z}})c^{-1}$.

PROOF. Let G' be the group scheme over $\mathrm{Spec}\mathbb{Z}$ that represents the group functor

$$R \mapsto \{\alpha \in (\mathrm{End}_{O_F}(A) \otimes R); \alpha'\alpha = 1\},$$

where R is a commutative ring and the map $\alpha \mapsto \alpha'$ is the Rosati involution induced by λ.

Choose a map $\phi \in \mathrm{Hom}_{O_F}(A_0, A) \otimes \mathbb{Q}$ such that $\phi^*\lambda = \lambda_0$. Then the element $c \in G_x(\mathbb{A}_f)$ has the property $(*)$: $\phi c_\ell \in \mathrm{Isom}((A_0(\ell), \lambda_0, \iota_0), (A(\ell), \lambda, \iota))$, $\forall\, \ell$. Note that $\alpha \in \mathrm{Aut}(A, \lambda, \iota)$ if and only if $\alpha \in G'(\mathbb{Q})$ and $\alpha_\ell \in \mathrm{Aut}(A(\ell), \lambda, \iota)$, $\forall\, \ell$.

The map ϕ gives an isomorphism $G_x(\mathbb{Q}) \to G'(\mathbb{Q})$ which sends β to $\phi\beta\phi^{-1} =: \alpha$. From $(*)$, we have $\alpha \in G'(\hat{\mathbb{Z}})$ if and only if $(\phi c)^{-1}\alpha(\phi c) \in G_x(\hat{\mathbb{Z}})$. Hence, $\alpha \in G'(\hat{\mathbb{Z}})$ if and only if $c^{-1}\beta c \in G_x(\hat{\mathbb{Z}})$. This completes the proof. ∎

(10.9) Corollary

$$\mathrm{mass}(\Lambda_x) := \sum_{(A,\lambda,\iota)\in\Lambda_x} \frac{1}{\mathrm{Aut}(A, \lambda, \iota)} = \mathrm{mass}(G_x(\hat{\mathbb{Z}})).$$

(10.10) The semi-simple algebra $C_x := \mathrm{End}_{O_F}(A_0) \otimes \mathbb{Q}$ is the centralizer of F into the simple algebra $M_g(\mathrm{End}^(E))$, where E is a supersingular elliptic curve. We have $C_x \simeq M_m(B)$, where B is a quaternion algebra over the totally real field F ramified at all real places and unramified at all primes not dividing p. From local invariants, one has $B \simeq F \otimes \mathrm{End}^0(E)$. Therefore,

$$\begin{aligned} &C_x \otimes \mathbb{R} \simeq M_m(\mathbb{H}) \\ &C_x \otimes \mathbb{Q}_\ell \simeq M_{2m}(F \otimes \mathbb{Q}_\ell), \quad \ell \neq p \\ &C_x \otimes \mathbb{Q}_p \simeq \prod_{\mathfrak{p}|p,\, f(\mathfrak{p}):\mathrm{odd}} M_m(B_\mathfrak{p}) \times \prod_{\mathfrak{p}|p,\, f(\mathfrak{p}):\mathrm{even}} M_{2m}(F_\mathfrak{p}). \end{aligned} \tag{33}$$

The Rosati involution induces the standard involution on $M_m(\mathbb{H})$ and $M_m(B_\mathfrak{p})$, which we will denote by $*$. It also induces the symplectic involution on $M_{2m}(F \otimes \mathbb{Q}_\ell)$ and $M_{2m}(F_\mathfrak{p})$.

Therefore, we have

$$
\begin{aligned}
&G_x(\mathbb{R}) \simeq \{g \in M_m(\mathbb{H}); g^* g = 1\},\\
&G_x(\mathbb{Q}_\ell) \simeq \mathrm{Sp}_{2m}(F \otimes \mathbb{Q}_\ell), \quad \ell \neq p\\
&G_x(\mathbb{Q}_p) \simeq \prod_{\mathfrak{p}|p,\, f(\mathfrak{p}):\mathrm{odd}} G_{\mathfrak{p}} \times \prod_{\mathfrak{p}|p,\, f(\mathfrak{p}):\mathrm{even}} \mathrm{Sp}_{2m}(F_{\mathfrak{p}}),
\end{aligned}
\tag{34}
$$

where $G_{\mathfrak{p}} = \{g \in M_m(B_{\mathfrak{p}}); g^* g = 1\}$. We will choose such isomorphisms and replace $\simeq$ by $=$.

(10.11) Lemma *There exists a left B-module V, a quaternion Hermittian from φ on V, and a maximal lattice L in the sense of Shimura (10.1) such that $G^\varphi \simeq G_x$ over $\mathbb{Q}$ and K_0 defined in (10.1) is*

$$K_0 = \prod_{\ell \neq p} \mathrm{Sp}_{2m}(O_F \otimes \mathbb{Q}_\ell) \times \prod_{\mathfrak{p}|p,\, f(\mathfrak{p}):odd} K_{\mathfrak{p}} \times \prod_{\mathfrak{p}|p,\, f(\mathfrak{p}):even} \mathrm{Sp}_{2m}(O_{\mathfrak{p}}), \tag{35}$$

where $O_{\mathfrak{p}}$ is the ring of integers in $F_{\mathfrak{p}}$ and $K_{0,\mathfrak{p}} = \{g \in M_m(O_{B_{\mathfrak{p}}}); g^* g = 1\}$.

PROOF. Let $V = B^{\oplus g}$ with $\varphi(\underline{x}, \underline{y}) = \sum x_i \bar{y}_i^\sigma$ and let $L = O_B^{\oplus g}$. ∎

(10.12) The method to compute $\mathrm{mass}(G_x(\hat{\mathbb{Z}}))$ is first to relate with the mass of the standard one K_0 and then to apply Shimura's mass formula. For any two open compact subgroups K_1, K_2 of $G_x(\mathbb{A}_f)$, of $G_x(\mathbb{Q}_p)$, or of $G_x(\mathbb{Q}_\ell)$, we denote $\mu_\bullet(K_2/K_1) := [K_1 : K_1 \cap K_2]^{-1}[K_2 : K_a \cap K_2]$, for $\bullet = f,\ p$ or ℓ.

We have the following properties

- $\mu_\ell(K_2/K_1) = 1$ for almost all ℓ.
- $\mu_\bullet(K_2/K_1) = [K_2 : K_1]$ if $K_1 \subset K_2$.
- If $K_1 = \prod_\ell K_{1,\ell}$ and $K_2 = \prod_\ell K_{2,\ell}$, where ℓ runs over all primes of $\mathbb{Q}$, then $\mu_f(K_2/K_1) = \prod_\ell \mu_\ell(K_{2,\ell}/K_{1,\ell})$.
- $\mathrm{mass}(K_1) = \mu_f(K_2/K_1) \cdot \mathrm{mass}(K_2)$.

(10.13) Lemma *Let K_1, K_2 be two maximal open compact subgroups of $\mathrm{Sp}_{2m}(F_v)$, where F_v is a non-archimedean local field. Then $\mu_v(K_2/K_1) = 1$.*

PROOF. This is obvious. ∎

(10.14) Theorem *The mass $\mathrm{mass}(\Lambda_x)$ denpends only on the Dieudonné module M_0 of x. Let K_{M_0} denote the group of automorphisms of the quasi-polarized $\mathcal{O}$-Dieudonné module M_0. Then*

$$\mathrm{mass}(\Lambda_x) = \mu_p(K_{0,p}/K_{M_0}) \cdot \mathrm{mass}(K_0).$$

(10.15) In the rest of this section we will concentrate the most interesting cases when M_0 is superspecial. There are two main reasons to do this. One is that there are only finitely many isomorphism classes of quasi-polarized superspecial $\mathcal{O}$-Dieudonné modules. It is possible to list all of them and calculate the groups of automorphisms. The other one, which is more important, is that the irreducible components of the supersingular locus is ralated to certain superspecial points. This connection has been achieved by canonical isogenies. Therefore, the computation of certain superspecial points will give the number of the irreducible components.

What we will do is to classify the Dieudonné modules in question and compute the groups of automorphisms. Some special cases of them has been done by Li and Oort, and the author. We start with these resuts.

In (10.16)-(10.21), we consider the case $m = g$ and $d = 1$, hence $B = \mathrm{End}^0(E)$.

(10.16) Proposition (LO, Proposition 6.1) *Let M be a W-rank $2g$ quasi-polarized superspecial Dieudonné module over k. Then one can decompose*

$$M \simeq M_1 \oplus M_2 \oplus \cdots \oplus M_d, \quad (\langle M_i, M_j \rangle = 0 \text{ if } i \neq j)$$

where each M_i is either of the following types:

(i) M_i is a W-rank two superspecial Dieudonné submodule generated by some x such that $\langle x, Fx \rangle = p^r \epsilon$ for some $r \in \mathbb{Z}$ and $\epsilon \in W^\times$ with $\epsilon^\sigma = -\epsilon$; or

(ii) M_i is a W-rank four superspecial Dieudonné submodule generated by some x, y such that $\langle x, y \rangle = p^r$ for some $r \in \mathbb{Z}$, and $\langle x, Fx \rangle = \langle y, Fy \rangle = \langle x, Fy \rangle = \langle y, Fx \rangle = 0$.

From the proof of the proposition above, we can deduce the following result.

(10.17) Proposition (E, Proposition 5.1) *Let M be a W-rank $2g$* separably *quasi-polarized superspecial Dieudonné module over k. Then $M \simeq M_1^{\oplus g}$ with M_1 of type (i) above.*

(10.18) Corollary *Let M be as in Propostion 10.17. Then the group K_M of the automorphisms of M is*

$$K_M = \{\alpha \in M_g(O_{B_p}); \alpha^* \alpha = 1\},$$

where B_p is the quaternion division algebra over $\mathbb{Q}_p$.

(10.19) Corollary *Let $x = (A_0, \lambda_0)$ be a separably polarized superspecial abelian variety over k and M_0 its Dieudonné module. Then $K_{M_0} = K_0$ and*

$$\mathrm{mass}(\Lambda_x) := \sum_{(A,\lambda)\in\Lambda_x} \frac{1}{\mathrm{Aut}(A,\lambda)} = \frac{(-1)^{g(g+1)/2}}{2^g} \left\{ \prod_{k=1}^{g} \zeta(1-2k) \right\} \cdot \prod_{k=1}^{g} \left\{ (p^k + (-1)^k \right\}.$$

In this case, Λ_x is the set of the isomorpshism classes of separably polarized superspecial abelian varieties over k such that the condition (ii) of (10.4) is satisfied.

(10.20) Corollary

$$\sum_{(A,\lambda)\in\Lambda} \frac{1}{\mathrm{Aut}(A,\lambda)} = \frac{(-1)^{g(g+1)/2}}{2^g} \left\{ \prod_{k=1}^{g} \zeta(1-2k) \right\} \cdot \prod_{k=1}^{g} \left\{ (p^k + (-1)^k \right\}, \tag{36}$$

where Λ is the set of the isomorpshism classes of principally polarized superspecial abelian varieties over k of dimension g.

PROOF. Take $x = (A_0, \lambda_0)$ to be a principally polarized superspecial abelian variety over k of dimension g. Then we have $\Lambda = \Lambda_x$. ∎

(10.21) In [LO, Sec. 4.9], the number of irreducible components of the supersingular locus of the $\mathcal{A}_{g,1} \otimes k$ is $\#\Lambda$. Here Λ is the set of the isomorphism classes of polarized superspecial abelian varieties (A, λ) of dimension g over k such that $\ker \lambda = A[F^{g-1}]$. They also show that

$$\#\Lambda = \begin{cases} H_g(p,1) & \text{if } g \text{ is odd,} \\ H_g(1,g) & \text{if } g \text{ is even.} \end{cases}$$

Here $H_g(p, 1)$ (resp. $H_g(1, g)$) is the principal genus (resp. the non-principal genus) of the Hermitian space B^g.

If g is odd, then we can divide the polarization by a power of p such that it is principal. This computation has been done. When g is even, by again dividing by a power of p, we can consider the case that the polarization λ satisfies the property $\ker \lambda = A[F]$. All we need to compute is the group K_M of automorphisms and compute $\mu_p(K_{0,p}/K_M)$.

From the previous results, it seems saying that we have $K_M = \mathrm{Aut}(N_g)$, where

$$N_g = O_{B_p}^{\oplus g} \cdot \gamma, \quad \gamma = \begin{pmatrix} I_{g/2} & 0 \\ 0 & \pi I_{g/2} \end{pmatrix} \cdot \xi \in \mathrm{GL}_g(B_p), \quad \xi \bar{\xi}^t = \text{anti-diag}\{1, 1, \dots, 1\}.$$

(10.22) Lemma $\mu_p(K_{0,p}/\,\mathrm{Aut}(N_g)) = 1$

PROOF. One can choose a left and right Haar measure μ on $G_x(\mathbb{Q}_p)$. Then $\mu_p(K_{0,p}/\,\mathrm{Aut}(N_g)) = \mu(K_{0,p})/\mu(\mathrm{Aut}(N_g))$. On the other hand, $N_g = \gamma^{-1} K_{0,p} \gamma$ and $\mu(\gamma^{-1} K_{0,p} \gamma) = ||g^{-1}|| \cdot ||g|| \mu(K_{0,p}) = \mu(K_{0,p})$. ∎

I am not sure if I have the computation wrong. The computation shows that one have one only mass formula not depending on the parity of g.

Let $\mathcal{A}_{g,1,n}$, $n \geq 3$, be the moduli scheme of principally polarized abelian varieties with full symplectice n-level structure. Let $\mathcal{S}_{g,1,n}$ be the supersingular locus of $\mathcal{A}_{g,1,n} \otimes \overline{\mathbb{F}_p}$.

(10.23) Proposition *The number of irreducible components of the supersingular locus $\mathcal{S}_{g,1,n}$ is*

$$\mathrm{Sp}(\mathbb{Z}/n\mathbb{Z}) \cdot \frac{(-1)^{g(g+1)/2}}{2^g} \left\{ \prod_{k=1}^{g} \zeta(1-2k) \right\} \cdot \prod_{k=1}^{g} \left\{ (p^k + (-1)^k \right\}.$$

PROOF. This is a reformulation of a result of Li and Oort by the mass formula.

(10.24) We turn to the focus on the Hilbert-Blumenthal cases. That is, the case that $m = 1$ and $d = g$. Let $(A_0, \lambda_0, \iota_0)$ be a superspecial point satisfying the Rapoport condition, and let M_0 be its covariant Dieudonné module. We have $M_0 = \oplus_{\mathfrak{p}|p} M_{\mathfrak{p}}$ and $K_{M_0} = \prod_{\mathfrak{p}|p} K_{M_{\mathfrak{p}}}$.

(10.25) Lemma *(1) If the residue degree $f(\mathfrak{p})$ is odd, then the a-number $a(M_{\mathfrak{p}}) = (e, e, \dots, e)$ and $K_{M_{\mathfrak{p}}} = K_{0,\mathfrak{p}}$.*

(2) If the residue degree $f(\mathfrak{p})$ is even, then there are at least two possibilities:

(i) the a-number $a(M_{\mathfrak{p}})$ is either $(2e, 0, 2e, 0, \dots, 2e, 0)$ or $(2e, 0, 2e, 0, \dots, 2e, 0)$. In either case, $K_{M_{\mathfrak{p}}} = K_{0,\mathfrak{p}}$;

(ii) the a-number $a(M_{\mathfrak{p}}) = (e, e, \dots, e)$ and

$$K_{M_{\mathfrak{p}}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(O_{F_{\mathfrak{p}}}); c \equiv 0 \mod \pi \right\}.$$

In this case we have $K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}} \simeq \mathbf{P}^1(k(\mathfrak{p}))$ and

$$\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}}) = \#\mathbf{P}^1(k(\mathfrak{p})) = N(\mathfrak{p}) + 1 = p^{f(\mathfrak{p})} + (-1)^{f(\mathfrak{p})}.$$

We will give a proof. In fact, we will use (i) and (ii) of (2).

(10.26) Corollary *Let $x = (A_0, \lambda_0, \iota_0)$ be a separably polarized superspecial O_F-abelian variety over k in the Rapoport locus, and let M_0 be its Dieudonné module. Then*

$$\mathrm{mass}(\Lambda_x) := \sum_{(A,\lambda,\iota)\in\Lambda_x} \frac{1}{\mathrm{Aut}(A, \lambda, \iota)} = \left[\frac{-1}{2}\right]^g \cdot \zeta_F(-1) \cdot \prod_{\mathfrak{p}|p} \{p^{f(\mathfrak{p})} + (-1)^{f(\mathfrak{p})}\} \tag{37}$$

In this case, Λ_x is the set of the isomorpshism classes of separably polarized superspecial O_F-abelian varieties over k in the Rapoport locus such that the condition (ii) of (10.4) is satisfied.

(10.27) Corollary

$$\sum_{(A,\lambda,\iota)\in\Lambda} \frac{1}{\mathrm{Aut}(A, \lambda, \iota)} = \left[\frac{-1}{2}\right]^g \cdot \zeta_F(-1) \cdot \prod_{\mathfrak{p}|p} \{p^{f(\mathfrak{p})} + (-1)^{f(\mathfrak{p})}\} \tag{38}$$

where Λ is the set of the isomorpshism classes of principally polarized superspecial O_F-abelian varieties over k of dimension g which satisfy the Rapoport condition.

PROOF. Take $x = (A_0, \lambda_0, \iota_0)$ to be a principally polarized superspecial O_F-abelian variety over k of dimension g in the Rapoport locus. Then we have $\Lambda = \Lambda_x$. ∎

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