AN INTEGRABLE SYSTEM OF K3-FANO FLAGS

D. MARKUSHEVICH

Abstract. Given a K3 surface S, we show that the relative intermediate Jacobian of the universal family of Fano 3-folds V containing S as an anticanonical divisor is a Lagrangian fibration.

Introduction

Beauville [B2] defined the moduli stack $\mathcal{F}_g^R$ of K3-Fano flags $S \subset V$, where S is a K3 surface and V an R-polarized Fano 3-fold containing S as an anticanonical divisor. Here R is an even integral lattice of signature $(1, \mathrm{rk}(R)-1)$ with a distinguished element ρ, and an R-polarization of a Fano variety V is an isomorphism $R \to \mathrm{Pic}(V)$ which sends ρ to $-K_V$ and such that the bilinear form on R is the pullback of the form $(D_1, D_2) \mapsto (-K_V \cdot D_1 \cdot D_2)$ on $\mathrm{Pic}(V)$. The subscript g is an integer parameter, defined by $2g-2 = (-K_V)^3$. Beauville also showed that the natural map $s_g^R : \mathcal{F}_g^R \to \mathcal{K}_g^R$ to the moduli stack $\mathcal{K}_g^R$ of R-polarized K3 surfaces with a very ample class of degree $2g-2$ is generically surjective.

The relative dimension of s_g^R at (S, V) is $b_3(V)/2$, and the fiber $\mathcal{F}_S^R := (s_g^R)^{-1}(S)$ is a smooth stack. Let $\mathcal{V} \to \mathcal{F}_S^R$ be the universal family of Fano 3-folds over $\mathcal{F}_S^R$ and $\mathcal{J} = J(\mathcal{V}/\mathcal{F}_S^R)$ its relative intermediate Jacobian. In this note, we prove that $\mathcal{J}$ is holomorphically symplectic locally over $\mathcal{F}_S^R$ and that the structure morphism $\mathcal{J} \to \mathcal{F}_S^R$ is a Lagrangian fibration.

This result has been proved earlier by different methods in some particular cases. Hassett–Tschinkel ([HT], Proposition 7.1) defined a Lagrangian fibration on the length-2 punctual Hilbert scheme $S^{[2]}$ of a K3 surface S which is a complete intersection of quadrics, and their construction implies the identification of $\mathcal{F}_S^R$, resp. $\mathcal{J}$ with an open subset in $\mathbb{P}^2$, resp. $S^{[2]}$. The authors of [IM] proved the result for $3 \le g \le 10$, S generic and $R = \mathbb{Z}\rho$. Their construction is a generalization of that of [HT]: $\mathcal{J}$ is identified with the relative Picard variety of a family of Lagrangian suvarieties of $S^{[2]}$.

The idea of constructing Lagrangian fibrations from a complete family of Fano 3-folds containing a fixed K3 surface appeared also in [Tyu], [B1], where the authors showed that for a K3-Fano flag (S, V), the moduli spaces of stable vector bundles on V are Lagrangian subvarieties of appropriate moduli spaces of vector bundles on S. In this way, a Lagrangian fibration $\mathcal{M} \to \mathcal{F}_S^R$ is constructed in [B1] for the K3 surface S which is a complete intersection of a quadric and a cubic ($g = 4$) and $R = \mathbb{Z} \cdot \frac{1}{2}\rho$. Its phase space $\mathcal{M}$ is a family of moduli spaces M_t of vector bundles on the cubic 3-folds V_t containing S, and each M_t is isomorphic to an open subset of the intermediate Jacobian $J(V_t)$. This fibration is a torsor under the associated intermediate Jacobian fibration $\mathcal{J} \to \mathcal{F}_S^R$, and it is plausible that the symplectic structure on $\mathcal{M}$ is induced by that of $\mathcal{J}$.

In Sect. 1, we recall the Donagi–Markman criteria for the existence of a quasi-Lagrangian and Lagrangian structures. In Sect. 2, we gather basic facts on the variations of Hodge structures necessary to verify the weak cubic condition of Donagi–Markman. We verify it in Sect. 3, which provides $\mathcal{J}$ with a quasi-Lagrangian structure. To get a genuine Lagrangian structure, we need variations of mixed Hodge structures, which are sketched in Sect. 4. Finally, in Sect. 5, we

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verify the sufficient condition of Donagi–Markman for the existence of a Lagrangian structure on $\mathcal{J}$.

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1. Donagi–Markman cubic

We will start by a reminder on the Kodaira-Spencer map. Let $f : \mathcal{V} \to B$ be a family of compact complex manifolds. The space of infinitesimal deformations of any special fiber $V = V_b$ of f ($b \in B$) over the Artinian ring $\mathbb{C}[\epsilon]/(\epsilon^2)$ is identified with $H^1(V, \mathcal{T}_V)$. Any tangent vector $a \in T_b B$ defines such an infinitesimal deformation, so we have the natural map $KS(f)_b : T_b B \to H^1(V, \mathcal{T}_V)$, called Kodaira–Spencer map. These maps fit into the bundle map $KS(f) : \mathcal{T}_B \to R^1 f_* \mathcal{T}_{\mathcal{V}/B}$, where $\mathcal{T}_{\mathcal{V}/B}$ denotes the vertical tangent bundle of f. Assume that the base B is Stein, then $KS(f)$ can be interpreted as the extension class of the tangent bundle sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{V}/B} \longrightarrow \mathcal{T}_{\mathcal{V}} \longrightarrow f^* \mathcal{T}_B \longrightarrow 0 \tag{1}$$

via the identifications

$$\mathrm{Ext}^1(f^*\mathcal{T}_B, \mathcal{T}_{\mathcal{V}/B}) = H^1(\mathcal{V}, f^*\mathcal{T}_B^\vee \otimes \mathcal{T}_{\mathcal{V}/B}) = H^0(B, \mathcal{T}_B^\vee \otimes R^1 f_* \mathcal{T}_{\mathcal{V}/B}).$$

Replacing the V_b by their intermediate Jacobians $J^k(V_b)$, we obtain a family $\pi = J^k f : \mathcal{X} \to B$ of complex tori, and its Kodaira–Spencer class $KS(\pi)$ is related to $KS(f)$ by means of the Gauss–Manin connection, see Sect. 2.

Given a family of abelian varieties or complex tori $\pi : \mathcal{X} \to B$, we will say that it is quasi-Lagrangian if $\dim B = \frac{1}{2} \dim \mathcal{X}$ and there exists, locally over B, a nondegenerate holomorphic 2-form ω on $\mathcal{X}$ such that the fibers of π are maximal isotropic with respect to ω. If moreover ω is closed, and hence symplectic, then we will say that π is Lagrangian. In [DM] Donagi and Markman determine conditions on π which are necessary and/or sufficient for the existence a (quasi-)Lagrangian structure. Let $\mathcal{E}$ denote the direct image $\pi_* \mathcal{T}_{\mathcal{X}/B}$ on B of the vertical tangent bundle of $\mathcal{X}$. As the fibers of π are isotropic, ω induces an isomorphism $j : \mathcal{T}_B^\vee \to \mathcal{E}$. If we assume, as before, that B is Stein, then the tangent bundle sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{X}/B} \longrightarrow \mathcal{T}_{\mathcal{X}} \longrightarrow \pi^* \mathcal{T}_B \longrightarrow 0 \tag{2}$$

determines the class

$$KS(\pi) \in H^0(B, \mathcal{T}_B^\vee \otimes R^1\pi_* \mathcal{T}_{\mathcal{X}/B}) = H^0(B, \mathcal{T}_B^\vee \otimes R^1 \pi_* \mathcal{O}_{\mathcal{X}} \otimes \mathcal{E}).$$

We will say that a complex torus $\mathbb{C}^g/\Lambda$ is quasi-polarized if Λ has a skew-symmetric bilinear form satisfying the first Riemann bilinear relation (the symmetry of the period matrix), but not necessariliy the second one (positive definiteness of the imaginary part of the period matrix). Assume that $\mathcal{X}/B$ is a family of quasi-polarized complex tori. Then the quasi-polarization defines an isomorphism $\theta : R^1\pi_*\mathcal{O}_{\mathcal{X}} \to \mathcal{E}$. The Donagi–Markman cubic associated to (π, θ, j) is the element $Q \in H^0(B, \mathcal{E}^{\otimes 3})$ defined by $Q = H^0(j \otimes \theta \otimes \mathrm{id}_{\mathcal{E}})(KS(\pi))$.

Lemma 1.1 (Donagi–Markman, [DM]). *Let $\pi : \mathcal{X} \to B$ be a family of quasi-polarized g-dimensional complex tori over a smooth Stein variety B of dimension g and $j : \mathcal{T}_B^\vee \to \mathcal{E}$ an isomorphism. Then there exists a quasi-Lagrangian structure on $\mathcal{X}/B$ inducing the given isomorphism $j : \mathcal{T}_B^\vee \to \mathcal{E}$ if and only if the associated Donagi–Markman cubic is symmetric, that is $Q \in H^0(B, \mathrm{Sym}^3(\mathcal{E}))$.*

This lemma gives a necessary condition for the existence of a Lagrangian structure on $\mathcal{X}/B$. Now we will state a sufficient condition. Let $\mathcal{H}_k(\mathcal{X}/B, \mathbb{Z})$ denote the local system of k-th homology groups of fibers of π with integer coefficients. The integration of 1-forms over 1-cycles on the fibers of π defines a natural embedding $\mathcal{H}_1(\mathcal{X}/B, \mathbb{Z}) \hookrightarrow \pi_*(\Omega^1_{\mathcal{X}/B})^\vee \simeq \mathcal{E}$.

Lemma 1.2 (Donagi–Markman, [DM])**.** *Under the assumptions of Lemma 1.1, assume that for any relative* 1*-cycle* $\gamma \in H^0(B, \mathcal{H}_1(\mathcal{X}/B, \mathbb{Z}))$*, its preimage* $j^{-1}(\gamma)$ *in* $H^0(B, \mathcal{T}_B^\vee)$ *is a closed* 1*-form on* B*. Then there exists a unique symplectic form* ω *on* $\mathcal{X}$ *with the following properties:*

(i) π *is a Lagrangian fibration;*

(ii) the induced isomorphism $\mathcal{T}_B^\vee \simeq \mathcal{E}$ *coincides with* j*;*

(iii) the zero section $O \subset \mathcal{X}$ *is Lagrangian, that is* $\omega|_O \equiv 0$*.*

2. Variations of Hodge structures

Let $f : \mathcal{V} \to B$ be a family of smooth projective or compact Kähler manifods of dimension n. Then the vector bundles $\mathcal{H}^k = R^k f_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B$ carry the structure of a variation of Hodge structures (VHS) of weight k. This means the following:

(1) $\mathcal{H}^k$ contains a local system $\mathcal{H}^k_{\mathbb{Z}}$ of integral lattices such that $\mathcal{H}^k = \mathcal{H}^k_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_B$. This, in particular, implies the existence of a natural real structure, defining the complex conjugation $s \to \overline{s}$ on the sections of $\mathcal{H}^k$, and the existence of a flat connection $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega^1_B$, called Gauss–Manin connection. The latter is defined by the requirement that the sections of $\mathcal{H}^k_{\mathbb{Z}}$ are flat.

(2) $\mathcal{H}^k$ possesses a decreasing filtration $\mathcal{H}^k = F^0\mathcal{H}^k \supset F^1\mathcal{H}^k \supset \ldots \supset F^k\mathcal{H}^k \supset F^{k+1}\mathcal{H}^k = 0$ by holomorphic subbundles such that $F^i\mathcal{H}^k \cap \overline{F^{k-i-1}\mathcal{H}^k} = 0$.

(3) The Griffiths transversality condition: $\nabla(F^i\mathcal{H}^k) \subset F^{i-1}\mathcal{H}^k \otimes \Omega^1_B$.

For each fiber $V_b = V$ of f,

$$\mathcal{H}^k|_b = H^k(V, \mathbb{C}), \quad F^i\mathcal{H}^k|_b = \bigoplus_{p \geq i} H^{p,k-p}(V, \mathbb{C}), \quad \mathcal{H}^k_{\mathbb{Z}}|_b = H^k(V, \mathbb{Z})/(\text{torsion}).$$

The Gauss-Manin connection is only $\mathbb{C}$-linear, but when restricted to F^k, it induces an $\mathcal{O}_B$-linear map $\overline{\nabla} : F^k \to (F^0/F^k) \otimes \Omega^1_B$, whose image is in fact contained in $(F^{k-1}/F^k) \otimes \Omega^1_B$. By [Gr], the graded version of the Gauss-Manin connection $\mathrm{gr}^i \nabla : F^i/F^{i+1} \to F^{i-1}/F^i \otimes \Omega^1_B$ acts by the contraction with the Kodaira–Spencer class $KS(f) \in H^0(B, \mathcal{T}_B^\vee \otimes R^1 f_* \mathcal{T}_{\mathcal{V}/B})$. Explicitly, we can use the Dolbeault isomorphism to identify $F^i\mathcal{H}^k|_b$ with $\bigoplus_{p \geq i} H^{k-p}(V, \Omega^p)$, and for each $a \in T_bB$, the graded covariant derivative $\mathrm{gr}^p \nabla_a$ in the direction of a acts on $\xi \in H^q(V, \Omega^p)$ by the formula

$$\mathrm{gr}^p \nabla_a : \xi \mapsto (KS(f)|_b(a)) \lrcorner \xi \in H^{q+1}(V, \Omega^{p-1}), \;\; H^1(V, \mathcal{T}_V) \times H^q(V, \Omega^p) \xrightarrow{\lrcorner} H^{q+1}(V, \Omega^{p-1}). \quad (3)$$

To a VHS $(\mathcal{H}^{2k-1}, \mathcal{H}^{2k-1}_{\mathbb{Z}}, F^\bullet)$ of odd weight $2k-1$, one can associate a family of complex tori $\pi : \mathcal{J}^k \to B$ of relative dimension $\frac{1}{2} \operatorname{rk} \mathcal{H}^{2k-1}$:

$$\mathcal{J}^k = \mathcal{H}^{2k-1}/(\mathcal{H}^{2k-1}_{\mathbb{Z}} + F^k\mathcal{H}^{2k-1})$$

If $\mathcal{H}^{2k-1}$ comes from a morphism f as above, $\mathcal{J}^k = \mathcal{J}^k(\mathcal{V}/B)$ is called the relative k-th intermediate Jacobian of f. In this case we can also write

$$\mathcal{J}^k \cong (F^{n-k+1}\mathcal{H}^{2n-2k+1})^\vee / \mathcal{H}_{2n-2k+1}(\mathcal{V}/B, \mathbb{Z}).$$

If $n = 2k-1$, then the k-th intermediate Jacobian has a natural quasi-polarization, given by the intersection form on $\mathcal{H}^{2k-1}_{\mathbb{Z}}$. If in addition the Hodge structure on $\mathcal{H}^{2k-1}$ is of height 1, that is $F^{k+1} = 0$, $F^{k-1} = \mathcal{H}^{2k-1}$, then it also satisfies the second Riemann bilinear relation and hence is a polarization.

From now on, we are assuming that $n = 2k-1$. Let $\mathcal{X} = \mathcal{J}^k$. We are going to describe the extension class $KS(\pi)$ of (2) in terms of the Gauss–Manin connection $\overline{\nabla} : F^k \to (F^0/F^k)\otimes\Omega^1_B$. The natural pairing $F^k \times (F^0/F^k) \to \mathcal{H}^{2n} \simeq \mathbb{C}$ identifies F^0/F^k with the dual of F^k, and $F^0/F^k \cong \pi_*\mathcal{T}_{\mathcal{X}/B}$. As in the previous section, we will denote this bundle by $\mathcal{E}$. Thus we can interprete $\overline{\nabla}$ as a map $\overline{\nabla} : \mathcal{T}_B \to (\mathcal{E})^{\otimes 2}$.

Lemma 2.1. *The extension class of the tangent bundle sequence of the map $\pi : \mathcal{X} = \mathcal{J}^k \to B$ coincides with $\overline{\nabla}$.*

Proof. The VHS of weight 1 of π is easily reconstructed from the VHS of weight $2k-1$ of f. The undelying vector bundle $\mathcal{H}^1(\mathcal{X}/B, \mathbb{C})$ is just $\mathcal{H}^{2k-1}(\mathcal{V}/B, \mathbb{C})$, similarly for integer lattices we have $\mathcal{H}^1(\mathcal{X}/B, \mathbb{Z}) = \mathcal{H}^{2k-1}(\mathcal{V}/B, \mathbb{Z})$, and the Hodge filtration is given by $F^1\mathcal{H}^1(\mathcal{X}/B, \mathbb{C}) = F^k\mathcal{H}^{2k-1}(\mathcal{V}/B, \mathbb{C})$, $F^2\mathcal{H}^1(\mathcal{X}/B, \mathbb{C}) = 0$. Hence the Gauss–Manin connection on $\mathcal{H}^1(\mathcal{X}/B, \mathbb{C}) = \mathcal{H}^{2k-1}(\mathcal{V}/B, \mathbb{C})$ constructed from π coincides with that constructed from f. By the cited result of Griffiths, applied to π, the Gauss–Manin connection is given by the contraction with $KS(\pi)$. The component $\overline{\nabla} : F^1\mathcal{H}^1 \simeq \mathcal{E}^\vee \to \mathcal{H}^1/F^1\mathcal{H}^1 \simeq \mathcal{E}\otimes\Omega^1_B$ determines completely $KS(\pi)$, and coincides with it if intepreted as a map $\mathcal{T}_B \to \mathcal{E}^{\otimes 2}$. ☐

We will apply the above lemma in the case when the fibers of f are Fano 3-folds. Then $H^{3,0}(V_b) = 0$ for all $b \in B$, so the Hodge structure on $\mathcal{H}^3$ is of height 1 and $\mathcal{J} = \mathcal{J}^1(\mathcal{V}/B)$ is a family of polarized abelian varieties of dimension $b_3(V_b)/2$.

3. Deformations of K3-Fano flags

Recall that a Fano 3-fold is by definition a 3-dimensional nonsingular projective variety V with ample anticanonical divisor $-K_V$. Iskovskih [I1], [I2] classified Fano 3-folds with Picard number 1, and Mori–Mukai classified all the remaining ones [MM]; see also [IP]. There are 105 deformation classes of Fano 3-folds, and around half of them have $h^{2,1} = 0$, so that there is no integrable system of intermediate Jacobians associated to them. For the sequel, fix some class of Fano 3-folds V with $h^{2,1} \neq 0$ and consider a K3-Fano flag (S, V). This means that S is a K3 surface embedded in V as an anticanonical divisor. Let R be the Picard lattice of V, $(-K_V)^3 = 2g-2$; by Lefschetz Theorem, R is embedded into $\mathrm{Pic}(S)$. Denote by i the natural embedding $S \hookrightarrow V$.

According to [Ka], the deformation theory of a pair $D \subset X$, consisting of a connected compact manifold X and a normal crossing divisor D in it, is governed by the sheaf

$$\mathcal{T}_X(-\log D) = \{v \in \mathcal{T}_X \mid vI_D \subset I_D\},$$

where I_D denotes the ideal sheaf of D. Beauville [B2] gives a proof of this fact in the algebraic setting. Applying it to the moduli stack of K3-Fano flags $\mathcal{F}^R_g$, we obtain that the tangent space to $\mathcal{F}^R_g$ at (S, V) is canonically isomorphic to $H^1(V, \mathcal{T}_V(-\log S))$, and the obstruction space $H^2(V, \mathcal{T}_V(-\log S))$ is zero, so the first order deformations of (S, V) are unobstructed.

The space of infinitesimal deformations of the complex structure on S is $H^1(S, \mathcal{T}_S)$, and as $H^2(S, \mathcal{T}_S) = 0$, the infinitesimal deformations are also unobstructed. Let $\sigma : R \to H^1(S, \Omega^1_S)$ be the natural map, defined as the composition $R \xrightarrow{\sim} \mathrm{Pic}(V) \xrightarrow{i^*} \mathrm{Pic}(S) \xrightarrow{c_1} H^1(S, \Omega^1_S)$, where c_1 denotes the first Chern class. Fixing some generator $\alpha_S \in H^0(S, \Omega^2_S) \simeq \mathbb{C}$, we have the contraction isomorphism $\mathcal{T}_S \xrightarrow[\sim]{\cdot\lrcorner\alpha_S} \Omega^1_S$, which we can use to identify $H^1(S, \mathcal{T}_S)$ with $H^1(S, \Omega^1_S)$. Under this identification, the tangent space $T_{[S]}\mathcal{K}^R_g$ is the orthogonal complement to $\sigma(R)$ with respect to the natural bilinear form on $H^1(S, \Omega^1_S)$: $T_{[S]}\mathcal{K}^R_g = \sigma(R)^\perp \subset H^1(S, \mathcal{T}_S)$.

Further, Beauville shows that the image of the natural map $H^1(r) : H^1(V, \mathcal{T}_V(-\log S)) \to H^1(S, \mathcal{T}_S)$ induced by the restriction $r : \mathcal{T}_V(-\log S) \to \mathcal{T}_S$ coincides with $\sigma(R)^\perp$ and that

$H^1(r)$ is the differential of the forgetful morphism $s_g^R : \mathcal{F}_g^R \to \mathcal{K}_g^R$ at $[(S,V)]$. This implies, in particular, that s_g^R is a submersion. Hence the fiber $\mathcal{F}_S^R := (s_g^R)^{-1}([S])$ is a smooth stack whose tangent space at $[(S,V)]$ is $\ker H^1(r)$.

To identify $\ker H^1(r)$, we follow Beauville who considers the natural exact triple

$$0 \longrightarrow \mathcal{T}_V(-S) \longrightarrow \mathcal{T}_V(-\log S) \xrightarrow{r} \mathcal{T}_S \longrightarrow 0. \tag{4}$$

Choose a generator $\alpha_V \in H^0(V, \Omega_V^3(\log S)) \simeq \mathbb{C}$. Then we have the contraction isomorphism $\mathcal{T}_V(-S) \xrightarrow[\sim]{\cdot \lrcorner \alpha_V} \Omega_V^2$. The long exact cohomology sequence associated to (4) acquires the form

$$0 \to H^1(V, \Omega_V^2) \to H^1(V, \mathcal{T}_V(-\log S)) \xrightarrow{H^1(r)} H^1(S, \mathcal{T}_S) \to H^2(V, \Omega_V^2) \to 0$$

and shows that $T_{[(S,V)]}\mathcal{F}_S^R = \ker(H^1(r)) = H^1(V, \Omega_V^2)$. In particulat, $\dim \mathcal{F}_S^R = b_3(V)/2$.

Proposition 3.1. *Let R, S, $\mathcal{F}_S^R$ be as above. Let $f = \mathcal{V} \to B = \mathcal{F}_S^R$ be the universal family of Fano 3-folds having S as an anticanonical divisor and $\pi : \mathcal{J} = \mathcal{J}(\mathcal{V}/B) \to B$ its relative intermediate Jacobian. Let $b = [(V,S)]$ be a point of B. Fix a generator α_V of $H^0(V, \Omega_V^3(\log S))$ and denote by α the corresponding isomorphism $\cdot \lrcorner \alpha_V : \mathcal{T}_V(-S) \xrightarrow{\sim} \Omega_V^2$. Let $h_b = H^1(\alpha)$ denote the isomorphism $T_bB = H^1(\mathcal{T}_V(-S)) \xrightarrow{\sim} H^1(V, \Omega_V^2)$ induced by α.*

The following statements hold:

(i) The value $KS(f)_b$ of the Kodaira–Spencer class of f at b is the natural map $H^1(s) : T_bB = H^1(\mathcal{T}_V(-S)) \to H^1(\mathcal{T}_V)$ associated to the morphism s in the exact triple

$$0 \longrightarrow T_V(-S) \xrightarrow{s} T_V \longrightarrow T_V|_S \longrightarrow 0.$$

(ii) The tangent space at 0 to the intermediate Jacobian $J^2(V)$ being $H^1(V, \Omega_V^2)^\vee \cong H^2(V, \Omega^1)$, we define the Donagi–Markman cubic Q of π at b with the help of the isomorphism $j_b = ({}^t h_b)^{-1} : T_bB^\vee \to T_0J^2(V)$. Then Q, as a cubic form on $H^1(V, \Omega_V^2)$, coincides with the composition of the cup-product $H^1(V, \Omega_V^2)^{\otimes 3} \to H^3(V, (\Omega_V^2)^{\otimes 3})$ with the map $\mathrm{Tr} \circ H^3(\gamma)$, where γ is defined by

$$\gamma : (\Omega_V^2)^{\otimes 3} \to \Omega_V^3\ , \quad \xi_1 \otimes \xi_2 \otimes \xi_3 \mapsto (s\alpha^{-1}(\xi_1) \lrcorner \xi_2) \wedge \xi_3 \tag{5}$$

and Tr *is the canonical isomorphism $H^3(V, \Omega_V^3) \xrightarrow{\sim} \mathbb{C}$.*

Proof. (i) The inclusion s factors through $t : \mathcal{T}_V(-\log S) \hookrightarrow \mathcal{T}_S$, the map identifying $\mathcal{T}_V(-\log S)$ as the subsheaf of germs of vector fields tangent to S. The fact that $H^1(t)$ coincides with the map, associating to each first order deformation of the pair (S,V) the respective first order deformation of its second component V, is proved in the same way as a similar property of $H^1(r)$ in [B2], Proposition 1.1. The wanted assertion then follows from the relation $s = t|_{\ker H^1(r)}$.

(ii) By part (i), Lemma 2.1 and formula (3), the Kodaira-Spencer class of π at b is the morphism $e_b : T_bB = H^1(V, \Omega_V^2) \longrightarrow \mathrm{Hom}(H^1(V, \Omega_V^2), H^2(V, \Omega_V^1))$ given by $e_b : a \mapsto [v \mapsto \tilde{a} \lrcorner v]$, where $\tilde{a} = H^1(s \circ \alpha^{-1})(a)$. The Donagi–Markman cubic Q is the image of e_b under the natural isomorphism $\mathrm{Hom}(E, \mathrm{Hom}(E, E^\vee)) \xrightarrow{\sim} (E \otimes E \otimes E)^\vee$, where $E = H^1(V, \Omega_V^2)$ and $H^2(V, \Omega_V^1)$ is identified with $E^\vee$ via the pairing $H^1(V, \Omega_V^2) \otimes H^2(V, \Omega_V^1) \xrightarrow{\wedge} H^3(V, \Omega_V^3) = \mathbb{C}$. Hence Q is the map $E^{\otimes 3} \to H^3(V, \Omega_V^3) = \mathbb{C}\ ,\quad a \otimes v \otimes w \mapsto\mapsto (\tilde{a} \lrcorner v) \wedge w$, which was to be proved. ☐

Corollary 3.2. *Under the hypotheses and in the notation of Proposition 3.1, the Donagi–Markman cubic Q is symmetric, so π is a quasi-Lagrangian fibration.*

Proof. The fact that (5) is skew-symmetric in ξ_i is an easy exercise in linear algebra. Then the induced pairing on the cohomology $\otimes_{i=1}^3 H^{p_i}(V, \Omega_V^2) \to H^{p_1+p_2+p_3}(V, \Omega_V^3)$ is graded skew-symmetric, that is, the sign change resulting from the transposition of the first and the second arguments is $(-1)^{(p_1+1)(p_2+1)}$, and similarly for the transposition of the second and the third arguments. It remains to apply Lemma 1.1. ☐

Remark 3.3. If we replace V by a Calabi–Yau 3-fold and choose for α_V a generator of $H^0(V,\Omega^3_V)$, then the cubic form constructed in part (ii) of Proposition 3.1 will be nothing else but the Yukawa coupling, well known from the superstring theory compactified on Calabi–Yau spaces.

In order to get a genuine symplectic structure on $\mathcal{J}$, we will have to choose the isomorphism $j:\mathcal{T}_B^\vee \to \mathcal{E}$ with more care. This choice is naturally explained in terms of the variation of the mixed Hodge strucuture on the cohomology of $V\setminus S$.

4. Variations of mixed Hodge structures

A mixed Hodge structure (MHS) on a finite-dimensional $\mathbb{C}$-vector space $H=H_{\mathbb{C}}$ is the following set of data:

- An integer lattice $H_{\mathbb{Z}}\subset H$ such that $H=H_{\mathbb{Z}}\otimes\mathbb{C}$.
- A decreasing filtration $F^\bullet H$ such that $F^0H=H$ and $F^iH=0$ for $i\gg 0$ (Hodge filtration).
- An increasing filtration $W_\bullet H$ defined over $\mathbb{Q}$ (weight filtration) such that $W_{-1}H=0$, $W_mH=H$ for $m\gg 0$, and such that $F^\bullet$ induces on $\operatorname{Gr}_m H=W_mH/W_{m-1}H$ a pure Hodge structure of weight m for any $m\geq 0$.

Here the induced Hodge filtration on $\operatorname{Gr}_m H$ is given by

$$F^p\operatorname{Gr}_m H=(F^pH\cap W_mH+W_{m-1}H)/W_{m-1}H.$$

Deligne [De] introduced a natural MHS on the cohomology of a nonsingular algebraic variety Y depending functorially on Y. It is described as follows. Let $\overline{Y}$ be a nonsingular complete variety containing Y such that $\overline{Y}\setminus Y=D$ is a divisor with simple normal crossings. Then $H^\bullet(Y,\mathbb{C})=\mathbf{H}^\bullet(\overline{Y},\Omega^\bullet_{\overline{Y}}(\log D))$. The Hodge and weight filtrations on the hypercohomology are defined via the respective filtrations on the complex of logarithmic differential forms. For the Hodge filtration, we have

$$F^p(\Omega^\bullet_{\overline{Y}}(\log D))=\Big(\Omega^p_{\overline{Y}}(\log D)\xrightarrow{d}\Omega^{p+1}_{\overline{Y}}(\log D)\xrightarrow{d}\dots\ \Big)[p],$$

where $K^\bullet[p]$ denotes the complex obtained from $K^\bullet$ by the shift p steps to the right (so that $\Omega^p_{\overline{Y}}(\log D)$ is placed on the spot number p). The subcomplex $W_k(\Omega^\bullet_{\overline{Y}}(\log D))\subset\Omega^\bullet_{\overline{Y}}(\log D)$, by defintion, consists of logarithmic forms with at most k poles. Then

$$F^pH^n(Y,\mathbb{C})=\operatorname{im}\mathbf{H}^n(F^p(\Omega^\bullet_{\overline{Y}}(\log D)))\ ,\quad W_{k+n}H^n(Y,\mathbb{C})=\operatorname{im}\mathbf{H}^n(W_k(\Omega^\bullet_{\overline{Y}}(\log D))),$$

the images being taken under the maps induced on the hypercohomology by the natural inclusions of the complexes. It is proved that $W_\bullet$ is defined over $\mathbb{Q}$ and that the resulting MHS does not depend on the completion $\overline{Y}$ of Y. Moreover, the spectral sequence of the Hodge filtration on the complex $\Omega^\bullet_{\overline{Y}}(\log D)$ converging to the hypercohomology of the latter degenerates at E_1, so that

$$F^pH^n(Y,\mathbb{C})/F^{p+1}H^n(Y,\mathbb{C})\simeq H^{n-p}(\overline{Y},\Omega^p_{\overline{Y}}(\log D)).$$

In the particular case when D is a smooth hypersurface in Y, the weight filtration has only two non-trivial graded factors, and the MHS is easily described via the Gysin exact sequence

$$\dots\to H^{q-2}(D)\xrightarrow{i_*}H^q(\overline{Y})\xrightarrow{u^*}H^q(Y)\xrightarrow{\operatorname{Res}}H^{q-1}(D)\to H^{q+1}(\overline{Y})\to\dots\ ,\tag{6}$$

where $Y=\overline{Y}\setminus D$, the coefficients of the cohomology groups are in $\mathbb{Z}$ or in a field, $i:D\hookrightarrow\overline{Y}$, $u:Y\hookrightarrow\overline{Y}$ are natural inclusions, i_* is the Gysin homomorphism induced by i, and Res is the residue (or the tube) map, see [G]. Here $W_{q-1}H^q(Y)=0$, $W_{q+1}H^q(Y)=H^q(Y)$, ${}_W\mathrm{gr}_q\,H^q(Y)=W_qH^q(Y)=u^*H^q(\overline{Y})$ is a pure Hodge structure of weight q, a quotient of the one on $H^q(\overline{Y})$, and ${}_W\mathrm{gr}_{q+1} H^q(Y) = W_{q+1}H^q(Y)/W_qH^q(Y) \cong \mathrm{im}(\mathrm{Res})$ is a pure Hodge substructure of $H^{q-1}(D)$ with weight shifted by 2.

A variation of mixed Hodge structures (VMHS) over a base B is a quadruple $(\mathcal{H}, \mathcal{H}_{\mathbb{Z}}, \mathcal{F}^\bullet, \mathcal{W}_\bullet)$ such that:

– $\mathcal{H}$ is a holomorphic vector bundle on B.
– $\mathcal{H}_{\mathbb{Z}} \subset \mathcal{H}$ is a local system of free $\mathbb{Z}$-modules such that $\mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_B \simeq \mathcal{H}$.
– $\mathcal{F}^\bullet$ is a decreasing filtration of $\mathcal{H}$ by holomorphic subbundles.
– $\mathcal{W}_\bullet$ is an increasing filtration of $\mathcal{H}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Z}} \otimes \mathbb{Q}$ by local systems of constant rank.
– For each $b \in B$, the lattice $H_{\mathbb{Z}} := \mathcal{H}_{\mathbb{Z}}|_b$ and the filtrations $W_kH := \mathcal{W}_k|_b$, $F^pH := \mathcal{F}^p|_b$ define on $H := \mathcal{H}|_b$ a MHS.
– The Gauss–Manin connection ∇ on $\mathcal{H}$ associated to the local system $\mathcal{H}_{\mathbb{Z}} \otimes \mathbb{C} \subset \mathcal{H}$ satisfies the transversality condition $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_B$.

Any family of pairs $(\overline{Y}, D)$ as above defines a VMHS on the associated bundle of cohomology groups $H^n(\overline{Y} \setminus D, \mathbb{C})$. By a family of pairs we mean a smooth proper morphism $f : \overline{\mathcal{Y}} \to B$ with connected fibers and a normal crossing divisor $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_r$ in $\overline{\mathcal{Y}}$ such that for any k-uple $(i_1, \ldots i_k)$ of distinct indices from $\{1, \ldots, r\}$, the scheme-theoretic intersection $\mathcal{D}_{i_1} \cap \ldots \cap \mathcal{D}_{i_k}$, if nonempty, is smooth over B and is of codimension k in $\overline{\mathcal{Y}}$. We denote the corresponding VMHS by $(\mathcal{H}^n(f, \mathcal{D}), \mathcal{H}^n_{\mathbb{Z}}(f, \mathcal{D}), \mathcal{F}^\bullet\mathcal{H}^n(f, \mathcal{D}), \mathcal{W}_\bullet\mathcal{H}^n(f, \mathcal{D}))$.

As we have already mentioned in Section 3, the space of infinitesimal deformations of a pair $(\overline{Y}, D)$ is canoncially isomorphic to $H^1(\overline{Y}, \mathcal{T}_{\overline{Y}}(-\log D))$, thus a family of pairs defines the Kodaira–Spencer maps

$$KS(f, \mathcal{D})_b : T_bB \longrightarrow H^1(\overline{Y}_b, \mathcal{T}_{\overline{Y}_b}(-\log D_b)) \ \text{ for any } \ b \in B$$

which fit into a morphism of vector bundles $KS(f, \mathcal{D}) : \mathcal{T}_B \to R^1f_*(\mathcal{T}_{\overline{\mathcal{Y}}/B}(-\log \mathcal{D}))$. Furthermore, there is a natural contraction map

$$\mathcal{T}_{\overline{Y}_b}(-\log D_b) \times \Omega^p_{\overline{Y}_b}(\log D_b) \xrightarrow{\ \lrcorner\ } \Omega^{p-1}_{\overline{Y}_b}(\log D_b),$$

and the p-th graded piece of the Gauss–Manin connection on $\mathcal{H}^n(f, \mathcal{D})$ with respect to the Hodge filtration $\mathrm{gr}^p_F \nabla_b : (\mathcal{F}^p/\mathcal{F}^{p+1})|_b \to (\mathcal{F}^{p-1}/\mathcal{F}^p) \otimes \Omega^1_B|_b$ is nothing but the contraction

$$H^{n-p}(\overline{Y}_b, \Omega^p_{\overline{Y}_b}(\log D_b)) \xrightarrow{\lrcorner KS(f,\mathcal{D})_b} H^{n-p+1}(\overline{Y}_b, \Omega^{p-1}_{\overline{Y}_b}(\log D_b)) \otimes \Omega^1_B|_b$$

with the class $KS(f, \mathcal{D})_b$, considered as an element of $H^1(\overline{Y}_b, \mathcal{T}_{\overline{Y}_b}(-\log D_b) \otimes \Omega^1_B|_b)$.

5. Symplectic structure

We are resuming some of the notation of Sect. 3: let S be a K3 surface, R an even integral lattice of signature $(1, \mathrm{rk}(R) - 1)$ with a distinguished element ρ, $\rho^2 = 2g - 2$, and $B = \mathcal{F}^R_S$ the moduli stack of Fano 3-folds V with $(\mathrm{Pic}\, V, -K_V) \simeq (R, \rho)$ containing S as an anticanonical divisor, the quadratic form on $\mathrm{Pic}\, V$ being defined by $(D_1, D_2) \mapsto (-K_V \cdot D_1 \cdot D_2)$. Assume that B is of dimension > 0 and denote by $f : \mathcal{V} \to B$ the universal family over B.

Theorem 5.1. *The relative Jacobian $\pi : \mathcal{J} \to B$ of the universal family $f : \mathcal{V} \to B$ is a Lagrangian fibration.*

Proof. In order to compute the MHS on $H^3(Y)$ for $Y = V \setminus S$, we apply (6) to the pair $(\overline{Y}, D) = (V, S)$:

$$0 \to H^3(V) \xrightarrow{u^*} H^3(Y) \xrightarrow{\mathrm{Res}} H^2(S) \xrightarrow{i_*} H^4(V) \to 0. \tag{7}$$

We will denote by $h^{p,q}$, b_k the Hodge, resp. Betti numbers of V. The Hodge filtration on $H^3(Y)$ is given by

$$F^4 = 0,\ F^3 = H^0(\Omega^3_V(S)) \simeq \mathbb{C},\ F^2 = F^3 \oplus H^1(\Omega^2_V(\log S)) \simeq \mathbb{C}^{h^{2,1}+21-b_2},$$
$$F^1 = F^2 \oplus H^2(\Omega^1_V(\log S)) = H^3(Y,\mathbb{C}) \simeq \mathbb{C}^{b_3+22-b_2}.$$

Further,

$$W_3 = H^3(V) \subset H^3(Y),\ W_3 \cap F^3 = 0,\ W_3 \cap F^2 = H^1(\Omega^2_V),$$
$$W_3 \cap F^1 = H^1(\Omega^2_V) \oplus H^2(\Omega^1_V) = H^3(V),$$

and the Hodge structure of ${}_W\mathrm{gr}_4\, H^3(Y)$ coincides with that of the lattice of transcendental cycles $T_S = \ker i_* = (\operatorname{Pic} S)^\perp$ with weight shifted by 2, where the orthogonal complement is taken with respect to the intersection form on $H^2(S,\mathbb{Z})$.

To get a symplectic structure on $\mathcal{J}$, we will apply the criterion of Lemma 1.2. We are using the same isomorphisms $j_b : T_bB^\vee \to T_0J(V_b)$ as in Proposition 3.1, dependind on the choice of a generator α_{V_b} of $H^3(\Omega^3_{V_b}(S))$. To define a global isomorphism $j : \mathcal{T}_B^\vee \to \mathcal{E}$, where $\mathcal{E}$ is realized as the restriction of $\mathcal{T}_{\mathcal{J}/B}$ to the zero section of $\mathcal{J}/B$, we should fix some section $\alpha_{\mathcal{V}}$ of the Hodge bundle $\mathcal{F}^3$. For instance, we will pick up a generator α_S of $H^0(\Omega^2_S)$ and determine $\alpha_{\mathcal{V}}$ by the requirement that $\mathrm{Res}_S(\alpha_{V_b}) = \alpha_S$ for all $b \in B$; this provides a global trivialization of $\mathcal{F}^3$ over B.

For any local section γ of $\mathcal{H}_3(\mathcal{V}/B,\mathbb{Z})$, we want $j^{-1}(\gamma)$ to be a closed 1-form on B, where $\mathcal{H}_3(\mathcal{V}/B,\mathbb{Z})$ is naturally embedded into $\mathcal{E} = (F^2\mathcal{H}_3(\mathcal{V}/B))^\vee$ via the integration map $\gamma \mapsto \int_\gamma$. The 1-form $j^{-1}(\gamma)$ is determined by its values on the tangent vectors $a \in T_bB = H^1(V, \mathcal{T}_{V_b}(-S))$:

$$j^{-1}(\gamma)(a) = \int_{\gamma_b} a \lrcorner\, \alpha_{V_b}.$$

If we try to mimic the proof of Theorem 7.7 in [DM], then in order to verify the closedness of $j^{-1}(\gamma)$, we should find a function g, defined locally on B, whose differential is $j^{-1}(\gamma)$, and the natural candidate for g would be $b \mapsto g(b) = \int_{\gamma_b} \alpha_{V_b}$. The problem is that $F^3H^3(Y)$ does not couple with $H_3(V)$ (here $V = V_b$, $Y = Y_b = V_b \setminus S$), so we have to correct the definition of g in lifting γ_b to $H_3(Y)$. Applying the Poincaré–Lefschetz duality to the cohomology exact sequence of the pair (V,S), we obtain an exact sequence for the homology of Y similar to (7):

$$0 \to H_4(V) \to H_2(S) \to H_3(Y) \to H_3(V) \to 0.$$

It shows that γ can be lifted to a local section of $\mathcal{H}_3((\mathcal{V}\setminus\mathbb{S})/B,\mathbb{Z})$, where $\mathbb{S} = S \times B \subset \mathcal{V}$. We denote the lifted section by the same symbol γ, then the above formula for g provides a well-defined function on B. Its differential at a point b is computed in terms of the Gauss–Manin connection of the VMHS associated to f:

$$d_bg(a) = \int_{\gamma_b} \nabla_a \alpha_{\mathcal{V}}.$$

We have $\nabla_a\alpha_{\mathcal{V}} \equiv a\lrcorner\,\alpha_V \mod F^3H^3(Y)$ and $a\lrcorner\,\alpha_V \in H^1(\Omega^2_V) \subset W_3$. Since this holds at any $b \in B$ and for any $a \in T_bB$, the subbundle $E = \mathcal{W}_3 \oplus \mathcal{F}^3 \subset \mathcal{H}^3((\mathcal{V}\setminus\mathbb{S})/B)$ is ∇-invariant. Picking up a flat basis of $\mathcal{W}_3$ and completing it by $\alpha_{\mathcal{V}}$ to a basis of $\mathcal{W}_3 \oplus \mathcal{F}^3$, we obtain a basis of E in which the matrix of $\nabla|_E$ has the following form:

$$C = \left[\begin{array}{c|c} \mbox{\Large 0} & \begin{matrix} c_{1n} \\ \vdots \\ c_{nn} \end{matrix} \end{array}\right], \quad c_{kn} \in H^0(\Omega^1_B).$$

We have $j^{-1}(\gamma) = dg - gc_{nn}$. Thus we can assure the closedness of $j^{-1}(\gamma)$ if we manage to replace $\alpha_{\mathcal{V}}$ by another basis of $\mathcal{F}^3$ for which $c_{nn} = 0$. The integrability condition $dC = C \wedge C$ implies that $dc_{nn} = 0$, hence c_{nn} has a primitive $\int c_{nn}$ locally over B. Let us set $\tilde{\alpha}_{\mathcal{V}} = \exp(-\int c_{nn})\alpha_{\mathcal{V}}$. Then if we use $\tilde{\alpha}_{\mathcal{V}}$ in place of $\alpha_{\mathcal{V}}$ to define j, and if we denote the thus modified j by $\tilde{j}$, then $\tilde{j}^{-1}(\gamma)$ is closed. $\square$

References

[B1] A. Beauville, *Vector bundles on the cubic threefold*, A. Bertram (ed.), Symposium in honor of C. H. Clemens, University of Utah, March 2000, Contemp. Math. **312**, 71–86 (2002).

[B2] A. Beauville, *Fano threefolds and $K3$ surfaces*, The Fano Conference, 175-184, Univ. Torino, 2004.

[De] P. Deligne, *Théorie de Hodge, II,* Publ. Math. IHES **40**, 5–57 (1971).

[DM] R. Donagi, E. Markman, *Spectral covers, algebraically completely integrable Hamiltonian systems, and moduli of bundles*, in M. Francaviglia (ed.), Integrable systems and quantum groups, Lecture Notes in Math. **1620**, 1–119 (1996).

[G] M. L. Green, *Infinitesimal methods in Hodge theory*, Algebraic cycles and Hodge theory (Torino, 1993), 1–92, Lecture Notes in Math. **1594**, Springer, Berlin, 1994.

[Gr] P. Griffiths, *Periods of integrals on algebraic manifolds II. Local study of the period mapping*, Amer. J. Math. **90**, 805–865 (1968).

[HT] B. Hassett, Yu. Tschinkel, *Abelian fibrations and rational points on symmetric products*, International J. Math. **11**, 1163–1176 (2000).

[I1] V. A. Iskovskih, *Fano 3-folds I, II*, Izv. Akad. Nauk SSSR Ser. Mat. **41**, 516–562 (1977); **41**, 469–506 (1978).

[I2] V. A. Iskovskih, *Double projection from a line onto Fano 3-folds of the first kind*, (Russian) Mat. Sb. **180**, 260–278 (1989); Engl. transl. in Math. USSR-Sb. **66** , 265–284 (1990).

[IM] A. Iliev, L. Manivel, *Prime Fano threefolds and integrable systems*, preprint math.AG/0606211.

[IP] V. A. Iskovskih, Yu. G. Prokhorov, *Fano varieties*, Algebraic geometry, V, 1–247, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999.

[Ka] M. Kashiwara, *The asymptotic behavior of a variation of polarized Hodge structure*, Publ. Res. Inst. Math. Sci. **21**, 853–875 (1985).

[MM] S. Mori, S. Mukai, *Classification of Fano 3-folds with $B_2 \geq 2$*, Manuscripta Math. **36**, 147–162 (1981/82). *Erratum,* Manuscripta Math. **110**, 407 (2003).

[Tyu] A.N. Tyurin, *Non-abelian analogues of Abel's theorem*, Izv. Math. **65**, 123–180 (2001).

D. M.: Mathématiques - bât. M2, Université Lille 1, F-59655 Villeneuve d'Ascq Cedex, France

E-mail address: markushe@math.univ-lille1.fr