Maximal subgroups of non locally finite p-groups

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Abstract

Kaplansky's conjecture claims that the Jacobson radical $\mathcal{J}K[G]$ of a group algebra $K[G]$, where K is a field of characteristic $p > 0$, coincides with its augmentation ideal $\mathcal{A}K[G]$ if and only if G is a locally finite p-group. By a theorem of Passman, if G is finitely generated and $\mathcal{J}K[G] = \mathcal{A}K[G]$ then any maximal subgroup of G is normal of index p. In the present paper, we consider three series of finitely generated infinite p-groups (hence not locally finite p-groups), Grigorchuk's 2-groups, GGS-groups, and EGS-groups. We prove that their maximal subgroups are nonetheless normal of index p.

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*Partially supported by RFFI, grant r2001 Ural N 01-01-96404, and by Russian Ministry of Education, grant E00-1.0-150"

Introduction

Let K be a field and let G be a group. The group algebra $K[G]$ is called semiprimitive if its Jacobson radical $\mathcal{J}K[G]$ is trivial.

The study of semiprimitivity of group algebras seeks to establish all groups G with semiprimitive group algebras $K[G]$. More generally, it is desirable to understand the structure of the Jacobson radical $\mathcal{J}K[G]$. The answer is well-known in case of finite groups, due to a result of Maschke [9]. The result states that if the characteristic of K is zero, then $K[G]$ is always semiprimitive, and if the characteristic is some $p > 0$ then $K[G]$ is semiprimitive iff G has no elements of order p. However, the situation with infinite groups is much more complex; some of the main results in that direction can be found in [10].

Suppose now that K is a field of characteristic $p > 0$ and let G be a finitely generated group. If $\mathcal{A}K[G]$ is the augmentation ideal of $K[G]$ then it is a maximal right ideal of $K[G]$ and therefore contains the Jacobson radical $\mathcal{J}K[G]$. There is the following conjecture of Kaplansky.

Conjecture. If $\text{char}K = p > 0$ and G is a finitely generated group, then $\mathcal{J}K[G] = \mathcal{A}K[G]$ if and only if G is a finite p-group.

The "if" direction is obtained easily (see [11]), so the real concern is with the converse direction. Theorem 6.3 of [11] states that if G is a finitely generated group, K is a field of characteristic $p > 0$, and $\mathcal{J}K[G] = \mathcal{A}K[G]$, then, in particular, G is a p-group and any maximal subgroup of G is normal of index p. Thus, should G turn out to be infinite, it would a be finitely generated, infinite p-group. Such groups are called *Burnside groups* for being counterexamples to the unrestricted Burnside problem.

Many examples of such groups are presently known. Among them are the groups constructed by Golod [4, 5], Grigorchuk's 2-groups [6], and so-called GGS-groups [3, 1] generalizing the examples by Gupta and Sidki [8].

In the present paper we study maximal subgroups of three classes of Burnside groups, the Grigorchuk's groups, GGS-groups, and EGS-groups [13]. For such groups to have a maximal subgroup of infinite index would be in accordance with Kaplansky's conjecture. However, it turned out that all their maximal subgroups are normal of index p; in particular, this answers question (1) of [16].

This main result is stated in Theorem 5.1. Some intermediate results given in Theorems 4.1, 4.2, and 4.3 may also be of independent interest.

The groups studied in the paper all act as automorphism groups of regular p-trees; in fact, they are just infinite (meaning all their proper quotients are finite) and belong to the class of so-called *branch groups*. The class of just infinite branch groups is interesting in itself. Indeed, it is known ([7]) that every finitely generated infinite group can be mapped onto a just infinite group. In its turn, an abstract just infinite group either is a branch group or contains a normal subgroup of finite index which is isomorphic to the direct product of a finite number of copies of a group L, where L is either simple or hereditarily just infinite. There is also a similar result for profinite groups; refer to [7, 18] for more details.

Acknowledgments The crucial part of this research has been carried out in the supportive environment of Max Planck Institute for Mathematics in Bonn whose staff I would like to thank for their kind support and hospitality. I also would like to thank A.V. Rozhkov for his attention to this work, R.I. Grigorchuk for attracting my interest to the problem, Donald Passman for pointing out the connections to the group representation theory, and A.V. Ershov for productive discussions, which were very helpful in clarifying the topological aspects of this research.

1 Main definitions

Let us start with introducing some notions from the theory of branch groups, which provides a language very suitable for dealing with the groups we intend to consider.

Definition 1.1. A group G is said to be a *branch group*, if there exist two decreasing sequences of subgroups $(L_i)_{i\in N}$, $(M_i)_{i\in N}$ and a sequence of integers $(k_i)_{i\in N}$ such that

$$L_0 = M_0 = G, \quad k_0 = 1,$$

$$\cap_{i\in N} M_i = 1,$$

and for each i the following four conditions hold,

(1) M_i is a normal finite-index subgroup of G;

(2) M_i is the direct product of k_i copies of the subgroup L_i, i.e. there are subgroups $L_i^{(1)}$, …, $L_i^{(k_i)}$ of G such that
$$M_i = L_i^{(1)} \times \ldots \times L_i^{(k_i)}, \tag{1.1}$$
and each of the factors is isomorphic to L_i;

(3) k_i properly divides k_{i+1}, i.e. $m_{i+1} = \frac{k_{i+1}}{k_i} \geq 2$, and the product decomposition (1.1) of M_{i+1} refines the product decomposition (1.1) of M_i, meaning that each factor $L_i^{(j)}$ of M_i contains m_{i+1} factors of M_{i+1}, namely the factors $L_{i+1}^{(l)}$ for $l = (j-1)m_{i+1} + 1$, …, jm_{i+1};

(4) conjugations by elements of G transitively permute the factors in the product decomposition (1.1).

Notice that the set of subgroups $L_i^{(j)}$ ordered by inclusion forms a locally finite rooted spherically homogeneous tree T. The group G acts by automorphisms on T and is transitive on each layer of T (comprised by all subgroups $L_i^{(j)}$ for a fixed i).

In general, this action may not be faithful. However, in case of *just infinite* branch groups, which are what we are primarily interested in, it is faithful. Otherwise, as is easily seen, the kernel of the action would be a non-trivial normal subgroup of infinite index. Thus, a just infinite branch group is a group of automorphisms of a rooted spherically homogeneous tree.

Let us specify what conditions a group of tree automorphisms must meet in order to be branch. Notice that if $A = (A_1, A_2, \ldots)$ is a sequence of finite sets of cardinality 2 or greater, then the set of all finite sequences

$$a_1 a_2 \ldots a_n, \ a_i \in A_i, \ n \in N,$$

forms a rooted spherically homogeneous tree, where the sequences correspond to vertices, the root is the empty sequence, and two vertices are joined by an edge if they have the form $a_1 a_2 \ldots a_n$ and $a_1 a_2 \ldots a_n a_{n+1}$ for some n and some $a_i \in A_i$. For example, the tree corresponding to a branch group G can be constructed over a sequence of arbitrary sets A_i such that the cardinality of A_i is equal to m_i.

The number n is called the *length* of a vertex $u = a_1 a_2 \ldots a_n$ and is denoted by $|u|$. The set of all vertices of length n is called the nth level of T.

Of special interest is the case when all sets A_i are equal to one set X. In this case the tree is called *regular*, and the set of its vertices is precisely the set of all words in the alphabet X.

Just as we constructed a tree T over an infinite sequence of sets $A = (A_1, A_2, \ldots)$, we can construct a tree over a finite sequence $(A_1, A_2, \ldots, A_n)$. This tree will be denoted by $T_{[n]}$; geometrically, it is obtained by removing everything that lies below the nth level.

Suppose that $u = \hat{a}_1 \hat{a}_2 \ldots \hat{a}_n$ is a vertex. The set of all vertices $\hat{a}_1 \hat{a}_2 \ldots \hat{a}_n a_{n+1} a_{n+2} \ldots a_{n+m}$, where $m \in N$ and a_{n+i} range over the set A_{n+i}, forms a subtree of T. We will denote that subtree by T_u. It is easy to see that T_u is naturally isomorphic to the tree $T_{(n)}$ constructed over the sequence $(A_{n+1}, A_{n+2}, \ldots)$, via the map

$$\hat{a}_1 \hat{a}_2 \ldots \hat{a}_n a_{n+1} a_{n+2} \ldots a_{n+m} \mapsto a_{n+1} a_{n+2} \ldots a_{n+m},$$

This map also allows to identify subtrees T_v for all vertices v of length n, with one fixed tree $T_{(n)}$.

Of course, if T is regular then for any vertex u T_u is isomorphic to T.

Consider now an arbitrary subgroup X in $\operatorname{Aut} T$ and a vertex v of T. The *rigid stabilizer* of v in X is the subgroup

$$\operatorname{rist}_X(v) = \{x \in X | \text{ for any } u \in T \setminus T_v \ u^x = u\}.$$

We also denote by $\operatorname{rist}_X(n)$ the subgroup $\prod_{|v|=n} \operatorname{rist}_X(n)$. This is, of course, a normal subgroup in X.

We are now ready to give an equivalent definition of a just-infinite branch group.

Definition 1.2. Let G be a group of automorphisms of a spherically homogenous tree T. G is called a *branch group*, if it is transitive on each level, and for every integer $n \geq 1$ $\operatorname{rist}_G(n)$ has finite index in G.

There are several notions that are particularly useful in studying branch groups. Let G be a branch group and let v be a vertex of T. The *stabilizer* of v in G is the subgroup

$$\operatorname{st}_G(v) = \{x \in G | v^x = v\}.$$

Denote also by $\mathrm{st}_G(n)$ the subgroup $\bigcap_{|v|=n} \mathrm{st}_G(v)$, which keeps all vertices of level n fixed.

Subgroups $\mathrm{st}_G(n)$ are called *principal congruence subgroups* in G. A subgroup of G which contains a principal congruence subgroup is called a *congruence subgroup*.

Definition 1.3. Group G is said to possess the *congruence property* if any its finite index subgroup is a congruence subgroup.

Notice that to every $x \in \mathrm{st}_{\mathrm{Aut}\,T}(v)$ we can assign a unique automorphism $x_v \in \mathrm{Aut}\,T_v$ which is obtained by taking the restriction of x onto the subtree T_v. Moreover, for each vertex v of length n x_v can be considered as an element of $\mathrm{Aut}\,T_{(n)}$. Thus, for each vertex v there is a fixed homomorphism $\varphi_v : \mathrm{st}_{\mathrm{Aut}\,T}(v) \to \mathrm{Aut}\,T_{(|v|)}$. We will denote $\varphi_v(\mathrm{st}_G(v))$ by G_v and call it an *upper companion group of G (at vertex v).*

In general, for different vertices u, v of the same length G_u and G_v are conjugate in $\mathrm{Aut}\,T_{(|u|)}$ but may not coincide. Let us fix for each pair of vertices u, v of the same length a conjugating element $\psi_{u,v}$ such that $G_u^{\psi_{u,v}} = G_v$, and $\psi_{u,u} = 1$ for any u. Nonetheless, in the examples we will be considering G_u and G_v do coincide for any two vertices of the same length, and all elements $\psi_{u,v}$ will be equal to the trivial element.

Definition 1.4. A branch group G acting on a regular tree T is *fractal*, if for each vertex u of T $G_u = G$.

As will be evidenced many times throughout this paper, the property of a group to be fractal is very helpful in inductive proofs.

It is clear that the direct product of φ_v taken over all vertices v of length n defines a monomorphism of $\mathrm{st}_{\mathrm{Aut}\,T}(n)$ into the direct product of $|A_1||A_2|\dots|A_n|$ copies of $\mathrm{Aut}\,T_{(n)}$. This product map will be denoted by Φ_n.

Of course, collections of maps φ_u and Φ_n vary with and are determined by the tree T. Since at each given moment a tree in question will be fixed, we will keep the same notation for maps on different trees.

The above maps give a way to describe some, but not all, automorphisms of T. A more general way is given by an observation that for any vertex u of length n x induces a permutation $x(u)$ of A_{n+1}, which acts according to the rule $a \mapsto ((ua)^x)_{n+1}$. Such permutations are called *accompanying* permutations of x. It is easy to see that they fully determine x.

Let us now describe our major examples.

1.1 Grigorchuk's groups

The periodic Grigorchuk's groups G_ω are known as the first examples of groups of intermediate growth [6]. They are just-infinite branch groups ([6]) acting on the infinite binary tree. That tree can be constructed as a tree over a sequence of sets $(X, X, \dots)$ for $X = \{0, 1\}$.

Groups G_ω are parameterized by infinite sequences $\omega = (\omega_1, \omega_2, \dots)$ such that for each i $\omega_i \in \{0, 1, 2\}$, and each of the numbers 0, 1, 2 occurs in ω infinitely many times. The set of all such ω is denoted by Ω_0. The expression $\sigma\omega$ denotes the sequence $(\omega_2, \omega_3, \dots)$. Clearly, $\sigma\omega$ is also in Ω_0.

Each Grigorchuk's group G_ω, $\omega \in \Omega_0$, is generated by four automorphisms a, b_ω, c_ω, d_ω. (Although just three of these automorphisms are actually enough). The only nontrivial accompanying permutation of a is at the root vertex and is the cyclic permutation, denoted by π, of two elements. The other three automorphisms have infinitely many nontrivial accompanying permutations, but all of them are located in vertices $u_n = \underbrace{11\dots1}_n 0$, $n = 0, 1, 2, \dots$. Permutation $b_\omega(u_n)$ is trivial if $\omega_{n+1} = 2$ and is equal to π otherwise. Likewise, permutation $c_\omega(u_n)$ is trivial if $\omega_{n+1} = 1$ and is equal to π otherwise, and permutation $d_\omega(u_n)$ is trivial if $\omega_{n+1} = 0$ and is equal to π otherwise.

Thus, for any vertex u of the form $u = \underbrace{11\dots1}_n$ or $u = \underbrace{11\dots1}_n 0 = u_n$ generators b_ω, c_ω, d_ω are in $\mathrm{st}_{G_\omega}(u)$, and we have

$$\varphi_u(b_\omega) = \begin{cases} a, & \text{if } u = u_n \text{ for some } n \text{ and } \omega_{n+1} \neq 2 \\ b_{\sigma_n \omega}, & \text{if } u = \underbrace{11\dots1}_n \end{cases}$$

$$\varphi_u(c_\omega) = \begin{cases} a, & \text{if } u = u_n \text{ for some } n \text{ and } \omega_{n+1} \neq 1 \\ c_{\sigma_n \omega}, & \text{if } u = \underbrace{11\dots1}_n \end{cases}$$

$$\varphi_u(d_\omega) = \begin{cases} a, & \text{if } u = u_n \text{ for some } n \text{ and } \omega_{n+1} \neq 0 \\ d_{\sigma_n \omega}, & \text{if } u = \underbrace{11\dots1}_n \end{cases}$$

In particular, for any vertex u $\varphi_u(G_\omega) = G_{\sigma^n \omega}$ for $n = |u|$.

It also follows from the above definitions that exactly one of generators b_ω, c_ω, d_ω has trivial accompanying permutation at $u_0 = (0)$. That generator will be called the *d-generator* of G_ω. Now, let $m(\omega)$ be the maximal number n such that the finite sequence $\omega_1\omega_2\dots\omega_n$ is constant. Suppose that y_ω is the generator of G_ω such that $y_{\sigma^{m(\omega)}\omega}$ is the d-generator of $G_{\sigma^{m(\omega)}\omega}$. Then y_ω is called the *c-generator* of G_ω. That of generators b_ω, c_ω, d_ω which is neither the d-generator nor the c-generator is called *b-generator.*

Directly applying Proposition 1 from [12], we get the following fact.

Lemma 1.1. Let G_ω be a periodic Grigorchuk's group. Then

$$G_\omega/[G_\omega, G_\omega] \cong < a > \times < b_\omega, c_\omega > \cong Z_2 \times Z_2 \times Z_2.$$

Corollary 1.1. *Let x_ω be the d-generator of G_ω and let y_ω be its c-generator. Then the factorgroup of G_ω by the normal closure N of x_ω is isomorphic to the group generated by a and y_ω.*

Proof. It is sufficient to show that the intersection $< a, y_\omega > \cap N$ is trivial. Notice that N is contained in $\mathrm{st}_{G_\omega}(m(\omega)+1)$. On the other hand, the intersection $< a, y_\omega > \cap\, \mathrm{st}_{G_\omega}(m(\omega)+1)$ is generated by element $(ay_\omega)^{2^{m(\omega)+1}}$. Since the order of this element is equal to 2, it is sufficient to prove that the element does not belong to N.

The image of $(ay_\omega)^{2^{m(\omega)+1}}$ under $\Phi_{m(\omega)+1}$ equals $(\underbrace{y_{\sigma^{m(\omega)+1}\omega}, y_{\sigma^{m(\omega)+1}\omega}, \ldots, y_{\sigma^{m(\omega)+1}\omega}}_{2^{m(\omega)+1}})$. However, since x_ω is the d-generator of G_ω, for any vertex u of length $m(\omega)$ $\varphi_u(N)$ is in the normal closure of $x_{\sigma^{m(\omega)}\omega}$. And since $x_{\sigma^{m(\omega)}\omega}$ is not the d-generator of $G_{\sigma^{m(\omega)}\omega}$, for any vertex u of length $m(\omega)+1$ $\varphi_u(N)$ is in the normal closure of the set $\{a, x_{\sigma^{m(\omega)+1}\omega}\}$. Hence the image of any $\varphi_u(N)$ in the factor-group of $G_{\sigma^{m(\omega)+1}\omega}$ by its derived subgroup is in the subgroup

$$(< a, x_{\sigma^{m(\omega)+1}\omega} > [G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}])/[G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}].$$

By Lemma 1.1 we get that the image of $y_{\sigma^{m(\omega)+1}\omega}$ is not in this subgroup, hence $(ay_\omega)^{2^{m(\omega)+1}}$ is not in N. ∎

Let us introduce length function $\partial_\omega()$ on each G_ω. Each element of G_ω can be represented by a word of the form,

$$a^\epsilon x_1 a \ldots a x_m a^\mu,$$

where x_1, ..., $x_m \in \{b_\omega, c_\omega, d_\omega\}$, and ϵ, μ are in GF(2). The length of such a word, by definition, is equal to m. The length $\partial_\omega(x)$ is defined as the minimum of lengths of words representing x.

1.2 Periodic GGS-groups

The term GGS-group was introduced by Gilbert Baumslag [3] and refers to Rostislav ***G***rigorchuk, Narain ***G***upta and Said ***S***idki. Those groups act on regular p-trees, where $p \geq 3$. We will always assume that p is prime.

Each GGS-group is fully defined by a vector

$$\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{p-1}) \in \underbrace{\mathrm{GF}(p) \oplus \mathrm{GF}(p) \oplus \ldots \oplus \mathrm{GF}(p)}_{p-1} \tag{1.2}$$

in the following way.

Construct the regular p-tree T_p as the tree over the sequence $(X, X, \ldots)$, $X = \{0, 1, 2, \ldots, p-1\}$. Each GGS-group $G_{\bar{\alpha}}$ acting on T_p is generated by two automorphisms a and b. The only accompanying permutation of a is at the root and is the cyclic permutation $(01\ldots p-1)$, i.e. a cyclically permutes edges coming out of the root. The second generator is in the stabilizer of level 1, and

$$\Phi_1(b) = (a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_{p-1}}, b).$$

The vector (1.2) is called the *accompanying vector* of G.

By [17], all GGS-groups are just infinite branch; they are periodic if and only if $\sum_{i=1}^{p-1} \alpha_i = 0$. Notice that for any vertex u $\varphi_u(G_{\bar{\alpha}}) = G_{\bar{\alpha}}$. Hence all $G_{\bar{\alpha}}$ are fractal as well.

By Proposition 1 from [12], we have

Lemma 1.2. Let $G_{\bar{\alpha}}$ be a GGS-group. Then

$$G_{\bar{\alpha}}/[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \cong < a > \times < b > \cong Z/pZ \times Z/pZ.$$

Later on we will need the following length function on $G_{\bar{\alpha}}$. Each element $x \in G_{\bar{\alpha}}$ can be represented (not uniquely) by a word of the form,

$$a^{\delta_1} b^{\beta_1} \ldots a^{\delta_m} b^{\beta_m} a^{\delta_{m+1}},$$

where $\delta_2, \ldots, \delta_m, \beta_1, \ldots, \beta_m \in \mathrm{GF}(p) \setminus \{0\}$ and $\delta_1, \delta_{m+1} \in \mathrm{GF}(p)$. The number m will be called the length of such a word. The length $\partial_{G_{\bar{\alpha}}}(x)$ is defined as the minimal length of all words representing x.

Also, denote by $\partial_a(x)$ the sum of all δ_i and by $\partial_b(x)$ the sum of all β_i in some representation of x. By Lemma 1.2, the sums are independent of the choice of a particular representation. Indeed, pair $(\partial_a(x), \partial_b(x))$ is the image of x under the composition of the natural projection $G_{\bar{\alpha}} \to G_{\bar{\alpha}}/[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$ and the isomorphisms $G_{\bar{\alpha}}/[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \cong < a > \times < b > \cong Z/pZ \times Z/pZ$ of Lemma 1.2. Thus, $\partial_a(x)$ and $\partial_b(x)$ are well-defined.

1.3 EGS-groups

The term EGS-group stands for "extended Gupta-Sidki group", but the word "extended" should not be understood in the usual algebraic sense. While each EGS-group contains some specific GGS-group as a subgroup, it is not an extension of it. However, in a certain sense a periodic GGS-group, or, more precisely, its accompanying vector, does define a unique EGS-group.

Let $G_{\bar{\alpha}}$ be a periodic GGS-group with generators a and b and accompanying vector $(\alpha_1, \alpha_2, \ldots, \alpha_{p-1})$. The corresponding EGS-group $\Gamma_{\bar{\alpha}}$ is generated by automorphisms a, b, and the automorphism c such that

$$\Phi_1(c) = (c, a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_{p-1}}).$$

The group $G_{\bar{\alpha}}$ is then called the *associated GGS-group* of $\Gamma_{\bar{\alpha}}$.

Let us establish a few needed properties of EGS-groups. First of all, by Theorem 2 of [14], all EGS-groups are periodic. Just as with GGS-groups, for any vertex u $\varphi_u(\Gamma_{\bar{\alpha}}) = \Gamma_{\bar{\alpha}}$.

Lemma 1.3. All EGS-groups are just infinite branch groups.

Proof. Consider a EGS-group $\Gamma_{\bar{\alpha}}$ with accompanying vector $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{p-1})$. Let us prove first that for any vertex u $\mathrm{rist}_{\Gamma_{\bar{\alpha}}}(u)_u$ contains the third term $\gamma_3(\Gamma_{\bar{\alpha}})$ of the lower central series of $\Gamma_{\bar{\alpha}}$. To this end, we will need to check that $\bar{\alpha}$ is not *ε-diagonal*, i.e. it is not proportional to a vector of form $(\varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1})$, where ε would be a unity root of pth degree in the field $\mathrm{GF}(p^n)$ for a natural n.

By definition of a EGS-group, $\alpha_1 + \ldots + \alpha_{p-1} = 0$. But the sum of coordinates of a vector $(\varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1})$ is equal to

$$\varepsilon + \varepsilon^2 + \ldots + \varepsilon^{p-1} = \begin{cases} \varepsilon \frac{1-\varepsilon^{p-1}}{1-\varepsilon}, & \text{if } \varepsilon \neq 1 \\ p-1, & \text{if } \varepsilon = 1. \end{cases}$$

Thus, if $\varepsilon = 1$, this sum is not equal to 0. If $\varepsilon \neq 1$, then clearly $1 - \varepsilon^{p-1} \neq 0$, and the sum is not equal to 0, either.

We obtain that $\bar{\alpha}$ is not ε-diagonal. This, together with Theorem 2.7.1 of [15], implies that $\Phi_1(\gamma_3(G_{\bar{\alpha}}))$ contains the direct product $\underbrace{\gamma_3(G_{\bar{\alpha}}) \times \ldots \times \gamma_3(G_{\bar{\alpha}})}_{p}$ (recall that $G_{\bar{\alpha}}$ is the associated GGS-group of $\Gamma_{\bar{\alpha}}$). By the same theorem, we obtain that if $F_{\bar{\alpha}}$ is the subgroup of $\Gamma_{\bar{\alpha}}$ generated by elements a and c, then $\Phi_1(\gamma_3(F_{\bar{\alpha}}))$ contains the direct product $\underbrace{\gamma_3(F_{\bar{\alpha}}) \times \ldots \times \gamma_3(F_{\bar{\alpha}})}_{p}$.

Notice that $\gamma_3(\Gamma_{\bar{\alpha}})$ is the normal closure in $\Gamma_{\bar{\alpha}}$ of the following set, $[[a,b],a]$, $[[a,b],b]$, $[[a,b],c]$, $[[a,c],a]$, $[[a,c],b]$, $[[a,c],c]$, $[[b,c],a]$, $[[b,c],b]$, $[[b,c],c]$. It follows from the above that $\Phi_1(\gamma_3(G_{\bar{\alpha}}))$ contains elements $([[a,b],a],\underbrace{1,\ldots,1}_{p-1})$, $([[a,b],b],\underbrace{1,\ldots,1}_{p-1})$, $([[a,c],a],\underbrace{1,\ldots,1}_{p-1})$, $([[a,c],c],\underbrace{1,\ldots,1}_{p-1})$. Further, $\Phi_1([b^a,c]) = ([b,c],\underbrace{1,\ldots,1}_{p-1})$. Hence $\Phi_1(\gamma_3(\Gamma_{\bar{\alpha}}))$ contains elements $([[b,c],a],\underbrace{1,\ldots,1}_{p-1})$, $([[b,c],b],\underbrace{1,\ldots,1}_{p-1})$, $([[b,c],c],\underbrace{1,\ldots,1}_{p-1})$ as well.

Finally, assume that n is a number such that $\alpha_n \neq 0$. Then

$$\Phi_1([[b^a, b^{a^{p-n+1}}],c]) = ([[b,a^{\alpha_n}],c],\underbrace{1,\ldots,1}_{p-2},[[a^{\alpha_n},b],a^{\alpha_n}]),$$

$$\Phi_1([[c, b^{a^{p-n+1}}],b^a]) = ([[c,a^{\alpha_n}],b],\underbrace{1,\ldots,1}_{p-2},[[a^{\alpha_n},b],a^{\alpha_n}]).$$

Thus, we can conclude that for one, and hence for any, vertex u of length 1 $\mathrm{rist}_{\Gamma_{\bar{\alpha}}}(u)_u$ contains $\gamma_3(\Gamma_{\bar{\alpha}})$. Moreover, $\Phi_1(\gamma_3(\Gamma_{\bar{\alpha}}))$ contains the direct product $\underbrace{\gamma_3(\Gamma_{\bar{\alpha}}) \times \ldots \times \gamma_3(\Gamma_{\bar{\alpha}})}_{p}$. Hence, we have the desired inclusion for all vertices.

Since $\Gamma_{\bar{\alpha}}$ is periodic and finitely generated, $\gamma_3(\Gamma_{\bar{\alpha}})$ has finite index. Therefore for every n $\Phi_n^{-1}(\underbrace{\gamma_3(\Gamma_{\bar{\alpha}}) \times \ldots \times \gamma_3(\Gamma_{\bar{\alpha}})}_{p^n})$ has finite index in $\Gamma_{\bar{\alpha}}$. Indeed, it has finite index in $\mathrm{st}_{\Gamma_{\bar{\alpha}}}(n)$, because the image $\Phi_n(\mathrm{st}_{\Gamma_{\bar{\alpha}}}(n))$ is contained in the direct product of p^n copies of $\Gamma_{\bar{\alpha}}$. But $\mathrm{st}_{\Gamma_{\bar{\alpha}}}(n)$ obviously has finite index in $\Gamma_{\bar{\alpha}}$, so we get that $\Phi_n^{-1}(\underbrace{\gamma_3(\Gamma_{\bar{\alpha}}) \times \ldots \times \gamma_3(\Gamma_{\bar{\alpha}})}_{p^n})$ has finite index in $\Gamma_{\bar{\alpha}}$.

Thus, we have proven that $\Gamma_{\bar{\alpha}}$ is a branch group. However, $\Gamma_{\bar{\alpha}}$ is periodic and finitely generated, hence $\gamma_3(\Gamma_{\bar{\alpha}})$ is also periodic and finitely generated. Therefore, index $|\gamma_3(\Gamma_{\bar{\alpha}}) : [\gamma_3(\Gamma_{\bar{\alpha}}), \gamma_3(\Gamma_{\bar{\alpha}})]|$ is finite. Then by Theorem 4 of [7] $\Gamma_{\bar{\alpha}}$ is just infinite. ∎

In particular, we have that all $\Gamma_{\bar{\alpha}}$ are fractal.

Lemma 1.4. Let $\Gamma_{\bar{\alpha}}$ be a EGS-group. Then

$$\Gamma_{\bar{\alpha}}/[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}] \cong <a> \times <b> \times <c> \cong Z/pZ \times Z/pZ \times Z/pZ.$$

Proof. Consider a group $H = <a,b,c | a^p, b^p, c^p>$, the free product of three copies of Z/pZ. Each element x of H is uniquely represented in the form

$$a^{s_1} b^{\beta_1} c^{\delta_1} \ldots a^{s_m} b^{\beta_m} c^{\delta_m}, \tag{1.3}$$

where s_i, β_i, $\delta_i \in \mathrm{GF}(p)$, and for each $i = 1, 2, \ldots, m-1$ either β_i or δ_i is not equal to 0, for each $i = 1, 2, \ldots, m-1$ either δ_i or s_{i+1} is not equal to 0, and for each $i = 2, 3, \ldots, m$ either s_i or β_i is not equal to 0. The amount of non-zero numbers among β_1, β_2, ..., β_m, δ_1, δ_2, ..., δ_m is called the *length* of x and is denoted by $\partial(x)$. Denote the sum of all β_i by $\partial_b(x)$ and the sum of all δ_i by $\partial_c(x)$.

Consider the subgroup $L(H)$ that consists of all elements x such that $s_1+s_2+\ldots+s_m = 0$. This is clearly a normal subgroup generated by elements b, b^a, ..., $b^{a^{p-1}}$, c, c^a, ..., $c^{a^{p-1}}$. Moreover, each element of $L(H)$ is uniquely represented in the form

$$b^{\beta_1 a^{s_1}} c^{\delta_1 a^{t_1}} \ldots b^{\beta_m a^{s_m}} c^{\delta_m a^{t_m}}, \tag{1.4}$$

where s_i, t_i β_1, $\delta_m \in \mathrm{GF}(p)$, and β_2, ..., β_m, δ_1, ..., δ_{m-1} are in $\mathrm{GF}(p) \setminus \{0\}$. Let us denote by $\partial_{b,i}(x)$ the sum of the β_j such that $s_j = i$, and by $\partial_{c,i}(x)$ the sum of the δ_j such that $t_j = i$.

Consider a map $\tilde{\Phi} : L(H) \to \underbrace{H \times \ldots \times H}_{p}$ acting via the rule:

$$\tilde{\Phi}(b) = (a^{\alpha_1}, \ldots, a^{\alpha_{p-1}}, b),$$

$$\tilde{\Phi}(c) = (c, a^{\alpha_1}, \ldots, a^{\alpha_{p-1}}),$$

and $\tilde{\Phi}(b^{a^i})$ and $\tilde{\Phi}(c^{a^i})$ are obtained from, respectively, $\tilde{\Phi}(b^{a^{i-1}})$ and $\tilde{\Phi}(c^{a^{i-1}})$ by one cyclic shift to the right.

Let us show first that

$$\tilde{\Phi}^{-1}(\underbrace{[H, H] \times \ldots \times [H, H]}_{p}) \le [H, H].$$

Consider an arbitrary element $x \in L(H)$. Suppose that $\tilde{\Phi}(x) = (x_1, x_2, \ldots, x_{p-1}, x_0)$. Then $\partial_b(x_i) = \partial_{b,i}(x)$ and $\partial_c(x_i) = \partial_{c,i}(x)$. Hence

$$\sum_{i \in \mathrm{GF}(p)} \partial_b(x_i) = \partial_b(x)$$

and

$$\sum_{i \in \mathrm{GF}(p)} \partial_c(x_i) = \partial_c(x).$$

Suppose that x is not in $[H, H]$. Then either $\partial_b(x)$ or $\partial_c(x)$ is not equal to zero. By the above equalities, not all of numbers $\partial_b(x_i)$, $\partial_c(x_i)$ are equal to zero. Hence $\tilde{\Phi}(x)$ is not in $([H, H] \times \ldots \times [H, H])$, which is equivalent to saying that $\tilde{\Phi}^{-1}([H, H] \times \ldots \times [H, H]) \le [H, H]$.

Consider now the following sequence of subgroups. $K_0 = 1$; $K_n = \tilde{\Phi}^{-1}(K_{n-1} \times \ldots \times K_{n-1})$, $n \ge 1$. Take $K = \cup_{n=0}^{\infty} K_n$. It is easy to show that K is a normal subgroup of H. It follows, for instance, from Proposition 1 of [14] that $\Gamma_{\bar{\alpha}}$ is isomorphic to the factor-group H/K. Hence $\Gamma_{\bar{\alpha}}/[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}] \cong H/[H, H]$, and we have the desired statement. ∎

Later on we will need a length function $\partial_{\Gamma_{\bar{\alpha}}}()$ on each $\Gamma_{\bar{\alpha}}$. Since $\Gamma_{\bar{\alpha}}$ is a homomorphic image of H from the above lemma, each its element x can be represented (not uniquely) by a word of form (1.3). Define the length $\partial_{\Gamma_{\bar{\alpha}}}(x)$ of x to be the minimal length of all such words (as defined in the proof of Lemma 1.4) representing x.

Notice that the restriction of length function $\partial_{\Gamma_{\bar{\alpha}}}()$ to the associated GGS-group gives length function $\partial_{G_{\bar{\alpha}}}()$ introduced in Section 1.2.

Also, for some word (1.3) representing x, denote the sum of all s_i by $\partial_a(x)$, the sum of all β_i by $\partial_b(x)$, and the sum of all δ_i by $\partial_c(x)$. By Lemma 1.4, these numbers do not depend on the choice of a particular word representing x. Indeed, the triple $(\partial_a(x), \partial_b(x), \partial_c(x))$ is the image of x under the composition of the natural projection $\Gamma_{\bar{\alpha}} \to \Gamma_{\bar{\alpha}}/[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$ and the isomorphisms $\Gamma_{\bar{\alpha}}/[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}] \cong < a > \times < b > \times < c > \cong Z/pZ \times Z/pZ \times Z/pZ$. Hence, the numbers are well-defined.

1.4 Scheme of the proof

The key ingredient in studying maximal subgroups of infinite index is a simple observation that their presence in a group is equivalent to that group having proper subgroups everywhere dense in the profinite topology (Lemma 4.1 and Proposition 5.1). It is such subgroups that we will in fact attempt to locate throughout the paper (ultimately failing at that).

The main idea of the proof that our examples 1.1, 1.2, and 1.3 do not contain proper everywhere dense subgroups is to study possible upper companion groups of such subgroups. We will eventually establish that their companion groups are also everywhere dense (in the corresponding upper companion groups of their respective groups, see Proposition 4.1), and that for an everywhere dense subgroup to be proper, all of its upper companion groups have to be proper as well (Lemmas 4.2, 4.4, and 4.7).

Two lengthy sections, 2 and 3, contain technical results (Corollaries 2.1 and 3.2) necessary for proving the reverse, i.e. that any everywhere dense subgroup has an upper companion group coinciding with the corresponding upper companion group of the whole group (Propositions 4.2, 4.3, and 4.4). This allows us to establish Theorems 4.1, 4.2, and 4.3, which easily yield the main result of the paper, Theorem 5.1.

2 Upper companion groups of subgroups in Grigorchuk's groups: one special case

We start with Grigorchuk's 2-groups G_ω. Let H_ω denote the following group:

$$< a, b_\omega, c_\omega, d_\omega | a^2, b_\omega^2, c_\omega^2, d_\omega^2, b_\omega c_\omega d_\omega > .$$

Up to isomorphisms, these are just an infinite collection of exemplars of the free product $Z_2 * (Z_2 \times Z_2)$.

A generator of a group H_ω is called its d-generator, c-generator, or b-generator, respectively, if it is the d-generator, c-generator, or b-generator, of the corresponding group G_ω.

Each element z of H_ω is uniquely represented in the form

$$a^i x_1 a x_2 a \dots a x_m a^j, \tag{2.1}$$

where $x_1, x_2, \dots, x_m \in \{b_\omega, c_\omega, d_\omega\}$, $i, j \in \mathrm{GF}(2)$. Denote by $\partial_\omega(z)$ the number m (it will be called the *length* of z), by $\partial_{b_\omega}(z)$ the number of occurrences of b_ω in z, by $\partial_{c_\omega}(z)$ the number of occurrences of c_ω in z, by $\partial_{d_\omega}(z)$ the number of occurrences of d_ω in z, and by $\partial_a(z)$ the number of occurrences of a in z.

We say that z has an *inner occurrence* of the d-, c-, or b-generator, if among $x_2, \dots, x_{m-1}$ of (2.1) there is the d-, c-, or b-generator, respectively. z has a *boundary occurrence* of the d-, c-, or b-generator, if among x_1, x_m there is the d-, c-, or b-generator, respectively.

Denote by $L(H_\omega)$ the set of all z such that $\partial_a(z)$ is even (clearly, it is a subgroup of index 2). It is easy to see that $L(H_\omega)$ is generated by elements b_ω, c_ω, d_ω, b_ω^a, c_ω^a, d_ω^a; in fact, it is the free product of the subgroups generated by these elements.

Define a pair of maps $\tilde\varphi_1^\omega$, $\tilde\varphi_2^\omega : L(H_\omega) \to H_{\sigma\omega}$, in the following way:

$$\left\{\begin{array}{rcl} \tilde\varphi_1^\omega(b_\omega^a) &=& b_{\sigma\omega} \\ \tilde\varphi_1^\omega(c_\omega^a) &=& c_{\sigma\omega} \\ \tilde\varphi_1^\omega(d_\omega^a) &=& d_{\sigma\omega} \end{array}\right. , \quad \left\{\begin{array}{rcl} \tilde\varphi_2^\omega(b_\omega) &=& b_{\sigma\omega} \\ \tilde\varphi_2^\omega(c_\omega) &=& c_{\sigma\omega} \\ \tilde\varphi_2^\omega(d_\omega) &=& d_{\sigma\omega} \end{array}\right.$$

Besides, the image of the d-generator under $\tilde\varphi_1^\omega$ is equal to 1, and the images of the other two generators are equal to a. Likewise, the image of the d-generator conjugated by a under $\tilde\varphi_2^\omega$ is equal to 1, and the images of the a-conjugates of the other two-generators are equal to a. For instance, if d_ω is the d-generator, c_ω is the c-generator, and b_ω is the b-generator, then we have

$$\left\{\begin{array}{rcl} \tilde\varphi_1^\omega(b_\omega) &=& a \\ \tilde\varphi_1^\omega(c_\omega) &=& a \\ \tilde\varphi_1^\omega(d_\omega) &=& 1 \\ \tilde\varphi_1^\omega(b_\omega^a) &=& b_{\sigma\omega} \\ \tilde\varphi_1^\omega(c_\omega^a) &=& c_{\sigma\omega} \\ \tilde\varphi_1^\omega(d_\omega^a) &=& d_{\sigma\omega} \end{array}\right. , \quad \left\{\begin{array}{rcl} \tilde\varphi_2^\omega(b_\omega) &=& b_{\sigma\omega} \\ \tilde\varphi_2^\omega(c_\omega) &=& c_{\sigma\omega} \\ \tilde\varphi_2^\omega(d_\omega) &=& d_{\sigma\omega} \\ \tilde\varphi_2^\omega(b_\omega^a) &=& a \\ \tilde\varphi_2^\omega(c_\omega^a) &=& a \\ \tilde\varphi_2^\omega(d_\omega^a) &=& 1 \end{array}\right.$$

Obviously, the extensions by homomorphism of these maps to the whole $L(H_\omega)$ are well-defined.

Let us introduce a map

$$\Theta_\omega : [H_\omega, H_\omega] \to [H_{\sigma\omega}, H_{\sigma\omega}],$$

which will play a key part in the sequel. It acts via the rule

$$\Theta_\omega(z) = a\tilde\varphi_1^\omega(z) a \tilde\varphi_2^\omega(z).$$

We need, of course, to verify that the map is well-defined, i.e. the image of a derived subgroup $[H_\omega, H_\omega]$ under it is contained in the derived subgroup $[H_{\sigma\omega}, H_{\sigma\omega}]$. Indeed, it is easy to see that $\{1, a, b_\omega, c_\omega, d_\omega, ab_\omega, ac_\omega, ad_\omega\}$ is a Schreier system of representatives for $[H_\omega, H_\omega]$. Hence $[H_\omega, H_\omega]$ is generated by elements $[a, b_\omega] = (ab_\omega)^2$, $[a, c_\omega] = (ac_\omega)^2$, $[a, d_\omega] = (ad_\omega)^2$.

Each of these elements has the property that its images under $\tilde{\varphi}_1^\omega$ and $\tilde{\varphi}_2^\omega$ belong to the same coset of $[H_{\sigma\omega}, H_{\sigma\omega}]$. Since the factor-group $H_{\sigma\omega}/[H_{\sigma\omega}, H_{\sigma\omega}]$ is of exponent 2, the product of those images belongs to $[H_{\sigma\omega}, H_{\sigma\omega}]$. Certainly, the same is true for any other element of $[H_\omega, H_\omega]$.

Let us consider how maps Θ_ω interact with the corresponding length functions. First we show that in some specific cases applying the maps causes length to decrease.

Lemma 2.1. Let z belong to $[H_\omega, H_\omega]$. Then the following inequality holds

$$\partial_{\sigma\omega}(\Theta_\omega(z)) \leq \partial_\omega(z).$$

In addition,

1) if z has at least one inner occurrence of the d-generator (this always happens when $\partial_{d_\omega}(z) \geq 3$), then

$$\partial_{\sigma\omega}(\Theta_\omega(z)) \leq \partial_\omega(z) - 1;$$

2) if $\partial_{\sigma\omega}(\Theta_\omega(z)) = \partial_\omega(z)$, then $\partial_{b_{\sigma\omega}}(\Theta_\omega(z)) = \partial_{b_\omega}(z)$, $\partial_{c_{\sigma\omega}}(\Theta_\omega(z)) = \partial_{c_\omega}(z)$, $\partial_{d_{\sigma\omega}}(\Theta_\omega(z)) = \partial_{d_\omega}(z)$;

3) if at least one of the numbers $\partial_{b_\omega}(z)$, $\partial_{c_\omega}(z)$, $\partial_{d_\omega}(z)$ is greater than or equal to 3, then there exists n such that $\partial_{\sigma^{n+1}\omega}(\Theta_{\sigma^n\omega} \circ \ldots \circ \Theta_\omega(z)) < \partial_\omega(z)$.

Proof. Inequality $\partial_{\sigma\omega}(\Theta_\omega(z)) \leq \partial_\omega(z)$ and 2) follow from the fact that to each occurrence of b_ω, c_ω, d_ω in z corresponds exactly one occurrence of, respectively, $b_{\sigma\omega}$, $c_{\sigma\omega}$, $d_{\sigma\omega}$ in either $\tilde{\varphi}_1^\omega(z)$ or $\tilde{\varphi}_2^\omega(z)$. Statement 1) is verified directly. Let us prove 3).

Without losing any generality, we can assume that d_ω is the d-generator of H_ω, andc_ω is the c-generator. Since at least one of the numbers $\partial_{d_\omega}(z)$, $\partial_{c_\omega}(z)$, $\partial_{b_\omega}(z)$ is greater than or equal to 3, we have three cases to consider.

If $\partial_{d_\omega}(z) \geq 3$, then by 1) we have that $\partial_{\sigma\omega}(\Theta_\omega(z)) < \partial_\omega(z)$. Suppose that $\partial_{d_\omega}(z) \leq 2$, but $\partial_{c_\omega}(z) \geq 3$. Apply to z subsequently Θ_ω, $\Theta_{\sigma\omega}$, ..., $\Theta_{\sigma^{m(\omega)-1}\omega}$ (recall that $m(\omega)$ is the maximal length of a constant initial segment of ω). Then either upon application of these maps we receive an element of smaller length, or by 2) we get

$$\partial_{c_{\sigma^{m(\omega)}\omega}}(\Theta_{\sigma^{m(\omega)-1}\omega} \circ \ldots \circ \Theta_\omega(z)) = \partial_{c_\omega}(z) \geq 3.$$

Since $c_{\sigma^{m(\omega)}\omega}$ is the d-generator of $H_{\sigma^{m(\omega)}\omega}$, it follows from 1) that

$$\partial_{\sigma^{m(\omega)+1}\omega}(\Theta_{\sigma^{m(\omega)}\omega} \circ \ldots \circ \Theta_\omega(z)) < \partial_\omega(z).$$

If both $\partial_{d_\omega}(z) \leq 2$ and $\partial_{c_\omega}(z) \leq 2$, we have that $\partial_{b_\omega}(z) \geq 3$. Let $n(\omega)$ be the minimal number such that the segment $(\omega_1, \ldots, \omega_{n(\omega)})$ of ω contains all three numbers 0,1,2. Notice that in this case $b_{\sigma^{n(\omega)-1}\omega}$ is the d-generator of $H_{\sigma^{n(\omega)-1}\omega}$. Apply to z the maps Θ_ω, $\Theta_{\sigma\omega}$, ..., $\Theta_{\sigma^{n(\omega)-2}\omega}$. Then either upon application of these maps we receive an element of smaller length, or by 2) we get

$$\partial_{b_{\sigma^{n(\omega)-1}\omega}}(\Theta_{\sigma^{n(\omega)-2}\omega} \circ \ldots \circ \Theta_\omega(z)) = \partial_{b_\omega}(z) \geq 3.$$

Since $b_{\sigma^{n(\omega)-1}\omega}$ is the d-generator of $H_{\sigma^{n(\omega)-1}\omega}$, it follows from 1) that

$$\partial_{\sigma^{n(\omega)}\omega}(\Theta_{\sigma^{n(\omega)-1}\omega}\circ\ldots\circ\Theta_\omega(z))<\partial_\omega(z).$$

This concludes the proof of the lemma. ∎

Let us consider a general case now.

Proposition 2.1. *Let $\Theta_\omega:[H_\omega,H_\omega]\to[H_{\sigma\omega},H_{\sigma\omega}]$ be the maps introduced above. Then for any $z\in[H_\omega,H_\omega]$ there exists n such that*

$$\Theta_{\sigma^n\omega}\circ\ldots\circ\Theta_\omega(z)\in<(ay_{\sigma^{n+1}\omega})^2>,$$

where $y_{\sigma^{n+1}\omega}$ is the d-generator of $H_{\sigma^{n+1}\omega}$.

Proof. The proof is by induction on length $\partial_\omega(z)$. As above, assume that d_ω is the d-generator and c_ω is the c-generator.

The base of induction: $\partial_\omega(z)\le 2$**.** If $\partial_\omega(z)=0$ and thus $z=1$, there's nothing to prove. Obviously, no element of $[H_\omega,H_\omega]$ can have length 1. For $\partial_\omega(z)=2$ we have 6 cases: $(ad_\omega)^2$, $(d_\omega a)^2$, $(ac_\omega)^2$, $(c_\omega a)^2$, $(ab_\omega)^2$, $(b_\omega a)^2$. Now, since d_ω is the d-generator, $(ad_\omega)^4=1$ and hence $(d_\omega a)^2=(ad_\omega)^2$.

Consider the remaining cases:

$$\Theta_\omega:(ac_\omega)^2\mapsto(ac_{\sigma\omega})^2,\ (c_\omega a)^2\mapsto(c_{\sigma\omega}a)^2,$$

$$(ab_\omega)^2\mapsto(ab_{\sigma\omega})^2,\ (b_\omega a)^2\mapsto(b_{\sigma\omega}a)^2.$$

Thus,

$$\Theta_{\sigma^{m(\omega)-1}\omega}\circ\ldots\circ\Theta_{\sigma\omega}\circ\Theta_\omega:(ac_\omega)^2\mapsto(ac_{\sigma^{m(\omega)}\omega})^2,\ (c_\omega a)^2\mapsto(c_{\sigma^{m(\omega)}\omega}a)^2,$$

$$\Theta_{\sigma^{n(\omega)-2}\omega}\circ\ldots\circ\Theta_{\sigma\omega}\circ\Theta_\omega:(ab_\omega)^2\mapsto(ab_{\sigma^{n(\omega)}\omega})^2,\ (b_\omega a)^2\mapsto(b_{\sigma^{n(\omega)-1}\omega}a)^2.$$

Since by definitions of $m(\omega)$, $n(\omega)$ $c_{\sigma^{m(\omega)}\omega}$ and $b_{\sigma^{n(\omega)-1}\omega}$ are the d-generators of $H_{\sigma^{m(\omega)}\omega}$ and $H_{\sigma^{n(\omega)-1}\omega}$, respectively, we get the base of induction.

The inductive step. In carrying out the inductive step, the following remark will be important to us. An element $z\in H_\omega$ belongs to $[H_\omega,H_\omega]$ if and only if $\partial_{b_\omega}(z)\equiv\partial_{c_\omega}(z)\equiv\partial_{d_\omega}(z)(\mathrm{mod}\,2)$ (this is easily derived from the definition of H_ω).

We have several possibilities depending on the length of z.

Case 1. Assume that $\partial_\omega(z)=3$. Then by the preceding remark there are twelve possibilities, $c_\omega ad_\omega ab_\omega$, $b_\omega ad_\omega ac_\omega$, $d_\omega ab_\omega ac_\omega$, $d_\omega ac_\omega ab_\omega$, $c_\omega ab_\omega ad_\omega$, $b_\omega ac_\omega ad_\omega$, $ac_\omega ad_\omega ab_\omega a$, $ab_\omega ad_\omega ac_\omega a$, $ac_\omega ab_\omega ad_\omega a$, $ab_\omega ac_\omega ad_\omega a$, $ad_\omega ab_\omega ac_\omega a$, $ad_\omega ac_\omega ab_\omega a$. Let us consider them all.

We have

$$\Theta_\omega(c_\omega ad_\omega ab_\omega)=1,$$

$$\Theta_\omega(b_\omega ad_\omega ac_\omega)=1,$$

$$\Theta_\omega(d_\omega ab_\omega ac_\omega)=(ac_{\sigma\omega})^2,$$

$$\Theta_\omega(d_\omega ac_\omega ab_\omega)=(ab_{\sigma\omega})^2.$$

The result of applying our maps to the next two elements depends on $m(\omega)$. Indeed, $\Theta_\omega(c_\omega ab_\omega ad_\omega) = b_{\sigma\omega}ac_{\sigma\omega}ad_{\sigma\omega}$. If $m(\omega)=1$, the resulting element has an inner occurrence of the d-generator. If $m(\omega)>1$, then $\Theta_{\sigma\omega}(b_{\sigma\omega}ac_{\sigma\omega}ad_{\sigma\omega}) = c_{\sigma^2\omega}ab_{\sigma^2\omega}ad_{\sigma^2\omega}$. Thus

$$\Theta_{\sigma^{m(\omega)-1}\omega}\circ\ldots\circ\Theta_\omega(c_\omega ab_\omega ad_\omega) = \begin{cases} b_{\sigma^{m(\omega)}\omega}ac_{\sigma^{m(\omega)}\omega}ad_{\sigma^{m(\omega)}\omega}, & \text{if } m(\omega)=2k+1 \\ c_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}ad_{\sigma^{m(\omega)}\omega}, & \text{if } m(\omega)=2k \end{cases}$$

Since $c_{\sigma^{m(\omega)}\omega}$ is the d-generator, in the first case after applying $\Theta_{\sigma^{m(\omega)}\omega}$ the length will decrease by Lemma 2.1. In the second case we have

$$\Theta_{\sigma^{m(\omega)}\omega}(c_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}ad_{\sigma^{m(\omega)}\omega}) = (ad_{\sigma^{m(\omega)+1}\omega})^2.$$

In both cases we can use the inductive assumption.

Similarly, $\Theta_\omega(b_\omega ac_\omega ad_\omega) = c_{\sigma\omega}ab_{\sigma\omega}ad_{\sigma\omega}$. If $m(\omega)=1$, we have $\Theta_{\sigma\omega}(c_{\sigma\omega}ab_{\sigma\omega}ad_{\sigma\omega}) = (ad_{\sigma^2\omega})^2$. If $m(\omega)>1$, then $\Theta_{\sigma\omega}(c_{\sigma\omega}ab_{\sigma\omega}ad_{\sigma\omega}) = b_{\sigma^2\omega}ac_{\sigma^2\omega}ad_{\sigma^2\omega}$. Thus

$$\Theta_{\sigma^{m(\omega)-1}\omega}\circ\ldots\circ\Theta_\omega(b_\omega ac_\omega ad_\omega) = \begin{cases} c_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}ad_{\sigma^{m(\omega)}\omega}, & \text{if } m(\omega)=2k+1 \\ b_{\sigma^{m(\omega)}\omega}ac_{\sigma^{m(\omega)}\omega}ad_{\sigma^{m(\omega)}\omega}, & \text{if } m(\omega)=2k \end{cases}$$

Since $c_{\sigma^{m(\omega)}\omega}$ is the d-generator, in the latter case after applying $\Theta_{\sigma^{m(\omega)}\omega}$ we get an element of smaller length by Lemma 2.1. In the first case we have $\Theta_{\sigma^{m(\omega)}\omega}(c_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}ad_{\sigma^{m(\omega)}\omega}) = (ad_{\sigma^{m(\omega)+1}\omega})^2$. Hence in both cases we can use the inductive assumption.

Further, we have

$$\Theta_\omega(ac_\omega ad_\omega ab_\omega a) = 1,$$

$$\Theta_\omega(ab_\omega ad_\omega ac_\omega a) = 1,$$

$$\Theta_\omega(ac_\omega ab_\omega ad_\omega a) = (ac_{\sigma\omega})^2,$$

$$\Theta_\omega(ab_\omega ac_\omega ad_\omega a) = (ab_{\sigma\omega})^2.$$

Now let us consider $ad_\omega ab_\omega ac_\omega a$. We have

$$\Theta_\omega(ad_\omega ab_\omega ac_\omega a) = ad_{\sigma\omega}ac_{\sigma\omega}ab_{\sigma\omega}a.$$

Thus, if $m(\omega)=1$, the latter element has an inner occurrence of the d-generator, therefore upon applying to it $\Theta_{\sigma^2\omega}$ we get an element of smaller length. And if $m(\omega)>1$, then $\Theta_{\sigma^2\omega}(ad_{\sigma\omega}ac_{\sigma\omega}ab_{\sigma\omega}a) = ad_{\sigma^2\omega}ab_{\sigma^2\omega}ac^2_\sigma\omega a$. Therefore,

$$\Theta_{\sigma^{m(\omega)-1}\omega}\circ\ldots\circ\Theta_\omega(ad_\omega ab_\omega ac_\omega a) = \begin{cases} ad_{\sigma^{m(\omega)}\omega}ac_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}a, & \text{if } m(\omega)=2k+1 \\ ad_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}ac_{\sigma^{m(\omega)}\omega}a, & \text{if } m(\omega)=2k. \end{cases}$$

Since $c_{\sigma^{m(\omega)}\omega}$ is the d-generator, in the former case by Lemma 2.1 we get a decrease in the length right after applying $\Theta_{\sigma^{m(\omega)}\omega}$. In the latter case we have $\Theta_{\sigma^{m(\omega)}\omega}(ad_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}ac_{\sigma^{m(\omega)}\omega}a) = (ad_{\sigma^{m(\omega)+1}\omega})^2$, and we can apply the inductive assumption.

Finally, $\Theta_\omega(ad_\omega ac_\omega ab_\omega a) = ad_{\sigma\omega}ab_{\sigma\omega}ac_{\sigma\omega}a$. Again, if $m(\omega) = 1$, then $\Theta_{\sigma\omega}(ad_{\sigma\omega}ab_{\sigma\omega}ac_{\sigma\omega}a) = (ad_{\sigma^2\omega})^2$. If $m(\omega) > 1$, then $\Theta_{\sigma\omega}(ad_{\sigma\omega}ab_{\sigma\omega}ac_{\sigma\omega}a) = ad_{\sigma^2\omega}ac_{\sigma^2\omega}ab_{\sigma^2\omega}a$. Therefore,

$$\Theta_{\sigma^{m(\omega)-1}\omega}\circ\ldots\circ\Theta_\omega(ad_\omega ac_\omega ab_\omega a) = \begin{cases} ad_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}ac_{\sigma^{m(\omega)}\omega}a, & \text{if } m(\omega) = 2k+1 \\ ad_{\sigma^{m(\omega)}\omega}ac_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}a, & \text{if } m(\omega) = 2k \end{cases}$$

Since $c_{\sigma^{m(\omega)}\omega}$ is the d-generator, in the latter case by Lemma 2.1 we get a decrease in the length right after applying $\Theta_{\sigma^{m(\omega)}\omega}$. In the former case we have $\Theta_{\sigma^{m(\omega)}\omega}(ad_{\sigma^{m(\omega)}\omega}ab_{\sigma^{m(\omega)}\omega}ac_{\sigma^{m(\omega)}\omega}a) = (ad_{\sigma^{m(\omega)+1}\omega})^2$. In both cases, we can use the inductive assumption. Thus, we have considered all possibilities for $\partial_\omega(z) = 3$.

Case 2. Assume that $\partial_\omega(z) = 4$. If at least one of the numbers $\partial_{b_\omega}(z)$, $\partial_{c_\omega}(z)$, $\partial_{d_\omega}(z)$ is greater than 2, then by Lemma 2.1 there exists n such that $\partial_{\sigma^{n+1}\omega}(\Theta_{\sigma^n\omega}\circ\ldots\circ\Theta_\omega(z)) < \partial_\omega(z)$. If each of these numbers does not exceed 2, then one of them must be zero and the other two equal to 2. If $\partial_{d_\omega}(z) = 0$, then either $\partial_{\sigma^{m(\omega)}}(\Theta_{\sigma^{m(\omega)-1}\omega}\circ\ldots\circ\Theta_\omega(z)) < \partial_\omega(z)$, or $\partial_{c_{\sigma^{m(\omega)}\omega}}(\Theta_{\sigma^{m(\omega)-1}\omega}\circ\ldots\circ\Theta_\omega(z)) = \partial_{c_\omega}(z) = 2$. Therefore it suffices to consider only the cases when $\partial_\omega(z) = 4$ and $\partial_{d_\omega}(z) = 2$.

By Lemma 2.1, if z has an inner occurrence of d_ω, then $\partial_{\sigma\omega}(\Theta_\omega(z)) < \partial_\omega(z)$. Hence we should only look at those instances when d_ω has only boundary occurrences. There are just four such elements: $d_\omega c_\omega^a c_\omega d_\omega^a$, $d_\omega^a c_\omega c_\omega^a d_\omega$, $d_\omega b_\omega^a b_\omega d_\omega^a$, $d_\omega^a b_\omega b_\omega^a d_\omega$. Let us consider them in order.

Case 2.1: $z = d_\omega c_\omega^a c_\omega d_\omega^a$**.** We have that $\Theta_\omega(d_\omega c_\omega^a c_\omega d_\omega^a) = c_{\sigma\omega}^a d_{\sigma\omega} d_{\sigma\omega}^a c_{\sigma\omega}$. If $m(\omega) > 1$, the latter element has an inner occurrence of the d-generator, so upon applying to it $\Theta_{\sigma^2\omega}$ we get an element of smaller length.

Let $m(\omega) = 1$. Then $c_{\sigma\omega}$ is the d-generator, and

$$\Theta_{\sigma\omega}(c_{\sigma\omega}^a d_{\sigma\omega} d_{\sigma\omega}^a c_{\sigma\omega}) = c_{\sigma^2\omega}^a d_{\sigma^2\omega} d_{\sigma^2\omega}^a c_{\sigma^2\omega}.$$

Now, if $\omega_3 = 0$, i.e. if $d_{\sigma^2\omega}$ is the d-generator again, the obtained element has an inner occurrence of the d-generator. If ω_3 is equal to 1 or 2, then

$$\Theta_{\sigma^2\omega}(c_{\sigma^2\omega}^a d_{\sigma^2\omega} d_{\sigma^2\omega}^a c_{\sigma^2\omega}) = c_{\sigma^3\omega}^a d_{\sigma^3\omega} d_{\sigma^3\omega}^a c_{\sigma^3\omega}.$$

Let n be the minimal number such that $\omega_n = 0$ and $n > 1$. Then by the above remarks

$$\Theta_{\sigma^{n-2}\omega}\circ\ldots\circ\Theta_\omega(d_\omega c_\omega^a c_\omega d_\omega^a) = c_{\sigma^{n-1}\omega}^a d_{\sigma^{n-1}\omega} d_{\sigma^{n-1}\omega}^a c_{\sigma^{n-1}\omega},$$

and the resulting element has an inner occurrence of the d-generator.

Case 2.2: $z = d_\omega^a c_\omega c_\omega^a d_\omega$**.** We have

$$\Theta_\omega(d_\omega^a c_\omega c_\omega^a d_\omega) = d_{\sigma\omega}^a c_{\sigma\omega} c_{\sigma\omega}^a d_{\sigma\omega}.$$

Obviously,

$$\Theta_{\sigma^{m(\omega)-1}\omega}\circ\ldots\circ\Theta_\omega(d_\omega^a c_\omega c_\omega^a d_\omega) = d_{\sigma^{m(\omega)}\omega}^a c_{\sigma^{m(\omega)}\omega} c_{\sigma^{m(\omega)}\omega}^a d_{\sigma^{m(\omega)}\omega},$$

and the resulting element has an inner occurrence of the d-generator.

Case 2.3: $z = d_\omega b^a_\omega b_\omega d^a_\omega$. Then we have

$$\Theta_\omega(d_\omega b^a_\omega b_\omega d^a_\omega) = b^a_{\sigma\omega} d_{\sigma\omega} d^a_{\sigma\omega} b_{\sigma\omega}.$$

If $m(\omega) > 1$, the latter element has an inner occurrence of the d-generator, hence by applying $\Theta_{\sigma^2\omega}$ to it we get an element of smaller length.

Suppose that $m(\omega) = 1$. Then $c_{\sigma\omega}$ is the d-generator, so

$$\Theta_{\sigma\omega}(b^a_{\sigma\omega} d_{\sigma\omega} d^a_{\sigma\omega} b_{\sigma\omega}) = b^a_{\sigma^2\omega} d_{\sigma^2\omega} d^a_{\sigma^2\omega} b_{\sigma^2\omega}.$$

If $\omega_3 = 0$, i.e. if $d_{\sigma^2\omega}$ is the d-generator, the latter element has an inner occurrence of the d-generator. If ω_3 equals 1 or 2, then

$$\Theta_{\sigma^2\omega}(b^a_{\sigma^2\omega} d_{\sigma^2\omega} d^a_{\sigma^2\omega} b_{\sigma^2\omega}) = b^a_{\sigma^3\omega} d_{\sigma^3\omega} d^a_{\sigma^3\omega} b_{\sigma^3\omega}.$$

Let n be the minimal number such that $\omega_n = 0$ and $n > 1$. Then by the above

$$\Theta_{\sigma^{n-2}\omega} \circ \ldots \circ \Theta_\omega(d_\omega b^a_\omega b_\omega d^a_\omega) = b^a_{\sigma^{n-1}\omega} d_{\sigma^{n-1}\omega} d^a_{\sigma^{n-1}\omega} b_{\sigma^{n-1}\omega},$$

and the resulting element has an inner occurrence of the d-generator.

Case 2.4: $z = d^a_\omega b_\omega b^a_\omega d_\omega$. Here we have

$$\Theta_\omega(d^a_\omega b_\omega b^a_\omega d_\omega) = d^a_{\sigma\omega} b_{\sigma\omega} b^a_{\sigma\omega} d_{\sigma\omega}.$$

Since ω_2 equals either 0 or 1,

$$\Theta_{\sigma\omega}(d^a_{\sigma\omega} b_{\sigma\omega} b^a_{\sigma\omega} d_{\sigma\omega}) = d^a_{\sigma^2\omega} b_{\sigma^2\omega} b^a_{\sigma^2\omega} d_{\sigma^\omega}).$$

Thus,

$$\Theta_{\sigma^{n(\omega)-2}\omega} \circ \ldots \circ \Theta_\omega(d^a_\omega b_\omega b^a_\omega d_\omega) = d^a_{\sigma^{n(\omega)-1}\omega} b_{\sigma^{n(\omega)-1}\omega} b^a_{\sigma^{n(\omega)-1}\omega} d_{\sigma^{n(\omega)-1}\omega},$$

and the resulting element has an inner occurrence of the d-generator. Thus, all four possibilities have been considered.

Case 3. We proceed to assume that $\partial_\omega(z) = 5$. The latter condition, however, implies that at least one of the following numbers $\partial_{b_\omega}(z)$, $\partial_{c_\omega}(z)$, $\partial_{d_\omega}(z)$ is greater than 2. Indeed, since z is in the derived subgroup, all three numbers have to be equal modulo 2. As their sum equals 5, they all have to be odd, and therefore one of them has to be greater than 2. Then by Lemma 2.1 there is an n such that $\partial_{\sigma^{n+1}\omega}(\Theta_{\sigma^n\omega} \circ \ldots \circ \Theta_\omega(z)) < \partial_\omega(z)$, and we can apply the inductive assumption.

Case 4. Suppose that $\partial_\omega(z) = 6$. Then either one of the numbers $\partial_{b_\omega}(z)$, $\partial_{c_\omega}(z)$, $\partial_{d_\omega}(z)$ is greater than 2, and by Lemma 2.1 there is n such that $\partial_{\sigma^{n+1}\omega}(\Theta_{\sigma^n\omega} \circ \ldots \circ \Theta_\omega(z)) < \partial_\omega(z)$ or $\partial_{d_\omega}(z) = \partial_{c_\omega}(z) = \partial_{b_\omega}(z) = 2$. In the latter case, suppose z has inner occurrences of d_ω. Then we have $\partial_{\sigma\omega}(\Theta_\omega(z)) < \partial_\omega(z)$. Hence we should only consider the cases when z has only boundary occurrences of d_ω. Then there are twelve possibilities.

Consider the following table.

N	z	$\Theta_\omega(z)$	N	z	$\Theta_\omega(z)$
4.1	$d_\omega b^a_\omega b_\omega c^a_\omega c_\omega d^a_\omega$	$b^a_{\sigma\omega} c_{\sigma\omega} d^a_{\sigma\omega} d_{\sigma\omega} b^a_{\sigma\omega} c_{\sigma\omega}$	4.7	$d^a_\omega b_\omega b^a_\omega c_\omega c^a_\omega d_\omega$	$d^a_{\sigma\omega} b_{\sigma\omega} c^a_{\sigma\omega} b_{\sigma\omega} c^a_{\sigma\omega} d_{\sigma\omega}$
4.2	$d_\omega b^a_\omega c_\omega b^a_\omega c_\omega d^a_\omega$	$b^a_{\sigma\omega} b_{\sigma\omega} d^a_{\sigma\omega} d_{\sigma\omega} c^a_{\sigma\omega} c_{\sigma\omega}$	4.8	$d^a_\omega b_\omega c^a_\omega b_\omega c^a_\omega d_\omega$	$d^a_{\sigma\omega} c_{\sigma\omega} c^a_{\sigma\omega} b_{\sigma\omega} b^a_{\sigma\omega} d_{\sigma\omega}$
4.3	$d_\omega b^a_\omega c_\omega c^a_\omega b_\omega d^a_\omega$	$b^a_{\sigma\omega} c_{\sigma\omega} d^a_{\sigma\omega} d_{\sigma\omega} c^a_{\sigma\omega} b_{\sigma\omega}$	4.9	$d^a_\omega c_\omega c^a_\omega b_\omega b^a_\omega d_\omega$	$d^a_{\sigma\omega} c_{\sigma\omega} b^a_{\sigma\omega} c_{\sigma\omega} b^a_{\sigma\omega} d_{\sigma\omega}$
4.4	$d_\omega c^a_\omega c_\omega b^a_\omega b_\omega d^a_\omega$	$c^a_{\sigma\omega} b_{\sigma\omega} d^a_{\sigma\omega} d_{\sigma\omega} c^a_{\sigma\omega} b_{\sigma\omega}$	4.10	$d^a_\omega c_\omega b^a_\omega c_\omega b^a_\omega d_\omega$	$d^a_{\sigma\omega} b_{\sigma\omega} b^a_{\sigma\omega} c_{\sigma\omega} c^a_{\sigma\omega} d_{\sigma\omega}$
4.5	$d_\omega c^a_\omega b_\omega c^a_\omega b_\omega d^a_\omega$	$c^a_{\sigma\omega} c_{\sigma\omega} d^a_{\sigma\omega} d_{\sigma\omega} b^a_{\sigma\omega} b_{\sigma\omega}$	4.11	$d^a_\omega b_\omega c^a_\omega c_\omega b^a_\omega d_\omega$	$d^a_{\sigma\omega} c_{\sigma\omega} b^a_{\sigma\omega} b_{\sigma\omega} c^a_{\sigma\omega} d_{\sigma\omega}$
4.6	$d_\omega c^a_\omega b_\omega b^a_\omega c_\omega d^a_\omega$	$c^a_{\sigma\omega} b_{\sigma\omega} d^a_{\sigma\omega} d_{\sigma\omega} b^a_{\sigma\omega} c_{\sigma\omega}$	4.12	$d^a_\omega c_\omega b^a_\omega b_\omega c^a_\omega d_\omega$	$d^a_{\sigma\omega} b_{\sigma\omega} c^a_{\sigma\omega} c_{\sigma\omega} b^a_{\sigma\omega} d_{\sigma\omega}$

In cases 4.1-4.5 we have inner occurrences of $c_{\sigma\omega}$ and $d_{\sigma\omega}$. Since c_ω is the c-generator, in those cases we always (irrespective of ω) have an inner occurrence of a d-generator.

Now consider case 4.6. If $\omega_2 \neq 1$, we have an inner occurrence of the d-generator. Suppose that $\omega_2 = 1$ and let $n > 2$ be the smallest integer exceeding 2 and such that $\omega_n \neq 1$. It is easy to see that $\Theta_{\sigma^{n-2}\omega} \circ \ldots \circ \Theta_\omega(z)$ has the form

$$c^a_{\sigma^{n-1}\omega} * * * * c_{\sigma^{n-1}\omega},$$

where exactly two stars are from the set $\{d_{\sigma^{n-1}\omega}, d^a_{\sigma^{n-1}\omega}\}$, and the other two are from $\{b_{\sigma^{n-1}\omega}, b^a_{\sigma^{n-1}\omega}\}$. Hence the resulting element has an inner occurrence of a d-generator.

If $m(\omega) = 1$, then in cases 4.7-4.12 $\Theta_\omega(z)$ has an inner occurrence of the d-generator. Otherwise the equalities

$$\Theta_{\sigma\omega} \circ \Theta_\omega(d^a_\omega b_\omega b^a_\omega c_\omega c^a_\omega d_\omega) = d^a_{\sigma^2\omega} c_{\sigma^2\omega} c^a_{\sigma^2\omega} b_{\sigma^2\omega} b^a_{\sigma^2\omega} d_{\sigma^2\omega},$$

$$\Theta_{\sigma^2\omega} \circ \Theta_{\sigma\omega} \circ \Theta_\omega(d^a_\omega b_\omega b^a_\omega c_\omega c^a_\omega d_\omega) = d^a_{\sigma^3\omega} c_{\sigma^3\omega} b^a_{\sigma^3\omega} c_{\sigma^3\omega} b^a_{\sigma^3\omega} d_{\sigma^3\omega},$$

$$\Theta_{\sigma^3\omega} \circ \Theta_{\sigma^2\omega} \circ \Theta_{\sigma\omega} \circ \Theta_\omega(d^a_\omega b_\omega b^a_\omega c_\omega c^a_\omega d_\omega) = d^a_{\sigma^4\omega} b_{\sigma^4\omega} b^a_{\sigma^4\omega} c_{\sigma^4\omega} c^a_{\sigma^4\omega} d_{\sigma^4\omega}$$

and

$$\Theta_{\sigma\omega} \circ \Theta_\omega(d^a_\omega b_\omega c^a_\omega c_\omega b^a_\omega d_\omega) = d^a_{\sigma^2\omega} b_{\sigma^2\omega} c^a_{\sigma^2\omega} c_{\sigma^2\omega} b^a_{\sigma^2\omega} d_{\sigma^2\omega}$$

imply that for any element of the form $d^a_{\sigma^{m(\omega)}\omega} * * * * d_{\sigma^{m(\omega)}\omega}$, where the stars are in one-to-one correspondence with the set $\{b_{\sigma^{m(\omega)}\omega}, b^a_{\sigma^{m(\omega)}\omega}, c_{\sigma^{m(\omega)}\omega}, c^a_{\sigma^{m(\omega)}\omega}\}$, $\Theta_{\sigma^{m(\omega)-1}\omega} \circ \ldots \circ \Theta_\omega(z)$ has a similar form and therefore has an inner occurrence of the d-generator.

Case 5. Finally, suppose that $\partial_\omega(z) > 6$. Then at least one of the numbers $\partial_{b_\omega}(z)$, $\partial_{c_\omega}(z)$, $\partial_{d_\omega}(z)$ is greater than 2, so by Lemma 2.1 there is n such that $\partial_{\sigma^{n+1}\omega}(\Theta_{\sigma^n\omega} \circ \ldots \circ \Theta_\omega(z)) < \partial_\omega(z)$ This (finally) proves the proposition. ■

Proposition 2.1 allows to introduce a "norm" $||\ ||_\omega$ on $[H_\omega, H_\omega]$.

Definition 2.1. We put $||z||_\omega$ to be the smallest n such that $\Theta_{\sigma^{n-1}\omega} \circ \ldots \circ \Theta_\omega(z)$ belongs to the subgroup generated by $(ay_{\sigma^n\omega})^2$, where $y_{\sigma^n\omega}$ is the d-generator of $H_{\sigma^n\omega}$.

Theorem 2.1. *Let x_ω, y_ω be two different elements of the set $\{b_\omega, c_\omega, d_\omega\}$, let z be an arbitrary element of $[H_\omega, H_\omega]$, and let $X = < x_\omega, y_\omega az >$. Then there exists n such that*

$$\tilde{\varphi}_2^{\sigma^{n-1}\omega}(L(\tilde{\varphi}_2^{\sigma^{n-2}\omega}(L(\ldots(\tilde{\varphi}_2^{\omega}(L(X)))\ldots)))) = H_{\sigma^n\omega}.$$

Proof. We will assume that d_ω is the d-generator and c_ω is the c-generator. There are three possibilities, $< b_\omega, d_\omega az >=< b_\omega, c_\omega az >$, $< c_\omega, d_\omega az >=< c_\omega, b_\omega az >$, $< d_\omega, c_\omega az >=< d_\omega, b_\omega az >$.

The proof is by induction on $||z||_\omega$ simultaneously for all H_ω. The base of induction is $||z||_\omega = 0$, i.e. $z = 1$ or $z = (ad_\omega)^2$.

Let $z = 1$. Consider the subgroup $< b_\omega, d_\omega az >$. Since $\tilde{\varphi_2}^\omega((d_\omega a)^2) = d_{\sigma\omega}$, $\tilde{\varphi_2}^\omega(b_\omega^{d_\omega a}) = a$, $\tilde{\varphi_2}^\omega(b_\omega) = b_{\sigma\omega}$, we have $\tilde{\varphi_2}^\omega(L(X)) \ni d_\omega, a, b_\omega$, i.e. $\tilde{\varphi_2}^\omega(L(X)) \geq H_{\sigma\omega}$.

Now consider the subgroup $< c_\omega, d_\omega az >$. Since $\tilde{\varphi_2}^\omega((d_\omega a)^2) = d_{\sigma\omega}$, $\tilde{\varphi_2}^\omega(c_\omega^{d_\omega a}) = a$, $\tilde{\varphi_2}^\omega(c_\omega) = c_{\sigma\omega}$, we have $\tilde{\varphi_2}^\omega(L(X)) \ni d_\omega, a, c_\omega$, i.e. $\tilde{\varphi_2}^\omega(L(X)) \geq H_{\sigma\omega}$.

Finally, consider $< d_\omega, c_\omega az >$. Notice that the order of $c_\omega a$ is equal to $2^{m(\omega)+2}$, and

$$\tilde{\varphi}_2^{\sigma^{m(\omega)}\omega} \ldots \tilde{\varphi}_2^{\sigma\omega} \tilde{\varphi}_2^{\omega}((c_\omega a)^{2^{m(\omega)+1}}) = c_{\sigma^{m(\omega)+1}\omega}.$$

Further,

$$\tilde{\varphi}_2^{\sigma^{m(\omega)-1}\omega} \ldots \tilde{\varphi}_2^{\sigma\omega} \tilde{\varphi}_2^{\omega}(d_\omega^{(c_\omega a)^{2^{m(\omega)}}}) = d^a_{\sigma^{m(\omega)}\omega}.$$

Hence $\tilde{\varphi}_2^{\sigma^{m(\omega)}\omega}(L(\ldots \tilde{\varphi}_2^{\sigma\omega}(L(\tilde{\varphi}_2^{\omega}(L(X)))) \ldots)) \ni c_{\sigma^{m(\omega)+1}\omega}, a, d_{\sigma^{m(\omega)+1}\omega}$, i.e. $\tilde{\varphi}_2^{\sigma^{m(\omega)}\omega}(L(\ldots \tilde{\varphi}_2^{\sigma\omega}(L(\tilde{\varphi}_2^{\omega}(L(X)))) \ldots)) \geq H_{\sigma\omega}$.

Let us now consider the case when $z = (ad_\omega)^2$. We have that $< b_\omega, d_\omega a(ad_\omega)^2 >=< b_\omega, ad_\omega >$, this case has already been considered. Next, $< c_\omega, d_\omega a(ad_\omega)^2 >=< c_\omega, ad_\omega >$. This case also has been already considered.

Finally, $< d_\omega, c_\omega a(ad_\omega)^2 >=< d_\omega, b_\omega ad_\omega >=< d_\omega, b_\omega a >$. Again this case has been considered, and the base of induction is completed.

Let us proceed to the inductive step. Consider the subgroup $< b_\omega, d_\omega az >=< b_\omega, c_\omega az >$. Since $\tilde{\varphi_2}^\omega(b_\omega) = b_{\sigma\omega}$, $\tilde{\varphi_2}^\omega((c_\omega az)^2) = c_{\sigma\omega} a\Theta_\omega(z)$, we get that $\tilde{\varphi_2}^\omega(L(X)) \geq< b_{\sigma\omega}, c_{\sigma\omega} a\Theta_\omega(z) >$. Since, obviously, $||\Theta_\omega(z)||_{\sigma\omega} < ||z||_\omega$, we can apply the inductive assumption.

Next consider the subgroup $< c_\omega, d_\omega az >=< c_\omega, b_\omega az >$. Since $\tilde{\varphi_2}^\omega(c_\omega) = c_{\sigma\omega}$, $\tilde{\varphi_2}^\omega((b_\omega az)^2) = b_{\sigma\omega} a\Theta_\omega(z)$, we have that $\tilde{\varphi_2}^\omega(L(X)) \geq< c_{\sigma\omega}, b_{\sigma\omega} a\Theta_\omega(z) >$, and we can apply the inductive assumption.

Finally, consider the subgroup $< d_\omega, b_\omega az >$. Since $\tilde{\varphi_2}^\omega(d_\omega) = d_{\sigma\omega}$, $\tilde{\varphi_2}^\omega((b_\omega az)^2) = b_{\sigma\omega} a\Theta_\omega(z)$, we have that $\tilde{\varphi_2}^\omega(L(X)) \geq< d_{\sigma\omega}, b_{\sigma\omega} a\Theta_\omega(z) >$, and we can apply the inductive assumption. ∎

Returning to groups G_ω, we have th following important fact.

Corollary 2.1. *Let x_ω, y_ω be two different generators in $\{b_\omega, c_\omega, d_\omega\}$ in the generating set of G_ω, $z \in [G_\omega, G_\omega]$. Then there exists n such that $X_u = G_{\sigma^n\omega}$, where $X =< x_\omega, y_\omega az >$, $u = \underbrace{11\ldots1}_{n}$.*

Proof. Notice that every G_ω is a factor-group of H_ω. Let $\varepsilon_\omega : H_\omega \to G_\omega$ be the natural projection. Then by definition of homomorphisms $\tilde{\varphi}_1^\omega$, $\tilde{\varphi}_2^\omega$ the following diagrams are commutative:

$$\begin{array}{ccccc} & L(H_\omega) & \stackrel{\varepsilon_\omega}{\to} & \mathrm{st}_{G_\omega}(1) & \\ \tilde{\varphi}_1^\omega & \downarrow & & \downarrow & \varphi_0 \\ & H_\omega & \stackrel{\varepsilon_\omega}{\to} & G_\omega & \end{array} \qquad \begin{array}{ccccc} & L(H_\omega) & \stackrel{\varepsilon_\omega}{\to} & \mathrm{st}_{G_\omega}(1) & \\ \tilde{\varphi}_2^\omega & \downarrow & & \downarrow & \varphi_1 \\ & H_\omega & \stackrel{\varepsilon_\omega}{\to} & G_\omega & \end{array}$$

Hence the required statement follows from Theorem 2.1. ■

Remark 2.1. Applying the same reasoning as in the above proof, we obtain from Proposition 2.1 that if z be an arbitrary element of $[G_\omega, G_\omega]$, then there exists n such that $\Theta_{\sigma^n\omega} \circ \ldots \circ \Theta_\omega(z) \in < (ay_{\sigma^{n+1}\omega})^2 >$, where $y_{\sigma^{n+1}\omega}$ is the d-generator of $G_{\sigma^{n+1}\omega}$. Hence a "norm" similar to the one of Definition 2.1 can be introduced on $[G_\omega, G_\omega]$ as well.

3 Upper companion groups of subgroups in GGS-groups: one special case

Let $G_{\bar{\alpha}}$ be a periodic GGS-group with accompanying vector

$$\bar{\alpha} = (\alpha_1, \ldots, \alpha_{p-1}) \in \underbrace{\mathrm{GF}(p) \times \ldots \times \mathrm{GF(p)}}_{p-1}.$$

Since $G_{\bar{\alpha}}$ is transitive on each layer of vertices, there exists (in general, not unique) n such that $\alpha_n \neq 0$. Fix some such number for each vector $\bar{\alpha}$ and denote it by $n(\bar{\alpha})$. Then, up to replacement of b with its power, we can assume that $\alpha_{n(\bar{\alpha})} = n(\bar{\alpha})$ for any GGS-group.

Let us fix a vector $\bar{\alpha}$ defining a periodic GGS-group for the duration of this Section. For simplicity, we will write just n for the number $n(\bar{\alpha})$.

Let us consider a group

$$H =< a, b | a^p, b^p >,$$

i.e., the free product $Z/pZ * Z/pZ$. Any element z of H is uniquely represented in the form

$$a^{s_1} b^{\beta_1} \ldots a^{s_m} b^{\beta_m} a^{s_{m+1}}, \tag{3.1}$$

where s_1, $s_{m+1} \in \mathrm{GF}(p)$, and s_2, …, s_m, β_1, …, $\beta_m \in \mathrm{GF}(p) \setminus \{0\}$. We define the *length* of z to be m and denote it by $\partial(z)$.

The set of all z such that $s_1 + s_2 + \ldots + s_{m+1} = 0$ is a subgroup (of finite index p), which will be denoted by $L(H)$. If $X \leq H$, then $L(X)$ is defined to be the intersection $X \cap L(H)$.

Notice that any $z \in L(H)$ admits a unique representation in the form

$$b^{\beta_1 a^{i_1}} \ldots b^{\beta_m a^{i_m}}, \tag{3.2}$$

where β_1, …, $\beta_m \in \mathrm{GF}(p) \setminus \{0\}$, i_1, …, $i_m \in \mathrm{GF}(p)$, and $i_j \neq i_{j+1}$ for $j = 1, 2, \ldots, m-1$. Here m is equal to $\partial(z)$ (as defined above). Finally, let $\partial_i(z)$ denote the number of i_j such that $i_j = i$.

Define two homomorphisms, $\tilde{\varphi}_n$ and $\tilde{\varphi}_p$, acting from $L(H)$ to H as follows:

$$\tilde{\varphi}_n(b^{a^i}) = \begin{cases} b, & i = n \\ a^{\alpha_{n-i}}, & \text{otherwise} \end{cases}$$

$$\tilde{\varphi}_p(b^{a^i}) = \begin{cases} b, & i = 0 \\ a^{\alpha_{p-i}}, & \text{otherwise} \end{cases}$$

The indices above are taken modulo p. Now, since $L(H)$ is the free product of the cyclic groups generated by b^{a^i}, $i \in \mathrm{GF}(p)$, the above maps can be extended to the whole $L(H)$.

Theorem 3.1. *Let $z \in [H, H]$ and let K be the subgroup of H generated by b and $a^n z$. Then there exists N (which depends on z) such that*

$$\tilde{\varphi}_p(L(\tilde{\varphi}_p(L(\ldots(\tilde{\varphi}_p(L(K)))\ldots)))) = H$$

($\tilde{\varphi}_p$ is applied N times.)

Proof. Note first that

$$\tilde{\varphi}_p(b^{z^{-1}a^{-n}}) = \tilde{\varphi}_n(z)a^n\tilde{\varphi}_n(z)^{-1}. \tag{3.3}$$

Therefore, if we introduce a function $\Theta_{n(\bar{\alpha})} : [H, H] \to [H, H]$:

$$\Theta_{n(\bar{\alpha})}(z) = a^{-n}\tilde{\varphi}_n(z)a^n\tilde{\varphi}_n(z)^{-1},$$

then by (3.3) we have that

$$\tilde{\varphi}_p(L(< b, a^n z >)) \geq < b, a^n\Theta_{n(\bar{\alpha})}(z) > . \tag{3.4}$$

Moreover, it is obvious that

$$\partial(\Theta_{n(\bar{\alpha})}(z)) \leq 2\partial_n(z), \tag{3.5}$$

which we can use to proceed by induction on $\partial(z)$.

The base of induction: $\partial(z) \leq 2$. Since z is in $[H, H]$, either $\partial(z) = 2$, or $\partial(z) = 0$ and $z = 1$. In the latter case there is nothing to prove, and we assume that $\partial(z) = 2$. Then z has the form,

$$z = b^{\alpha a^i} b^{-\alpha a^j},$$

where $\alpha \in \mathrm{GF}(p) \setminus \{0\}$, $i, j \in \mathrm{GF}(p)$ and $i \neq j$. Thus, there are three possibilities.

(a) If neither i nor j are equal to n, then $\tilde{\varphi}_n(z) = a^\beta$ for some $\beta \in \mathrm{GF}(p)$. In this case $\Theta_{n(\bar{\alpha})}(z) = 1$, and $\tilde{\varphi}_p(L(K)) \geq < b, a^n >= H$.

(b) Suppose that $i = n$, i.e.

$$z = b^{\alpha a^n} b^{-\alpha a^j}.$$

Then $j \neq n$, and

$$\tilde{\varphi}_n(z) = b^\alpha a^{-\alpha\alpha_{n-j}}.$$

Although $\Theta_{n(\bar{\alpha})} = b^{\alpha a^n} b^{-\alpha}$, we have that $\tilde{\varphi}_p(b^{z^{-1}a^{-n}}) = b^\alpha a^n b^{-\alpha}$, and $\tilde{\varphi}_p(L(K)) \geq < b, b^\alpha a^n b^{-\alpha} >=< b, a^n >= H$.

(c) Suppose that $j = n$, i.e.
$$z = b^{\alpha a^i} b^{-\alpha a^n}.$$
Then $i \neq n$, and
$$\tilde{\varphi}_n(z) = a^{\alpha\alpha_{n-i}} b^{-\alpha},$$
$$\Theta_{n(\bar{\alpha})}(z) = b^{-\alpha a^{n-\alpha\alpha_{n-i}}} b^{\alpha a^{-\alpha\alpha_{n-i}}}.$$
If neither $n - \alpha\alpha_{n-i}$ nor $-\alpha\alpha_{n-i}$ are equal to n, then we have (a). If $n - \alpha\alpha_{n-i} = n$, i.e. $\alpha_{n-i} = 0$, then we have (b).

Finally, assume that $-\alpha\alpha_{n-i} = n$, i.e. α_{n-i} is invertible and $\alpha = -\frac{n}{\alpha_{n-i}}$. Then
$$\Theta_{n(\bar{\alpha})}(z) = b^{-\alpha a^{2n}} b^{\alpha a^n}. \tag{3.6}$$
Again,
$$\tilde{\varphi}_n(\Theta_{n(\bar{\alpha})}(z)) = a^{-\alpha\alpha_{p-n}} b^{\alpha},$$
which implies
$$\Theta^2_{n(\bar{\alpha})}(z) = b^{\alpha a^{n+\alpha\alpha_{p-n}}} b^{-\alpha a^{\alpha\alpha_{p-n}}}.$$
If, as above, neither of the numbers $n + \alpha\alpha_{p-n}$, $-\alpha a^{\alpha\alpha_{p-n}}$ is equal n, then we have (a). If $n + \alpha\alpha_{p-n}$ equals n, we have (b).

Assume that $\alpha\alpha_{p-n} = n$, i.e. α_{p-n} is invertible and $\alpha = \frac{n}{\alpha_{p-n}}$. Then we should only consider $z = b^{\frac{n}{\alpha_{p-n}} a^{2n}} b^{-\frac{n}{\alpha_{p-n}} a^n}$.

In this case we see that
$$\tilde{\varphi}_p(z) = a^n b^{-\frac{n}{\alpha_{p-n}}},$$
and thus
$$\Theta_{n(\bar{\alpha})}(z) = b^{-\frac{n}{\alpha_{p-n}}} a^n b^{\frac{n}{\alpha_{p-n}}} a^{-n} = b^{-\frac{n}{\alpha_{p-n}}} b^{\frac{n}{\alpha_{p-n}} a^{-n}}.$$
Since n is invertible, $-n$ is not equal to n, and we get case (a). This completes the base of induction.

The inductive step. Here again we have three main cases. To describe them, we need the following definition.

Definition 3.1. We say that z *is of type one*, if $\partial_n(z) \neq \frac{\partial(z)}{2}$, *is of type two*, if $\partial_n(z) = \frac{\partial(z)}{2}$ and $\beta_1 = n$ (see (3.2)), and *is of type three*, if $\partial_n(z) = \frac{\partial(z)}{2}$ and $\beta_{\partial(z)} = n$.

Case 1: z is of type one. Then there are two possibilities: $\partial_n(z) < \frac{\partial(z)}{2}$ and $\partial_n(z) > \frac{\partial(z)}{2}$.

If $\partial_n(z) < \frac{\partial(z)}{2}$, then by (3.5)
$$\partial(\Theta_{n(\bar{\alpha})}(z)) \leq 2\partial_n(z) < \partial(z),$$
and we can apply the inductive assumption.

The second possibility is that $\partial_n(z) > \frac{\partial(z)}{2}$. It implies that $\partial(z)$ is odd and $\partial_n(z) = \frac{\partial(z)+1}{2}$. If this is the case, we apply the observation that
$$\tilde{\varphi}_p((a^n z)^p) = \tilde{\varphi}_n(z)\tilde{\varphi}_n(z^{a^{-n}}) \dots \tilde{\varphi}_n(z^{a^{-(p-1)n}}),$$

which follows from equality (true for any $x \in L(H)$)

$$\tilde{\varphi}_p(x^{a^{-n}}) = \tilde{\varphi}_n(x).$$

Thus, we get

$$\tilde{\varphi}_p(b^{(a^n z)^{p-1}}) = \tilde{\varphi}_p(b^{z^{-1}a^{-n}(a^n z)^p}) = (\tilde{\varphi}_n(z)a^n\tilde{\varphi}_n(z)^{-1})^{\tilde{\varphi}_n(z)\tilde{\varphi}_n(z^{a^{-n}})\ldots\tilde{\varphi}_n(z^{a^{-(p-1)n}})}$$

$$= \tilde{\varphi}_n(z^{a^{-(p-1)n}})^{-1}\ldots\tilde{\varphi}_n(z^{a^{-n}})^{-1}a^n\tilde{\varphi}_n(z^{a^{-n}})\ldots\tilde{\varphi}_n(z^{a^{-(p-1)n}}).$$

Since n is invertible, we have

$$\partial(\tilde{\varphi}_n(z^{a^{-n}})\ldots\tilde{\varphi}_n(z^{a^{-(p-1)n}})) \leq \partial_1(z) + \ldots + \partial_{n-1}(z) + \partial_{n+1}(z) + \ldots + \partial_p(z) = \partial(z) - \partial_n(z).$$

Therefore,

$$\partial(a^{-n}\tilde{\varphi}_p(b^{(a^n z)^{p-1}})) \leq 2(\partial(z) - \partial_n(z)) = 2\partial(z) - (\partial(z)+1) < \partial(z).$$

Since

$$\tilde{\varphi}_p(L(K)) \geq < b, a^n(a^{-n}\tilde{\varphi}_p(b^{(a^n z)^{p-1}})) >,$$

and $a^{-n}\tilde{\varphi}_p(b^{(a^n z)^{p-1}})$ is in $[H,H]$, we can apply the inductive assumption.
Case 2: z is of type two. Then z has the form,

$$z = b^{\beta_1 a^n}b^{\gamma_1 a^{i_1}}b^{\beta_2 a^n}b^{\gamma_2 a^{i_2}}\ldots b^{\beta_m a^n}b^{\gamma_m a^{i_m}}.$$

Therefore

$$\tilde{\varphi}_n(z) = b^{\beta_1}a^{\delta_1}\ldots b^{\beta_m}a^{\delta_m}$$

for some $\delta_1,\ldots,\delta_m \in \mathrm{GF}(p)$. Now,

$$\tilde{\varphi}_p(b^{z^{-1}a^{-n}b^{\beta_1}}) = b^{-\beta_1}\tilde{\varphi}_n(z)a^n\tilde{\varphi}_n(z)^{-1}b^{\beta_1} = a^{\delta_1}b^{\beta_2}\ldots a^{\delta_{m-1}}b^{\beta_m}a^n b^{-\beta_m}a^{-\delta_{m-1}}\ldots b^{-\beta_2}a^{-\delta_1}.$$

Thus,

$$\partial(a^{-n}\tilde{\varphi}_p(b^{z^{-1}a^{-n}b^{\beta_1}})) \leq 2m-2 = \partial(z)-2,$$

and we can use the inductive assumption as before.

Case 3: z is of type three. Then z has the form,

$$z = b^{s_1 a^{\gamma_1}}b^{\beta_1 a^n}b^{s_2 a^{\gamma_2}}b^{\beta_2 a^n}\ldots b^{s_m a^{\gamma_m}}b^{\beta_m a^n}.$$

Let us write

$$\tilde{\varphi}_n(z) = a^{\delta_1}b^{\beta_1}a^{\delta_2}b^{\beta_2}\ldots a^{\delta_m}b^{\beta_m}.$$

Lemma 3.1. Let z be of type three, and $\tilde{\varphi}_n(z) = a^{\delta_1}b^{\beta_1}a^{\delta_2}b^{\beta_2}\ldots a^{\delta_m}b^{\beta_m}$. Then $\Theta_{n(\bar{\alpha})}(z)$ is of type three if and only if $\delta_i = (-1)^i n$ for $i = 1,2,\ldots,m$.

Proof. If z is of type three, then

$$z = b^{s_1 a^{\gamma_1}} b^{\beta_1 a^n} b^{s_2 a^{\gamma_2}} b^{\beta_2 a^n} \dots b^{s_m a^{\gamma_m}} b^{\beta_m a^n} = a^{-\gamma_1} b^{s_1} a^{\gamma_1 - n} b^{\beta_1} a^{n-\gamma_2} \dots a^{n-\gamma_m} b^{s_m} a^{\gamma_m - n} b^{\beta_m} a^n.$$

Of course, vector

$$(-\gamma_1, \gamma_1 - n, n - \gamma_2, \gamma_2 - n, \dots, n - \gamma_m, \gamma_m - n, n) \in \underbrace{\mathrm{GF}(p) \times \dots \times \mathrm{GF}(p)}_{2m+1} \tag{3.7}$$

is uniquely determined by z. Vice versa, any arbitrary element $w \in H$,

$$w = a^{\mu_1} b^{t_1} \dots a^{\mu_l} b^{t_l} a^{\mu_{l+1}},$$

is determined by a pair of vectors

$$(\mu_1, \dots, \mu_{l+1}) \in (\mathrm{GF}(p) \setminus \{0\}) \times \underbrace{\mathrm{GF}(p) \times \dots \times \mathrm{GF}(p)}_{l-1} \times (\mathrm{GF}(p) \setminus \{0\}),$$

$$(t_1, \dots, t_l) \in \underbrace{(\mathrm{GF}(p) \setminus \{0\}) \times \dots \times (\mathrm{GF}(p) \setminus \{0\})}_{l-1}.$$

As follows from (3.7), w is of type three if and only if the following conditions hold:

(i) l is even;

(ii) $\mu_{l+1} = n$;

(iii) $\mu_{2i-1} + \mu_{2i} = 0$, $i = 2, 3, \dots, \frac{l}{2}$;

(iv) $\mu_1 + \mu_2 = -n$.

Let us now calculate $\Theta_{n(\bar{\alpha})}(z)$ for z of type three:

$$\Theta_{n(\bar{\alpha})}(z) = a^{\delta_1 - n} b^{\beta_1} a^{\delta_2} \dots a^{\delta_m} b^{\beta_m} a^n b^{-\beta_m} a^{-\delta_m} \dots a^{-\delta_2} b^{-\beta_1} a^{-\delta_1}.$$

We will consider two cases.

1) m is odd. Then conditions (i)-(iv) are equivalent to the following system of equations:

$$\left\{\begin{array}{rcr} -\delta_1 & = & n \\ -\delta_2 - \delta_3 & = & 0 \\ & \cdots & \\ -\delta_{m-1} - \delta_m & = & 0 \\ n + \delta_m & = & 0 \\ & \cdots & \\ \delta_3 + \delta_4 & = & 0 \\ \delta_1 + \delta_2 - n & = & -n \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcr} \delta_1 & = & -n \\ \delta_1 + \delta_2 & = & 0 \\ \delta_2 + \delta_3 & = & 0 \\ \delta_3 + \delta_4 & = & 0 \\ & \cdots & \\ \delta_{m-2} + \delta_{m-1} & = & 0 \\ \delta_{m-1} + \delta_m & = & 0 \\ \delta_m + n & = & 0 \end{array}\right.$$

The latter system is obtained from the former by multiplying some equations by -1 and then changing the order of equations; the second equation of the latter system is obtained from the last equation of the former system by eliminating $-n$.

It is easy to see now that the latter system is equivalent to $\delta_i = (-1)^i n$ for all $i = 1, 2, \ldots, m$. We are done with the first case.

2) The second case is when m is even. Then conditions (i)-(iv) are equivalent to the following system of equations:

$$\left\{\begin{array}{rcl} -\delta_1 & = & n \\ -\delta_2 - \delta_3 & = & 0 \\ & \cdots & \\ -\delta_{m-2} - \delta_{m-1} & = & 0 \\ -\delta_m + n & = & 0 \\ \delta_{m-1} + \delta_m & = & 0 \\ & \cdots & \\ \delta_3 + \delta_4 & = & 0 \\ \delta_1 + \delta_2 - n & = & -n \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \delta_1 & = & -n \\ \delta_1 + \delta_2 & = & 0 \\ \delta_2 + \delta_3 & = & 0 \\ \delta_3 + \delta_4 & = & 0 \\ & \cdots & \\ \delta_{m-2} + \delta_{m-1} & = & 0 \\ \delta_{m-1} + \delta_m & = & 0 \\ \delta_m - n & = & 0 \end{array}\right.$$

Just as before, this is equivalent to $\delta_i = (-1)^i n$ for all $i = 1, 2, \ldots, m$. Since we have considered all possible cases, the lemma is proven. ■

Corollary 3.1. *Let $\Theta_{n(\bar{\alpha})}(z) = b^{s_1 a^{\gamma_1}} b^{\beta_1 a^n} b^{s_2 a^{\gamma_2}} b^{\beta_2 a^n} \ldots b^{s_m a^{\gamma_m}} b^{\beta_m a^n}$ be of type three. Then:*

1) if m is even, then

$$\gamma_i = \begin{cases} 2n, & i = 1, \ldots, \frac{m}{2} \\ 0, & i = \frac{m}{2} + 1, \ldots, m \end{cases}$$

2) if m is odd, then

$$\gamma_i = \begin{cases} 2n, & i = 1, \ldots, \frac{m+1}{2} \\ 0, & i = \frac{m+3}{2}, \ldots, m \end{cases}$$

The corollary is verified directly.

We will consider the following three cases separately: $m = 2$, $m = 3$, and $m \geq 4$.

Case 3.1: $m = 2$**.** In this case z has the form

$$z = b^{s_1 a^{\gamma_1}} b^{\beta_1 a^n} b^{s_2 a^{\gamma_2}} b^{\beta_2 a^n},$$

and therefore

$$\tilde{\varphi}_n(z) = a^{\delta_1} b^{\beta_1} a^{\delta_2} b^{\beta_2}.$$

If $\Theta_{n(\bar{\alpha})}(z)$ is of type one or two, we go to Cases 1 or, respectively, 2 above. Assume that $\Theta_{n(\bar{\alpha})}(z)$ is of type three. Then by Lemma 3.1

$$\delta_1 = -n, \ \delta_2 = n,$$

and by Corollary 3.1

$$\Theta_{n(\bar{\alpha})}(z) = b^{\beta_1 a^{2n}} b^{\beta_2 a^n} b^{-\beta_2} b^{-\beta_1 a^n}.$$

Let us again consider $\tilde{\varphi}_n(\Theta_{n(\bar{\alpha})}(z))$:

$$\tilde{\varphi}_n(\Theta_{n(\bar{\alpha})}(z)) = a^{\beta_1 \alpha_{p-n}} b^{\beta_2} a^{-\beta_2 n} b^{-\beta_1}.$$

If $\Theta^2_{n(\bar{\alpha})}(z)$ is of type one or two, we go to Cases 1 or 2 again. Suppose that $\Theta^2_{n(\bar{\alpha})}(z)$ is of type three. Then by Lemma 3.1

$$\beta_1\alpha_{p-n} = -n,\ -\beta_2 n = n.$$

Clearly, when α_{p-n} is not invertible, this is impossible, so $\Theta^2_{n(\bar{\alpha})}(z)$ will necessarily be of either type one or type two. Assume that α_{p-n} is invertible, then $\Theta^2_{n(\bar{\alpha})}(z)$ is of type three if and only if

$$\beta_1 = -\frac{n}{\alpha_{p-n}},\ \beta_2 = -1.$$

In this case

$$\Theta^2_{n(\bar{\alpha})}(z) = b^{-a^{2n}} b^{\frac{n}{\alpha_{p-n}}a^n} b^{-\frac{n}{\alpha_{p-n}}} b^{a^n},$$

and therefore

$$\tilde{\varphi}_n(\Theta^2_{n(\bar{\alpha})}(z)) = a^{-\alpha_{p-n}} b^{\frac{n}{\alpha_{p-n}}} a^{-\frac{n^2}{\alpha_{p-n}}} b.$$

By Lemma 3.1, for $\Theta^3_{n(\bar{\alpha})}(z)$ to be of type three, it is necessary that

$$-\alpha_{p-n} = -n,\ -\frac{n^2}{\alpha_{p-n}} = n.$$

Since n is invertible, $-n$ is not equal to n. Therefore, the equalities contradict each other. Hence, $\Theta^3_{n(\bar{\alpha})}(z)$ is always of either type one or type two, and we can proceed to either Case 1 or Case 2.

Case 3.2: $m = 3$**.** Then z has form

$$z = b^{s_1 a^{\gamma_1}} b^{\beta_1 a^n} b^{s_2 a^{\gamma_2}} b^{\beta_2 a^n} b^{s_3 a^{\gamma_3}} b^{\beta_3 a^n},$$

and therefore

$$\tilde{\varphi}_n(z) = a^{\delta_1} b^{\beta_1} a^{\delta_2} b^{\beta_2} a^{\delta_3} b^{\beta_3}.$$

By Lemma 3.1 $\Theta_{n(\bar{\alpha})}(z)$ is of type three if and only if $\delta_1 = -n$, $\delta_2 = n$, and $\delta_3 = -n$. If this is the case, then

$$\Theta_{n(\bar{\alpha})}(z) = b^{\beta_1 a^{2n}} b^{\beta_2 a^n} b^{\beta_3 a^{2n}} b^{-\beta_3 a^n} b^{-\beta_2} b^{-\beta_1 a^n},$$

and

$$\tilde{\varphi}_n(\Theta_{n(\bar{\alpha})}(z)) = a^{\beta_1\alpha_{p-n}} b^{\beta_2} a^{\beta_3\alpha_{p-n}} b^{-\beta_3} a^{-\beta_2 n} b^{-\beta_1}.$$

Again, $\Theta^2_{n(\bar{\alpha})}(z)$ is of type three if and only if

$$\alpha_{p-n}\beta_1 = -n,\ \beta_3\alpha_{p-n} = n,\ -\beta_2 n = -n.$$

If α_{p-n} is not invertible, this is impossible; assume that it is. Then for

$$\beta_1 = -\frac{n}{\alpha_{p-n}},\ \beta_2 = 1,\ \beta_3 = \frac{n}{\alpha_{p-n}}$$

$\Theta^2_{n(\bar{\alpha})}(z)$ is of type three, and

$$\Theta^2_{n(\bar{\alpha})}(z) = b^{a^{2n}} b^{-\frac{n}{\alpha_{p-n}}a^n} b^{\frac{n}{\alpha_{p-n}}a^{2n}} b^{-\frac{n}{\alpha_{p-n}}a^n} b^{\frac{n}{\alpha_{p-n}}} b^{-a^n}.$$

Furthermore,

$$\tilde{\varphi}_n(\Theta^2_{n(\bar{\alpha})}(z)) = a^{\alpha_{p-n}} b^{-\frac{n}{\alpha_{p-n}}} a^n b^{-\frac{n}{\alpha_{p-n}}} a^{\frac{n^2}{\alpha_{p-n}}} b^{-1},$$

and we see that $\Theta^3_{n(\bar{\alpha})}(z)$ is of type three if and only if $\alpha_{p-n} = -n$. If this is so, then

$$\tilde{\varphi}_n(\Theta^2_{n(\bar{\alpha})}(z)) = a^{-n} b a^n b a^{-n} b^{-1},$$

and we have

$$\Theta^3_{n(\bar{\alpha})}(z) = b^{a^{2n}} b^{a^n} b^{-a^{2n}} b^{a^n} b^{-1} b^{-a^n}.$$

Notice that

$$\tilde{\varphi}_n(\Theta^3_{n(\bar{\alpha})}(z)) = a^{-n} b a^n b a^{-n} b^{-1},$$

and therefore $\Theta^4_{n(\bar{\alpha})}(z) = \Theta^3_{n(\bar{\alpha})}(z)$.

The end conclusion is that if z is of type 3 and has length 6 (i.e. $m = 3$) then either one of the elements $\Theta_{n(\bar{\alpha})}(z)$, $\Theta^2_{n(\bar{\alpha})}(z)$, $\Theta^3_{n(\bar{\alpha})}(z)$ is not of type three, or $\Theta^3_{n(\bar{\alpha})}(z)$ and all subsequent $\Theta^N_{n(\bar{\alpha})}(z)$ are equal to $b^{a^{2n}} b^{a^n} b^{-a^{2n}} b^{a^n} b^{-1} b^{-a^n}$. Therefore we should check the statement of the theorem specifically for $z = b^{a^{2n}} b^{a^n} b^{-a^{2n}} b^{a^n} b^{-1} b^{-a^n}$.

If $z = b^{a^{2n}} b^{a^n} b^{-a^{2n}} b^{a^n} b^{-1} b^{-a^n}$, then

$$\tilde{\varphi}_p(z) = a^{2\alpha_{p-n}} b^{-1} a^{-\alpha_{p-n}}.$$

Because we assume that $\alpha_{p-n} = -n$ (i.e., $\alpha_{p-n} = -\alpha_n$), we have

$$\tilde{\varphi}_p(b^{-a^n z}) = a^n (b^{a^{2n}} b^{-a^n}).$$

And since

$$\tilde{\varphi}_p(L(K)) \geq < b, \tilde{\varphi}_p(b^{-a^n z}) >,$$

we can apply the inductive assumption.

Case 3.3: $m \geq 4$.

Lemma 3.2. Let z be of type three, and let $\partial(z) \geq 8$. Then at least one of the elements $\Theta_{n(\bar{\alpha})}(z)$, $\Theta^2_{n(\bar{\alpha})}(z)$, $\Theta^3_{n(\bar{\alpha})}(z)$ is not of type three.

Proof. As before, there are two cases: when m is even and when m is odd.

Case 3.3.1: m **is even.** Let m be $2k$, $k \geq 2$,

$$\tilde{\varphi}_n(z) = a^{\delta_1} b^{\beta_1} \ldots a^{\delta_{2k}} b^{\beta_{2k}}.$$

By Lemma 3.1, $\Theta_{n(\bar{\alpha})}(z)$ is of type three if and only if $\delta_i = (-1)^i n$. If these equalities hold, then

$$\Theta_{n(\bar{\alpha})}(z) = b^{\beta_1 a^{2n}} b^{\beta_2 a^n} \ldots b^{\beta_{2k-1} a^{2n}} b^{\beta_{2k} a^n} b^{-\beta_{2k}} b^{-\beta_{2k-1} a^n} \ldots b^{-\beta_2} b^{-\beta_1 a^n},$$

and

$$\tilde{\varphi}_n(\Theta_{n(\bar{\alpha})}(z)) = a^{\beta_1 \alpha_{p-n}} b^{\beta_2} \ldots a^{\beta_{2k-1} \alpha_{p-n}} b^{\beta_{2k}} a^{-\beta_{2k} n} b^{-\beta_{2k-1}} \ldots a^{-\beta_2 n} b^{-\beta_1}$$

(remember that $\alpha_n = n$).

Suppose $\Theta^2_{n(\bar{\alpha})}(z)$ is of type three. Then, in particular, $-\beta_4 n = -n$, i.e.

$$\beta_4 = 1, \tag{3.8}$$

and

$$\Theta^2_{n(\bar{\alpha})}(z) = b^{\beta_2 a^{2n}} b^{\beta_4 a^n} \dots b^{-\beta_3 a^{2n}} b^{-\beta_1 a^n} b^{\beta_1} b^{\beta_3 a^n} \dots b^{-\beta_4} b^{-\beta_2 a^n}.$$

Since

$$\tilde{\varphi}_n(\Theta^2_{n(\bar{\alpha})}(z)) = a^{\beta_2 \alpha_{p-n}} b^{\beta_4} \dots a^{-\beta_3 \alpha_{p-n}} b^{-\beta_1} a^{\beta_1 n} b^{\beta_3} \dots a^{-\beta_4 n} b^{-\beta_2}$$

and m is even, then $\Theta^3_{n(\bar{\alpha})}(z)$ is of type three only if $-\beta_4 n = n$, i.e. only if

$$\beta_4 = -1.$$

But this contradicts to (3.8), therefore $\Theta^3_{n(\bar{\alpha})}(z)$ cannot have type three.
Case 3.3.2: m is odd. Let m be $2k+1$, $k \geq 2$,

$$\tilde{\varphi}_n(z) = a^{\delta_1} b^{\beta_1} \dots a^{\delta_{2k+1}} b^{\beta_{2k+1}}.$$

By Lemma 3.1, $\Theta_{n(\bar{\alpha})}(z)$ is of type three if and only if $\delta_i = (-1)^i n$. If these equalities take place, then

$$\Theta_{n(\bar{\alpha})}(z) = b^{\beta_1 a^{2n}} b^{\beta_2 a^n} \dots b^{\beta_{2k} a^n} b^{\beta_{2k+1} a^{2n}} b^{-\beta_{2k+1} a^n} b^{-\beta_{2k}} \dots b^{-\beta_2} b^{-\beta_1 a^n},$$

and

$$\tilde{\varphi}_n(\Theta_{n(\bar{\alpha})}(z)) = a^{\beta_1 \alpha_{p-n}} b^{\beta_2} \dots b^{\beta_{2k}} a^{\beta_{2k+1} \alpha_{p-n}} b^{-\beta_{2k+1}} a^{-\beta_{2k} n} \dots a^{-\beta_2 n} b^{-\beta_1}$$

(remember that $\alpha_n = n$).

Suppose $\Theta^2_{n(\bar{\alpha})}(z)$ is of type three. Then, in particular, $-\beta_4 n = n$, i.e.

$$\beta_4 = -1, \tag{3.9}$$

and

$$\Theta^2_{n(\bar{\alpha})}(z) = b^{\beta_2 a^{2n}} b^{\beta_4 a^n} \dots b^{-\beta_3 a^n} b^{-\beta_1 a^{2n}} b^{\beta_1 a^n} b^{\beta_3} \dots b^{-\beta_4} b^{-\beta_2 a^n}.$$

Further, since

$$\tilde{\varphi}_n(\Theta^2_{n(\bar{\alpha})}(z)) = a^{\beta_2 \alpha_{p-n}} b^{\beta_4} \dots b^{-\beta_3} a^{-\beta_1 \alpha_{p-n}} b^{\beta_1} a^{\beta_3 n} \dots a^{-\beta_4 n} b^{-\beta_2}$$

and m is odd, then $\Theta^3_{n(\bar{\alpha})}(z)$ is of type three only if $-\beta_4 n = -n$, i.e. only if

$$\beta_4 = 1.$$

But this contradicts to (3.9), therefore $\Theta^3_{n(\bar{\alpha})}(z)$ cannot have type three.

Since we have considered all possibilities, the lemma is proven. ■

This completes the inductive step, so the theorem is proven. ■

Remark 3.1. Notice that the proof works as well in case of a non-prime p and a GGS-group $G_{\bar{\alpha}}$ for which there exists an *invertible* number n such that α_n is also invertible.

Let us return to the GGS-group $G_{\bar{\alpha}}$ we'd started with.

Corollary 3.2. *Let z be an arbitrary element of $[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$, and let K be the subgroup of $G_{\bar{\alpha}}$ generated by b and $a^n z$. Then there exists a vertex u such that $K_u = G_{\bar{\alpha}}$.*

Proof. Notice that $G_{\bar{\alpha}}$ is a factor-group of H. Let $\varepsilon_{\bar{\alpha}} : H \to G_{\bar{\alpha}}$ be the natural projection. Then by definition of homomorphisms $\tilde{\varphi}_n$, $\tilde{\varphi}_p$ the following diagrams are commutative:

$$\begin{tikzcd} L(H) \arrow[r, "\varepsilon_{\bar{\alpha}}"] \arrow[d, "\tilde{\varphi}_n"'] & \mathrm{st}_{G_{\bar{\alpha}}}(1) \arrow[d, "\varphi_{n-1}"] \\ H \arrow[r, "\varepsilon_{\bar{\alpha}}"] & G_{\bar{\alpha}} \end{tikzcd} \qquad \begin{tikzcd} L(H) \arrow[r, "\varepsilon_{\bar{\alpha}}"] \arrow[d, "\tilde{\varphi}_p"'] & \mathrm{st}_{G_{\bar{\alpha}}}(1) \arrow[d, "\varphi_{p-1}"] \\ H \arrow[r, "\varepsilon_{\bar{\alpha}}"] & G_{\bar{\alpha}} \end{tikzcd}$$

Hence the required statement follows from Theorem 3.1. ■

4 Dense subgroups

As noted in Section 1.4, we intend to study maximal subgroups of our groups in question via investigating their subgroups everywhere dense in the profinite topology. One such subgroup clearly exists in any group; it is the group itself. This section is devoted to the question of existence of proper everywhere dense subgroups.

Let us give a criterion for a subgroup to be everywhere dense. Let G be a just infinite group and let M be an arbitrary subgroup. For every normal subgroup N in G denote by ε_N the natural projection $G \to G/N$. Then M is everywhere dense in the profinite topology if and only if for any normal subgroup N of G

$$\varepsilon_N(M) = \varepsilon_N(G). \tag{4.1}$$

Thus, in order to find a proper everywhere dense subgroup, we need to find a proper subgroup satisfying condition (4.1). This condition tells us how it acts on all finite trees $T_{[n]}$. The next natural question to ask is probably how it acts on subtrees, i.e. how large its upper companion groups are. These things probably do not determine the subgroup entirely, but they are good for the beginning.

In the previous two sections we have already established some results about upper companion groups of subgroups of some special form. Below we develop those results. First, we need the following lemma.

Lemma 4.1. Let $G \leq \mathrm{Aut}\, T$ be a group acting transitively on each layer of T. Then any everywhere dense proper subgroup M of G has infinite index.

Proof. Suppose an everywhere dense subgroup M has finite index. Then it contains a normal subgroup N which is also of finite index. Since by definition $\varepsilon_N(M) = \varepsilon_N(G)$, we have $M = G$. ■

Proposition 4.1. *Let T be a rooted spherically periodic tree, let $G \leq \mathrm{Aut}\, T$ be a just infinite group acting transitively on each layer of T, and let M be an everywhere dense subgroup of G. Then we have* *(a) M is transitive on each layer of vertices;*

(b) for any vertex u of T M_u is everywhere dense in G_u.

Proof. Let us prove (a). Since for any natural number n G is transitive on the set of vertices of level n, the factor-group $G/\operatorname{st}_G(n)$ acts transitively on the level n vertices of the finite tree $T_{[n]}$. Then by (4.1)

$$\varepsilon_{\operatorname{st}_G(n)}(M) = \varepsilon_{\operatorname{st}_G(n)}(G) = G/\operatorname{st}_G(n).$$

Hence $\varepsilon_{\operatorname{st}_G(n)}(M)$ is also transitive on the level n vertices of the finite tree $T_{[n]}$. Since $\varepsilon_{\operatorname{st}_G(n)}(M) = M/\operatorname{st}_M(n)$, M is transitive on the set of vertices of level n.

Let us prove (b) now. Take a normal subgroup $N \lhd G_u$ and consider the subgroup

$$\bar{N} = G \cap \Phi_{|u|}^{-1}\Big(\prod_{v\colon |v|=|u|} N^{\psi_{u,v}}\Big).$$

This is a normal subgroup, since N is normal in G_u. Then by (4.1)

$$\varepsilon_{\bar{N}}(M) = \varepsilon_{\bar{N}}(G).$$

Since $\bar{N} \leq \operatorname{st}_G(u)$, we also have that

$$\varepsilon_{\bar{N}}(\operatorname{st}_M(u)) = \varepsilon_{\bar{N}}(\operatorname{st}_G(u)). \tag{4.2}$$

Denote ε_N^u the natural projection $G_u \to G_u/N$. Since $\bar{N}$ is in the kernel of the composition map $\varepsilon_N^u \circ \varphi_u : \operatorname{st}_G(u) \to \varepsilon_N^u(G_u)$, this map can be factorized to a map $\hat{\varphi}_u : \varepsilon_{\bar{N}}(\operatorname{st}_G(u)) \to \varepsilon_N^u(G_u)$ such that

$$\hat{\varphi}_u \circ \varepsilon_{\bar{N}} = \varepsilon_N^u \circ \varphi_u.$$

Hence, by (4.2) we have

$$\varepsilon_N^u(M_u) = \hat{\varphi}_u(\varepsilon_{\bar{N}}(\operatorname{st}_M(u))) = \hat{\varphi}_u(\varepsilon_{\bar{N}}(\operatorname{st}_G(u))) = \varepsilon_N^u(G_u),$$

which is what we need. ■

Now we proceed to separate treatment of our three examples. However, we more or less follow the same general scheme. We first prove that any upper companion group of a proper everywhere dense subgroup is a proper subgroup (and therefore is also a proper everywhere dense subgroup, by the above proposition), see Lemmas 4.2, 4.4, and 4.7. In cases of Grigorchuk's groups and GGS-groups we then prove that each everywhere dense subgroup has an upper companion group containing a subgroup from Corollary 2.1 and Corollary 3.2, respectively, and thus has an upper companion group which is not proper. In order to establish the same fact for EGS-groups, we use the fact that every EGS-group contains a GGS-group.

4.1 Grigorchuk's groups

Lemma 4.2. Let M an everywhere dense proper subgroup of G_ω. Then for any vertex u of T $\mathrm{st}_M(u)_u$ is a proper subgroup of $G_{\sigma^n\omega}$, $n = |u|$.

Proof. Assume the contrary, i.e. that there exists a vertex u such that $\mathrm{st}_M(u)_u = G_{\sigma^n\omega}$, $n = |u|$. By Proposition 4.1 it is sufficient to assume that $|u| = 1$.

Since M is everywhere dense, M is transitive on the set of vertices of length 1. Therefore $\mathrm{st}_M(v)_v$ also equals $G_{\sigma\omega}$, where v is the other vertex of length 1.

Let $M_1 = \mathrm{rist}_M(u)_u$. Since $\mathrm{rist}_M(u) \lhd \mathrm{st}_M(u)$, we have $M_1 \lhd \mathrm{st}_M(u)_u = G_{\sigma\omega}$. As any nontrivial normal subgroup of G_ω has finite index, either M_1 has finite index in $G_{\sigma\omega}$ or $M_1 = 1$.

If M_1 is a finite index subgroup of $G_{\sigma\omega}$, the following subgroup

$$< \mathrm{rist}_M(u), \mathrm{rist}_M(v) >,$$

has finite index in the direct product $\Phi_1^{-1}(G_{\sigma\omega} \times G_{\sigma\omega})$, hence has finite index in $\mathrm{st}_{G_\omega}(1)$. This implies that index $|G_\omega : M|$ is finite, in contradiction with Lemma 4.1.

Suppose $M_1 = 1$. In this case $\mathrm{rist}_M(1) = 1$. This means that $\mathrm{st}_M(1)$ can be represented in the following way,

$$\mathrm{st}_M(1) \cong \{(x, x^\phi) | x \in G_{\sigma\omega}\},$$

where $\phi : G_{\sigma\omega} \to G_{\sigma\omega}$ is an automorphism.

By definition of an everywhere dense subgroup there exists $y \in G \cap \Phi_\omega^{-1}([G_{\sigma\omega}, G_{\sigma\omega}] \times [G_{\sigma\omega}, G_{\sigma\omega}])$ such that $d_\omega y \in M$, where d_ω is the d-generator of G_ω. Hence $(\varphi_0(y))^\phi = d_{\sigma\omega}\varphi_1(y)$, and $\varphi_0(y), \varphi_1(y) \in [G_{\sigma\omega}, G_{\sigma\omega}]$. This means that the automorphism of $G_{\sigma\omega}/[G_{\sigma\omega}, G_{\sigma\omega}]$ induced by ϕ takes the trivial coset to coset $d_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$. This does contradict Lemma 1.1. Thus, the lemma is proven. ■

Lemma 4.3. Let M be an everywhere dense subgroup of G_ω. Then there exist an element $x \in M$ and a vertex $u \in T$ such that X_u contains one of the elements $b_{\sigma^n\omega}$, $c_{\sigma^n\omega}$, $d_{\sigma^n\omega}$, where $n = |u|$ and X is the cyclic subgroup generated by x.

Proof. Let x belong to one of the following cosets $b_\omega[G_\omega, G_\omega]$, $c_\omega[G_\omega, G_\omega]$, $d_\omega[G_\omega, G_\omega]$. Let us prove by induction on $\partial_\omega(x)$ that there exists a vertex u such that X_u contains one of the elements $b_{\sigma^n\omega}$, $c_{\sigma^n\omega}$, $d_{\sigma^n\omega}$, $n = |u|$.

Without losing any generality, we can assume that d_ω is the d-generator and c_ω is the c-generator. The base of induction is when $\partial_\omega(x) = 1$, in which case the statement is obvious.

Let $\partial_\omega(x) = m > 1$. By choice, $x = y_\omega z$, where $y_\omega \in \{b_\omega, c_\omega, d_\omega\}$ and $z \in [G_\omega, G_\omega]$. Consider elements $\varphi_0(x)$, $\varphi_1(x)$, $\varphi_0(z)$ and $\varphi_1(z)$. It follows from Lemma 1.1 that the latter two elements belong to the same coset of $[G_{\sigma\omega}, G_{\sigma\omega}]$, namely, to one of the cosets $[G_{\sigma\omega}, G_{\sigma\omega}]$, $d_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$, $ab_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$, $ac_{\sigma\omega}[G_{\sigma\omega}, G_{\sigma\omega}]$. Thus, depending on y_ω, we get twelve possibilities, listed in the following table (each row corresponds to a fixed choice of

y_ω and each column to a fixed choice of the coset of $\varphi_0(z)$; the cells contain the cosets of pair $(\varphi_0(x), \varphi_1(x))$).

	1	$d_{\sigma\omega}$	$ac_{\sigma\omega}$	$ab_{\sigma\omega}$
b_ω	$(a, b_{\sigma\omega})$	$(ad_{\sigma\omega}, c_{\sigma\omega})$	$(c_{\sigma\omega}, ad_{\sigma\omega})$	$(b_{\sigma\omega}, a)$
c_ω	$(a, c_{\sigma\omega})$	$(ad_{\sigma\omega}, b_{\sigma\omega})$	$(c_{\sigma\omega}, a)$	$(b_{\sigma\omega}, ad_{\sigma\omega})$
d_ω	$(1, d_{\sigma\omega})$	$(d_{\sigma\omega}, 1)$	$(ac_{\sigma\omega}, ab_{\sigma\omega})$	$(ab_{\sigma\omega}, ac_{\sigma\omega})$

Thus, if $x \in b_\omega[G_\omega, G_\omega] \cup c_\omega[G_\omega, G_\omega]$, one of the elements $\varphi_0(x)$, $\varphi_1(x)$ also belongs to $b_{\sigma\omega}[G_\omega, G_\omega] \cup c_{\sigma\omega}[G_\omega, G_\omega]$. Since both $\varphi_0(x)$, $\varphi_1(x)$ a strictly smaller length, we can apply the inductive assumption.

Further, if $x \in d_\omega[G_\omega, G_\omega]$, then either one of $\varphi_0(x)$, $\varphi_1(x)$ is in $d_{\sigma\omega}[G_\omega, G_\omega]$, and we use the inductive assumption, or one of the elements $\varphi_0(x)$, $\varphi_1(x)$ is in $ac_{\sigma\omega}[G_\omega, G_\omega]$, and the other in $ab_{\sigma\omega}[G_\omega, G_\omega]$. In this case we consider $x^{2^{m(\omega)}}$.

Clearly, $x^{2^{m(\omega)}} \in \mathrm{st}_{G_\omega}(m(\omega)+1)$. By definition of $m(\omega)$ $c_{\sigma^{m(\omega)}\omega}$ is the d-generator of $G_{\sigma^{m(\omega)}\omega}$, therefore there is at least one vertex u of length $(m(\omega)+1)$ such that $\varphi_u(x^{2^{m(\omega)}})$ belongs to the coset $c_{\sigma^{m(\omega)+1}\omega}[G_{\sigma^{m(\omega)+1}\omega}, G_{\sigma^{m(\omega)+1}\omega}]$.

Let us prove by induction on $n = 1, 2, \ldots, m(\omega)$ that for any vertex u of length $n+1$ $\partial_{\sigma^{n+1}\omega}(\varphi_u(x^{2^n}))$ is strictly less than $\partial_\omega(x)$. Notice that for any ω, for any $t \in G_\omega \setminus \mathrm{st}_{G_\omega}(1)$ and for any $j = 0, 1$

$$\partial_{\sigma\omega}(\varphi_j(t^2)) \le \partial_\omega(t). \tag{4.3}$$

Let us apply inequality (4.3) to $\sigma\omega$ and an element $t = \varphi_i(x)$, where $i = 0$ or 1. We have

$$\partial_{\sigma^2\omega}(\varphi_j(\varphi_i(x)^2)) \le \partial_{\sigma\omega}(\varphi_i(x)) < \partial_\omega(x), \tag{4.4}$$

where the latter inequality follows from the fact that for any $x \in \mathrm{st}_{G_\omega}(1)$ $\partial_{\sigma\omega}(\varphi_i(x)) \le \frac{\partial_\omega(x)+1}{2}$, and $\frac{\partial_\omega(x)+1}{2}$ is strictly less than $\partial_\omega(x)$ as soon as $\partial_\omega(x) > 1$.

Inequality (4.4) means that for any vertex u of length 2

$$\partial_{\sigma^2\omega}(\varphi_u(x^2)) < \partial_\omega(x).$$

Thus, the desired statement is true for $n = 1$. Suppose now that $m(\omega) > 1$. Let v be a vertex of length $m(\omega)$ and u be either $v0$ or $v1$. Again, applying (4.3) to $\sigma^{m(\omega)}\omega$ and $t = \varphi_v(x^{2^{m(\omega)-1}})$ we get

$$\partial_{\sigma^{m(\omega)+1}\omega}(\varphi_u(x^{2^{m(\omega)}})) \le \partial_{\sigma^{m(\omega)}\omega}(\varphi_v(x^{2^{m(\omega)-1}})) < \partial_\omega(x).$$

Indeed, for any $n < m(\omega)$ x^{2^n} is not in $\mathrm{st}_{G_\omega}(n+2)$ and therefore $\varphi_v(x^{2^{m(\omega)-1}})$ is not in $\mathrm{st}_{G_{\sigma^{m(\omega)}\omega}}(1)$. Thus, we can apply the inductive assumption to obtain that the length of $\varphi_u(x^{2^{m(\omega)}})$ is strictly less than the length of x. Now we can apply the (initial) inductive assumption to prove the lemma. ∎

Proposition 4.2. *Let M be an everywhere dense subgroup of a Grigorchuk's 2-group G_ω. Then there exists a vertex u such that $M_u = G_{\sigma^{|u|}\omega}$.*

Proof. By Lemma 4.3, there exists a vertex v such that M_v contains an element $x_{\sigma^m\omega}$ from the set $\{b_{\sigma^m\omega}, c_{\sigma^m\omega}, d_{\sigma^m\omega}\}$, $m = |v|$. Since M_v is everywhere dense, it contains an element of the form $y_{\sigma^m\omega}az$, where $y_{\sigma^m\omega} \neq x_{\sigma^m\omega}$ is from the set $\{b_{\sigma^m\omega}, c_{\sigma^m\omega}, d_{\sigma^m\omega}\}$, $z \in [G_{\sigma^m\omega}, G_{\sigma^m\omega}]$. Then by Corollary 2.1 there is a vertex w such that $(M_v)_w = G_{\sigma^n\omega}$, where $n = |w| + m$. Hence, there is a vertex $u = vw$ such that $M_u = G_{\sigma^n\omega}$. ∎

Theorem 4.1. *No G_ω contains everywhere dense proper subgroups.*

Proof. Follows from contradiction between Proposition 4.2 and Lemma 4.2. ∎

4.2 GGS-groups

Lemma 4.4. Let $G_{\bar{\alpha}}$ be a GGS-group and let M be its everywhere dense proper subgroup. Then for each $u \in T$ M_u is a proper subgroup of $G_{\bar{\alpha}}$.

Proof. By Proposition 4.1 it is sufficient to consider $|u| = 1$.

Let u be an arbitrary vertex of length 1. Suppose $M_u = G_{\bar{\alpha}}$. Then, since by Proposition 4.1 M is transitive on the set of vertices of length 1, we have that for every other vertex v of length 1 M_v also equals $G_{\bar{\alpha}}$.

Let us prove by induction on $n = 1, 2, \ldots, p-1$ that M has a trivial intersection with any of the following subgroups of $\operatorname{Aut} T$:

$$\prod_{u \in U} \operatorname{rist}_{\operatorname{Aut} T}(u),$$

where U is some set of vertices of length 1 and the cardinality of U is equal to n.

The base of induction is $n = 1$. Consider $M_u \cap \operatorname{rist}_{\operatorname{Aut} T}(u)_u =: L$. Clearly, $L \lhd G_{\bar{\alpha}}$ and $M \geq \Phi_1^{-1}(\underbrace{L \times \ldots \times L}_{p})$. If $L \neq 1$, then, since $G_{\bar{\alpha}}$ is just infinite, L has finite index in $G_{\bar{\alpha}}$, which implies $\Phi_1^{-1}(\underbrace{L \times \ldots \times L}_{p})$ has finite index in $G_{\bar{\alpha}}$. This, however, contradicts Lemma 4.1. Thus, $L = 1$ and

$$M \cap \operatorname{rist}_{\operatorname{Aut} T}(u) = 1.$$

Denote

$$M(n, U) := M \cap \prod_{u \in U} \operatorname{rist}_{\operatorname{Aut} T}(u),$$

where U is some set of vertices of length 1 and the cardinality of U is equal to n. Assume now that for $k = 1, \ldots, n$ and for each set U $M(k, U) = 1$. Let $k = n+1$ and choose U of cardinality $n+1$.

Suppose that $M(n+1, U)$ is not trivial. Choose an $i \in \{0, 1, \ldots, p-1\}$ such that $\varphi_i(M(n+1, U)) \neq 1$ and a j such that $\varphi_{i-j}(M(n+1, U)) = 1$ and $\varphi_{i+j}(M(n+1, U)) \neq 1$ (such a j exists because $n+1 \geq 2$ and p is prime; indeed, starting with a couple of nontrivial subgroups $\varphi_k(M(n+1, U))$ and $\varphi_{k+l}(M(n+1, U))$, we consider subgroups $\varphi_{k-l}(M(n+1, U))$, $\varphi_{k-2l}(M(n+1, U))$, …, $\varphi_{k-(p-1)l}(M(n+1, U))$. Since numbers k, $k-l$, $k-2l$, …, $k-(p-1)l$

are all the numbers $0, 1, 2, \dots, p-1$ but written in a different order, sooner or later we will meet a nontrivial subgroup). Denote $\varphi_i(M(n+1,U))$ by L_1 and $\varphi_{i+j}(M(n+1,U))$ by L_2.

Notice that since every element of $G_{\bar{\alpha}}$ acts either trivially or transitively on the set of all vertices of length 1, for any vertex u of length 1 each set $M(n+1,U)$ is normal in $\mathrm{st}_M(u)$. Thus, since for every such u $M_u = G_{\bar{\alpha}}$, L_1 and L_2 are normal in $G_{\bar{\alpha}}$. Since M is everywhere dense, it contains an element of the form $a^j x$, $x \in \mathrm{st}_{G_{\bar{\alpha}}}(1)$. Since L_1 is normal in $G_{\bar{\alpha}}$, we have

$$\varphi_{i+j}(M(n+1,U)^{a^j x}) = L_1^{\varphi_{i+j}(x)} = L_1.$$

It follows from the similar equality

$$\varphi_i(M(n+1,U)^{a^j x}) = \varphi_{i-j}(M(n+1,U))^{\varphi_i(x)} = 1$$

that subgroups $M(n+1,U)$ and $M(n+1,U)^{a^j x}$ commute. Indeed,

$$[M(n+1,U), M(n+1,U)^{ax}] \leq M(n,U'),$$

for some set U' of cardinality n (U' is contained in $U \setminus \{i\}$). But $M(n,U')$ is trivial by inductive assumption, so L_1 and L_2 commute as well.

Since L_1 and L_2 are normal and by assumption nontrivial, they both have finite index in $G_{\bar{\alpha}}$. Therefore their intersection also has finite index in $G_{\bar{\alpha}}$. However, it is a finite index subgroup in a finitely generated group and therefore is finitely generated. But it is also Abelian and a subgroup of a p-group, hence, it is finite. This implies that $G_{\bar{\alpha}}$ is finite, which is a contradiction. Thus, $M(n+1,U)$ is trivial. The induction is finished.

Thus, the intersection of M with any $\prod_{u\in U} \mathrm{rist}_{\mathrm{Aut}\,T}(u)$, where the cardinality of U is $p-1$, is trivial. This means that M consists of elements of the form:

$$(g, g^{\psi_1}, g^{\psi_2}, \dots, g^{\psi_{p-1}}),$$

where $g \in G_{\bar{\alpha}}$, $\psi_1, \psi_2, \dots, \psi_{p-1}$ are some automorphisms of $G_{\bar{\alpha}}$. Let us consider those automorphisms.

Being everywhere dense, M contains elements of the form bx_1, $b^a x_2$, …, $b^{a^{p-1}} x_p$, where x_1, x_2, …, x_p are in $G_{\bar{\alpha}} \cap \Phi_1^{-1}(\underbrace{[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \times \ldots \times [G_{\bar{\alpha}}, G_{\bar{\alpha}}]}_{p})$. Thus,

$$\Phi_1(bx_1) = (a^{\alpha_1} x_{11}, a^{\alpha_2} x_{12}, \dots, bx_{1p}),$$

$$\Phi_1(b^a x_2) = (bx_{21}, a^{\alpha_1} x_{22}, \dots, a^{\alpha_{p-1}} x_{2p}),$$

$$\dots$$

$$\Phi_1(b^{a^{p-1}} x_p) = (a^{\alpha_2} x_{p1}, a^{\alpha_3} x_{p2}, \dots, a^{\alpha_1} x_{pp}),$$

where all $x_{ij} \in [G_{\bar{\alpha}}, G_{\bar{\alpha}}]$. Let $\bar{\psi}_1$ be the induced automorphism $G_{\bar{\alpha}}/[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \to G_{\bar{\alpha}}/[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$. Then we have

$$\begin{array}{ccc} a^{\alpha_1}[G_{\bar{\alpha}}, G_{\bar{\alpha}}] & \mapsto & a^{\alpha_2}[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \\ a^{\alpha_2}[G_{\bar{\alpha}}, G_{\bar{\alpha}}] & \mapsto & a^{\alpha_3}[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \\ & \cdots & \\ a^{\alpha_{p-2}}[G_{\bar{\alpha}}, G_{\bar{\alpha}}] & \mapsto & a^{\alpha_{p-1}}[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \\ a^{\alpha_{p-1}}[G_{\bar{\alpha}}, G_{\bar{\alpha}}] & \mapsto & b[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \\ b[G_{\bar{\alpha}}, G_{\bar{\alpha}}] & \mapsto & a^{\alpha_1}[G_{\bar{\alpha}}, G_{\bar{\alpha}}] \end{array}$$

If $\alpha_1 = 0$, we obtain a contradiction right away. Suppose $\alpha_1 \neq 0$. Then $\bar{\psi}_1$ takes $a[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$ to $a^{\frac{\alpha_2}{\alpha_1}}[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$. Hence it takes $\alpha_{p-1}[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$ to $a^{\frac{\alpha_2 \alpha_{p-1}}{\alpha_1}}[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$, not to $b[G_{\bar{\alpha}}, G_{\bar{\alpha}}]$. So we get a contradiction anyway, and this proves the lemma. ∎

Lemma 4.5. Let M be an everywhere dense subgroup of a GGS-group $G_{\bar{\alpha}}$. Then there exist an element $x \in M$ and a vertex $u \in T$ such that $X_u \ni b$, where X is the cyclic subgroup generated by x.

Proof. By definition of an everywhere dense subgroup M contains an element x of the form $b^i y$, $y \in [G_{\bar{\alpha}}, G_{\bar{\alpha}}]$, $i \neq 0$. Let us prove by induction on $n = \partial_{G_{\bar{\alpha}}}(x)$ that for every such x there exists a vertex u such that $X_u \ni b$, where X is as above.

The base of induction when $\partial_{G_{\bar{\alpha}}}(x) = 1$ is obvious. Suppose $\partial_{G_{\bar{\alpha}}}(x) = n > 1$. Notice that $\partial_b(\varphi_0(x)) + \partial_b(\varphi_1(x)) + \ldots + \partial_b(\varphi_{p-1}(x)) = \partial_b(x) \neq 0$. Therefore there exists j such that $\partial_b(\varphi_j(x)) \neq 0$.

Observe that

$$\partial_{G_{\bar{\alpha}}}(\varphi_j(x)) \leq \frac{\partial_{G_{\bar{\alpha}}}(x) + 1}{2} < \partial_{G_{\bar{\alpha}}}(x).$$

Thus, if $\varphi_j(x)$ is in $\mathrm{st}_{G_{\bar{\alpha}}}(1)$, then $\varphi_j(x)$ has the desired form $b^k y'$, $k \neq 0$, $y' \in [G_{\bar{\alpha}}, G_{\bar{\alpha}}]$, and we can apply the inductive assumption.

Suppose that $\varphi_j(x)$ is not in $\mathrm{st}_{G_{\bar{\alpha}}}(1)$. However,

$$\partial_b(\varphi_l(\varphi_j(x)^p)) = \partial_b(\varphi_j(x)) \neq 0$$

for all $l = 0, 1, \ldots, p-1$ and

$$\partial_a(\varphi_l(\varphi_j(x)^p)) = (\alpha_1 + \ldots + \alpha_{p-1})\partial_b(\varphi_j(x)) = 0,$$

since $G_{\bar{\alpha}}$ is periodic. Thus, $\varphi_l(\varphi_j(x)^p)$ has the desired form. And since

$$\partial_{G_{\bar{\alpha}}}(\varphi_l(\varphi_j(x)^p)) \leq \partial_{G_{\bar{\alpha}}}(\varphi_j(x)) < \partial_{G_{\bar{\alpha}}}(x),$$

we can apply the inductive assumption. ∎

Proposition 4.3. *Let M be an everywhere dense subgroup of a GGS-group $G_{\bar{\alpha}}$. Then there exists a vertex u such that $M_u = G_{\bar{\alpha}}$.*

Proof. By Proposition 4.5 there exists a vertex v such that $M_v \ni b$. By Proposition 4.1 M_v is everywhere dense as well, hence it contains an element of the form $a^{n(\bar{\alpha})}z$, $z \in [G_{\bar{\alpha}}, G_{\bar{\alpha}}]$. Thus, M_v contains a subgroup generated by elements b and $a^{n(\bar{\alpha})}z$. By Corollary 3.2 there exists a vertex w such that $(M_v)_w = G_{\bar{\alpha}}$. In other words, if $u = vw$, then $M_u = G_{\bar{\alpha}}$. ∎

This leads us to the central conclusion about GGS-groups.

Theorem 4.2. *No GGS-group contains everywhere dense proper subgroups.*

Proof. Indeed, there is a contradiction between Proposition 4.3 and Lemma 4.4, which proves the theorem. ■

4.3 EGS-groups

Lemma 4.6. Let M be an everywhere dense subgroup of $\Gamma_{\bar{\alpha}}$. Then there exists a proper everywhere dense subgroup M' of $\Gamma_{\bar{\alpha}}$ such that $M' \cap G_{\bar{\alpha}}$ either is everywhere dense in the profinite topology of $G_{\bar{\alpha}}$ or is a subgroup of finite index in $G_{\bar{\alpha}}$.

Proof. If $M \cap G_{\bar{\alpha}}$ coincides with $G_{\bar{\alpha}}$, then this intersection is everywhere dense, and there is nothing to prove. Suppose that $M \cap G_{\bar{\alpha}}$ is a proper subgroup of $G_{\bar{\alpha}}$. Denote by M' a maximal subgroup in $\Gamma_{\bar{\alpha}}$ containing M. This is, of course, a proper everywhere dense subgroup, and $M' \cap G_{\bar{\alpha}}$ is either a maximal subgroup of $G_{\bar{\alpha}}$ or $G_{\bar{\alpha}}$ itself.

Suppose that $M' \cap G_{\bar{\alpha}}$ is a proper subgroup of infinite index. Then by Proposition 5.1 it is everywhere dense in $G_{\bar{\alpha}}$. ■

Lemma 4.7. Let $\Gamma_{\bar{\alpha}}$ be a EGS-group and let M be its everywhere dense proper subgroup. Then for each $u \in T$ M_u is a proper subgroup of $\Gamma_{\bar{\alpha}}$.

Proof. The proof repeats word-for-word that of Lemma 4.4. ■

Lemma 4.8. Let M be an everywhere dense subgroup of a EGS-group $\Gamma_{\bar{\alpha}}$. Then there exist an element $x \in M$ and a vertex $u \in T$ such that $X_u \ni c$, where X is the cyclic subgroup generated by x.

Proof. By definition of an everywhere dense subgroup, M contains an element x of the form $c^{\alpha}y$, where y belongs to the subgroup generated by b and the derived subgroup $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$ and $\alpha \neq 0$. Let us prove by induction on $n = \partial_{\Gamma_{\bar{\alpha}}}(x)$ that for every such x there exists a vertex u such that $X_u \ni c$.

The base of induction is $\partial_{\Gamma_{\bar{\alpha}}}(x) \leq 2$. If $\partial_{\Gamma_{\bar{\alpha}}}(x) = 1$, the statement is obvious. Suppose $\partial_{\Gamma_{\bar{\alpha}}}(x) = 2$. Then either $x = c^{\alpha a^i} b^{\beta a^j}$ or $x = b^{\beta a^j} c^{\alpha a^i}$.

Let us consider two separate cases: when $j = i+1$ and when $j \neq i+1$. Assume that $j = i+1$. Then $\varphi_i(x) = c^{\alpha} b^{\beta}$ or $\varphi_i(x) = b^{\beta} c^{\alpha}$, respectively, and $\varphi_0(\varphi_i(x)) = c^{\alpha} a^{\alpha_1 \beta}$ or $\varphi_0(\varphi_i(x)) = a^{\alpha_1 \beta} c^{\alpha}$, respectively. Thus, $\varphi_0 \circ \varphi_0 \circ \varphi_i(x^p) = c^{\alpha}$, which is what we need.

Assume that $j \neq i+1$. Then $\varphi_i(x) = c^{\alpha} a^{\beta \alpha_{j-i-1}}$ or $\varphi_i(x) = a^{\beta \alpha_{j-i-1}} c^{\alpha}$, respectively (index $j-i-1$ is taken modulo p). If $\alpha_{j-i-1} = 0$, we already have the desired element. If $\alpha_{j-i-1} \neq 0$, then $\varphi_0(\varphi_i(x^p)) = c^{\alpha}$.

Suppose $\partial_{\Gamma_{\bar{\alpha}}}(x) = n > 1$. Observe that either

$$\partial_{\Gamma_{\bar{\alpha}}}(\varphi_i(x)) \leq \partial_{\Gamma_{\bar{\alpha}}}(x) - 1,$$

or else x has the form

$$x = c^{\gamma_1 a^i} b^{\beta_1 a^{i+1}} \ldots c^{\gamma_m a^i} b^{\beta_m a^{i+1}}, \tag{4.5}$$

where γ_1, β_m, $i \in \mathrm{GF}(p)$, and $\gamma_2, \ldots, \gamma_m, \beta_1, \ldots, \beta_{m-1} \in \mathrm{GF}(p)$. (Notice that $\gamma_1 + \ldots + \gamma_m = \alpha$.)

In the latter case we have that

$$\varphi_i(x) = c^{\gamma_1} b^{\beta_1} \ldots c^{\gamma_m} b^{\beta_m},$$

and therefore

$$\partial_{\Gamma_{\bar{\alpha}}}(\varphi_0(\varphi_i(x))) \leq m \leq \frac{\partial_{\Gamma_{\bar{\alpha}}}(x) + 1}{2} < \partial_{\Gamma_{\bar{\alpha}}}(x),$$

while $\partial_c(\varphi_0(\varphi_i(x))) = \gamma_1 + \ldots + \gamma_m = \alpha$ and $\partial_a(\varphi_0(\varphi_i(x))) = \alpha_1(\beta_1 + \ldots + \beta_m)$. If $\alpha_1(\beta_1 + \ldots + \beta_m) = 0$, we can apply the inductive assumption.

Suppose that $\alpha_1(\beta_1 + \ldots + \beta_m) \neq 0$. Then $\partial_c(\varphi_0 \circ \varphi_0 \circ \varphi_i(x^p)) = \alpha$ and $\partial_a(\varphi_0 \circ \varphi_0 \circ \varphi_i(x^p)) = 0$. Notice also that

$$\partial_{\Gamma_{\bar{\alpha}}}(\varphi_0 \circ \varphi_0 \circ \varphi_i(x^p)) \leq \partial_{\Gamma_{\bar{\alpha}}}(\varphi_0(\varphi_i(x))) < \partial_{\Gamma_{\bar{\alpha}}}(x).$$

Hence, we can apply the inductive assumption.

Suppose that x does not have the form (4.5). This means, in particular, that for any i

$$\partial_{\Gamma_{\bar{\alpha}}}(\varphi_i(x)) \leq \frac{\partial_{\Gamma_{\bar{\alpha}}}(x) + 1}{2} < \partial_{\Gamma_{\bar{\alpha}}}(x).$$

Notice that $\partial_c(\varphi_0(x)) + \partial_c(\varphi_1(x)) + \ldots + \partial_c(\varphi_{p-1}(x)) = \partial_c(x) \neq 0$. Therefore there exists j such that $\partial_c(\varphi_j(x)) \neq 0$.

If $\varphi_j(x)$ is in $\mathrm{st}_{\Gamma_{\bar{\alpha}}}(1)$, $\varphi_j(x)$ has the desired form $c^\beta z$, where $\beta \neq 0$ and z is in the subgroup generated by b and $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$. Therefore we can apply the inductive assumption.

Now assume that $\varphi_j(x)$ is not in $\mathrm{st}_{\Gamma_{\bar{\alpha}}}(1)$. However,

$$\partial_c(\varphi_k(\varphi_j(x)^p)) = \partial_c(\varphi_j(x)) \neq 0$$

for all $k = 0, 1, \ldots, p-1$ and

$$\partial_a(\varphi_k(\varphi_j(x)^p)) = (\alpha_1 + \ldots + \alpha_{p-1})\partial_a(\varphi_j(x)) = 0,$$

since $\Gamma_{\bar{\alpha}}$ is periodic.

Thus, $\varphi_k(\varphi_j(x^p))$ has the desired form. And since

$$\partial_{\Gamma_{\bar{\alpha}}}(\varphi_k(\varphi_j(x^p))) \leq \partial_{\Gamma_{\bar{\alpha}}}(\varphi_j(x)) < \partial_{\Gamma_{\bar{\alpha}}}(x),$$

we can apply the inductive assumption. ∎

Proposition 4.4. *Let M be an everywhere dense subgroup of a EGS-group $\Gamma_{\bar{\alpha}}$. Then there exists a vertex u such that $M_u = \Gamma_{\bar{\alpha}}$.*

Proof. By Lemma 4.6, we can assume that $M \cap G_{\bar{\alpha}}$ either has a finite index in $G_{\bar{\alpha}}$ or is everywhere dense in it. Suppose that $M \cap G_{\bar{\alpha}}$ has finite index. Then it contains a normal subgroup N. By Proposition 4.1 of [13], there is a vertex u such that N_u, and hence M_u, contains $G_{\bar{\alpha}}$. Now, if $M \cap G_{\bar{\alpha}}$ is everywhere dense in $G_{\bar{\alpha}}$, then by Theorem 4.2, it has to coincide with $G_{\bar{\alpha}}$. In particular, we get that for every vertex u M_u contains $G_{\bar{\alpha}}$.

Let us choose a vertex u such that M_u contains $G_{\bar{\alpha}}$. By Proposition 4.1, M_u is everywhere dense in $\Gamma_{\bar{\alpha}}$. By Proposition 4.8 there exists a vertex v such that $(M_u)_v \ni c$. Thus, M_w contains a subgroup generated by c and $G_{\bar{\alpha}}$, where $w = uv$. This means that $M_w = \Gamma_{\bar{\alpha}}$. ■

Theorem 4.3. *No EGS-group contains everywhere dense proper subgroups.*

Proof. Indeed, there is a contradiction between Proposition 4.4 and Lemma 4.7, which proves the theorem. ■

5 Maximal subgroups

It is important to notice in studying maximal subgroups of our examples that all of them are just infinite p-groups for a prime p. Hence it is very easy to describe which of their maximal subgroups are normal. Those are exactly all subgroups of index p (for the suitable p).

Indeed, a normal maximal subgroup of either of those groups cannot be of infinite index, because the groups are just infinite, and so any their normal subgroup has finite index. On the other hand, suppose G is a p-group from one of the series 1.1, 1.2, or 1.3. Any maximal subgroup M of finite index in G contains a normal subgroup N of finite index. The factor-group G/N by that subgroup is a finite p-group, and the image of M/N in that factor-group is again a maximal subgroup. But the maximal subgroups of finite p-groups are exactly the subgroups of index p. Hence M has index p in the whole group. Reversing this argument shows that every subgroup of index p is maximal in G.

Moreover, being a finite p-group, G/N is nilpotent. Since every subgroup of a nilpotent group is subnormal, M/N, being of prime index, is normal in G/N. Thus, M is normal in G.

It follows that the existence of non-normal maximal subgroups in the groups under study is equivalent to the existence of maximal subgroups of infinite index. In Section 1.4 we mentioned that it is, in its turn, equivalent to the existence of proper everywhere dense subgroups. The following easy proposition provides a precise proof to this claim.

Proposition 5.1. *Any maximal subgroup M of infinite index in a group G is everywhere dense in the topology generated by all subgroups of finite index in G.*

Proof. Let N be a normal subgroup of finite index. M is maximal, therefore the difference $M \setminus N$ is non-empty, and $MN = G$. Hence,

$$\varepsilon_N(G) = G/N = MN/N \cong M/(M \cap N) = \varepsilon_N(M).$$

Since $\varepsilon_N(M) \leq \varepsilon_N(G)$, we have $\varepsilon_N(M) = \varepsilon_N(G)$. ■

Corollary 5.1. *Neither Grigorchuk's groups nor GGS-groups nor EGS-groups contain maximal subgroups of infinite index.*

Combining this corollary with the above discussion, we obtain our main result.

Theorem 5.1. *All maximal subgroups of Grigorchuk's 2-groups, GGS-groups, and EGS-groups are normal of prime index.*

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