High genus GW invariants of blowups along curves and surfaces

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Abstract:

In this paper, we established a relation between high genus Gromov-Witten invariants of a symplectic manifold M and its blowups along smooth curves and surfaces.

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1 Introduction

Over the last a few years, many mathematicians contributed their efforts to establish the mathematical foundation of the theory of quantum cohomology or GW(Gromov-Witten) invariants. In 1995, Ruan-Tian [R1, RT] first established for the semipositive symplectic manifolds. Recently the semipositivity condition has been removed by many authors. Now the focus is on the calculations and applications. Many Fano manifolds were computed. We think that it is important to study the change of GW invariants under surgery.

Some recent research indicated that there is a deep amazing relation between quantum cohomology and birational geometry. The quantum minimal model conjecture [R2, R3] leads to attempt to find quantum cohomology of a minimal model without knowing minimal model. This problem requires a thorough understanding of blowup type formula of GW invariants and quantum cohomology.

According to McDuff [M] the blowup operation in symplectic geometry amounts to a removal of an open symplectic ball followed by a collapse of some boundary directions. Lerman [L] gave a generalization of blowup construction, "the symplectic cut".

Let M be a compact symplectic manifold of dimension $2n$, $\tilde{M}$ be the blowup of M along a smooth submanifold. Denote by $p : \tilde{M} \longrightarrow M$ the natural projection. Denote by $\Psi^M_{(A,g)}(\alpha_1, \cdots, \alpha_m)$ the genus g GW invariant of M. The authors refer the interested reader to [RT] for the definition of GW invariants.

In [H1, H2], we considered the changes of genus zero GW invarints under blowup along a smooth curves and surfaces. In this paper, We will generalize some results about the cases of blowing up along curves and surfaces to the case of any genus. If the blown-up submanifolds are smooth curves with nonzero genus or special genus zero curves, we can show the following theorems:

Theorem 1: Suppose that C is a smooth curve in M such that either its genus $g_0 \geq 1$ or $g_0 = 0$ and $C_1(M)(C) \geq 0$, where $C_1(M)$ denotes the first Chern classes of M and its normal bundle respectively. $A \in H_2(M)$, $\alpha_i \in H^*(M)$, $1 \leq i \leq m$. Then

$$\Psi^M_{(A,g)}(\alpha_1, \ldots, \alpha_m) = \Psi^{\tilde{M}}_{(p!(A),g)}(p^*\alpha_1, \ldots, p^*\alpha_m)$$

About the changes of GW-invariants of blow-up of symplectic manifold along a smooth surface, in this paper, we will assume that the symplectic manifold is semipositive. Our result is

Theorem 2: Suppose that M is a semipositive compact symplectic manifold and S is a smooth surface in M. If $A \in H_2(M)$, $\alpha_i \in H^*(M)$, $1 \leq i \leq m$, satisfy either $deg\alpha_i > 2$ or $deg\alpha_i \leq 2$ and support away from S. Then

$$\Psi^M_{(A,g)}(\alpha_1, \cdots, \alpha_m) = \Psi^{\tilde{M}}_{(p!(A),g)}(p^*\alpha_1, \cdots, p^*\alpha_m),$$

where $p!(A) = PDp^*PD(A)$.

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2 Review of Gluing Formula

The proofs of our results depend strongly on the gluing formula for symplectic cutting developed by Li and Ruan [LR]. Let $\overline{M}^+$, $\overline{M}^-$ be the symplectic cuttings of M along $N \subset M$. Here N is the common boundary of $M^{\pm}$ with a

S^1-action and a generic level set of a local periodic hamiltonian function. The quotient $Z = N/S^1$ is the famous symplectic reduction, see [LR], p9-10 for details. Li and Ruan defined **Log GW invariants** $\Psi^{(\overline{M}^{\pm},Z)}$ of $\overline{M}^{\pm}$ relative Z by counting the number of log stable holomorphic maps intersecting Z at finitely many points with prescribed tangency. In this section, we want to extend Li and Ruan's gluing formula to the case with more ends. Li and Ruan have pointed out that their proof works in this case, see Remark 5.8 of [LR]. Therefore, here we only state the results.

Let Z_i, $i = 1, \cdots, l$, be real codimension two symplectic submanifolds of M. By a result of Guillemian and Sternberg [GS], the symplectic structure of each Z_i is modeled on a neighborhood of Z in either $\overline{M}^{+}$ or $\overline{M}^{-}$.

Let $K = \{0, \cdots, 0|k_1^{(1)}, \cdots, k_{\nu_1}^{(1)}| \cdots |k_1^{(l)}, \cdots, k_{\nu_l}^{(l)}\}$ with m-many zeros be a set of nonnegative integers with $k_j^i > 0$. Denote by $\mathcal{M}_A(g, Z_1, \cdots, Z_l, K)$ the space of genus g pseudo-holomorphic maps u, which represent the homology class A, such that u has marked points $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+\nu_1}, \cdots, x_{m+\sum_{k=1}^{l}\nu_k})$ with the property u is tangent to Z_i at $x_{m+\sum_{k=1}^{i-1}\nu_k+j}$ with order k_j^i and $f(x_j) \notin Z_i$ for $0 \leq j \leq m$ and $\sum_{j=1}^{\nu_i} k_j^{(i)} = Z_i^*(A)$, where Z_i^* is the Poincare dual to Z_i. We compactify $\mathcal{M}_A(g, Z_1, \cdots, Z_l, K)$ by $\overline{\mathcal{M}}_A(g, Z_1, \cdots, Z_l, K)$, the space of log stable maps. We have evaluation maps

$$e_k : \overline{\mathcal{M}}_A(g, Z_1, \cdots, Z_l, K) \longrightarrow M,$$

for marked points $x_1, \cdots, x_m$ and

$$e_j^i : \overline{\mathcal{M}}_A(g, z_1, \cdots, Z_l, K) \longrightarrow Z_i, \quad i = 1, \cdots, l, \quad j = 1, \cdots, \nu_i,$$

for marked points $x_{m+\sum_{k=1}^{i-1}\nu_k+j}$. Roughly the Log GW invariants may be defined as

$$\Psi_{(A,g,K)}^{(M,Z_1,\cdots,Z_l)}(\alpha_1, \cdots, \alpha_m, \beta_j^i) = \int_{\overline{\mathcal{M}}_A(g,Z_1,\cdots,Z_l,K)} \Pi_{k=1}^m e_k^* \alpha_k \wedge \Pi_{i,j}(e_j^i)^* \beta_j^i,$$

for $\alpha_i \in H^*(M)$, $\beta_j^i \in H^*(Z_i)$. To be precise, we need to construct a virtual neighborhood of $\overline{\mathcal{M}}_A(g, Z_1, \cdots, Z_l, K)$. Then we take the integrand over the normalization of the virtual neighborhood.

Suppose Z is a common real codimension two symplectic submanifold of $\overline{M}^{\pm}$. Next we will state the gluing formula of Log GW invariants with many ends. Clearly other ends do not influence our gluing when we are gluing along some end Z. Therefore, to simplify the notations, we just drop data from other ends when we write the gluing formula for some end Z

Given $u^{\pm} \in \mathcal{M}_{A^{\pm}}(\overline{M}^{\pm}, Z, Z_1^{\pm}, \cdots, Z_{l^{\pm}}^{\pm}, K^{\pm})$ where $K^{\pm}$ have $m^{\pm}$-many zeros respectively and $m = m^+ + m^-$. We consider the linearization of $\overline{\partial}$-operator

$$D_{u^{\pm}} = D\partial_J(u^{\pm}) : C^{\infty}((u^{\pm})^*TM^{\pm}) \longrightarrow \Omega^{0,1}((u^{\pm})^*TM^{\pm}).$$

If we choose a proper weighted Sobolev norm, we can define the Fredholm index $IndD_{u^{\pm}}$. see section 3.4 of [LR] for details. We also can glue u^+ and u^- along the end Z to obtain a pseudo-holomorphic map u in M. Suppose that $[u] = A$. Then

Proposition 2.1:([LR], Theorem 3.23)

$$IndD_{u^+} + IndD_{u^-} = 2(n-1)\nu + 2C_1(A) + 2(n-3)(1-g) + 2m,$$

where C_1 is the first Chern class of M.

Suppose that the homology classes of u^+, u^-, u are A^+, A^-, A respectively. If $(u^{+\prime}, u^{-\prime})$ is another representative and glue to u', by Lemma 2.11 of [LR], we have $[u'] = [u]$. Denote by $C = \{A^+, g^+, K^+; A^-, g^-, K^-\}$ the gluing component. The gluing formula counted the contribution of the gluing components to Log GW invariant of M. Denote by Ψ_C the contribution of C.

Choose a homology bass $\{\beta_b\}$ of $H^*(Z, R)$. Let (δ_{ab}) be its intersection matrix. There is a map

$$\pi : M \longrightarrow \overline{M}^+ \cup_Z \overline{M}^-,$$

where $\overline{M}^+ \cup_Z \overline{M}^-$ is the union of $\overline{M}^{\pm}$ along Z. For the gluing component $C = \{A^+, g^+, m^+, K^+; A^-, g^-, m^-, K^-\}$ where $K^{\pm} = (0, \cdots, 0|k_1, \cdots, k_\nu|(k_j^i)^{\pm})$ with $m^{\pm}$-many zeros, we have

Proposition 2.2: Let $\alpha_i^{\pm}$ be differential forms with $\deg \alpha_i^+ = \deg \alpha_i^-$ even. Suppose that $\alpha_i^+|_Z = \alpha_i^-|_Z$ and $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{M}^+ \cup_Z \overline{M}^-, R)$. Let $\alpha_i = \pi(\alpha_i^+ \cup_Z \alpha_i^-)$. We have the following gluing formula

$$\begin{aligned} \Psi_C^{(M, Z_i^+, Z_k^-)}(\alpha_1, \cdots, \alpha_m|(\beta_j^i)^+|(\beta_j^k)^-) &= |K| \sum_{I,J} \Psi_{(A^+, g^+, K^+)}^{(\overline{M}^+, Z, Z_i^+)}(\alpha_i^+, \beta_I, (\beta_j^i)^+) \\ &\times \ \delta^{I,J} \Psi_{(A^-, g^-, K^-)}^{(\overline{M}^-, Z, Z_k^-)}(\alpha_i^-, \beta_J, (\beta_j^k)^-), \end{aligned}$$

where we associate $\beta_i \delta^{ij} \beta_j$ to every intersection with Z and put $|K| = k_1 \cdots k_\nu$, $\delta^{I,J} = \delta^{i_1, j_1} \cdots \delta^{i_\nu, j_\nu}$. Denote by $\Psi_{(A^{\pm}, g^{\pm}, K^{\pm})}^{(\overline{M}^{\pm}, Z, Z_i^{\pm})}(\alpha_i^{\pm}, \beta_I, (\beta_j^i)^{\pm})$ the product of Log GW invariants corresponding to each component.

This Proposition is an easy extension of Theorem 5.8 of [LR] to Log GW invariant. Their proof works in this case. Therefore we omit the proof.

Proposition 2.3: For $C=\{A^{\pm},g^{\pm},(0,\cdots,0)\}$, we have

$$\Psi_C(\alpha_i^{\pm})=\Psi^{(\overline{M}^{\pm},Z)}_{(A^{\pm},g^{\pm},(0,\cdots,0))}(\alpha_i^{\pm}).$$

3 Proof of main theorem

Proof of Theorem 1: Since C is a smooth curve in M, the normal bundle N_C is a symplectic vector bundle. By symplectic neighborhood theorem, there is a tubular neighborhood $\mathcal{N}_\delta(C)$ of C which is symplectomorphic to the normal bundle N_C. We perform the symplectic cutting as in section 2.1 of [H1]. We obtained

$$\overline{M}^{+}=\mathbf{P}(N_C\oplus\mathcal{O}),\quad \overline{M}^{-}=\tilde{M}.$$

¿From the divisor property and skew symmetry of GW-invariants, without loss of generality, we may assume that $deg\alpha_i>2$, $1\leq i\leq m$. Therefore, if we choose a sufficiently small $\delta>0$, we may also assume $\alpha_i^{+}=0$.

Similar to the proof of Theorem 1.2 of [H1], we first consider the contribution of each component to the GW-invariants. Therefore, we consider the component

$$C=\{A^{+},g^{+},m^{+},K^{+};A^{-},g^{-},m^{-},K^{-}\}.$$

¿From Proposition 2.1, we have

$$IndD_{u^{+}}+IndD_{u^{-}}=2(n-1)\nu+2C_1(A)+2(n-3)(1-g)+2m.$$

We will use the same convention as [H1]. We assume $u^{\pm}:\Sigma^{\pm}\longrightarrow M^{\pm}$ may have $l^{\pm}$ connected components $u_i^{\pm}:\Sigma_i^{\pm}\longrightarrow M^{\pm}$, $i=1,\ldots,l^{\pm}$. Suppose $\Sigma_i^{\pm}$ have arithemetic genus $g_i^{\pm}$, $g^{\pm}=\Sigma g_i^{\pm}$, with $m_i^{\pm}$ marked points and $m^{\pm}=\Sigma m_i^{\pm}$. The same calculation as in the proof of Theorem 1.5 of [H1] shows

$$\begin{aligned}
IndD_{u^{+}} &\geq 2(n-3)(l^{+}-g^{+})+2(n-1)\sum k_i+2\nu+2m^{+},\\
IndD_{u^{-}} &\leq 2C_1(A)+(2n-6)(1-l^{+}+g^{+}-g)\\
&\quad -2(n-1)(\nu-\sum k_i)-2\nu+2m^{-}.
\end{aligned}$$

Since $\alpha_i^{+}=0$, $1\leq i\leq m$, if $m^{+}>0$, we have for any $\beta_b\in H^*(Z)$

$$\Psi^{(\overline{M}^{-},Z)}_{(A^{+},g^{+},m^{+},K^{+})}(\alpha_i^{+},\beta_b)=0.$$

This implies $\Psi_C=0$ except $m^{-}=m$. So we may assume $m^{-}=m$. We also may assume

$$\sum deg\alpha_i=2C_1(A)+2(n-3)(1-g)+2m.$$

Otherwise, Theorem 1 follows from the degree reason.

About the contribution of a gluing component, we have the following claim:

Claim: There exists a symplectic cutting such that the contribution $\Psi_C = 0$ excpt $C = \{A^-, g, m\}$. Here no $K^\pm$ means $K^\pm = \{0, \cdots, 0\}$.

First of all, we prove that $\Psi_C = 0$ for the gluing component C with $g^+ = 0$ and $\nu > 0$. In fact, we have

$$\begin{aligned}\sum deg(\alpha_i^-) &= 2C_1(A) + 2(n-3)(1-g) + 2m \\ &> 2C_1(A) + 2(n-3)(1-l^+ + g^+ - g) \\ &\quad +2(n-1)(\nu - \sum k_i) - 2\nu + 2m \\ &\geq IndD_{u^-},\end{aligned}$$

since $\nu > 0$, $k_i > 0$, $n \geq 3$. Therefore, by the definition of Log GW-invariants, we have for any $\beta_b \in H^*(Z)$

$$\Psi^{(\overline{M}^-, Z)}_{(A^-, g^-, m, K^-)}(\alpha_i^-, \beta_b) = 0.$$

Therefore, from Proposition 2.2, we have $\Psi_C = 0$.

Next, we consider the contribution of gluing components with $g^+ > 0$. From the connectness of stable maps and the assumption of our theorem, it follows that $g^+ > 0$ implies $\nu > 0$.Therefore, we only consider the component C with $g^+ > 0$ and $\nu > 0$. By the gluing formula, we have

$$\Psi_C = \sum_{I,J} |K| \Psi^{(\overline{M}^-, Z)}_{(A^-, g^-, K^-)}(\alpha_1^-, \cdots, \alpha_m^-, \beta_I)\delta^{I,J}\Psi^{(\overline{M}^+, Z)}_{(A^+, g^+, K^+)}(\beta_J).$$

We perform the symplectic cutting for $\overline{M}^+$ along the curve C. Then the same arguement as above shows

$$\begin{aligned}\Psi_C &= \sum_{I,J} |K| \Psi^{(\overline{M}^-, Z)}_{(A^-, g^-, K^-)}(\alpha_1^-, \cdots, \alpha_m^-, \beta_I)\delta^{I,J} \\ &\quad \times \sum_{i,j} |K_1| \Psi^{(\overline{(\overline{M}^+)}^-, Z, Z_1)}_{((A^+)^-, g_1^-, K_1^-)}(\beta_J, \beta_i)\delta^{i,j}\Psi^{(\overline{(\overline{M}^+)}^+, Z_1)}_{((A^+)^+, g_1^+, K_1^+)}(\beta_j) \\ &= \sum_{i,j} |K_1| \Psi^{(\overline{M}^-, Z_1)}_{(A_2^-, g_2^-, K_1^-)}(\alpha_1^-, \cdots, \alpha_m^-, \beta_i)\delta^{i,j}\Psi^{(\overline{M}^+, Z_1)}_{((A^+)^+, g_1^+, K_1^+)}(\beta_j) \\ &= \Psi_{C'}, \quad \text{for } C' = \{(A^+)^+, g_1^+, 0, K_1^+; A_2^-, g_2^-, m, K_1^-\}\end{aligned}$$

where $g_1^+ < g^+$. In the second equality we use the gluing formula and the fact that A_2^- is the gluing class of A^- and $(A^+)^-$, and the identifications

$\overline{M}^+ \cong \overline{\overline{M}^+}^+ \cong \mathbf{P}(N_C \oplus \mathcal{O})$ and the fact that the symplectic gluing of $\overline{M}^-$ and $(\overline{M^+})^-$ also recovers $\overline{M}^-$. Therefore, we may reduce the contribution of C to that of another component $C' = \{(A^+)^+, g_1^+, 0, K_1^+; A_2^-, g_2^-, m, K_1^-\}$ with $g_1^+ < g^+$ by choosing a smaller symplectic cutting. By the finiteness of the genus g^+, after finite steps, we may reduce the component C to the case $g^+ = 0$. Since we only have finite many gluing components with nonzero contribution, see [LR], so our claim follows from the argument of the case $g^+ = 0$ and $\nu > 0$.

So now it remains to show

$$\Psi^{\tilde{M}}_{(p!(A),g)}(p^*\alpha_1, \ldots, p^*\alpha_m) = \Psi^{(\overline{M}^-, Z)}_{(A^-, g, m)}(\alpha_1^-, \ldots, \alpha_m^-).$$

To prove this, we perform the symplectic cutting for $\tilde{M}$ around E. Therefore, we have

$$\overline{\tilde{M}}^+ = \mathbf{P}(N_E \oplus \mathcal{O}), \quad \overline{\tilde{M}}^- \cong \tilde{M}.$$

Now we use the gluing theorem to prove the contribution of stable J-holomorphic curves in $\tilde{M}$ which touch the exceptional divisor E to the GW-invariants of $\tilde{M}$ is zero. We consider the component

$$C = \{p!(A)^+, g^+, m^+, K^+; p!(A)^-, g^-, m^-, K^-\}.$$

Since $\alpha_i^+ = 0$, $1 \le i \le m$, we have $\Psi_C = 0$ except

$$C = \{p!(A)^+, g^+, 0, K^+; p!(A)^-, g^-, m, K^-\}.$$

The same index calculation as in the proof of Theorem 1.5 of [H1] shows

$$\begin{aligned} IndD_{u^+} &\ge 2(n-3)(l^+ - g^+) + 2(2n-3)\sum k_i + 2\nu \\ IndD_{u^-} &\le 2C_1(A) + (2n-6)(1 - l^+ + g^+ - g) \\ &\quad +(2n-2)(\nu - \sum k_i) - 2(n-2)\sum k_i + 2m. \end{aligned}$$

The same argument as in the first part of our proof shows that the contribution of J-holomorphic curves to the GW-invariant is nonzero only if it doesn't touch the exceptional divisor E, i. e. $C = \{p!(A)^-, g, m\}$. Therefore, we have

$$\Psi^{\tilde{M}}_{(p!(A),g)}(p^*\alpha_1, \ldots, p^*\alpha_m) = \Psi^{(\overline{\tilde{M}}^-, Z)}_{(p!(A)^-, g, m)}((p^*\alpha_1)^-, \ldots, (p^*\alpha_m)^-).$$

The rest of the proof is the same as that of the proof of Theorem 1.2 of [H1]. So we omit it here. This proves Theorem 1.

Proof of Theorem 2: Since S is a smooth surface, the normal bundle N_S is a symplectic vector bundle. By symplectic neighborhood theorem, there is a tubular neighborhood $\mathcal{N}_\delta(S)$ of S which is symplectomorphic to the normal bundle N_S. We perform the symplectic cutting as in section 2.1 of [H1]. We obtain

$$\overline{M}^+ = \mathbf{P}(N_S \oplus \mathcal{O}), \quad \overline{M}^- = \tilde{M}.$$

We may assume $\alpha_i^+ = 0$ if we choose a sufficiently small $\delta > 0$ because of the assumption of α_i.

Similar to the proof of Theorem 1, we first consider the contribution of each component to the GW-invariants. Therefore, we consider the component

$$C = \{A^+, g^+, m^+, K^+; A^-, g^-, m^-, K^-\}.$$

¿From Proposition 2.1, we have

$$IndD_{u^+} + IndD_{u^-} = 2(n-1)\nu + 2C_1(A) + 2(n-3)(1-g) + 2m.$$

We assume $u^\pm : \Sigma^\pm \longrightarrow \overline{M}^\pm$ has $l^\pm$ connected components $u_i^\pm : \Sigma_i^\pm \longrightarrow \overline{M}^\pm$, $i = 1, \ldots, l^\pm$. Suppose $\Sigma_i^\pm$ has arithemetic genus $g_i^\pm$, $g^\pm = \sum g_i^\pm$ with $m^\pm$ marked points and $m^\pm = \sum m_i^\pm$.

The same index calculation as in the proofs of Theorem 1.6 of [H1] and Theorem of [H2] shows

$$IndD_{u^+} = 2(n-3)(l^+ - g^+) + 2(n-2)\sum k_i + 2\nu + 2m^+.$$

Therefore, we have

$$\begin{aligned} IndD_{u^-} &= 2C_1(A) + 2(n-3)(1 - l^+ + g^+ - g) \\ &\quad + 2(n-2)(\nu - \sum k_i) + 2m^-. \end{aligned}$$

The same argument as in the proof of Theorem 1 shows that the contribution of the component C to the GW-invariant of M is nonzero only if C is the form

$$C = \{A^+, g^+, 0, K^+; A^-, g^-, m, K^-\}.$$

We also assume

$$\sum deg\alpha_i = 2C_1(A) + 2(n-3)(1-g) + 2m.$$

The same argument as in the proof of theorem 1 shows that there is a symplectic cutting such that $\Psi_C = 0$ except $C = \{A^-, g, m\}$. ¿From the gluing theorem, we have

$$\Psi^M_{(A,g)}(\alpha_1, \ldots, \alpha_m) = \Psi^{(\overline{M}^-, Z)}_{(A^-, g, m)}(\alpha_1^-, \ldots, \alpha_m^-).$$

Now it remains to prove

$$\Psi^{\tilde{M}}_{(p!(A),g)}(p^*\alpha_1,\ldots,p^*\alpha_m) = \Psi^{(\overline{M}^-,Z)}_{(A^-,g,m)}(\alpha_1^-,\ldots,\alpha_m^-).$$

To prove this, we perform the symplectic cutting for $\tilde{M}$ around E as in the proof of Theorem 1. Therefore, we have

$$\overline{\tilde{M}}^+ = \mathbf{P}(N_E \oplus \mathcal{O}), \quad \overline{\tilde{M}}^- \cong \tilde{M}.$$

We also use the gluing theorem to prove that the contribution of stable J-holomorphic curves in $\tilde{M}$ which touch the exceptional divisor E to the GW-invariant of $\tilde{M}$ is zero. We consider the component

$$C = \{p!(A)^+, g^+, m^+, K^+; p!(A)^-, g^-, m^-, K^-\}.$$

Since $\alpha_i^+ = 0$, $1 \leq i \leq m$, we have $\Psi_C = 0$ except

$$C = \{p!(A)^+, g^+, 0, K^+; p!(A)^-, g^-, m, K^-\}.$$

The similar calculation to that in the proof of Theorem 1 shows

$$\begin{aligned} IndD_{u^+} &= 2(n-3)(l^+ - g^+) + 2\nu + 2(2n-5)\sum k_i, \\ IndD_{u^-} &= 2C_1(A) + 2(n-3)(1 - l^+ + g^+ - g) \\ &\quad +2(n-2)(\nu - \sum k_i) - 2(n-3)\sum k_i + 2m. \end{aligned}$$

The same argument as in the proof of Theorem 1 shows

$$\Psi^{\tilde{M}}_{(p!(A),g)}(p^*\alpha_1,\ldots,p^*\alpha_m) = \Psi^{(\overline{\tilde{M}}^-,Z)}_{(p!(A)^-,g,m)}((p^*\alpha_1)^-,\ldots,(p^*\alpha_m)^-).$$

The rest of the proof is the same as that of the proof Theorem 1.2 of [H1]. So we omit it. This completes the proof of Theorem 2.

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