ON THE 2-TYPE OF AN ITERATED LOOP SPACE

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Let $(\Omega^n X)_0$ be the path component of the n-fold iterated loop space of a space X and let L^n be the functor which carries X to the 2-type of $(\Omega^n X)_0$. It is well known that the space $L^n(X)$ for $n \geq 2$ splits as a product of Eilenberg-MacLane spaces

$$(*) \qquad L^n(X) \simeq K(\pi_{n+1}X, 1) \times K(\pi_{n+2}X, 2);$$

see for example Arlettaz [1]. We give an algebraic proof of this fact in (5.8). Is it possible to choose the homotopy equivalence (*) natural in X? As a main result we prove that this is not possible. We identify algebraically the associated obstruction, which is non-trivial on the subcategory consisting of all spaces X which are one point unions of $(n+1)$-dimensional spheres. The method of proof relies on the description of algebraic functors λ^n which are equivalent to the functors L^n above. For this we use the crossed module $\lambda(Y)$ of a space Y which is an algebraic model of the 2-type of Y. This yields the functor λ^n by the crossed module

$$\lambda^n(X) = \lambda(\Omega^n X)_0$$

of $(\Omega^n X)_0$. We show that the functor λ^n is determined by the boundary d_{n+1} of the Moore chain complex of a simplicial group G associated to X. Moreover for $n \geq 1$ the functor λ^n is equivalent to a functor $\delta\overline{\lambda}^n$ where $\overline{\lambda}^n(X)$ is a 'reduced quadratic module' which is stable for $n \geq 2$. These results are used in [5] for the construction of algebraic models of certain homotopies and homotopy types.

§1 The Obstruction

Let $\underline{\underline{C}}$ be a small category and let $D : \underline{\underline{C}}^{op} \times \underline{\underline{C}} \to \underline{\underline{Ab}}$ be a bifunctor to the category of abelian groups. A <u>derivation</u> $\Delta : \underline{\underline{C}} \to D$ is function which carries each morphism $f : A \to B$ in $\underline{\underline{C}}$ to an element $\Delta(f) \in D(A, B)$ such that for a composition gf of morphisms in $\underline{\underline{C}}$ one has

$$\Delta(gf) = g_*\Delta(f) + f^*\Delta(g).$$

This is an <u>inner derivation</u> if there exists a function ∇ which carries each object A in $\underline{\underline{C}}$ to an element $\nabla(A) \in D(A, A)$ such that

$$\Delta(f) = f_* \nabla(A) - f^* \nabla(B).$$

Let $Der(\underline{\underline{C}}, D)$ and $Ider(\underline{\underline{C}}, D)$ be the sets of derivations and inner derivations respectively. By adding pointwise these sets are abelian groups and the quotient group

$$H^1(\underline{\underline{C}}, D) = Der(\underline{\underline{C}}, D)/Ider(\underline{\underline{C}}, D)$$

is the first cohomology of $\underline{\underline{C}}$ with coefficients in D; compare IV. 7.6 in [2]. The obstruction for the natural splittability of the functor L^n in the introduction is canonically an element in such a cohomology group.

Let $\underline{\underline{spaces}}$ be the category of connected CW-spaces with basepoint and basepoint preserving maps. Moreover let $\underline{\underline{C}}$ be a subcategory of the homotopy category $\underline{\underline{spaces}} / \simeq$. We obtain a bifunctor

$$H^2(\pi_{n+1}, \pi_{n+2}) : \underline{\underline{C}}^{op} \times \underline{\underline{C}} \to \underline{\underline{Ab}} \tag{1.2}$$

which carries (X, Y) to the second cohomology $H^2(\pi_{n+1}X, \pi_{n+2}Y)$ of the group $\pi_{n+1}X$ with coefficients in $\pi_{n+2}Y$. Recall that the functor $L^n : \underline{\underline{C}} \to \underline{\underline{\text{spaces}}}/ \simeq$ carries a space X in $\underline{\underline{C}}$ to the 2-type of $(\Omega^n X)_0$.

(1.3) Lemma. *Let $n \geq 2$. Then the functor $L^n : \underline{\underline{C}} \to \underline{\underline{spaces}}/ \simeq$ admits for $X \in \underline{\underline{C}}$ a natural splitting*

$$L^n(X) \simeq K(\pi_{n+1}X, 1) \times K(\pi_{n+2}X, 2)$$

if and only if an obstruction element

$$O(L^n|\underline{\underline{C}}) \in H^1(\underline{\underline{C}}, H^2(\pi_{n+1}, \pi_{n+2}))$$

vanishes.

Proof. There is a fibration sequence

$$K(\pi_{n+2}X, 2) \xrightarrow{i} L^n X \xrightarrow{q} K(\pi_{n+1}X, 1)$$

which is natural in X obtained by the Postnikov tower for $L^n X$. Let

$$s_X : K(\pi_{n+1}X, 1) \to L^n X$$

be a map with $qs_X \simeq 1$. Such a map exists for $n \geq 2$, [1]. Then we get for $f : X \to Y \in \underline{\underline{C}}$ the diagram in $\underline{\underline{\text{spaces}}} / \simeq$

$$\begin{array}{ccc} L^n X & \xleftarrow{s_X} & K(\pi_{n+1}X, 1) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (\pi_{n+1}f)_*} \\ L^n Y & \xleftarrow{s_Y} & K(\pi_{n+1}Y, 1) \end{array}$$

which needs not to commute. The difference element

$$\Delta'(f) = f_* s_X - s_Y(\pi_{n+1} f)_*$$

obained by loop addition in $L^n Y$ satisfies $q\Delta'(f) = 0$. Thus there is a unique element

$$\Delta(f) \in [K(\pi_{n+1}X, 1), K(\pi_{n+2}Y, 2)] = H^2(\pi_{n+1}X, \pi_{n+2}Y)$$

with $i\Delta(f) = \Delta'(f)$. One readily checks that Δ is a derivation. Choosing different splittings s_X alters Δ only by an inner derivation. Hence we obtain a well defined cohomology class $O(L^n|\underline{\underline{C}}) = \{\Delta\}$ with the property in (1.3). q.e.d.

For an abelian group A let $\Lambda^2(A) = A \otimes A/\sim$ be the exterior square obtained by $a \otimes a \sim o$ and let $\hat{\otimes}^2(A) = A \otimes A/\approx$ be obtained by $a \otimes b + b \otimes a \approx 0$. If A is free abelian we have the short exact sequence

$$(1.4) \qquad 0 \to A \otimes \mathbb{Z}/2 \xrightarrow{i} \hat{\otimes}^2 A \xrightarrow{q} \Lambda^2 A \to 0$$

which is natural in A. Here q is the quotient map and i carries $a \otimes 1$ to $\{a \otimes a\}$. Let $\underline{\underline{ab}}$ be the category of finitely generated free abelian groups and for functors $F, G : \underline{\underline{ab}} \to \underline{\underline{Ab}}$ let $Ext^n_{\underline{\underline{ab}}}(F, G)$ be the group of extensions in the category of functors $\underline{\underline{ab}} \to \underline{\underline{Ab}}$ with natural transformations as morphisms. Then (1.4) represents the element

$$(1.5) \qquad \left\{\hat{\otimes}^2\right\} \in Ext^1_{\underline{\underline{ab}}}(\Lambda^2, \otimes\mathbb{Z}/2)$$

Here the right hand side is a cyclic group of order 2 as follows from

(1.6) Lemma. *The element* $\left\{\hat{\otimes}^2\right\}$ *is the generator in* $Ext^1_{\underline{\underline{ab}}}(\Lambda^2, \otimes\mathbb{Z}/2) = \mathbb{Z}/2$

Proof. We write $F = \otimes\mathbb{Z}/2$. Then (1.4) yields the long exact sequence

$$Hom\left(\hat{\otimes}^2, F\right) \to Hom(F, F) \to Ext^1(\Lambda^2, F) \to Ext^1\left(\hat{\otimes}^2, F\right)$$

Let $SP^2(A)$ be the symmetric square of A. Then we have for $A \in \underline{\underline{ab}}$ the natural short exact sequence

$$0 \to SP^2(A) \to \otimes^2 A \to \hat{\otimes}^2 A \to 0$$

which yields by (2.15) in [17] the isomorphism

$$Hom(SP^2, F) = Ext^0(SP^2, F) \cong Ext^1\left(\hat{\otimes}^2, F\right)$$

We now use the theory of quadratic modules in [4] to show that $Hom\left(\hat{\otimes}^2, F\right) = 0$ and $Hom(SP^2, F) = 0$. In fact, the quadratic modules associated to $\hat{\otimes}^2, F, SP^2$ are

$$\begin{array}{rcllllll} \hat{\otimes}^2 & = & (& \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z}/2 &) \\ F & = & (& \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 &) \\ SP^2 & = & (& \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} &) \end{array}$$

Compare [4]. Since $Hom(F, F) = \mathbb{Z}/2$ we obtain the proposition. The element $\left\{\hat{\otimes}^2\right\}$ is non trivial since there exists no retraction of $F \to \hat{\otimes}^2$ by $Hom\left(\hat{\otimes}^2, F\right) = 0$.
q.e.d.

By 3.11 in [17] we obtain the natural isomorphism

$$\chi : Ext^1_{\underline{\underline{ab}}}(\Lambda^2, \otimes\mathbb{Z}/2) = H^1(\underline{\underline{ab}}, Hom(\Lambda^2, \otimes\mathbb{Z}/2)). \tag{1.7}$$

Now let $\underline{\underline{C}} = \underline{\underline{S}}^{n+1}$ be the full homotopy category of one point unions of $(n+1)$-dimensional spheres. Then homology yields on isomorphism of categories $(n \geq 1)$

$$\underline{\underline{S}}^{n+1} = \underline{\underline{ab}}$$

which we use as an identification. Moreover the homotopy groups

$$\pi_{n+1}, \pi_{n+2} : \underline{\underline{ab}} = \underline{\underline{S}}^{n+1} \to \underline{\underline{Ab}}$$

carry $A \in \underline{\underline{ab}}$ to $\pi_{n+1}A = A$ and $\pi_{n+2}A = A \otimes \mathbb{Z}/2$. It is classical that for $A, B \in \underline{\underline{ab}}$ we have the binatural isomorphism

$$H^2(A, B \otimes \mathbb{Z}/2) = Hom(\Lambda^2 A, B \otimes \mathbb{Z}/2) \tag{1.8}$$

Hence the obstruction in (1.3) for $\underline{\underline{C}} = \underline{\underline{S}}^{n+1}$ is an element

$$O(L^n|\underline{\underline{S}}^{n+1}) \in H^1(\underline{\underline{ab}}, Hom(\Lambda^2, \otimes\mathbb{Z}/2))$$

where the right hand side is a cyclic group of order 2 by (1.7) and (1.6).

(1.9)<u>Theorem</u>. *The obstruction element $O(L^n|\underline{\underline{S}}^{n+1})$ is nontrivial. In fact, we have the equation*

$$O(L^n|\underline{\underline{S}}^{n+1}) = \chi\left\{\hat{\otimes}^2\right\}$$

where χ is the isomorphism in (1.7) and where $\hat{\otimes}^2$ is the extension element in (1.5).

We shall prove this result in §7 below.

§2 THE CROSSED MODULE OF A SPACE

Let $\underline{\underline{Gr}}$ be the category of groups and let $\underline{\underline{ab}}$ be the full subcategory of abelian groups. Let N and M be groups. An N-group (or an action of N on M) is a homomorphism h from N to the group of automorphisms of M. For $x \in M$, $\alpha \in N$ we denote the action by $x^\alpha = h(\alpha^{-1})(x)$. The action is trivial if $x^\alpha = x$ for all x, α. A crossed module $\partial : M \to N$ is a homomorphism in $\underline{\underline{Gr}}$ together with an action of N on M such that for $x, y \in M$, $\alpha \in N$ we have

$$(2.1) \qquad \begin{cases} \partial(x^\alpha) & = \alpha^{-1} x \alpha \\ x^{\partial y} & = y^{-1} x y \end{cases}$$

We say that the crossed module ∂ is free in degree 1 if N is free group. A morphism $\partial \to \partial'$ between crossed modules is a commutative diagram in $\underline{\underline{Gr}}$.

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ N & \xrightarrow[f]{} & N' \end{array}$$

where g is f-equivariant, that is $g(x^\alpha) = (gx)^{f(\alpha)}$. This is a weak equivalence if (f, g) induces isomorphisms $\pi_i(\partial) \cong \pi_i(\partial')$ for $i = 1, 2$ where

$$\begin{cases} \pi_1(\partial) & = \text{cokernel } (\partial) \\ \pi_2(\partial) & = \text{kernel } (\partial) \end{cases}$$

We point out that for ∂' there is always a weak equivalence $(g, f) : \partial \to \partial'$ where ∂ is free in degree 1.

Let $\underline{\underline{cross}}$ be the category of crossed modules and let $Ho(\underline{\underline{cross}})$ be its localization with respect to weak equivalences. It is well known ([20], [18], [3]) that there is an equivalence of categories

$$(2.2) \qquad \underline{\underline{types}}(2) \xrightarrow{\sim} Ho(\underline{\underline{cross}})$$

where the category $\underline{\underline{types}}$ (2) is the homotopy category of connected CW-spaces X with $\pi_i X = 0$ for $i \geq 3$. For any connected CW-space X we obtain its 2-type $P_2(X)$ by the second stage of the Postnikov tower of X. This yields the functor

$$(2.3) \qquad P_2' : \underline{\underline{spaces}}/\simeq \xrightarrow{P_2} \underline{\underline{types}}(2) \xrightarrow{\sim} Ho(\underline{\underline{cross}})$$

where $\underline{\underline{spaces}}/\simeq$ is the homotopy category of connected CW-spaces with basepoint. We now define the functor

$$(2.4) \qquad \lambda : \underline{\underline{spaces}} \longrightarrow \underline{\underline{cross}}$$

which induces a functor $Ho(\lambda)$ between homotopy categories such that $Ho(\lambda)$ is naturally isomorphic to the functor P_2' in (2.3). For a pointed connected CW-space

X let SX be the singular simplicial set of all simplexes $\Delta^n \to X$ which carry the 0-skeleton of Δ^n to the basepoint of X. Let $Y^n = |SX|^n$ be the n-skeleton of the realization $Y = |SX|$ and let

$$\pi_3(Y^3, Y^2) \xrightarrow{d_3} \pi_3(Y^2, Y^1) \xrightarrow{d_2} \pi_1(Y^1)$$

be part of the crossed chain complex of Y; see for example [3]. Then d_2 induces the crossed module

$$\lambda(X) : \text{cokernel } d_3 \to \pi_1(|SX|^1)$$

which is a functor in X.

(2.5) Definition. Let $\underline{\underline{C}}$ be a category with weak equivalences, let $\underline{\underline{K}}$ be a category and let $\lambda, \lambda' : \underline{\underline{K}} \to \underline{\underline{C}}$ be functors. We say that λ is equivalent to λ' if there exists a natural transformation $\tau : \lambda \to \lambda'$ such that $\tau_X : \lambda(X) \to \lambda'(X)$ is a weak equivalence in $\underline{\underline{C}}$ for all objects $X \in \underline{\underline{K}}$. More generally we say that λ is equivalent to λ' if there exists a finite chain of equivalences $\lambda \leftarrow \lambda_0 \to \lambda_1 \leftarrow \lambda_2 \ldots \leftarrow \lambda'$.

We shall construct simpler functors which are equivalent to the functor $(n \geq 0)$

$$\lambda^n : \underline{\underline{\text{spaces}}} \to \underline{\underline{\text{cross}}} \tag{2.6}$$

where λ^n carries a space X to the crossed module $\lambda(\Omega^n X)_0$.

§ 3 Crossed modules associated to simplicial groups

For a simplicial group G we define the Moore chain complex NG by

$$\begin{aligned} N_n(G) &= \bigcap_{i<n} \text{kernel}(d_i) \\ \partial_n : N_n(G) \to N_{n-1}(G),\ \partial_n &= \text{restriction of } d_n \end{aligned} \tag{3.1}$$

Here d_i, $0 \leq i \leq n$, are the face maps in G. The degeneracy maps in G are denoted by s_i. The subgroup image(∂_{n+1}) is normal in kernel(∂_n) so that the quotient group

$$\pi_n(G) = \frac{\text{kernel}\,\partial_n}{\text{image}\,\partial_{n+1}}$$

is defined. A map $f : G \to G'$ between simplicial groups is a weak equivalence if $f_* : \pi_i(G) \cong \pi_i(G')$ is an isomorphism for all i. Moreover ∂_n induces the exact sequence of groups

$$0 \to \pi_n(G) \to \text{cokernel}(\partial_{n+1}) \xrightarrow{\partial_n(G)} \text{kernel}(\partial_{n-1}) \to \pi_{n-1}(G) \to 0$$

(3.2) Lemma. *The homomorphism $\bar{\partial}_n(G)$ has the natural structure of a crossed module for $n \geq 1$.*

Proof. We define the action of $\alpha \in \text{kernel}(\partial_{n-1})$ on $\{y\} \in \text{cokernel}(\partial_{n+1})$, $y \in N_n(G)$, by

$$\{y\}^\alpha = \{s_{n-1}(\alpha)^{-1} y\, s_{n-1}(\alpha)\}$$

Since $d_n\, s_{n-1} = \text{identity}$ we have $\partial_n\{y\}^\alpha = \alpha^{-1} y \alpha$. Moreover we observe that for $x, y \in N_n(G)$ the element [9]

$$\{x, y\}_{n+1} = s_n(x^{-1}y^{-1}x)(s_{n-1}x)^{-1}(s_n y)(s_{n-1}x)$$

lies in $N_{n+1}(G)$ with

$$\partial_{n+1}\{x, y\}_{n+1} = x^{-1}y^{-1}x(s_{n-1}\partial_n x)^{-1}y(s_{n-1}\partial_n x)$$

Hence we obtain for $\alpha = \partial x$ the equation $\{y\}^{\partial x} = \{x^{-1} y\, x\}$ and therefore $\bar{\partial}_n = \bar{\partial}_n(G)$ is a crossed module. q.e.d.

Let $s\,\underline{\underline{Gr}}$ be the category of simplicial groups and let $(s\,\underline{\underline{Set}})_0$ be the category of simplicial sets K with $K_0 = *$. There are pairs of adjoint functors

$$(3.3) \qquad \underline{\underline{spaces}} \overset{|\,|}{\underset{S}{\leftrightarrows}} (s\,\underline{\underline{Set}})_0 \overset{G}{\underset{\bar{W}}{\rightleftarrows}} s\,\underline{\underline{Gr}}$$

together with adjunction maps which are weak equivalences. Here S is the reduced singular set in (2.4) and $|\;\;|$ is the realization. Moreover G and $\bar{W}$ are the functors of Kan; compare for example [10] and [22]. The functors in (3.3) induce equivalences of categories

$$\underline{\underline{spaces}}/\simeq \sim Ho(s\,\underline{\underline{Set}})_0 \sim Ho(s\,\underline{\underline{Gr}})$$

where Ho denotes the localization with respect to weak equivalences. Using (3.2) we obtain the functor

$$(3.4) \qquad \bar{\partial}_n : s\,\underline{\underline{Gr}} \to \underline{\underline{cross}}$$

which we compare with λ^{n-1} in (2.6) as follows.

(3.5) Theorem. *For $n \geq 1$ there are equivalences between functors:*

$$\lambda^{n-1} \sim \bar{\partial}_n\, GS : \underline{\underline{spaces}} \longrightarrow \underline{\underline{cross}}$$
$$\bar{\partial}_n \sim \lambda^{n-1}|\,|\bar{W} : s\,\underline{\underline{Gr}} \longrightarrow \underline{\underline{cross}}$$

The result generalizes the classical natural isomorphism

$$\begin{array}{ccccc} \pi_1(\Omega^{n-1}X)_0 & \cong & \pi_n X & \cong & \pi_{n-1}(GSX) \\ \| & & & & \| \\ \pi_1\lambda^{n-1}X & & & & \pi_1\,\bar{\partial}_n(GSX) \end{array}$$

Proof. We use functors P_n, $Q_n : s\,\underline{\underline{Gr}} \to s\,\underline{\underline{Gr}}$ together with natural transformations

$$Q_n(G) \xrightarrow{i} G \xrightarrow{p} P_n G$$

Here the Moore chain complexes are given by the following diagram

$$\begin{array}{ccccccccc} N\,Q_n(G) = (\ldots & \longrightarrow & N_{n+1}G & \longrightarrow & \ker\partial_n & \longrightarrow & 0 & \longrightarrow & \ldots) \\ \downarrow{\scriptstyle i_*} & & \downarrow\| & & \downarrow{\scriptstyle i} & & \downarrow & & \\ NG = (\ldots & \longrightarrow & N_{n+1}G & \longrightarrow & N_nG & \longrightarrow & N_{n-1}G & \longrightarrow & \ldots) \\ \downarrow{\scriptstyle p_*} & & \downarrow & & \downarrow{\scriptstyle p} & & \| & & \\ N\,P_n(G) = (\ldots & \longrightarrow & 0 & \longrightarrow & \operatorname{cok}\partial_{n+1} & \longrightarrow & N_{n-1}G & \longrightarrow & \ldots) \end{array}$$

The map i is the inclusion and p is the quotient map. The simplicial group $Q_n(G)$ is the simplicial subgroup of G generated by the subset $N\,Q_n(G) \subset G$. Moreover the functor p corresponds to the projection on page 227 of [8]. We clearly have

$$\begin{aligned} \bar{\partial}_n(G) &= \bar{\partial}_n(Q_{n-1}G) = \bar{\partial}_n(P_n(G)) \\ &= \bar{\partial}_n(Q_{n-1}P_n(G)) \end{aligned}$$

Therefore it suffices to construct an equivalence $\bar{\partial}_n(G) \sim \lambda^{n-1}|\bar{W}G|$ for simplicial groups G with $N_iG = 0$ for $i \neq n,\ n-1$. Such simplicial groups are classified for $n \geq 3$ by '<u>stable crossed modules</u>' M (see 3.4 [9]), and for $n = 2$ by a '<u>crossed module M of length 2</u>' which is <u>reduced</u>, that is $M_0 = 0$. Hence by the construction $\tilde{G}$ of [9] we have $G = \tilde{G}(M,n)$ for $n \geq 2$. Here $\bar{\partial}_n(\tilde{G}(M,n)) = \partial_M$ coincides with the underlying differential in M. By 3.6 [9] we know that for $n \geq 3$ the functors $M \longmapsto \tilde{G}(M, n-1)$ and $M \longmapsto G\,\tilde{G}(M,n)$ are equivalent. Similarly for $n = 2$ the functors $M \longmapsto G\,\tilde{G}(M,2)$ and $M \longmapsto \tilde{G}(\partial_M, 1)$ are equivalent. For $n = 1$ the proposition of (3.5) is well known; see for example 2.2.4 [8]. Moreover for $n \geq 2$ proposition (3.5) is a consequence of the equivalences above since for $U \in (s\,\underline{\underline{Set}})_0$ the functors $U \longmapsto |GU|$ and $U \longmapsto \Omega|U|$ are equivalent. q.e.d.

§ 4 Reduced and stable quadratic modules

A <u>reduced quadratic module</u> (ω, δ) is a diagram

$$M^{ab} \otimes M^{ab} \xrightarrow{\omega} L \xrightarrow{\delta} M \tag{4.1}$$

of homomorphism between groups such that the following properties hold. The group M has nilpotency degree 2 and the quotient map $M \twoheadrightarrow M^{ab}$ to the abelianization M^{ab} of M is denoted by $x \longmapsto \{x\}$. The composition $\delta\omega = w$ is the commutator map, that is

$$\delta\omega(\{x\} \otimes \{y\}) = x^{-1}y^{-1}xy$$

for $x, y \in M$. For $a \in L$, $x \in M$ we have

$$\omega(\{\delta a\} \otimes \{x\} + \{x\} \otimes \{\delta a\}) = 0$$

Commutators in L satisfy the formula $(a, b \in L)$

$$\omega(\{\delta a\} \otimes \{\delta b\}) = a^{-1}b^{-1}ab$$

We say that (ω, δ) is a stable quadratic module if in addition

$$\omega(\{x\} \otimes \{y\} + \{y\} \otimes \{x\}) = 0$$

is satisfied for $x, y \in M$. We say that (w, d) is free in degree 1 if $M = G/\Gamma_3 G$ where G is a free group and $\Gamma_3 G$ is the subgroup of triple commutators in G. A map $(l, m) : (\omega, \delta : L \to M) \to (\omega', \delta' : L' \to M')$ is a pair of homomorphism $l : L \to L'$, $m : M \to M'$ with $m\delta = \delta' l$ and $l\omega = \omega'(m^{ab} \otimes m^{ab})$. Let $\underline{\underline{rquad}}$ (resp. $\underline{\underline{squad}}$) be the corresponding catgeory of reduced (resp. stable) quadratic modules. We obtain a faithful functor

$$\delta : \underline{\underline{rquad}} \to \underline{\underline{cross}} \tag{4.2}$$

which carries (ω, δ) to the associated crossed module $\delta : L \to M$ with the action of $x \in M$ on $a \in L$ given by

$$a^x = a \cdot \omega(\{\delta a\} \otimes \{x\}).$$

One readily checks that δ is a well defined functor. A map in $\underline{\underline{rquad}}$ (resp. $\underline{\underline{squad}}$) is a weak equivalence if the induced map is $\underline{\underline{cross}}$ is a weak equivalence. Each object in rquad (resp. $\underline{\underline{squad}}$) is weakly equivalent to an object which is free in degree 1.

(4.3) Theorem. *For $n = 2$, resp. $n \geq 3$, there is a functor*

$$\mu_n : s\,\underline{\underline{Gr}} \to \underline{\underline{rquad}}, \text{ resp. } \underline{\underline{squad}},$$

such that $\delta\mu_n$ is equivalent to $\bar{\partial}_n$ in (2.4).

Proof. For $G' \in s\,\underline{\underline{Gr}}$ we obtain the free simplicial group $G'' = G\bar{W}(G')$. Let M be given by $Q_{n-1}P_n G'' = \tilde{G}(M, n)$ as in the proof of (3.5). There exists a weak equivalence $M \to M/P_3$ where M/P_3 is a reduced quadratic module obtained from M by dividing out triple Peiffer commutators; see IV.B.11 in [3]. Then μ_n carries G' to M/P_3 and the weak equivalences $M \to M/P_3$ and $G\bar{W}(G') \to G'$ induce natural weak equivalences

$$\bar{\partial}_n G' \xleftarrow{\sim} \bar{\partial}_n(G'') = \partial_M \xrightarrow{\sim} \delta(M(P_3)) = \delta\mu_n(G')$$

in *cross*. q.e.d.

(4.4) **Corollary.** *For $n = 1$, resp. $n \geq 2$, there is a functor*

$$\bar{\lambda}^n : \mathit{spaces} \to \mathit{rquad}(\text{resp. } \mathit{squad})$$

such that $\delta\bar{\lambda}^n$ is equivalent to λ^n in (2.6).

Proof. Let $\bar{\lambda}^n(X) = \mu_{n+1}(GSX)$. Then the corollary follows from (3.5) and (4.3). q.e.d.

On the level of homotopy categories the functors δ and $\overline{\lambda}^n$ are part of the commutative diagram in (4.7) below.

Let *types*$(n+2)$ be the homotopy category of CW- spaces X with $\pi_i X = 0$ for $i \leq n$ and $i > n+2$. Moreover let $\underline{k}(n)$ be the following algebraic category, $n \geq 0$.

$$\underline{k}(n) = \begin{cases} \mathit{cross} & n = 0 \\ \mathit{rquad} & n = 1 \\ \mathit{squad} & n \geq 2 \end{cases}$$

It is proved in [3] that there is an equivalence of categories:

$$\overline{\lambda}_n : \mathit{types}(n+2) \xrightarrow{\sim} Ho\underline{k}(n) \tag{4.5}$$

which for $n = 0$ is induced by λ in (2.4) and which for $n \geq 1$ is induced by $\overline{\lambda}^n$ in (4.4). The functor L^n in the introduction with $L^n(X) = P_2(\Omega^n X)_0$. has a factorization

$$L^n : \mathit{spaces}/\simeq \xrightarrow{P^n} \mathit{types}(n+2) \xrightarrow{\Omega^n} \mathit{type}(2) \tag{4.6}$$

Here P^n carries a space X to the $(n+2)$-type of the n-connected cover of X; see [24]. Moreover the following diagram of functors commuts up to natural isomorphism of functors, $n \geq 0$, with $L^n = \Omega^n P^n$.

$$\begin{array}{ccccc} \mathit{spaces}/\simeq & \xrightarrow{P^n} & \mathit{types}(n+2) & \xrightarrow{\Omega^n} & \mathit{types}(2) \\ \| & & \downarrow\sim & & \downarrow\sim \\ \mathit{spaces}/\simeq & \xrightarrow{\bar{\lambda}^n} & Ho(\underline{k}(n)) & \xrightarrow{\delta} & Ho(\mathit{cross}) \end{array} \tag{4.7}$$

Here the equivalences of categories show that Ω^n restricted to *types*$(n+2)$ can be replaced by the algebraic functor δ in the bottom row. The diagram shows that the obstruction element (1.3) satisfies

$$O(L^n|\underline{\underline{spaces}}/\simeq) = (P^n)^* O(\Omega^n|\underline{\underline{types}}(n+2))$$

$$(4.8) \qquad = (\overline{\lambda}^n)^* O(\delta)$$

where $O(\delta)$ is defined by the functor δ in (4.7) similarly as $O(L^n|\underline{\underline{C}})$ in (1.3). The advantage is that $O(\delta)$ can be computed algebraically.

§ 5 k-INVARIANTS

Each space X in the category $\underline{\underline{types}}(n+2)$, $n \geq 0$, determines a $\underline{k\text{-invariant}}$ k_X which is an element in the cohomology group of an Eilenberg-MacLane space:

$$(5.1) \qquad H^{n+3}(K(\pi_{n+1}, n+1), \pi_{n+2}) = \begin{cases} H^3(K(\pi_1, 1), \pi_2), & n = 0 \\ Hom(\Gamma\pi_2, \pi_3), & n = 1 \\ Hom(\pi_{n+1} \otimes \mathbb{Z}/2, \pi_{n+2}), & n \geq 2 \end{cases}$$

Here $\pi_i = \pi_i(X)$ is the homotopy group of X which is a $\pi_1 X$ -module. the computation of the cohomology group for $n \geq 1$ was achieved by Eilenberg-Mac Lane [13]. Here Γ is Whitehead's quadratic functor. The k-invariant k_X determines the homotopy type of X by the following classical result:

(5.2) <u>Lemma</u>. *Let $n \geq 0$ and let X, Y be spaces in $\underline{\underline{types}}(n+2)$ with homotopy groups $\pi_i = \pi_i(X), \pi_i' = \pi_i(Y)$. Then there is a homotopy equivalence $X \simeq Y$ if and only if there exists isomorphisms $\varphi_{n+1} : \pi_{n+1} \cong \pi'_{n+1}$ and $\varphi_{n+2} : \pi_{n+2} \cong \pi'_{n+2}$ (where φ_{n+2} is φ_{n+1}-eqivariant for $n = 0$) such that*

$$(\varphi_{n+2})_* k_X = (\varphi_{n+1})^* k_Y.$$

Moreover each element k in the cohomology (5.1) above is the k-invariant $k = k_X$ of a space X in $\underline{\underline{types}}(n+2)$.

Using the equivalence of categories in (4.5) an object X in $\underline{\underline{types}}(n+2)$ is completly determined by the object $A = \overline{\lambda}^n(X)$ in the algebraic category $\underline{\underline{k}}(n)$. Hence the k-invariant $k_X = k_A$ has to be computable in terms of A; this can be done as follows.

(5.3) <u>The k-invariant of a crossed module</u>

(Compare [20] , [14] or [15]) Let $\partial : L \longrightarrow M$ be a crossed module which is free in degree 1. Let $u : \pi_1(\partial) \longrightarrow M$ be a normalized set theoretic section of the quotient homomorphism $M \to \pi_1(\partial)$. Then, for $q_1, q_2 \in \pi_1(\partial)$, $u(q_1q_2)^{-1}u(q_1)u(q_2) \in \partial L$ and this element is a non abelian 2-cocycle. Using a homorphic section $\delta L \longrightarrow L$ of ∂ we get $v(q_1, q_2) \in L$ such that $\partial v(q_1, q_2) = u(q_1q_2)^{-1}u(q_1)u(q_2)$. Let w be defined by

$$w(q_1, q_2, q_3) = v(q_2, q_3)^{-1}v(q_1, q_2q_3)^{-1}v(q_1q_2, q_3)v(q_1, q_2)^{u(q_3)}.$$

Then $w(q_1, q_2, q_3) \in \pi_2(\partial)$, is a 3-cocycle and the image $k(\partial)$ of w in $H^3(\pi_1(\partial), \pi_2(\partial))$ is independant of the choices of u and v. The element $k(\partial) \in H^3(\pi_1(\partial), \pi_2(\partial))$ is called the k-invariant of the crossed module ∂. Clearly $k(\partial)$ depends only on the isomorphism type of ∂ in $Ho(\underline{\underline{cross}})$. Moreover for a space X in $\underline{\underline{types}}(2)$ we have

$$k_X = k(\lambda(X))$$

where $\lambda(X)$ is the crossed module in (2.4).

(5.4) The k-invariant of a reduced (stable) quadratic module (Compare [3]): Let (w, δ) be a reduced a reduced (resp. stable) quadratic module, which is free in degree 1. Then w determines a unique homomorphism $\varphi = k(w, \delta)$ by the following commutative diagram

$$\begin{array}{ccc}
M^{ab} \otimes M^{ab} & \xleftarrow[H]{} & \Gamma(M^{ab}) \\
| & & \downarrow q_* \\
| & & \Gamma\pi_1 \\
w \downarrow & & \downarrow \varphi \\
L & \longleftarrow & \pi_2 = \ker(\delta) \\
\delta \downarrow & & \\
M & \longrightarrow & \pi_1 = \operatorname{coker}(\delta)
\end{array}$$

Here H is the cross effect map of Γ and q_* is induced by the projection $q : M^{ab} \to \pi_1$ given by the cokernel of δ. Moreover φ factors uniquely

$$\varphi : \Gamma(\pi_1) \overset{\sigma}{\twoheadrightarrow} \pi_1 \otimes \mathbb{Z}/2 \overset{k(w,\delta)}{\longrightarrow} \pi_2$$

if (w, d) is stable. Here σ is the suspension map. The k-invariant $k(w, \delta)$ satisfies for $X \in \underline{\underline{types}}(n+2)$, $n \geq 1$, the equation

$$k_X = k(\overline{\lambda}^n X)$$

where $\overline{\lambda}^n$ is the equivalence in (4.5). Clearly $k(w, \delta)$ depends only on the isomorphism type of (w, δ) in $Ho(\underline{\underline{rquad}})$, resp. $Ho(\underline{\underline{squad}})$.

We now are ready to prove the following algebraic result:

(5.5) Theorem. *Let (w, δ) be a stable quadratic module. Then the k-invariant $k(\delta)$ of the associated crossed module δ in (4.2) is trivial.*

The theorem is a consequence of the following two lemmas.

(5.6)Lemma. *Let (w,δ) be a stable quadratic module. Then there exists a stable quadratic module (w',δ) such that the associated crossed modules δ of (w,δ) and (w',δ) coincide and such that $k(w',\delta)=0$.*

Proof of (5.6). We may assure that (w,δ) is free in degree 1. Let

$$\varphi = k(w,\delta) : \pi_1 \otimes \mathbb{Z}/2 \to \pi_2$$

be the k-invariant. We choose a bases B of the $\mathbb{Z}/2$-vector space $\pi_1 \otimes \mathbb{Z}/2$ and we define a symmetric bilinear map

$$\beta : \pi_1 \otimes \mathbb{Z}/2 \times \pi_1 \otimes \mathbb{Z}/2 \to \pi_2$$

by $\beta(e,e)=\varphi(e)$ for $e \in B$ and $\beta(e,f)=0$ for $e \neq f$ and $e,f \in B$. Then we obtain by the quotient map $q : M^{ab} \to \pi_1$ the map $(x,y \in M^{ab})$

$$w' : M^{ab} \otimes M^{ab} \to L$$

by $w'(x \otimes y) = w(x \otimes y) \cdot \beta(q(x),q(y))$. Since β maps to the kernel of δ, we clearly have $\delta w' = \delta w$. Moreover we have $(a \in L)$

$$w'(\{\delta a\} \otimes x) = w(\{\delta a\} \otimes x)$$

since $q\{\delta a\}=0$. This shows that (w',δ) is well defined and that the associated crossed module δ coincides with the associated crossed module of (w,δ). Clearly $k(w',\delta)=0$. q.e.d.

(5.7)Lemma. *Let (w',δ) be a stable crossed module with trivial k-invariant $k(w',\delta)=0$. Then the k-invariant $k(\delta)$ of the associated crossed module δ is trivial.*

Proof of (5.7). We may assume that (w',δ) is free in degree 1. Then we obtain the following commutative diagram in which rows and columns are exact

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \downarrow & & & & \\
0 & \longrightarrow & \Gamma M^{ab} & \longrightarrow & M^{ab}\otimes M^{ab} & \longrightarrow & \Lambda^2 M^{ab} & \longrightarrow & 0 & & \\
 & & {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle w'} & & \downarrow & & & & \\
0 & \longrightarrow & \pi_2 & \longrightarrow & L & \xrightarrow{\delta} & M & \longrightarrow & \pi_1 & & \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \pi_2 & \longrightarrow & L' & \xrightarrow{\partial} & M^{ab} & \longrightarrow & \pi_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

Here L' is the cokernel of w'. This shows that ∂ is a crossed module with the trivial action of M^{ab} on L' such that ∂ is weakly equivalent to ∂. Since image (∂) is free abelian we see that $k(\partial) = 0$ and hence

$$k(\delta') = k(\partial) = 0$$

This proves Lemma (5.7).

q.e.d.

Equivalently to (5.7) we obtain the following result, compare [1].

(5.8) Theorem. *The first k-invariant of a connected double loop space is trivial.*

Proof. We use (5.5) and diagram (4.7).

q.e.d.

§ 6 Reduced and stable 2-modules

A quite different algebraic proof of theorem (5.8) is related to a result of Deligne; see 1.4 of [11]. For this we embed the category of reduced quadratic modules into the larger category of reduced 2-modules:

(6.1) Definition. A reduced 2-module (ψ, ∂) is a group homomorphism $\partial : L \to M$ together with a map $\Psi : M \times M \to L$ such that the following properties hold for $x, y, z \in M$ and $a, b \in L$

$$\begin{aligned} \partial\Psi(x,y) &= x^{-1}y^{-1}xy \\ \Psi(\partial a, \partial b) &= a^{-1}b^{-1}ab \\ \Psi(\partial a, x)\cdot\Psi(x,\partial a) &= 1 \\ \Psi(x, yz) &= \Psi(x,z)\Psi(x,y)\Psi(y^{-1}x^{-1}yx, z) \\ \Psi(xy, z) &= \Psi(y^{-1}xy, y^{-1}zy)\psi(y,z) \end{aligned}$$

This is a stable 2-module if for $x, y \in M$

$$\Psi(x,y)\Psi(y,x) = 1$$

holds. Moreover Ψ, ∂ is strict if $\Psi(x,x) = 1$ for $x \in M$. The associated crossed module of (Ψ, ∂) is the crossed module $\partial : L \to M$ with the action of M on L given by

$$a^x = a \cdot \Psi(\partial a, x)$$

for $a \in L, x \in M$. A map between such 2-modules is a map of the associated crossed modules which is compatible with Ψ. This is a weak eqivalence if it is a weak equivalence for the associated crossed modules.

One readily checks that (w, δ) in (4.1) satisfies the properties in (6.1). This yields the inclusion of the category of reduced (resp. stable) quadratic modules into the category of reduced (resp. stable) 2-modules. We point out that the 2-modules above are special "crossed modules of length 2" in the sense of [9].

(6.2) Remark. One readily checks that a reduced (resp. stable) quadratic module (w, d) is strict in the sense of (6.1) if and only if the k-invariant $k(w, \delta)$ is trivial.

The next lemma is proved in [3].

(6.3) Lemma. *Each reduced (resp. stable) 2- module is weakly equivalent to a reduced (resp. stable) quadratic module.*

Hence the inclusion functors above induce equivalences of localized categories

$$\begin{cases} Ho(\underline{\underline{rquad}}) & \xrightarrow{\sim} Ho(\underline{\underline{reduced\,2-modules}}) \\ Ho(\underline{\underline{squad}}) & \xrightarrow{\sim} Ho(\underline{\underline{stable\,2-modules}}) \end{cases}$$

Lemma (3.2) implies the following corollary of theorem (5.5).

(6.5) Theorem. *Let (Ψ,δ) be a stable 2-module. Then the k-invariant $k(\partial)$ of the associated crossed module ∂ is trivial.*

This result can also be obtained by the following two lemmas which correspond to (5.6) and (5.7) respectively.

(6.6) Lemma. *Let (Ψ,∂) be a stable 2-module. Then there exists a stable 2-module (Ψ',∂) such that the associated crossed modules ∂ of (Ψ,∂) and (Ψ',∂) coincide and such that (Ψ',∂) is strict.*

(6.7) Lemma. *Let (Ψ',∂) be a strict stable 2-module. Then the k-invariant $k(\partial)$ of the associated crossed module ∂ is trivial.*

We now compare lemma (6.7) with a result of Deligne [11]. Using results of Isbell [16] and R.Brown-Spencer [7] a strict reduced 2-module can be identified with a strict Picard category. A Picard category is a symmetric monoidal category [19] enriched with commutativity data corresponding to the map Ψ. Now Deligne in 1.4 observed that strict Picard categories are homotopically trivial. This was used by Sinh [23] in the homotopy classification of Picard categories. Moreover Sinh [23] used a similar construction as in the proof of (5.6) for the construction of certain Picard categories. For a good survey on the relation between Picard categories and reduced 2-modules compare the review of J. Duskin on the paper "Cohomology with coefficients in symmetric cat-groups“ by Bullejos-Carrasco-Cegarra; see Math. reviews 1994 k:18014.

§ 7 Proof of theorem (1.9)

Let Z be and index set and let

$$X = \bigvee_Z S^{n+1} \tag{7.1}$$

be the one point union of spheres $S^{n+1}_e = S^{n+1}$ with $e \in Z$. Then $\pi_{n+1}(X) = A$ is the free abelian group generated by the set Z and $\pi_{n+2}(X) = A \otimes \mathbb{Z}/2$ for $n \geq 2$. Let G_A be the free group generated by Z and let

$$E_A = G_A/\Gamma_3(G_A) \tag{7.2}$$

be the quotient where $\Gamma_3(G_A)$ is the subgroup of triple commutators. We have the classical central extension of groups

(7.3) $$0 \to \Lambda^2(A) \xrightarrow{\chi} E_A \xrightarrow{q} A \longrightarrow 0$$

where q is the abelianization and where χ carries $q(x) \wedge q(y) \in \Lambda^2(A)$ with $x, y \in E_A$ to the commutator $x^{-1}y^{-1}xy$. Combining (7.3) with the short exact sequence (1.4) we obtain the stable quadratic module (w_A, δ_A) given by

(7.4) $$A \otimes A \xrightarrow{w_A} \hat{\otimes}^2 A \xrightarrow{\delta_A} E_A$$

where δ_A is the composition $\delta_A = \chi q : \hat{\otimes}^2(A) \to \Lambda^2(A) \to E_A$. We have

$$\begin{aligned} \pi_1 &= \text{cokernel}(\delta_A) = A \\ \pi_2 &= \text{kernel}(\delta_A) = A \otimes \mathbb{Z}/2 \end{aligned}$$

Moreover the equivalence of categories in (4.5) carries the $(n+2)$-type of X in (7.1) to the stable quadratic module (w_A, δ_A) in (7.4). This is proved in [3]. We now restrict the functor δ in (4.7) to the subcategory of objects of the form (w_A, δ_A) with $A \in \underline{\underline{ab}}$. Then δ carries (w_A, δ_A) to the crossed module δ_A given by the homomorphism δ_A above with the trivial action of E_A on $\hat{\otimes}^2 A$. Since A is free abelian we know that $\Lambda^2 A$ is free abelian and therefore we can choose a retraction $r : \hat{\otimes}^2 A \to A \otimes \mathbb{Z}/2$ of the inclusion i in (1.4). Using this retraction we obtain the weak equivalence of crossed modules

(7.5) $$\begin{array}{ccc} \hat{\otimes}^2 A & \xrightarrow{r} & A \otimes \mathbb{Z}/2 \\ \delta_A \downarrow & & \downarrow 0 \\ E_A & \xrightarrow{q} & A \end{array}$$

Here the right hand side is the trivial crossed module corresponding to a product of Eilenberg-MacLane spaces $K(A,1) \times K(A \otimes \mathbb{Z}/2, 2)$ in the category $\underline{\underline{types}}(2)$; compare (2.4). Hence the morphism (7.5) is via (2.4) the same as the choice of a homotopy equivalence $(*)$ in the introduction. The obstruction for the naturality of $(*)$ is therefore the same as the obstruction for the naturality of (7.5) in $Ho(\underline{\underline{cross}})$. But clearly this is the same as the obstruction for the naturality of the ratraction r and this obstruction is the element $\{\hat{\otimes}^2\}$ in $Ext^1_{\underline{\underline{ab}}}(\Lambda^2, \otimes\mathbb{Z}/2)$. This proves (1.9). q.e.d.

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