SINGULAR FANO COMPACTIFICATIONS OF $\mathbb{C}^3$ (I)

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Abstract. In this paper we study the singular Fano compactifications of $\mathbb{C}^3$ with small Gorenstein singularities and determine the structure of such a compactification in the case of the Fano index $r \geq 2$.

Notation

$N_{Y|X}$: normal bundle of Y in X

$c_1(\mathcal{F})$: first Chern class of $\mathcal{F}$

$\mathrm{h}^i(\mathcal{L}) := \dim_{\mathbb{C}} H^i(\cdot; \mathcal{L})$

$\mathrm{Bs}\,|\mathcal{L}|$: base locus of the linear system $|\mathcal{L}|$

$\mathrm{b}_i(X) := \dim_{\mathbb{R}} H_i(X; \mathbb{R})$

$\rho(X)$: Picard number of X

$\mathrm{mult}_A X$: multiplicity of X at a general point of A

K_X : canonical divisor of X

$\sim$: linear equivalence

$\cong$: isomorphism

$\mathbb{F}_n$: Hirzebruch surface of degree n

$\mathbb{Q}^n$: smooth quadric hypersurface in $\mathbb{P}^{n+1}$

$\mathbb{Q}_0^2$: quadric cone in $\mathbb{P}^3$

$\mathbb{Q}_0^3$: quadric hypersurface in $\mathbb{P}^4$ with an isolated singular point

V_5 : smooth Fano threefold of index two and degree 5 in $\mathbb{P}^6$ (see [16])

V_5^*: singular Fano threefold of index two and degree 5 in $\mathbb{P}^6$ with small Gorenstein singularities (see[3],[20])

V_{22} : smooth Fano threefold of index one and degree 22 in $\mathbb{P}^{13}$ (see [17])

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V_{22}^*: singular Fano threefold of index one and degree 22 in $\mathbb{P}^{13}$ with small Gorenstein singularities

Introduction. Let X be a compact complex manifold and Y a closed subvariety og X. We call the pair (X, Y) the analytic compactification of $\mathbb{C}^n$ if $X - Y$ is biholomorphic to $\mathbb{C}^n$.Then Hirzebruch proposed the following problem.

Problem ([15]). *Determine all smooth analytic compactifications (X, Y) of $\mathbb{C}^n$ with the second Betti number* $\mathrm{b}_2(X) = 1$.

For $n \leq 2$, we have the following:

(1) $(X, Y) \cong (\mathbb{P}^1, \infty)$ if $n = 1$

(2) $(X, Y) \cong (\mathbb{P}^2, L)$ if $n = 2$, where L is a line on $\mathbb{P}^2$ ([32]).

The problem is still open for $n \geq 4$. In the case of $n = 3$, under the assumption that X is projective, we obtain the following:

Theorem I ([6],[7],[12],[27],[28],[29]). *Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$ with the second Betti number $b_2(X) = 1$. Then Y is an ample divisor and $-K_X \sim rY$ $(r \in \mathbb{N}, 1 \leq r \leq 4)$, that is, X is a smooth Fano threefold with $\rho(X) = 1$. Moreover,*

(1) $r = 4 \Longrightarrow (X, Y) \cong (\mathbb{P}^3, \mathbb{P}^2)$.

(2) $r = 3 \Longrightarrow (X, Y) \cong (\mathbb{Q}^3, \mathbb{Q}_0^2)$.

(3) $r = 2 \Longrightarrow (X, Y) \cong (V_5, H_5^0)$ *or* (V_5, H_5^∞).

(4) $r = 1 \Longrightarrow (X, Y) \cong (V_{22}, H_{22}^0)$ *or* (V_{22}, H_{22}^∞),

Notation:

- H_5^0 (resp. H_5^∞) is a normal (resp. non-normal) hyperplane section of V_5 such that $\operatorname{Sing} H_5^0 = \{$a rational double point of A_4 type$\}$ (resp. $\operatorname{Sing} H_5^\infty = \ell$ (a line) with the normal bundle $N_{\ell|V_5} \cong \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1)$.
- H_{22}^0 (resp.H_{22}^∞) is a non-normal hyperplane section such that $\operatorname{Sing} H_{22}^0 = \operatorname{Sing} H_{22}^\infty = L$ (a line) with the normal bundle $N_{L|V_{22}} \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L(1)$, in particular, $\operatorname{mult}_L H_{22}^0 = 2$ and $\operatorname{mult}_L H_{22}^\infty = 3$.

On the other hand, in the case where X is non-projective, Peternell-Schneider obtain the following:

Theorem II ([9],[27],[28],[29]). *Let (X, Y) be a smooth analytic compactification of $\mathbb{C}^3$ with $\mathrm{b}_2(X) = 1$. Asuume that X is non-projective. Then*

(i) *X is Moishezon.*

(ii) *Y is a non- projective non-normal irreducible divisor.*

(iii) *$-K_X \sim rY$ $(r = 1, 2)$.*

Moreover assume that Y is nef. Then there is a small birational contraction $\Phi : X \longrightarrow V^*$ of X onto a Fano threefold V^* with small Gorenstein singularities such that

(a) *the exceptional set of Φ consists of finitely many smooth rational curves C_i supported in Y, and the normal bundle $N_{C_i|X}$ of C_i in X is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$ or $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.*

(b) *$\Delta := \Phi_* Y \in \operatorname{Pic} V^*$ is ample and $-K_{V^*} \sim r\Delta$, in particular, $\operatorname{Pic} V^* \cong \mathbb{Z}\mathcal{O}_{V^*}(\Delta)$.*

(c) *$V^* - \Delta \cong \mathbb{C}^3$ with $\mathrm{b}_2(V^*) = 1$.*

To determine the structure of (X, Y) in the case where Y is nef, one needs the study of the Fano compactifications of $\mathbb{C}^3$ with small Gorenstein singularities. For this we propose the following:

Problem A. *Determine all the singular compactifications (V^*, Δ) of $\mathbb{C}^3$ such that*

(i) *V^* is a Fano threefold with (non-empty) small Gorenstein singularities,*

(ii) *$\mathrm{b}_2(V^*) = 1$, and*

(iii) *$\Delta \in |\mathcal{O}_{V^*}(1)|$.*

Let (V^*, Δ) be as above. Then we have $K_{V^*} \sim -rH$ $(0 < r \in \mathbb{Z})$ for $H \in |\mathcal{O}_{V^*}(1)|$. The integer r is called the Fano-index of V^*. If $r \geq 4$, then $V^* \cong \mathbb{P}^3$ (in fact, $r = 4$). Since V^* has singularities, this case can be excluded, that is, we have only to consider the case of $1 \leq r \leq 3$. We remark that the condition (ii) $\mathrm{b}_2(V^*) = 1$ does not necessarily imply $\mathrm{b}_4(V^*) = 1$ even if V^* has mild singularities. In fact, there is an example such that $\mathrm{b}_2(V^*) = 1$ and $\mathrm{b}_4(V^*) = 2$. In particular, the fourth Betti number $\mathrm{b}_4(V^*)$ is equal to the number of irreducible components of the boundary divisor Δ. Thus assume that $\mathrm{b}_4(V^*) = 1$. Then, taking a small resolution $X := \widehat{V}^*$ of V^*, one has $\mathrm{b}_2(X) = \mathrm{b}_4(X) = 1$, which is the case in the later half of Theorem II.

On Problem A, the following is obtained.

Theorem III ([9],[10],[11]). *Assume that* $\mathrm{b}_4(V^*) = 1$ *and* Δ *is non-normal. Then* $1 \leq r \leq 2$ *and*

(1) If $r = 2$ *(that is,* $K_{V^*} \sim -2\Delta$*), then* $(V^*, \Delta) \cong (V_4^*, \Delta_4)$*, where* $V_4^* \subset \mathbb{P}^5$ *is a Fano threefold of degree 4 with index two, which is a complete intersection of two quadrics in* $\mathbb{P}^5$*, and* $\Delta_4 \in |\mathcal{O}_{V_4^*}(1)|$ *is a non-normal cone over a nodal quartic curve in* $\mathbb{P}^3$*. In particular,* $\operatorname{Sing} V_4^*$ *consists of a small Gorenstein singularity* $\{p\}$ *and* $C = \Phi^{-1}(p)$ *is a smooth rational curve with the normal bundle* $N_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$ *for a small resolution* $\Phi : X \longrightarrow V^*$*.*

(2) If $r = 1$*, then there exists a Fano threefold* $V_{18}^* \subset \mathbb{P}^{11}$ *of degree 18 (that is, the genus* $g = \frac{1}{2}(-K_{V_{18}^*})^3 + 1 = 10$*) and a non-normal hyperplane section* Δ_{18} *of* V_{18}^* *such that*

(i) $\mathrm{b}_2(V_{18}^*) = \mathrm{b}_4(V_{18}^*) = 1$.

(ii) V_{18}^* *a small Gorenstein singularity* $p \in \Delta_{18}$*.*

(iii) $V_{18}^* - \Delta_{18} \cong \mathbb{C}^3$.

Moreover let $\Phi : X \longrightarrow V_{18}^*$ *be a small resolution. Then* $C = \Phi^{-1}(p)$ *is a smooth rational curve with* $N_{C|V_{18}^*} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$

Now in this paper we shall consider only the case of $r \geq 2$. The result obtained is the following:

Main Theorem. *Assume that* $r \geq 2$*. Then*

(I) If $r = 3$*, then* $(V^*, \Delta) \cong (\mathbb{Q}_0^3, \Delta_2)$*, where* $\Delta_2 \sim \Delta_2^1 + \Delta_2^2$ *(as a Weil divisor) is a hyperplane section consisting of two planes such that* $\Delta_2^1 \cap \Delta_2^2$ *is a generating line passing through the vertex of* $\mathbb{Q}_0^3$*.*

(II) If $r = 2$*, then* $d = 4, 5$*. Moreover,*

(i) $d = 4 \Longrightarrow (V^*, \Delta) \cong (V_4^*, \Delta_4)$*, and* $\mathrm{b}_2(V_4^*) = \mathrm{b}_4(V_4^*) = 1$. (see Theorem III-(1)).

(ii) $d = 5 \Longrightarrow (V^*, \Delta) \cong (V_5^*, \Delta_5)$*, where* V_5^* *has a small hypersurface singularity of* A_1*-type,and* $\mathrm{b}_2(V_5^*) = 1$ *and* $\mathrm{b}_4(V_5^*) = 2$*.* $\Delta_5 = \Delta_5^1 + \Delta_5^2$ *is a reducible hyperplane section. Moreover, there is a compactification* (V_5^*, Δ_5) *of* $\mathbb{C}^3$ *such that* $\Delta_5^1 \cong \mathbb{F}_1$*,* Δ_5^2 *is a normal rational surface with a rational* *double point of A_1-type.*

Conjecture. *Assume that $r = 1$. Then* $\mathrm{b}_4(V^*) \leq 2$ *and*

(1) $V^* \cong V^*_{18}$ *if* $\mathrm{b}_4(V^*) = 1$ (see Theorem III-(2)),

(2) $V^* \cong V^*_{22}$ *if* $\mathrm{b}_4(V^*) = 2$.

(3) V^*_{18} *and* V^*_{22} *really exist as a compactification of* $\mathbb{C}^3$.

Remark 1. The Main Theorem gives the complete determination of non-projective Moishezon compactifications (X, Y) of $\mathbb{C}^3$ with $\mathrm{b}_2(X) = 1$ in the case where $r = 2$ and Y is nef. However in the case of $r = 1$ the structure of (V^*, Δ) is not known well. We need a careful investigation on the non-normal projective surfaces with trivial dualizing sheaves. This will be discussed in the forthcoming paper.

Remark 2. In the case that Y is not-nef, it is shown that for each index $r \leq 2$, there are infinitely many smooth non-projective Moishezon compactifications (X, Y) of $\mathbb{C}^3$ with $\mathrm{b}_2(X) = 1$ (cf.[10],[26]). The detail will be discussed elsewhere.

Remark 3. In Problem A, if we replace the condition (i) by

(i)' V^* has the Gorenstein terminal singularities,

then there are many types of such a singular Fano compactification of $\mathbb{C}^3$ as in Remark (1.2). In more general, the recent work due to H. Takagi ([35],[36], cf.[25]) enable us to study the $\mathbb{Q}$-Fano compactifications of $\mathbb{C}^3$ with $\mathbb{Q}$-factorial terminal singularities.

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1. Proof of Main Theorem.

Let (V^*, Δ) be as in Problem A. If $r = 3$, then it is well known that V^* is a quadric hypersurface in $\mathbb{P}^4$. Since V^* has a small singularity, we have $V^* \cong \mathbb{Q}^3_0$. The structure of $\mathbb{Q}^3_0$ as a compactification of $\mathbb{C}^3$ will be discussed in the section **2**.

Now assume that $r = 2$, that is ,$-K_{V^*} \sim 2H$. We put $d := (H^3) > 0$. Then we have $\mathrm{Bs}\,|H| = \emptyset$ if $d \geq 2$, and $\mathrm{Bs}\,|H|$ consists of one point on the smooth part of

V^* if $d = 1$ (cf.[33]).

1.1 Lemma. $\Delta \in |H|$ *is a non-normal divisor.*

Proof. Suppose that Δ is normal, hence it is irreducible. We take a small resolution $\Phi : X \longrightarrow V^*$ with the exceptional set $C := \mathrm{Exc}(\Phi) = \cup_{i \geq 1} C_i$, where each C_i is a smooth rational curve. Let $Y := \Phi^* \Delta$ be the proper transform of Δ in X. Then we have a non-projective smooth Moishezon compactification (X, Y) of $\mathbb{C}^3$ with $\mathrm{Pic}\, X \cong \mathrm{H}^2(X; \mathbb{Z}) \cong \mathbb{Z} c_1(\mathcal{O}_X(Y))$. By Peternell-Schneider[27], if Y is normal, then X is projective. Thus Y is non-normal. In particular, the non-normal locus C' of Y is contained in the exceptional set C. In our case, one has $\mathrm{H}^1(C; \mathbb{Z}) = 0$, $\mathrm{H}^2(\Delta; \mathbb{Z}) \cong \mathbb{Z}$, and $\mathrm{H}^2(Y; \mathbb{Z}) \cong \mathbb{Z}$. Then since $Y - C \cong \Delta - \Phi(C)$, we have the following exact sequence (cf.[2]):

$$\longrightarrow 0 = \mathrm{H}^1(C; \mathbb{Z}) \longrightarrow \mathrm{H}^2(\Delta; \mathbb{Z}) \longrightarrow \mathrm{H}^2(Y; \mathbb{Z}) \xrightarrow{r^*} \mathrm{H}^2(C; \mathbb{Z}) \longrightarrow \mathrm{H}^3(\Delta; \mathbb{Z})$$
$$\longrightarrow \mathrm{H}^3(Y; \mathbb{Z}) \longrightarrow 0$$

Since r^* is the zero-map , we obtain

$$\mathrm{b}_3(\Delta) = \mathrm{b}_2(C) + \mathrm{b}_3(Y) > \mathrm{b}_3(Y) \tag{1.1.1}$$

Let $M \longrightarrow \Delta$ be the minimal resolution. Then one has $\mathrm{b}_3(\Delta) = \mathrm{b}_3(M)$. Moreover let $\overline{Y} \xrightarrow{\nu} Y$ be the normalization with the non-normal locus $C' \subset C$, and let $\overline{C}'$ be the proper transform of C' in $\overline{Y}$. From the exact sequence (cf.[2]):

$$\longrightarrow \mathrm{H}^1(Y; \mathbb{Z}) \longrightarrow \mathrm{H}^1(\overline{Y}; \mathbb{Z}) \oplus \mathrm{H}^1(C'; \mathbb{Z}) \longrightarrow \mathrm{H}^1(\overline{C}'; \mathbb{Z}) \longrightarrow \mathrm{H}^2(Y; \mathbb{Z})$$
$$\longrightarrow \mathrm{H}^2(\overline{Y}; \mathbb{Z}) \oplus \mathrm{H}^2(C'; \mathbb{Z}) \longrightarrow \mathrm{H}^2(\overline{C}'; \mathbb{Z}) \longrightarrow \mathrm{H}^3(Y; \mathbb{Z}) \longrightarrow \mathrm{H}^3(\overline{Y}; \mathbb{Z}) \oplus 0 \longrightarrow 0$$

one has $\mathrm{b}_3(\overline{Y}) \leq \mathrm{b}_3(Y)$. Since $\mathrm{b}_3(\overline{Y}) = \mathrm{b}_3(\widehat{Y})$ for any resolution $\widehat{Y}$ of $\overline{Y}$, one has $\mathrm{b}_3(\overline{Y}) = \mathrm{b}_3(\widehat{Y}) = \mathrm{b}_3(M) = \mathrm{b}_3(\Delta)$. Thus by (1.1.1) we obtain $\mathrm{b}_3(\Delta) > \mathrm{b}_3(Y) \geq \mathrm{b}_3(\overline{Y}) = \mathrm{b}_3(\Delta)$. This is a contradiction. Therefore Δ is non-normal. $\square$

1.2 Remark. There is a singular Fano threefold V^* of $\mathrm{b}_2(V^*) = 1$ with hypersurface terminal (but not small !) singularities and a normal hyperplane section Δ such that $V^* - \Delta \cong \mathbb{C}^3$.

(Example 1.2.1). Let V_4^* be the complete intersection of two quadric hypersurfaces Q_1 and Q_2 in $\mathbb{P}^5$, where $Q_1 := \{x_1^2 = x_0 x_3 + x_3^2 + x_4 x_5\}$, $Q_2 := \{x_2^2 =$

$x_0x_3 + x_1x_5\}$, and set $\Delta_4 := \{x_5 = 0\}$. Then V_4^* is a singular Fano threefold of degree 4 with hypersurface terminal (not small) singularity . In particular, Δ_4 is a normal cone over a elliptic curve of degree 4 in $\mathbb{P}^3$, and $V_4^* - \Delta_4 \cong \mathbb{C}^3$, $\mathrm{b}_2(V_4^*) = \mathrm{b}_4(V_4^*) = 1$.

(Example 1.2.2). Let us consider a singular cubic threefold $V_3^* := \{x_0^2x_3 + x_1^2x_2 + x_2^3 + x_4^3 = 0\}$ in $\mathbb{P}^4$ and set $\Delta_3 := \{x_0 = 0\}$. Then we have $V_3^* - \Delta_3 \cong \mathbb{C}^3$ and $\mathrm{b}_2(V_4^*) = \mathrm{b}_4(V_4^*) = 1$, in particular, Δ_3 is a normal cone over a cubic curve in $\mathbb{P}^2$. In this case, the singularity of V_3^* is terminal but not small.

(Example 1.2.3). Let B be a quartic surface in $\mathbb{P}^3$ with the defining equation as follows;

$$B := \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \,|\, x_0x_1^3 + x_1x_2^3 + x_2x_3^3 = 0\}$$

Then the affine part $B^* := B \cap \{x_3 \neq 0\}$ is isomorphic to $\mathbb{C}^2$ and there is an algebraic automorphism $\theta \in \mathrm{Aut}\,\mathbb{C}^3$ such that $\theta(B^*)$ is a linear plane in $\mathbb{C}^3$ (see [37]). Let $\varphi : V_2 \longrightarrow \mathbb{P}^3$ be the double cover of $\mathbb{P}^3$ branched along B. We put $A := \varphi^*(x_3 = 0)$. Then (V, A) is a singular Fano compactification of $\mathbb{C}^3$ with index $r = 2$ and the degree 2, which is called a singular quartic double solid. In particulat A is a normal del Pezzo surface with exactly one rational double point p of E_7-type and (V, p) is a hypersurface terminal singularity of E_7-type but it is not small. Next we set $A^{'} := \varphi^*(x_1 = 0)$. Then $A^{'}$ is a non-normal del Pezzo surface whose normalization is isomorphic $\mathbb{Q}_0^2$ (cf.[1],[31]), and $V - A^{'} \cong \mathbb{C}^3$. Of course V has no small Gorenstein singularities.

(Example 1.2.4). Let us consider the normal quartic hypersurfaces:

$$B_1 := \{x_2^3x_3 + x_0^2x_1(x_0 + x_1) + x_2x_1^3 = 0\}$$
$$B_2 := \{x_2^3x_3 + x_0^2x_1^2 + x_2(x_1^3 + x_0^3) = 0\}$$
$$B_3 := \{x_2^3x_3 + x_0^4 + x_2x_1^3) = 0\}$$

Let $\varphi : V_2^i \longrightarrow \mathbb{P}^3$ be the double cover branched along B_i and put $A^i = \varphi^*(x_2 = 0)$. Then A^1 (resp. A^i, $i = 2, 3$) is the irreducible non-normal del Pezzo surface whose normalization is $\mathbb{Q}_0^2$ (resp. reducible surface consisting of two plane $\mathbb{P}^2$), and (V_2^i, A^i) is a singular Fano compactification of $\mathbb{C}^3$ with (not small) hypersurface terminal singularity with E_6-type if $i = 1$ and E_7-type if $i = 2, 3$ respectively.

1.3 Proposition. $4 \leq d \leq 5$.

Proof. Let (V^*, Δ) be as in Problem A. Let H be a fundamental (ample) divisor, that is, $-K_{V^*} \sim 2H$. It is well-known that the base point set $\mathrm{Bs}\,|H| = \emptyset$ if $d \geq 2$ and the degree $d = (H^3) \leq 9$, in particular, H is very ample if $d \geq 3$ (cf.[33]). By Iskovskih [16; (4.6),(4.7)]), if $d = 9$, then V^* is the cone over $\mathbb{P}^2$, and if $d = 8$, then V^* is the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$. On the other hand, by Fujita [5;(2.6)], if$V_d := V^*$ $(d \geq 5)$ has local complete intersection singularities, then V_d is not a cone. From this fact, the cases $d = 8, 9$ can be excluded. Moreover Fujita shows that if $d \geq 4$ and V^* is not a complete intersection (see [4;(2.9)]), then one has $V^* \cong V_6$ or V_5. Thus we have only to consider the case $d \leq 6$.

1.3.1 Claim. $d \neq 6$.

In fact, take a general hyperplane section H. Then H is a smooth del Pezzo surface of degree $d = 6$, that is H is obtained from $\mathbb{P}^2$ by blowing up 3-points in general position. There is a (-1)-curve ℓ in H, which is a line on V_6 since $(-K_H \cdot \ell) = (H \cdot \ell) = 1$. Let us consider the projection $\pi_\ell : V_6 \dashrightarrow \mathbb{P}^5$ from the line ℓ. Let $\sigma : V_6^{'} \longrightarrow V_6$ be the blowing up along ℓ with the exceptional divisor $L^{'}$. One can see easily that $L^{'} \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$ and $\mathrm{Bs}\,|\sigma^* H - L^{'}| = \emptyset$. Then one has a birational morphism $\phi := \phi_{|\sigma^* H - L^{'}|} : V_6^{'} \longrightarrow W = \phi(V_6^{'}) \subset \mathbb{P}^5$ with $\pi_\ell = \phi \circ \sigma^{-1}$ and has $\deg W = 3 = \mathrm{codim}\, W + 1$. Then it is well-known that $W \cong \mathbb{P}^2 \times \mathbb{P}^1$ or the cone over $\mathbb{F}_1 \subset \mathbb{P}^4$ (see [16]). If $W \cong \mathbb{P}^2 \times \mathbb{P}^1$, then one has $\rho(V_6^{'}) \geq 3$ since $\rho(W) = 2$. But this is impossible since $1 = \rho(V_6) = \rho(V_6^{'}) - 1 \geq 2$. If W is the cone over $\mathbb{F}_1$, one sees that $S := \phi(L^{'})$ does not pass through the vertex w of the cone W. Thus we have a surjective morphism $\pi : S \longrightarrow \mathbb{F}_1$ via the point-projection $W \dashrightarrow \mathbb{F}_1$ of W from the vertex w. This is also impossible since $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_2$ or $\mathbb{Q}_0^2$. Therefore we have $d \neq 6$. $\square$

1.3.2 Claim. $d \neq 3$.

It is proved in [9] that $d \neq 3$ if Δ is irreducible. We assume that Δ is reducible. Since Δ is a hyperplane section with $H^2 \cdot \Delta = 3$, as a Wei divisor, one has $\Delta \sim \Delta_1 + \Delta_2 + \Delta_3$, $\Delta \sim \Delta_1 + 2\Delta_2$, or $\Delta \sim \Delta_1 + Q$, where $\Delta_i \cong \mathbb{P}^2$ is a plane and Q is an irreducible quadric surface. Take a small singularity v of $V^* = V_3 \subset \mathbb{P}^4$. Let $\sigma : B_v(\mathbb{P}^4) \longrightarrow \mathbb{P}^4$ be the blowing up with center v with the exceptional divisor $\sigma^{-1}(v) = E \cong \mathbb{P}^3$. Let $\overline{V}_3(\sim \sigma^* V_3 - 2E)$ be the proper transform of V_3 and set $\overline{D} := \mathcal{O}_{\overline{V}_3}(E) \in \mathrm{Pic}\, \overline{V}_3$. Then $\overline{D}$ is an irreducible reduced quadric in $E \cong \mathbb{P}^3$. An easy computation shows that $(\overline{D})^3_{\overline{V}_3} = 2$ and $(\sigma^*H - \overline{D})^3 = 1$. Sinve $\mathrm{Bs}\,|\sigma^*H - \overline{D}| = \emptyset$, we have a birational morphism $\varphi := \varphi_{|\sigma^*H-\overline{D}|} : \overline{V}_3 \longrightarrow \mathbb{P}^3$ associated with the point projection $V_3 \dashrightarrow \mathbb{P}^3$ from the singularity v. Then $Q_0 := \varphi(\overline{D})$ is a quadric hypersurface. By construction the exceptional set $\mathrm{Exc}(\varphi)$ is the proper transform of the set of lines on V_3 passing through the point $v \in \Delta$ ($v \in \Delta_i$ for some i or $v \in Q$). Take a line $v \in \ell_v \subset V_3$. Since Δ has a singularity at v and $(\Delta \cdot \ell_v) = 1$, the line ℓ_v is contained in Δ. Thus $\mathrm{Exc}(\varphi)$ is contained in the proper transform of Δ. Then we have $V_3 - \Delta \cong \mathbb{P}^3 - (\overline{Q}_0 \cup \varphi(\overline{\Delta}) \not\cong \mathbb{C}^3$. Thus $d \neq 3$. $\square$

1.3.3 Claim. $d \neq 2$.

Let $\Phi : X \longrightarrow V^*$ be the small resolution with the exceptional set C and Y the proper transform of Δ in X. Then (X, Y) is a smooth Moishezon compactification of $\mathbb{C}^3$ with $\mathrm{b}_2(X) = \mathrm{b}_2(Y)$. Let $\nu : \overline{\Delta} \longrightarrow \Delta$ be normalization with the non-normal locus E and $\overline{E}$ the pullback of E. We have the following exact sequence of cohomology over $\mathbb{Z}$:

$$
\begin{aligned}
&0 \longrightarrow \mathrm{H}^2(V^*;\mathbb{Z}) \xrightarrow{\Phi^*} \mathrm{H}^2(X;\mathbb{Z}) \xrightarrow{r^*} \mathrm{H}^2(C;\mathbb{Z}) \\
&\longrightarrow \mathrm{H}^3(V^*;\mathbb{Z}) \longrightarrow \mathrm{H}^3(X;\mathbb{Z}) \longrightarrow 0
\end{aligned}
\tag{1.3.3.1}
$$

$$
\begin{aligned}
&\longrightarrow \mathrm{H}^{i-1}(\overline{E};\mathbb{Z}) \longrightarrow \mathrm{H}^i(\Delta;\mathbb{Z}) \xrightarrow{\gamma_i^*} \mathrm{H}^i(\overline{\Delta};\mathbb{Z}) \oplus \mathrm{H}^i(E;\mathbb{Z}) \\
&\longrightarrow \mathrm{H}^i(\overline{E};\mathbb{Z}) \longrightarrow \mathrm{H}^{i+1}(\Delta;\mathbb{Z}) \longrightarrow
\end{aligned}
\tag{1.3.3.2}
$$

(cf.[2]) If Δ is irreducible, Δ is non-normal del Pezzo surface of degree 2. According to the classification by Reid [32](cf.[1]), there are following possibilities of $(\overline{\Delta}, \overline{E})$.

(i) $(\mathbb{P}^1 \times \mathbb{P}^1, e_0 + e_1)$

(ii) $(\mathbb{P}^1 \times \mathbb{P}^1, e_2)$

(iii) $(\mathbb{Q}^2_0, g_1 + g_2)$

(iv) $(\mathbb{Q}^2_0, g_0)$

(v) $(\mathbb{Q}^2_0, s_\infty)$

where e_0, e_1 (resp. e_2) are two different rulings (resp.a diagonal) of $\mathbb{P}^1 \times \mathbb{P}^1$, and g_i (resp. s_∞) is a generating line (resp. an infinite section) of $\mathbb{Q}^2_0$. From (1.3.3.2), one has $\mathrm{b}_3(\Delta) = 0$ in the cases (i),(ii),(iv),(v) and $\mathrm{b}_3(\Delta) = 1$ in the case (iii). On the other hand, since Y is irreducible and $(Y \cdot C_i) = 0$ for an irreducible component C_i of C, one has $\mathrm{b}_2(V^*) = \mathrm{b}_2(X)$ and $\mathrm{b}_3(V^*) = \mathrm{b}_2(C) + \mathrm{b}_3(X)$. Since $\mathrm{b}_3(V^*) = \mathrm{b}_3(\Delta)$, we have $\mathrm{b}_2(C) = \mathrm{b}_3(V^*) = 1$ and $\mathrm{b}_3(X) = 0$ by (1.3.3.1). This shows that V^* has a Gorenstein singularity with the small resolution whose exceptional set is a smooth rational curve and $(\overline{\Delta}, \overline{E})$ is of type (iii). Since the morphism $\varphi = \varphi_{|H|} : V^* \longrightarrow \mathbb{P}^3$ is of degree two, the ramification divisor $B \subset \mathbb{P}^3$ is a normal quartic hypersurface since V^* has isolated singularities. Then the restriction $\varphi : \Delta \longrightarrow \mathbb{P}^2$ is a surjective morphism of degree two. By lifting of the morphism φ to the normalization $\overline{\Delta}$, one has the morphism $\mathbb{Q}_0 \cong \overline{\Delta} \xrightarrow{\varphi} \mathbb{P}^2$ of degree two branched along a normal quartic surface. Thuis implies the canonical bundle $K_{\overline{\Delta}} = -\varphi^*\ell$, where ℓ is a line on $L \cong \mathbb{P}^2$. This cannot happen, because $K^2_{\overline{\Delta}} = 8$. Therefore Δ is not irreducible.

Next, we shall consider the case that Δ is reducible. Since $H^3 = H^2 \cdot \Delta = 2$, one has $\Delta := \Delta_1 + \Delta_2$ (a a Weil divisor), where Δ_i is a plane $\mathbb{P}^2$. In this case one has $\mathrm{b}_3(V^*) = \mathrm{b}_3(\Delta) = 0$ and $\mathrm{b}_2(X) = \mathrm{b}_4(V^*) = 2$, hence $\mathrm{b}_2(C) = 1$ by (1.3.3.1). Thus V has exactly one Gorenstein singularity with small resotution whose exceptional set is a smooth rational curve. Since $\Delta \in |\mathcal{O}_{V^*}(1)|$, there is a linear plane $L = \mathbb{P}^2$ and a morphism $\Delta \longrightarrow L$ of degree two branched along the quartic curve $B_0 := L \cap B = 2Q$, where Q is a smooth quadric curve on L. Then the normalization $\overline{\Delta}$ consists of two disjoint component $\overline{\delta}_i = \mathbb{P}^2 (i = 1, 2)$ such that each $\overline{\Delta}_i$ is somorphic to L. Then we have $Y = \varphi^*\Delta = Y_1 + Y_2$. Since $(Y \cdot C) = 0$, one has $(Y_1 \cdot C) + (Y_2 \cdot C) = 0$. If $(Y_i \cdot C) = 0$, then C is homologous to 0, that is, r^* in (1.3.3.1) is zero-map, which is absurd since $\mathrm{b}_2(C) = 1$. Thus we may assume that $(Y_1 \cdot C) < 0$, hence $C \subset Y_1$, in particular, $C \not\subset Y_2$. This yields $\mathrm{b}_2(Y) = 2$, we have $\mathrm{b}_2(Y_1) = 2$, $\mathrm{b}_2(Y_2) = 1$ and $\mathrm{b}_2(Y_1 \cap Y_2) = 1$. One sees that $\gamma := Y_1 \cap Y_2$ is a smooth quadric curve in $Y_2 \cong \mathbb{P}^2$. $p = \operatorname{Sing} V^*$ is on the smooth quadric curve $\gamma \cong \Delta_1 \cap \Delta_2$. Since $\gamma_2 = Y_1|_{Y_2}$ (resp. $\gamma_1 = Y_2|_{Y_1}$) is a smooth rational curve, the Cartier divisor Y_1 has at worst A_n-singularity at the point $q \in C - \gamma$. In particular $(C \cdot \gamma_1)_{Y_1} = 1$. Now Y_1 is obtained from $\mathbb{P}^2$ by blowing up at $n+1$-(infinitely near) points on the smooth quadric $\Delta \cap \Delta_2$ whose exceptional curve is a linear tree of the (-2)-curve. Then we have

(1) $Y_1^2 Y_2 = (Y_1|_{Y_2})^2 = (\gamma_2^2) = 4$

(2) $Y^1 Y_2^2 = (Y_2|_{Y_1})^2 = (\gamma_1^2) = 3 - n$ for $n \geq 1$

(3) $Y_1^3 = (K_{Y_1} + 2\gamma_1)^2 = -n$

(4) $Y_2^3 = (K_{Y_2} + 2\gamma_2)^2 = 1$

Then we have $2 = (\varphi^*\Delta) = (Y_1+Y_2)^3 = 22-4n$, that is $n = 5$. In particular, since $C \subset Y_1$ passes through the rational double point of A_5-type, we have the normal bundle $N_{C|X} \cong \mathcal{O}_{P^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, hence the local defining equation of $(V^*, p) \approx \{xy + z^2 + t^{2k}\}$(cf.[30]). Now since the morphism $\varphi = \varphi_{|H|} : V^* \longrightarrow \mathbb{P}^3$ is of degree two, the ramification divisor $B \subset \mathbb{P}^3$ is a normal quartic hypersurface since V^* has isolated singularities. Let L be a hyperplane of $\mathbb{P}^3$ such that $\varphi^* L = \Delta$ and set $D = B \cap L$, $B_0 = B - D$. Since $V^* - \Delta \cong \mathbb{C}^3 \xrightarrow{\varphi} \mathbb{C}^3 \cong \mathbb{P}^3 - L$ is a double cover of $\mathbb{C}^3$ with smooth branched locus B_0, by the well-known Smith theorem [34], B_0 is a $\mathbb{Z}_2$-homology plane, hence it is $\mathbb{Q}$-homology plane, that is, $\mathrm{H}_i(B_0; \mathbb{Q}) = 0$ for $i > 0$. Then it is known that B is a rational surface and the boundary D consists of $\mathrm{b}_2(B)$ rational curves with no cycles (cf.[23]). Thus B is a normal rational quartic hypersurface with exactly one minimally elliptic singularity p by Umezu [38]. Let $\mu : \widehat{B} \longrightarrow B$ be the minimal resolution and Z be the fundamental cycle associated with the minimally elliptic singular point p. Then one has $K_{\widehat{B}} \sim -Z$ and $Z^2 = -1, -2, -3$, in particular, $Z^2 = -3$ if $\mathrm{mult}_p B = 3$. and $Z^2 = -1, -2$ if $\mathrm{mult}_p B = 2$ (see [21]). Since V has exactly one small Gorenstein singularity of A_1-type at $p := \varphi^{-1}(p)$, the local defining equation of (V, p) is analytically equivalent to $w^2 = xy + z^{2k}$ (see [19],[22], [30]). Let $f(x, y, z) = 0$ be the local defining equation of the germ (B, p) with $p = (0, 0, 0)$, which is given by Laufer [21]. The defining equation of (V, p) can be also written as $w^2 = f(x, y.z)$. Comparing with these two defining equations, one sees that $f(x, y, z) = 0$ has the multiplicity two at the origin $p = (0, 0, 0)$, that is, $\mathrm{mult}_p B = 2$. For each defining function $f(x, y, z)$ of (B, p) in the Table 1 and Table 2 in [21], we can easily check whether $f(x, y, z) \approx xy + z^{2k}$ (analytic equivalence) or not (see also Theorem 1.1 in Katz [19]). Then we conclude that there are no such analytic functions in the Table. $\square$

1.3.4 Claim. $d \neq 1$.

If Δ is irreducible, then the normalization $\overline{\Delta}$ is isomorphic to $\mathbb{P}^2$ ([1],[31]). From (1.3.3.2), we have $\mathrm{b}_3(\Delta) = 0$, hence $\mathrm{b}_3(V_1^*) = 0$. Thus we obtain $\mathrm{b}_2(C) = \mathrm{b}_3(V_1^*) = 0$ by (1.3.3.1), which is a contradiction since C is a curve. On the other hand since $H^3 = 1$, Δ cannot be decomposed into two Weil divisors, thai is, Δ is irreducible.

This cannot occur in our case, because Δ is reducible. Thus we have $d \neq 1$. This completes the proof. $\square$

1.4 Remark. In the case of $r = 2$, we assume that Δ is irreducible. Then from the classification of non-normal del Pezzo surfaces [1],[31] and the exact sequences (1.3.3.1), (1.3.3.2), one can show that $\mathrm{b}_3(V^*) = \mathrm{b}_2(C) = 1$, hence V^* has exacily one small Gorenstein singularity of A_1-type,and Δ is a non-normal cone over a nodal rational curve of degree d in $\mathbb{P}^{d-1}$ if $d = 4, 5$.

One can also prove that Δ is not irreducible if $d = 5$. In fact, assume that Δ is irreducible. Then we have $\mathrm{b}_4(V^*) = 1$. On the other hand, let H_5 be a general hyperplane section of V_5^*. Then there is a line ℓ on H_5. Taking a projection $V_5^* \dashrightarrow \mathbb{P}^4$ from the line ℓ, one sees that V_5^* is birational equivalent to the singular quadric $\mathbb{Q}_0^3$ since V_5^* has a small singularty. This implies that $\mathrm{b}_4(V_5^*) = 2$, which is a contradiction. Since $\mathrm{b}_4(X) = \mathrm{b}_2(X) = 2$ for the small resolution $X \xrightarrow{\varphi} V_5^*$. Since $\mathrm{b}_4(X) = \mathrm{b}_4(Y) = 2$, $Y := \varphi^*\Delta$ consists of two irreducible components, and so is Δ. In the case of $d = 4$, Δ is irreducible since $\mathrm{b}_4(V_4^*) = 1$ (cf.[9]).

1.5 Corollary. *Assume that $r = 2$ and Δ is reducible. Then $d = 5$, that is, V^* is a Fano threefold $V_5^* \subset \mathbb{P}^6$ of degree 5 and $\Delta := \Delta_5$ is a hyperplane section consisting of two irreducible Weil divisors.*

2. Singular Fano threefolds $\mathbb{Q}_0^3$ and V_5^* as a compactification of $\mathbb{C}^3$.

Let $\mathbb{P}^4$ be the 4-dimensional complex projective space with the homogeneous coodinate system $(x_0 : x_1 : x_2 : x_3 : x_4)$ and $\mathbb{Q}_0^3$ a quadric cone defined by $\mathbb{Q}_0^3 := \{x_0x_3 = x_1x_2\}$ in $\mathbb{P}^4$. Then the singular point of $\mathbb{Q}_0^3$ is the vertex $p := (0 : 0 : 0 : 0 : 1) \in \mathbb{P}^4$ of the cone. Let Q_∞ be the hyperplane section defined by $Q_\infty := \{x_4 = 0\}$. Then Q_∞ $(\cong \mathbb{P}^1 \times \mathbb{P}^1)$ is a smooth quadric hypersurface in $\mathbb{P}^3$. Let $\Delta_2 := \{x_0 = 0\}$ be the hyperplane section of $\mathbb{Q}_0^3$. Then we have $\Delta_2 := \Delta_2^{(1)} + \Delta_2^{(2)}$, where $\Delta_2^{(i)} := \{x_0 = x_i = 0\}$ $(i = 1, 2)$ is a smooth $\mathbb{Q}$-Cartier divisor isomorphic to $\mathbb{P}^2$. We put $g := \Delta_2^{(1)} \cap \Delta_2^{(2)} \cong \mathbb{P}^1$ (a generating line of $\mathbb{Q}_0^3$). It is easy to see that $\mathbb{Q}_0^3 - \Delta_2 \cong \mathbb{C}^3$. Let $\Phi : X \longrightarrow \mathbb{Q}_0^3$ be a small resolution with the exceptional set $C := \Phi^{-1}(p) \cong \mathbb{P}^1$. Let $\alpha : B_p(\mathbb{P}^4) \longrightarrow \mathbb{P}^4$ be the blowing up of $\mathbb{P}^4$ with the center p and $\overline{\mathbb{Q}_0^3}$ the proper transform of $\mathbb{Q}_0^3$. Then the restriction $\alpha : \overline{\mathbb{Q}_0^3} \longrightarrow \mathbb{Q}_0^3$ is a resolution of the singularity p with exceptional set $\alpha^{-1}(p) = \overline{Q}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $\beta : B_C(X) \longrightarrow X$

be the blowing up of X with the center C. Then we have $B_C(X) \cong \overline{\mathbb{Q}_0^3}$ and the birational morphism α is factorized as $\alpha = \Phi \circ \beta$, that is, $\alpha : \overline{\mathbb{Q}_0^3} \xrightarrow{\beta} X \xrightarrow{\Phi} \mathbb{Q}_0^3$. In particular, one has the normal bundle $N_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. From $\mathbb{Q}_0^3 - \Delta_2 \cong \mathbb{C}^3$, one has $\mathrm{b}_i(\mathbb{Q}_0^3) = \mathrm{b}_i(\Delta_2)$ for $i > 0$ (cf. [7],[27]). On the other hand, since $\mathrm{b}_i(\Delta_2) = \mathrm{b}_i(\Delta_2^{(1)}) + \mathrm{b}_i(\Delta_2^{(2)}) - \mathrm{b}_i(g)$, one has easily $\mathrm{b}_1(\mathbb{Q}_0^3) = \mathrm{b}_3(\mathbb{Q}_0^3) = 0$, $\mathrm{b}_2(\mathbb{Q}_0^3) = 1$ and $\mathrm{b}_4(\mathbb{Q}_0^3) = 2$. Thus we have

2.1 Proposition. $(\mathbb{Q}_0^3, \Delta_2)$ *is a singular Fano compactification of* $\mathbb{C}^3$ *of index* $r = 3$ *with a small hypersurface singularity of* A_1*-type, in particular,* $\mathrm{b}_2(\mathbb{Q}_0^3) = 1$ *and* $\mathrm{b}_4(\mathbb{Q}_0^3) = 2$.

Next we shall give a construction of a singular Fano threefold V_5^* as a compactification of $\mathbb{C}^3$. Let $(\mathbb{Q}_0^3, \Delta_2, \Delta_2^{(i)}, Q_\infty, g)$ be as above. Let us consider a twisted cubic curve $\gamma : \mathbb{P}^1 \longrightarrow Q_\infty \subset \mathbb{P}^3$ defined by $\gamma(u : v) = (u^3 : u^2v : uv^2 : v^3)$. We set $s := \{x_0 = x_1 = x_4 = 0\}$, $f := \{x_0 = x_2 = x_4 = 0\}$ and $\gamma := \gamma(\mathbb{P}^1) \subset Q_\infty$. One sees that s, f are two different rulings of $Q_\infty \cong \mathbb{P}^1 \times \mathbb{P}^1$ with $\operatorname{Pic} Q_\infty \cong \mathbb{Z}f \oplus \mathbb{Z}s$. Then we have a linear equivalence $\gamma \sim s + 2f$. By construction one has $\Delta_2^{(2)} \cap \gamma = f \cap \gamma = q$ and $\Delta_2^{(1)} \cap \gamma = s \cap \gamma = 2q$, where $q = (0 : 0 : 0 : 1 : 0) = s \cap f \cap g$. Let $\pi : \widehat{\mathbb{Q}_0^3} \longrightarrow \mathbb{Q}_0^3$ be the blowing up of $\mathbb{Q}_0^3$ along γ with the exceptional set $\widehat{S} := \pi^{-1}(\gamma)$.

2.2 Lemma. $\widehat{S} \cong \mathbb{F}_1$

Proof. Let us consider the exact sequence:

$$0 \longrightarrow N_{\gamma|Q_\infty} \longrightarrow N_{\gamma|\mathbb{Q}_0^3} \longrightarrow N_{Q_\infty|\mathbb{Q}_0^3}|_\gamma \longrightarrow 0$$

By a theorem of Grothendieck, one has $N_{\gamma|\mathbb{Q}_0^3} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ $(a \leq b,\ a, b \in \mathbb{Z})$. Since $N_{\gamma|Q_\infty} \cong \mathcal{O}_{\mathbb{P}^1}(4)$ and $N_{Q_\infty|\mathbb{Q}_0^3}|_\gamma \cong \mathcal{O}_{\mathbb{P}^1}(3)$,one has $N_{\gamma|\mathbb{Q}_0^3} \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$, that is, $\widehat{S} \cong \mathbb{F}_1$. $\square$

On the other hand, one has $K_{\widehat{\mathbb{Q}_0^3}} \sim \pi^* K_{\mathbb{Q}_0^3} + \widehat{S} \sim -3\pi^* Q_\infty + \widehat{S} \sim -3\widehat{Q}_\infty - 2\widehat{S}$, where $\widehat{Q}_\infty \cong Q_\infty$ is the proper transform of Q_∞ in $\widehat{\mathbb{Q}_0^3}$. The adjunction formula implies that $K_{\widehat{Q}_\infty} \sim (K_{\widehat{\mathbb{Q}_0^3}} + \widehat{Q}_\infty)|_{\widehat{Q}_\infty} \sim -2\widehat{Q}_\infty|_{\widehat{Q}_\infty} - 2\widehat{s} - 4\widehat{f}$, where $\widehat{s}$ (resp. $\widehat{f}$) is the proper transfom of s (resp. f). Since $K_{\widehat{Q}_\infty} \sim -2\widehat{s} - 2\widehat{f}$, we obtain $\widehat{Q}_\infty|_{\widehat{Q}_\infty} \sim -\widehat{f}$. An easy computation shows

2.3 Lemma.

(2.3.1) $\widehat{Q}_\infty^3 = (\widehat{Q}_\infty|_{\widehat{Q}_\infty} \cdot \widehat{Q}_\infty|_{\widehat{Q}_\infty})^2 = (-\widehat{f})^2 = 0$

(2.3.2) $\widehat{S} \cdot \widehat{Q}_\infty^2 = (\widehat{S}|_{\widehat{Q}_\infty} \cdot \widehat{Q}_\infty|_{\widehat{Q}_\infty}) = -1$

(2.3.3) $(\widehat{S})^2 \cdot \widehat{Q}_\infty = (\widehat{S}|_{\widehat{Q}_\infty})^2 = 4$

(2.3.4) $\widehat{S}^3 = -7$

(2.3.5) $(2\pi^* Q_\infty - \widehat{S})^3 = (\widehat{S} + 2\widehat{Q}_\infty)^3 = 5$

2.4 Lemma.

(1) $h^0(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + 2\widehat{Q}_\infty)) = 7$

(2) $\mathrm{Bs}\,|\widehat{S} + 2\widehat{Q}_\infty| = \emptyset$

Proof. (1). Let us consider the exact sequences:

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}} \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S}) \longrightarrow \mathcal{O}_{\widehat{S}}(\widehat{S}) \longrightarrow 0 \tag{2.4.1}$$

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S}) \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + \widehat{Q}_\infty) \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(\widehat{S} + \widehat{Q}_\infty) \longrightarrow 0 \tag{2.4.2}$$

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + \widehat{Q}_\infty) \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + 2\widehat{Q}_\infty) \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(\widehat{S} + 2\widehat{Q}_\infty) \longrightarrow 0 \tag{2.4.3}$$

We put $\widehat{S}|_{\widehat{S}} \sim a\Sigma + bF$ $(a, b \in \mathbb{Z})$, where Σ (resp. F) is the negative section (resp. a fiber) of $\widehat{S} \cong \mathbb{F}_1$. Since $(\widehat{S} \cdot F) = -1$, we have $a = -1$. By (2.3.4), $-1 - 2b = -7$, that is, $b = 3$. Thus we have $\widehat{S}|_{\widehat{S}} \sim -\Sigma + 3F$. It is easy to see that $h^1(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}) = 0$, $h^0(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S})) = h^0(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}) = 1$, $h^1(\mathcal{O}_{\widehat{S}}(\widehat{S})) = h^1(\mathcal{O}_{\widehat{S}}(K_{\widehat{S}} + \Sigma + 6F)) = 0$ (Kodaira vanishing theorem). This yields $h^1(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S})) = 0$ by (2.4.1). Moreover since $\mathcal{O}_{\widehat{Q}_\infty}(\widehat{S}+\widehat{Q}_\infty) = \mathcal{O}_{\widehat{Q}_\infty}(\widehat{s}+\widehat{f})$, one has $h^0(\mathcal{O}_{\widehat{Q}_\infty}(\widehat{S}+\widehat{Q}_\infty)) = 4$ and $h^1(\mathcal{O}_{\widehat{Q}_\infty}(\widehat{S}+\widehat{Q}_\infty) = 0$. Then we have $h^0(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + \widehat{Q}_\infty)) = 5$ by (2.3.2) since $h^0(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S})) = 1$. Taking into consideration that $h^0(\mathcal{O}_{\widehat{Q}_\infty}(\widehat{S} + 2\widehat{Q}_\infty)) = h^0(\mathcal{O}_{\widehat{Q}_\infty}(\widehat{s})) = 2$, one sees that $h^0(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + 2\widehat{Q}_\infty)) = 7$ and there is a surjection

$$H^0(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + 2\widehat{Q}_\infty)) \twoheadrightarrow H^0(\mathcal{O}_{\widehat{Q}_\infty}(\widehat{S} + 2\widehat{Q}_\infty)). \tag{2.4.4}$$

(2). Since $\mathrm{Bs}\,|\mathcal{O}_{\widehat{Q}_\infty}(\widehat{S} + 2\widehat{Q}_\infty)| = \mathrm{Bs}\,|\mathcal{O}_{\widehat{Q}_\infty}(\widehat{s})| = \emptyset$, one sees that $\mathrm{Bs}\,|\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + 2\widehat{Q}_\infty)| = \emptyset$ on $\widehat{Q}_\infty$ by (2.4.4). Next let $Q(\cong \mathbb{P}^1 \times \mathbb{P}^1)$ be any smooth hyperplane section of $\mathbb{Q}_0^3$ such that $p \notin Q$ and set $\widehat{Q} = \pi^* Q$. Then $\widehat{Q}$ is a smooth del Pwzzo surface of degree $K_{\widehat{Q}}^2 = 5$. Let us consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\pi^* Q - \widehat{S}) \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\pi^* Q - \widehat{S}) \longrightarrow \mathcal{O}_{\widehat{Q}}(2\pi^* Q - \widehat{S}) \longrightarrow 0$$

We have $\mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(\pi^*Q - \widehat{S}) \cong \mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(\widehat{Q}_\infty)$, $\mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(2\pi^*Q - \widehat{S}) \cong \mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(2\widehat{Q}_\infty + \widehat{S})$ and $\mathcal{O}_{\widehat{Q}}(2\pi^*Q - \widehat{S}) \cong \mathcal{O}_{\widehat{Q}}(-K_{\widehat{Q}})$. Since $h^1(\mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(\widehat{Q}_\infty)) = 0$, one has a surjection

$$H^0(\widehat{\mathbb{Q}^3_0}; \mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(2\pi^*Q - \widehat{S}) \twoheadrightarrow H^0(\widehat{Q}; \mathcal{O}_{\widehat{Q}}(2\pi^*Q - \widehat{S}))$$

Since $\mathrm{Bs}\,|-K_{\widehat{Q}}| = \emptyset$, we have $\mathrm{Bs}\,|2\pi^*Q - \widehat{S}| = \emptyset$ on $\widehat{Q}$. This shows that $\mathrm{Bs}\,|2\widehat{Q}_\infty + \widehat{S}| = \emptyset$. $\square$

Let $\widehat{\Phi} : \widehat{\mathbb{Q}^3_0} \longrightarrow \mathbb{P}^6$ be a morphism defined by the linear system $|\widehat{S} + 2\widehat{Q}_\infty|$. We put $V_5^* := \widehat{\Phi}(\widehat{\mathbb{Q}^3_0}) \subset \mathbb{P}^6$. Since $(\widehat{S} + 2\widehat{Q}_\infty)^3 = 5$, one has $\deg V_5^* = 5$ in $\mathbb{P}^6$. Let us denote by $\mathrm{Exc}(\widehat{\Phi})$ the exceptional set of $\widehat{\Phi}$.

2.5 Lemma. $\mathrm{Exc}(\widehat{\Phi}) = \widehat{Q}_\infty$.

Proof. Since $(\widehat{S} + 2\widehat{Q}_\infty)^2 \cdot \widehat{Q}_\infty = 0$, $\mathrm{Exc}(\Phi) \supset \widehat{Q}_\infty$. Now we have $\mathrm{Pic}\,\widehat{\mathbb{Q}^3_0} \cong \mathbb{Z}\mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(\widehat{Q}_\infty) \oplus \mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(\widehat{S})$. Then $(\widehat{S} + 2\widehat{Q}_\infty)^2 \cdot (a\widehat{S} + b\widehat{Q}_\infty) = 0$ for $a,\ b \in \mathbb{Z}$ if and only if $a = 0$. Therefore there are no exceptional divisor of $\widehat{\Phi}$ other than $\widehat{Q}_\infty$. Since $(\widehat{S} \cdot F) = -1$, $(\widehat{S} \cdot \Sigma) = 4$, $(\widehat{Q}_\infty \cdot F) = 1$, $(\widehat{Q}_\infty \cdot \Sigma) = 1$,one sees easily that $(\widehat{S} + 2\widehat{Q}_\infty) \cdot (a\Sigma + bF) = 0$ $(b \geq a \geq 0)$ if and only if $a + 2b = 0$, that is, $a = b = 0$. This shows that $\widehat{S}$ contained no exceptional curve of $\widehat{\Phi}$, therefore we have the claim. $\square$

2.6 Lemma. *V_5^* is a Fano threefold with a hypersurface terminal singularity $p := \widehat{\Phi}(\pi^{-1}(p))$ of A_1-type and smooth along the line $E := \Phi(\widehat{Q}_\infty)$, in particular, $\widehat{\Phi} : \widehat{\mathbb{Q}^3_0} \longrightarrow V_5^*$ is the blowing up of V_5^* with the center E.*

Proof. Let H be a hyperplane section of V_5^*. Since $\widehat{\Phi}{*}H \sim \widehat{S}{+}2\widehat{Q}_\infty$ and $(\Phi^*H \cdot \widehat{f}) = (H \cdot E) = 1$, one sees that E is a line on V_5^*. Since $\widehat{Q}_\infty|_{\widehat{Q}_\infty} \sim -\widehat{f}$, V_5^* is smooth along E. This shows that $\widehat{\mathbb{Q}^3_0}$ is the blowing up of V_5^* with center E. We set $S := \widehat{\Phi}_*\widehat{S}$. Then we have $K_{V_5^*} \sim -2S$ and $\Phi^*S \sim \widehat{S} + 2\widehat{Q}_\infty$. Since $(\widehat{S} + 2\widehat{Q}_\infty)^3 = 5$, S is a non-normal hyperplane section of V_5^* with the non-normal locus E,in particular, one sees that $\mathrm{mult}_E\, S = 2$. This also shows that V^* is a Fano threefold of index $r = 2$. Since $\widehat{\Phi}$ is isomorphic near the vertex p of $\mathbb{Q}^3_0$, V^* has the same singularity at $p := \widehat{\Phi}(\pi^{-1}(p))$. $\square$

2.7 Remark. It is easy to see that the restriction $\nu := \widehat{\Phi}|_{\widehat{S}} : \widehat{S} \longrightarrow S$ is the normalization with $\nu^{-1}(E) = \Sigma$. By construction $\nu|_\Sigma : \Sigma \longrightarrow E$ is a double covering. It is easy to verify that $\mathrm{b}_2(V_5^*) = 1$, $\mathrm{b}_3(V_5^*) = 0$, $\mathrm{b}_4(V_5^*) = 2$, in particular, $\operatorname{Pic} V_5^* \cong \mathbb{Z}\mathcal{O}_{V_5^*}(S)$. The birational morphism $\overline{\Phi} := \widehat{\Phi} \circ \pi^{-1} : \mathbb{Q}_0^3 \dashrightarrow V_5^*$ is given by the linear system $|\mathcal{O}_{\mathbb{Q}_0^3}(2) - \gamma|$, since $\widehat{S} + 2\widehat{Q}_\infty \sim 2\pi^*(Q_\infty) - \widehat{S}$. We set $\Delta_5^{(i)} := \overline{\Phi}(\Delta_2^{(i)})$ for $i = 1,\ 2$ and $\Delta_5 = \Delta_5^{(1)} \cup \Delta_5^{(2)}$. Then $\Delta_5^{(i)}$ is an effective $\mathbb{Q}$-Cartier divisor on V_5^*. By construction one sees that $\Delta_5^{(1)}$(resp. $\Delta_5^{(2)}$) is a smooth rational surface (resp. rational surface with a rational double point of A_1-type). In particular, Δ_5 is a hyperplane section of V_5^*. In fact, we obtain $\widehat{\Phi}^*\Delta_5 \sim \widehat{\Phi}^* S \sim \widehat{S} + 2\widehat{Q}_\infty$.

2.8 Lemma. *(V_5^*, Δ_5) is a compactification of $\mathbb{C}^3$ with $\mathrm{b}_2(V_5^*) = 1$.*

Proof. We put $\gamma^{(0)} := \gamma \cap (\mathbb{Q}_0^3 - (\Delta_2^{(1)} \cup \Delta_2^{(2)})) \subset Q_\infty^{(0)} := Q_\infty \cap \{x_0 \neq 0\}$. We put $z_i := \dfrac{x_i}{x_0}$ for $1 \le i \le 4$. Then one sees that $W^{(0)} := \mathbb{Q}_0^3 - (\Delta_2^{(1)} \cup \Delta_2^{(2)}) \cong \mathbb{C}^3$, $Q_\infty^{(0)} \cong \mathbb{C}^2$ and $\gamma^0 \cong \mathbb{C}$. Then the defining equations of $W^{(0)}$, $Q_\infty^{(0)}$ and $\gamma^{(0)}$ are given as follows:

$$W^{(0)} = \{z_3 = z_1 z_2\} \subset \mathbb{C}^4 \tag{1}$$

$$Q_\infty^{(0)} = \{z_3 = z_1 z_2,\ z_4 = 0\} \tag{2}$$

$$\gamma^{(0)} = \{z_3 = z_1 z_2, z_2 = z_1^2, z_4 = 0\}, \tag{3}$$

where (z_1, z_2, z_3, z_4) is a holomorphic coordinates of $\mathbb{C}^4$.

Let us consider the following polynomial automorphism α

$$\theta_1 : \begin{cases} w_1 = z_1 \\ w_2 = z_2 - z_1^2 \\ w_3 = z_3 - z_1 z_2 \\ w_4 = z_4 \end{cases}$$

Then we obtain

$$W^* := \alpha(W^{(0)}) = \mathbb{C}^3 : (w_1, w_2, w_3) \tag{1}$$

$$Q_\infty^* := \alpha(Q_0^{(\infty)}) = \{(w_1, w_2, w_3) \in \mathbb{C}^3 | w_3 = 0\} \tag{2}$$

$$\gamma^* := \alpha(\gamma^{(0)}) = \{(w_1, w_2, w_3) \in \mathbb{C}^3 | w_2 = w_3 = 0\}, \tag{3}$$

Let $B_{\gamma^*}(\mathbb{C}^3)$ be the blowing up of $\mathbb{C}^3$ along γ^* and $\widehat{Q}_\infty^*$ be the proper transform of Q_∞^*. Then it is easy to see that $B_{\gamma^*}(\mathbb{C}^3) - \widehat{Q}_\infty^* \cong \mathbb{C}^3$. Then we obtain that $V_5^* - (\Delta_5^{(1)} \cup \Delta_5^{(2)}) \cong \widehat{\mathbb{Q}_0^3} - (\widehat{\Delta}_2^{(1)} \cup \widehat{\Delta}_2^{(2)} \cup \widehat{Q}_\infty) \cong B_{\gamma^*}(\mathbb{C}^3) - \widehat{Q}_\infty^* \cong \mathbb{C}^3$, where $\widehat{\Delta}_2^{(i)}$ is the proper transform of $\Delta_2^{(i)}$ in $\widehat{\mathbb{Q}_0^3}$ (cf.[8]). Thus (V_5^*, Δ_5) is a singular

Fano compactification of $\mathbb{C}^3$ of index two and $\mathrm{b}_2(V_5^*) = 1$. In particular, one has $\mathrm{b}_4(V_5^*) = \mathrm{b}_4(\Delta_5) = \mathrm{b}_4(\Delta_5^{(1)}) + \mathrm{b}_4(\Delta_5^{(2)}) = 2$ and $\mathrm{b}_1(V_5^*) = \mathrm{b}_3(V_5^*) = 0$. $\square$

Therefore we have finally the following

2.9 Proposition. (V_5^*, Δ_5) *is a singular Fano compactification of* $\mathbb{C}^3$ *of index* $r = 2$ *with a small hypersurface singularity of* A_1*-type, in particular,* $\mathrm{b}_2(V_5^*) = 1$ *and* $\mathrm{b}_4(V_5^*) = 2$.

2.10 Remark Let V be a Fano threefold of degree 5 in $\mathbb{P}^6$ with at worst hypersurface terminal singularities. Then V is smooth if $\mathrm{b}_2(V) = \mathrm{b}_4(V) = 1$ (cf.[3],[20]).

3. Fano threefold V_{22}^* of degree 22 in $\mathbb{P}^{13}$ with one small singularity of A_1-type.

Next we shall give a detailed structure of the non-normal del Pezzo surface S constructed in §2. Let us recall the normailization $\nu := \widehat{\Phi}|_{\widehat{S}} : \widehat{S} \longrightarrow S$ and the analytic inverse image $\nu^{-1}(E) = \Sigma$. Then anti-dualizing sheaf $\omega_S^{-1} = \mathcal{O}_S(S)$ is ample, that is, S is a non-normal del Pezzo surface with non-normal locus E.

(3.1) Lemma ([24; (3.35)]**).**

(i) $h^0(\mathcal{O}_\Sigma) = 1,\ h^1(\mathcal{O}_\Sigma) = 0$.

(ii) $\chi(\mathcal{O}_{\widehat{S}}) = 1,\ (\nu^*\omega_Y \cdot \Sigma) = -2$.

(iii) $(\omega_S \cdot E) = -1$ *and* E *is irreducible reduced, in particular,* $E \cong \mathbb{P}^1$.

(3.2) Lemma. $h^0(\nu^*\omega_S^{-1}) = h^0(\omega_S^{-1}) + 1$.

Proof. Let us consider an exact sequence (cf.[24; (3.34.2)]):

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \nu_*\mathcal{O}_{\widehat{S}} \longrightarrow \omega_S^{-1} \otimes \omega_E \longrightarrow 0 \tag{3.2.1}$$

By operating $\otimes\omega_S^{-1}$ on (1.2.1), we obtain

$$0 \longrightarrow \omega_S^{-1} \longrightarrow \nu_*\mathcal{O}_{\widehat{S}} \otimes \omega_S^{-1} \longrightarrow \omega_S^{-1} \otimes \omega_E \longrightarrow 0 \tag{3.2.2}$$

By the projection formula and the Serre duality theorem, we have:

$$\begin{aligned} H^0(S; \nu_*\mathcal{O}_{\widehat{S}} \otimes \omega_S^{-1}) &\cong H^0(S; \nu_*\mathcal{O}_{\widehat{S}}(\nu^*\omega_S^{-1})) \\ &\cong H^0(\widehat{S}; \nu^*\omega_S^{-1}) \end{aligned}$$

and

$$\begin{aligned} H^0(S;\omega_S^{-1}\otimes\omega_E) &\cong H^1(E;\mathcal{O}_E\otimes\omega_S) \\ &\cong H^1(E;\mathcal{O}_E(-1)) \\ &\cong H^0(E;\mathcal{O}_E) \\ &\cong H^0(\mathbb{P}^1;\mathcal{O}_{\mathbb{P}^1}) \\ &\cong \mathbb{C}. \end{aligned}$$

Since $H^1(S;\omega_S^{\otimes -n})=0$ for $n>0$ by Goto-Mori-Reid (cf.[24], [31]) we have an exact sequence

$$0\longrightarrow H^0(S;\omega_S^{-1})\xrightarrow{\nu^*} H^0(\widehat{S};\nu^*\omega_S^{-1})\longrightarrow H^0(E;\mathcal{O}_E)\longrightarrow 0 \tag{3.2.3}$$

This proves the lemma. $\square$

We set $\mathcal{L}:=2\widehat{Q}_\infty+\widehat{S}$. From the following exact sequence

$$0\longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty)\xrightarrow{\iota}\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\mathcal{L})\longrightarrow\mathcal{O}_{\widehat{S}}(\mathcal{L})\longrightarrow 0$$

one obtain the following:

$$0\longrightarrow \iota^*H^0(\widehat{\mathbb{Q}_0^3},\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\mathcal{L}))\longrightarrow H^0(\widehat{S},\mathcal{O}_{\widehat{S}}(\mathcal{L}))\longrightarrow H^1(\widehat{\mathbb{Q}_0^3},\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty))\longrightarrow 0 \tag{3.2.4}$$

(3.3) Claim. $H^0(\widehat{S},\mathcal{O}_{\widehat{S}}(\mathcal{L}))=H^0(\widehat{S},\mathcal{O}_{\widehat{S}}(\Sigma+3F))=H^0(\widehat{S},\nu^*\omega_S^{-1})=\mathbb{C}^7$.

In fact, one has $\mathcal{L}|_{\widehat{S}}=(2\widehat{Q}_\infty+\widehat{S})|_{\widehat{S}}\sim\Sigma+3F$. On the other hand, since $\nu^*\omega_S^{-1}$ is ample on $\widehat{S}\cong\mathbb{F}_1$ with $\deg\nu^*\omega_S^{-1}=5$, one has $\nu^*\omega_S^{-1}\sim\mathcal{O}_{\widehat{S}}(\Sigma+3F)$. $\square$

(3.4) Claim. $H^1(\widehat{\mathbb{Q}_0^3},\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty))\cong H^0(E,\mathcal{O}_E)$.

In fact, consider the following exact sequences

$$0\longrightarrow\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}\longrightarrow\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{Q}_\infty)\longrightarrow\mathcal{O}_{\widehat{Q}_\infty}(-\widehat{f})\longrightarrow 0 \tag{3.4.1}$$

$$0\longrightarrow\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{Q}_\infty)\longrightarrow\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty)\longrightarrow\mathcal{O}_{\widehat{Q}_\infty}(-2\widehat{f})\longrightarrow 0 \tag{3.4.2}$$

$$0\longrightarrow\mathcal{O}_{\widehat{Q}_\infty}(-\widehat{f})\longrightarrow\mathcal{O}_{\widehat{Q}_\infty}\longrightarrow\mathcal{O}_{\widehat{f}}\longrightarrow 0 \tag{3.4.3}$$

$$\text{(3.4.4)} \qquad 0 \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(-2\widehat{f}) \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(-\widehat{f}) \longrightarrow \mathcal{O}_{\widehat{f}} \longrightarrow 0$$

Since $\widehat{Q}_\infty \cong \mathbb{P}^1 \times \mathbb{P}^1$, one has

$$H^1(\widehat{Q}_\infty; \mathcal{O}_{\widehat{Q}_\infty}(-2\widehat{f})) \cong H^0(\widehat{f}; \mathcal{O}_{\widehat{f}}) \overset{(\Phi|_{\widehat{f}})_*}{\cong} H^0(E; \mathcal{O}_E).$$

by (2.12.4). On the other hand, one has easily $H^1(\widehat{\mathbb{Q}^3_0}, \mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(\widehat{Q}_\infty)) = 0$. This yields $H^1(\widehat{\mathbb{Q}^3_0}, \mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(2\widehat{Q}_\infty) \cong H^1(\widehat{Q}_\infty; \mathcal{O}_{\widehat{Q}_\infty}(-2\widehat{f}))$ by (3.4.2). Hence we have the claim $\square$

By (3.2.3) ,(3.3) and (3.4) , we obtain the following

(3.5) Lemma. $\iota^* H^0(\widehat{\mathbb{Q}^3_0}, \mathcal{O}_{\widehat{\mathbb{Q}^3_0}}(\mathcal{L})) \cong \nu^* H^0(S; \omega_S^{-1}) \cong \mathbb{C}^6$

Finally we shall give a basis $\{\nu^* h_0, \nu^* h_1, \dots, \nu^* h_5\}$ of $\nu^* H^0(S; \omega_S^{-1}) \cong \mathbb{C}^6$ explicitly below. Let $U_i := \{(u_i, v_i) \in \mathbb{C}^2\}$ $(i = 0, 1, 2, 3)$ be coordinates covering of $\widehat{S} \cong \mathbb{F}_1$ with

$$\begin{cases} u_1 = u_0^{-1} \\ v_1 = u_1 v_0, \end{cases} \begin{cases} v_2 = v_1^{-1} \\ u_2 = u_1, \end{cases} \begin{cases} u_3 = u_2^{-1} \\ v_3 = u_2^{-1} v_2, \end{cases} \begin{cases} v_0 = v_3^{-1} \\ u_0 = u_3, \end{cases}$$

on $U_i \cap U_{i+1} \cong \mathbb{C} \times \mathbb{C}^*$, $(i = 0, 1, 2)$ and $U_3 \cap U_0 \cong \mathbb{C} \times \mathbb{C}^*$ respectively.

Let F, F_∞ (resp. Σ, Σ_∞) be the fibers (resp. sections) defined as follows:

(1) $F \cap U_i = \{u_i = 0\}$ $(i = 0, 3)$ and $F_\infty \cap U_i = \{u_i = 0\}$ $(i = 1, 2)$.

(2) $\Sigma \cap U_i = \{v_i = 0\}$ $(i = 0, 1)$ and $\Sigma_\infty \cap U_i = \{v_i = 0\}$ $(i = 2, 3)$.

Taking into account that $\nu^* H^0(S; \omega_S^{-1}) \subset H^0(\widehat{S} : \mathcal{O}(\Sigma + 3F))$, we may assume that $\nu^* h_i$'s are written as follows:

$$\begin{cases} \nu^* h_0 &= u_0^3 v_0 = v_1 = 1 = u_3^3 \\ \nu^* h_1 &= u_0^2 v_0 = u_1 v_1 = u_2 = u_3^2 \\ \nu^* h_2 &= u_0 v_0 = u_1^2 v_1 = u_2^2 = u_3 \\ \nu^* h_3 &= v_0 = u_1^3 v_1 = u_2^3 = 1 \\ \nu^* h_4 &= u_0^2 = 1 = v_2 = u_3^2 v_3 \\ \nu^* h_5 &= 1 = u_1^2 = u_2^2 v_2 = v_3 \end{cases}$$

Then $\nu = \widehat{\Phi}|_{\widehat{S}} : \widehat{S} \longrightarrow S = V_5^* \cap \{z_6 = 0\} \subset \mathbb{P}^5$ is given by $(\nu^* h_0 : \nu^* h_0 : \nu^* h_1 : \dots : \nu^* h_5 : 0)$, where $(z_0 : z_1 : \dots : z_5 : z_6)$ be the homogeneous coordinates of $\mathbb{P}^6$. We set $x_i := \dfrac{z_i}{z_5}$ $(0 \le i \le 4)$. Then we have the local defining equation

$$\text{(3.6.1)} \qquad S^{(0)} := S \cap \{z_5 \neq 0\} \cong \{(x_2, x_3.x_4) \in \mathbb{C}^3 | x_2^2 = x_3^2 x_4\}$$

(3.6.2.)

$$E^{(0)} := E \cap \{z_5 \neq 0\} \cong \{(x_2, x_3.x_4) \in \mathbb{C}^3 | x_2 = x_3 = 0\}$$

To construct V_{22}^* we need the local defining equations of $S^{(0)}$ and $E^{(0)}$ in (3.6.1), (3.6.2). First we take a smooth rational curve $\Gamma \subset S \subset V_5^* \subset \mathbb{P}^6$ of degree 5 satisfying $\Gamma \cap E = \{t^*\}$ (double points), where $\nu^{-1}(t^*) \cap U_0 = \{(-1,0),(1,0)\}$. Such a rational curve Γ exists. In fact, consider a smooth rational curve $\widehat{\Gamma} \sim \Sigma + 3F$ in $\widehat{S}$ given as follows:

$$\widehat{\Gamma} \cap U_0 = \{v_0 = (u_0+1)^2\}$$
$$\widehat{\Gamma} \cap U_1 = \{u_1^3 v_1 = (u_1+1)^2\}$$
$$\widehat{\Gamma} \cap U_2 = \{u_2^3 = v_2(u_2+1)^2\}$$
$$\widehat{\Gamma} \cap U_3 = \{1 = v_3(u_3+1)^2\}$$

Then the curve $\Gamma := \nu(\widehat{\Gamma}) \subset S$ is desired. Then the local defining equation of Γ is given by $\Gamma^{(0)} := \Gamma \cap U_0 \cong \{x_2 = \frac{1}{2}(x_2 - x_3 - 1)x_3, x_4 = \frac{1}{4}(x_3 - x_4 - 1)^2\}$ in the affine part $U_0 \cong \mathbb{C}^2$.

After the following coordinates transformation on $\mathbb{C}^3$,

$$\theta_2 : \begin{cases} x = x_2 - \frac{1}{2}(x_3 - x_4 - 1)x_3 \\ y = \frac{1}{2}(x_3 - x_4 - 1) \\ z = x_4 - \frac{1}{4}(x_3 - x_4 - 1)^2 \end{cases}$$

one has the new local defining equations:

$$(\#) \quad \begin{aligned} S^{(0)} =& \{(x,y,z) \in \mathbb{C}^3 | x^2 + 2xy\{z + (y+1)^2\} - \{z + (y+1)^2\}z = 0\} \\ E^{(0)} =& \{(x,y,z) \in \mathbb{C}^3 | x = z + (y+1)^2 = 0\} \\ \Gamma^{(0)} =& \{(x,y,z) \in \mathbb{C}^3 | x = z = 0\} \\ t^* =& (0,-1,0) \end{aligned}$$

Now there are two lines F_1, F_2 in S passing through t^*, in fact, these are written as follows:

$$(F_1) : \begin{cases} x = 2(1-y^2) \\ y = 1 - y^2 \end{cases}, \quad (F_2) : \begin{cases} x = -2(1+y)^2 \\ y = 1 - y^2 \end{cases}$$

Then one has $F_1 \cap \Gamma = \{(0,1,0),(0,-1,0) = t^*\}$ and $F_2 \cap \Gamma = \{t^*\}$.

Let $\sigma : \overline{V_5}^* \longrightarrow V_5^*$ be the blowing up with center Γ and $\overline{D} = \mathrm{Exc}(\sigma)$ the exceptional set. Let $\overline{S}$ (resp. $\overline{E}$) be the proper transform of S (resp. E) in $\overline{V}_5^*$ and let $\overline{F}_i$ $(i = 1,2)$ be the proper transform of F_i in $\overline{S}$.

(3.7) Lemma. $\operatorname{Sing}\overline{S} = \overline{E} \cup \{\overline{t}^*\}$, *where the isolated singular point* $\overline{t}^* \in \overline{S}$ *is the rational double point of* A_1*-type.*

Proof. Look at the local defining equation (#), and let $\sigma_0 : B_{\Gamma^{(0)}}(\mathbb{C}^3) \longrightarrow \mathbb{C}^3$ be the blowing up of $\mathbb{C}^3$ along $\Gamma^{(0)} = \{x = z = 0\}$. Let $\{W_i \cong \mathbb{C}^3_{(u_i,v_i,w_i)}\}$ be an open covering of $B_{\Gamma^{(0)}}(\mathbb{C}^3) = W_1 \cup W_2$ satisfying the relation:

$$\begin{cases} x = u_1 = u_2 w_2 \\ y = v_1 = v_2 \\ z = u_1 w_1 = u_2 \\ w_1 w_2 = 1 \ \ on \ \ W_1 \cap W_2 \end{cases}$$

An easy computation shows that the local defining equations of $\overline{S}$ and $\overline{E}$ are given by

$$(\overline{S}) \ : \ \begin{cases} u_1 + 2v_1[u_1w_1 + (v_1+1)^2] = w_1[u_1w_1 + (v_1+1)^2]^2 \ \ on \ \ W_1 \\ [u_2 + (v_2+1)^2 - v_2w_2]^2 = w_2^2(u_2 + v_2^2) \ \ on \ \ W_2 \end{cases}$$

$(\overline{E})$: $w_2 = u_2 + (v_2+1)^2 = 0$ *on* W_2

It is easy to check that

$$\operatorname{Sing}\overline{S} \cap W_2 = \{w_2 = u_2 + (v_2+1)^2 = 0\} \cup \{(u_2, v_2, w_2) = (0, -1, 2) := \overline{t^*}\}.$$

Next we shall show that $\overline{S}$ has an (isolated) rational double point of A_1-type. Recall the local defining equation of $\overline{S}$ on W_2. The following coordinates transformation

$$\begin{cases} u = \dfrac{u_2 + (v_2+1)^2}{w_2} \\ v = v_2 \\ w = w_2 \end{cases}$$

yields

$$w^2(x-y)^2 = w^2(uw - 2v - 1).$$

Thus the defining equation of $\overline{S}$ near the point $(u, v, w) = (0, -1, 2)$ is given by

$$(u - v - 1)^2 = u(w - 2)$$

This shows that $\overline{S}$ has a rational double point of A_1-type at $\overline{t^*} \in W_1$. $\square$

(3.8) Corollary. *$\overline{S}|_{\overline{D}} = \overline{\Gamma} + 2\overline{G}_0$, where $\overline{\Gamma}$ is the closure of $\overline{S} \cap \sigma^{-1}(\Gamma - \{t^*\})$ in $\overline{D}$, and $\overline{G}_0$ is a fiber of $\mathbb{P}^1$-bundle $\overline{D} \longrightarrow \Gamma$ over a smooth rational curve Γ.*

Proof. We put $u_2 = 0$ in the above defining equation $(\overline{S})$ in W_2. $\square$

(3.9) Lemma. *On the surface $\overline{S}$ one has*

(1) $\overline{F}_1 \cap \overline{G}_0 = \{\overline{t^*}\}$, *in particular* $(\overline{F}_1 \cdot \overline{G}_0) = \frac{1}{2}$

(2) $(\overline{F}_1 \cdot \overline{\Gamma}) = 1$,

(3) $\overline{E} \cap \overline{F}_1 = \emptyset$.

(4) $\overline{t^*} \notin \overline{F}_2$.

Proof. The defining equation of the pull back $F_i^* := \sigma^* F_i$ of F_i is given by

$$(F_1^*) : \begin{cases} u_1 = 2(1 - v_1^2) \\ w_1 = \frac{1}{2} \end{cases} \bigcup \begin{cases} u_1 = 0 \\ v_1 = -1 \end{cases}$$
$$(F_2^*) : \begin{cases} u_2 = 2(1 - v_1^2) \\ w_2(1 - v_2) + 2(1 + v_2) = 0 \end{cases} \bigcup \begin{cases} u_2 = 0 \\ v_2 = -1, \end{cases}$$

where

$$\begin{aligned} \overline{G}_0 &= \{u_i = v_i + 1 = 0\}, \\ \overline{F}_1 &= \{u_1 - 2(1 - v_1^2) = w_1 - \frac{1}{2} = 0\} = \{w_2 - 2 = u_2 - 1 + v_2^2 = 0\}, \\ \overline{F}_2 &= \{u_1 + 2(v_1 + 1)^2 = 2w_1(v_1 + 1) + v_1 - 1 = 0\} \\ &= \{u_2 - 1 + v_2^2 = w_2(1 - v_2) + 2(1 + v_2) = 0\}. \end{aligned}$$

Since $\overline{F}_1$ and $\overline{G}_0$ intersects at the rational double point $\overline{t^*} \in \overline{S} - \overline{E}$ of A_1-type, one has $(\overline{F}_1 \cdot \overline{G}_0) = \frac{1}{2}$ via the minimal resolution. The rest follows directly from these local equations. $\square$

(3.10) Lemma. *$\overline{D} \cong \mathbb{F}_2$, $N_{\overline{\Gamma}|\overline{V}_5^*} \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6)$.*

Proof. Since $\overline{S} \sim \sigma^* S - \overline{D}$, one has $K_{\overline{V}_5^*} = \sigma^* K_{V_5^*} + \overline{D} \sim -2\overline{S} - \overline{D}$. By (3.7) one has $\overline{S}|_{\overline{D}} \sim 2\overline{G}_0 + \overline{\Gamma}$. Thus $K_{\overline{D}} = (K_{\overline{V}_5^*} + \overline{D})|_{\overline{D}} = -2\overline{S}|_{\overline{D}} = -2(\overline{\Gamma} + 2\overline{G}_0)$. Since $\overline{\Gamma}$ is a smooth rational curve in $\overline{D}$, the adjunction formula yields $-2 = (K_{\overline{D}} + \overline{\Gamma}) \cdot \overline{\Gamma} = -\overline{\Gamma}^2 - 4$. Hence $\overline{\Gamma}^2 = -2$, that is, $\overline{D} \cong \mathbb{F}_2$. On the other hand since $c_1(N_{\overline{\Gamma}|\overline{V}_5^*}) = 10$, one has the normal bundle $N_{\overline{\Gamma}|V_5^*} = \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6)$. $\square$

(3.11) Lemma.

(1) $(\overline{D} \cdot \overline{E}) = 2$

(2) $(\overline{S} \cdot \overline{E}) = -1$

(3) $(-K_{\overline{V}_5^*} \cdot \overline{E}) = (-K_{\overline{V}_5^*} \cdot \overline{F}_1) = 0$

(4) $(\overline{D})^3 = -8$

(5) $(\overline{S})^3 = -2$

(6) $(\overline{S})^2 \cdot \overline{D} = 2$

(7) $\overline{S} \cdot (\overline{D})^2 = 3$

(8) $(-K_{\overline{V}_5^*})^3 = 18$

Proof. (1): Since Γ is tangent to E at the point t^*, we have the claim. (2): $(\overline{S} \cdot \overline{E}) = (\sigma^* S - \overline{D}) \cdot \overline{E} = 1 - 2 = -1$. (3):$(-K_{\overline{V}_5^*} \cdot \overline{E}) = (2\sigma^* S - \overline{D}) \cdot \overline{E} = 2 - (\overline{D} \cdot \overline{E}) = 0$. $(-K_{\overline{V}_5^*} \cdot \overline{F}_1) = (2\sigma^* S - \overline{D}) \cdot \overline{F}_1 = 2 - (\overline{D} \cdot \overline{F}_1) = 2 - (\overline{D}|_{\overline{S}} \cdot \overline{F}_1)_{\overline{S}} = 2 - (\overline{\Gamma} + 2\overline{G}_0 \cdot \overline{F}_1)_{\overline{S}} = 2 - (1 + 2(\frac{1}{2}) = 0$. (4): $c_1(N_{\overline{\Gamma}|V_5^*}) = (-K_{V_5^*} \cdot \Gamma) - 2 = (2S \cdot \Gamma) - 2 = 10 - 2 = 8$. (5):$(\overline{S})^3 = (\sigma^* S - \overline{D})^3 = (\sigma^* S)^3 + (\sigma^* D \cdot \overline{D} \cdot \overline{D}) - (\overline{D})^3 = 5 - 15 + 8 = -2$. (6):$(\overline{S})^2 \cdot \overline{D} = (\sigma^* S - \overline{D})^2 \cdot \overline{D} = -2(\sigma^* S \cdot \overline{D} \cdot \overline{D}) + (\overline{D})^3 = 10 - 8 = 2$.(6): $\overline{S} \cdot (\overline{D})^2 = (\sigma^* S - \overline{D}) \cdot \overline{D}^2 = -5 + 8 = 3$. (7):$(-K_{\overline{V}_5^*})^3 = (2\overline{S} + \overline{D})^3 = 18$. $\square$

(3.12) Proposition. $\operatorname{Bs}|2\sigma^* S - \overline{D}| = \emptyset$.

Let $\lambda : \tilde{V}_5^* \longrightarrow \overline{V}_5^*$ be the blowing up with center $\overline{E}$ and $\lambda^{-1}(\overline{E}) := \tilde{L}$ the exceptional divisor. Let $\tilde{S}$ (resp. $\tilde{D}$) be the proper transform of $\overline{S}$ (resp. $\overline{D}$).

(3.12.1) Sublemma. *$\tilde{S}$ has a unique isolated singularity $\tilde{t^*} = \lambda^{-1}(\overline{t}^*)$, which is rational double point of A_1-type.*

Proof. $\operatorname{Sing} \overline{S} = \overline{E} \cup \{\overline{t^*}\}$. Thus we have only to show that $\operatorname{Sing} \tilde{S} = \{\tilde{t^*}\}$. In fact, in the chart $W_2 \cong \mathbb{C}^3_{(u_2, v_2, w_2)}$, we take a coordinates transformation

$$\begin{cases} u_2 := u_2 + (v_2 + 1)^2 \\ v_2 := v_2 \\ w_2 := w_2, \end{cases}$$

then the local defining equations of $\overline{S}$ and $\overline{E}$ change as follows: $(\overline{S}) : \quad (u_2 - v_2 w_2)^2 = w_2^2(u_2 - 2v_2 - 1)$, $(\overline{E}) : \quad u_2 = w_2 = 0$. Thus the proper transform $\tilde{S}$ of $\overline{S}$ is given by $(u_2 - v_2)^2 = u_2 w_2 - v_2 - 1$. This has only a rational double point of A_1-type as a singularity. We note that $\tilde{S}$ is smooth near $\tilde{\Sigma}$. $\square$

(3.12.2) Sublemma. $\mathrm{Bs}\,|2\sigma^*S - \overline{D}| = \emptyset$ *on* $\overline{S}$.

Proof. Let $\mu : M \longrightarrow \tilde{S}$ be the minimal resolution with $\mathrm{Exc}(\mu) = B$, where B is a smooth rational curve with $B^2 = -2$. We set $\lambda := \lambda|_{\tilde{S}} : \tilde{S} \longrightarrow \overline{S}$ and $\tau := \lambda \circ \mu$. We put $\mathcal{L} := (2\sigma^*S - \overline{D})|_{\overline{S}}$. Then M is a ruled surface over $\mathbb{P}^1$, $\psi : M \longrightarrow \mathbb{P}^1$, with only one singular fiber $\psi^{-1}(0) = \widehat{F}_1^+ \cup B \cup \widehat{G}_0^+$ with $\mathrm{Pic}\, M \cong \mathbb{Z}[\widehat{E}^+] \oplus \mathbb{Z}[\widehat{G}_0^+] \oplus \mathbb{Z}[\widehat{F}_1^+] \oplus \mathbb{Z}[B]$, where $\widehat{\Theta}^+$, $\widehat{G}_0^+$, $\widehat{\Gamma}^+$, $\widehat{E}^+$ be the proper transforms of $\overline{\Theta}$,$\overline{G}_0$,$\overline{F}_1$, $\overline{\Gamma}$, $\overline{E}$ respectively. One has easily

$$(\widehat{F}_1^+)^2 = (\widehat{G}_0^+)^2 = -1,\ (B)^2 = -2,\ (B \cdot \widehat{F}_1^+) = (B \cdot \widehat{G}_0^+) = 1,\ (\widehat{F}_1^+ \cdot \widehat{G}_0^+) = 0.$$

Then one has $\mathcal{L} \sim 2\overline{\Theta} - (2\overline{G}_0 + \overline{\Gamma})$, where $\overline{\Theta} = \sigma^*S|_{\overline{S}}$, which can be assumed to be irreducible. Then we have $\tau^*\mathcal{L} \sim 2\widehat{\Theta}^+ - 2\widehat{G}_0^+ - B - \widehat{\Gamma}^+$. Since $\widehat{\Theta}^+ \sim \widehat{\Gamma}^+ \sim \widehat{E}^+ + 5\widehat{G}_0^+ + 4B + 3\widehat{F}_1^+$, we have $\tau^*\mathcal{L} \sim \widehat{E}^+ + 3\widehat{G}_0^+ + 3B + 3\widehat{F}_1^+$, and $(\tau^*\mathcal{L} \cdot \widehat{E}^+) = (\tau^*\mathcal{L} \cdot \widehat{F}_1^+) = (\tau^*\mathcal{L} \cdot B) = 0$, $(\tau^*\mathcal{L} \cdot \widehat{G}_0)^+ = 1$. Thus we conclude that $\mathrm{Bs}\,|\tau^*\mathcal{L}|$ on M. This implies that $\mathrm{Bs}\,|\mathcal{L}| = \emptyset$ on $\overline{S}$. Next let us consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}(\sigma^*S) \longrightarrow \mathcal{O}(2\sigma^*S - \overline{D}) \longrightarrow \mathcal{O}_{\overline{S}}(2\sigma^*S - \overline{D}) \longrightarrow 0$$

Since $H^1(\overline{V}_5^*; \mathcal{O}(\sigma^*S)) = 0$, one has a surjection

$$H^0(\overline{V}_5^*; \mathcal{O}(2\sigma^*S - \overline{D}) \longrightarrow H^0(\overline{S}; \mathcal{O}(2\sigma^*S - \overline{D})) \longrightarrow 0$$

Since $\mathrm{Bs}\,|\mathcal{L}| = \mathrm{Bs}\,|\mathcal{O}_{\overline{S}}(2\sigma^*S - \overline{D})| = \emptyset$, we have the claim. □

(3.12.3) Sublemma. $-K_{\overline{V}_5^*} \sim 2\sigma^*S - \overline{D}$ *is nef and big on* $\overline{V}_5^*$.

Proof. Assume that there is a curve R on $\overline{V}_5^*$ with $(2\sigma^*S - \overline{D} \cdot R) < 0$. Since $2\sigma^*S - \overline{D} = \sigma^* + \overline{S}$, one has $(\overline{S} \cdot R) < 0$, that is $R \subset \overline{S}$. By (3.12.2), $\mathcal{L}$ is semi-ample on $\overline{S}$, hence $0 \leq (\mathcal{L} \cdot R)_{\overline{S}} = (2\sigma^*S - \overline{D}) \cdot R$. This is a contradiction. thus $2\sigma^*S - \overline{D}$ is nef. On the other hand one has $(2\sigma^*S - \overline{D})^3 = 18$ by (3.11)-(8). Thus we have the claim. □

(Proof of (3.12)).

First we have $h^0(\mathcal{O}_{\overline{V}_5^*}(2) \otimes \mathcal{I}_\Gamma) \geq 17$ by the counting method. In fact, take general 11 points of Γ. There any smooth quadric hypersurface in $\mathbb{P}^6$ passing through these

11 points always contain Γ since Γ is of degree 5 in $\mathbb{P}^6$. Thus a general member $\overline{Z}$ of the linear system $|2\sigma^*S - \overline{D}|$ is a smooth K-3 surfaces. We may assume that $|2\sigma^*S - \overline{D}|$ has no fixed component on $\overline{Z}$. Thus the nef big divisor $(2\sigma^*S - \overline{D})|_{\overline{Z}}$ has no base points. on $\overline{Z}$. From the exact sequence:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(2\sigma^*S - \overline{D}) \longrightarrow \mathcal{O}_{\overline{Z}}(2\sigma^*S - \overline{D}) \longrightarrow 0$$

we have an surjection

$$H^0(\overline{V}_5^*; \mathcal{O}(2\sigma^*S - \overline{D})) \longrightarrow H^0(\overline{Z}; \mathcal{O}_{\overline{Z}}(2\sigma^*S - \overline{D})) \longrightarrow 0,$$

since $H^1(\overline{V}_5^*; \mathcal{O}_{\overline{V}_5^*}) = 0$. Thus $|2\sigma^*S - \overline{D}|$ has no base points on $\overline{Z}$. Since $\overline{Z}$ is general, we have finally the base-point freeness of $|2\sigma^*S - \overline{D}|$. $\square$

(3.13)Lemma.

(1) $N_{\overline{E}|\overline{V}_5^*} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$
(2) $N_{\overline{F}_1|\overline{V}_5^*} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$

Proof. (1): We have $\tilde{S} \sim \lambda^*\overline{S} - 2\tilde{L}$, $\tilde{D} \sim \lambda^*\overline{D}$ and $K_{\tilde{V}_5^*} \sim \lambda^*K_{\overline{V}_5^*} + \tilde{L} \sim -2\tilde{S} - 3\tilde{L} - \tilde{D}$.We set $\tilde{\Sigma} := \tilde{S}|_{\tilde{L}} \subset \tilde{S} \cap \tilde{L}$, which is a smooth rational curve. Since $c_1(N_{\overline{E}|\overline{V}_5^*}) = -2 + (-K_{\overline{V}_5^*} \cdot \overline{E}) = -2$. one has $(\tilde{L})^3 = -2$. Thus $(\tilde{\Sigma})^2_{\tilde{L}} = (\tilde{S}|_{\tilde{L}})^2 = (\lambda^*\overline{S} - 2\tilde{L})^2 \cdot \tilde{L} = -4\lambda^*\overline{S} \cdot (\tilde{L})^2 + 4(\tilde{L})^3 = -4(\overline{S} \cdot \overline{E})(\tilde{T} \cdot \tilde{L}) + 8 = 4$, where $\tilde{T}$ is a general fiber of the $\mathbb{P}^1$-bundle $\tilde{L} \longrightarrow \overline{E}$. On the other hand since $K_{\tilde{V}_5^*} + \tilde{L} = -2\tilde{S} - 2\tilde{L} - \tilde{D} = -(\lambda^*\sigma^*S + \tilde{S})$, one has $K_{\tilde{L}} = -\lambda^*\sigma^*S|_{\tilde{L}} - \tilde{S}|_{\tilde{L}} = -(\sigma^*S \cdot \overline{E})\tilde{T} - \tilde{\Sigma} = -\tilde{T} - \tilde{\Sigma}$. In particular $(\tilde{T} \cdot \tilde{\Sigma}) = -\tilde{T} \cdot (K_{\tilde{L}} + \tilde{T}) = 2$.

Let $\tilde{\Delta}_0$ be the negative section of the $\mathbb{P}^1$-bundle $\tilde{L} \longrightarrow \overline{E}$. Then we can write as $\tilde{\Sigma} \sim 2\tilde{\Delta}_0 + a\tilde{T}$ for some $0 < a \in \mathbb{Z}$. Since $(\tilde{\Sigma})^2_{\tilde{L}} = 4$, one has $\tilde{\Delta}_0^2 + a = 1$. From the relation $(\tilde{\Sigma} \cdot \tilde{\Delta}_0) = 2\tilde{\Delta}_0^2 + a \geq 0$ and $0 \geq \tilde{\Delta}_0^2 = 1 - a$, one has $a = 1$, 2, hence

$$\tilde{L} \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & if \;\; a = 1 \\ \mathbb{F}_1 & if \;\; a = 2 \end{cases}$$

Since $c_1(N_{\overline{E}|\overline{V}_5^*}) \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$, $(m \leq n \in \mathbb{Z})$ with $m + n = -2$, $\tilde{L} \cong \mathbb{F}_{n-m} \cong \mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$, hence $m = n = -1$ and $a = 1$. This proves the assertion (1).

(2): Now $\overline{F}_1$ passes through the rational double point (of A_1-type) $\overline{t}^* \in \overline{S} - \overline{E} \subset \overline{V}_5^* - \{p^*\}$ and $(-K_{\overline{V}_5^*} \cdot \overline{F}_1) = 0$. Thus by [14], we have the assertion. $\square$

(3.14) Corollary. *Let $\tilde{F}_1$ (resp. $\tilde{G}_0$, $\tilde{\Gamma}$) be the proper transform of $\overline{F}_1$ (resp. $\overline{G}_0$, $\overline{\Gamma}$). Then*

(1) $(\tilde{\Sigma}^2)_{\tilde{S}} = -3$
(2) $(\tilde{F}_1^2)_{\tilde{S}} = (\tilde{G}_0^2)_{\tilde{S}} = -\frac{1}{2}$
(3) $(\tilde{F}_1 \cdot \tilde{G}_0)_{\tilde{S}} = \frac{1}{2}$
(4) $\tilde{D}|_{\tilde{S}} \sim \tilde{\Gamma} + 2\tilde{G}_0$

Proof. We have $(\tilde{\Sigma}^2)_{\tilde{S}} = (\tilde{L}|_{\tilde{S}} \cdot \tilde{L}|_{\tilde{S}})_{\tilde{S}} = (\tilde{L}^2 \cdot \tilde{S}) = \lambda^* \overline{S} - 2\tilde{L}(\tilde{L})^2 = -(\overline{S} \cdot \overline{E}) - 2(\tilde{L})^3) = 1 - 4 = -3$. This shows (1). By construction there is a $\mathbb{P}^1$-ruling $\tilde{\psi} : \tilde{S} \longrightarrow \mathbb{P}^1$ which has a unique singular fiber $\tilde{\psi}^{-1}(0) := \tilde{F}_1 \cup \tilde{G}_0$. Since $\mathrm{Sing}\, \tilde{S} = \tilde{t^*} = \tilde{F}_1 \cap \tilde{G}_0$ is the rational bouble point of A_1-type, one has the claims (2) and (3) via minimal resolution (see also (3.13). The claim (4) follows from the fact $\tilde{D} \cap \tilde{S} = \tilde{\Gamma} \cup \tilde{G}_0$.

(3.15) Corollary. *There is a birational contraction $\varphi : \tilde{V}_5^* \longrightarrow \overline{V}_5^+$ of $\tilde{L}$ to a smooth rational curve $\overline{E}^+ = \varphi(\tilde{L})$.*

Proof. Since $K_{\tilde{V}_5^*} \sim -2\tilde{S} - 3\tilde{L} - \tilde{D}$, $-2\tilde{\Delta}_0 - 2\tilde{T} \sim K_{\tilde{L}} \sim (K_{\tilde{V}_5^*} + \tilde{L})|_{\tilde{L}} = -2\tilde{S}|_{\tilde{L}} - 2\tilde{L}|_{\tilde{L}} - 2\tilde{T} = -4\tilde{\Delta}_0 - 4\tilde{T} - 2\tilde{L}|_{\tilde{L}} - \tilde{D}|_{\tilde{L}} = -2\tilde{\Sigma} - 2\tilde{L}|_{\tilde{L}}$. This yields $\tilde{L}|_{\tilde{L}} \sim -\tilde{\Delta}_0 - \tilde{T}$ .Thus $\tilde{L} \cong \mathbb{P}^1 \times \mathbb{P}^1$ contracts to another direction, that is $\tilde{\Delta}_0$-direction. This contraction morphism φ is desired. $\square$

We set $\overline{S}^+ = \varphi(\tilde{S})$, $\overline{D}^+ = \varphi(\tilde{D})$, $\overline{E}^+ := \varphi(\tilde{L})$, $\overline{F}_1^+ = \varphi(\tilde{F}_1)$ and $\overline{G}_0^+ := \varphi(\tilde{G}_0)$. We set $\chi_1 := \varphi \circ \lambda^{-1} : \overline{V}_5^* \dashrightarrow \overline{V}_5^+$, which is a birational map with $\overline{V}_5^* - \overline{E} \overset{\chi_1}{\cong} \overline{V}_5^+ - \overline{E}^+$. On the other hand we have easily $\tilde{S} \overset{\varphi}{\cong} \overline{S}^+$. Thus we have

(3.16) Lemma.

(1) $(K_{\overline{V}_5^+} \cdot \overline{E}^+) = (K_{\overline{V}_5^+} \cdot \overline{F}^+) = 0.$
(2) $K_{\overline{V}_5^+} \sim -2\overline{S}^+ - \overline{D}^+.$
(3) $(\overline{S}^+ \cdot \overline{E}^+) = 1.$
(4) $(\overline{D}^+ \cdot \overline{E}^+) = -2$, *hence* $\overline{E}^+ \subset \overline{D}^+$.
(5) $\mathrm{mult}_{\overline{E}^+} \overline{D}^+ = 2.$
(6) $\overline{S}^+|_{\overline{S}^+} = -(\overline{E}^+ + 2\overline{G}_0^+)$
(7) $\overline{D}^+|_{\overline{S}^+} = 2\overline{E}^+ + 2\overline{G}_0^+ + \overline{\Gamma}^+$, *where* $\overline{\Gamma}^+ = \overline{E}^+ + 5\overline{G}_0^+ + 3\overline{F}_1^+$ *in* $\overline{S}^+$.
(8) $(\overline{G}_0^+)^2_{\overline{S}^+} = (\overline{F}_1^+)^2_{\overline{S}^+} = -\frac{1}{2}$, $(\overline{G}_0^+ \cdot \overline{F}_1^+)_{\overline{S}^+} = \frac{1}{2}$, $(\overline{G}_0^+ \cdot \overline{E}^+) = 1$, $(\overline{E}^+)^2_{\overline{S}^+} = -3.$
(9) $(\overline{S}^+ \cdot \overline{F}_1^+) = -1$, $(\overline{D}^+ \cdot \overline{F}_1^+) = 2.$

Proof. (1): This follows from (3.14)-(3). (2): This follows from $K_{\overline{V}_5^*} \sim -2\overline{S} - \overline{D}$. (3):$(\overline{S}^+ \cdot \overline{E}^+) = -(\overline{S} \cdot \overline{E}) = 1$ by (3.14)-(2). (4): $0 = (-K_{\overline{V}_5^+} \cdot \overline{E}^+) = (\overline{S}^+ \cdot \overline{E}^+) + (\overline{D}^+ \cdot \overline{E}^+) = 2 + (\overline{D}^+ \cdot \overline{E}^+)$. (5):$\varphi^* K_{\overline{V}_5^+} = K_{\tilde{V}_5^*} - \tilde{L} = -2\tilde{S} - 4\tilde{L} - \tilde{D}$. On the other hand since $K_{\overline{V}_5^+} = -2\overline{S}^+ - \overline{D}^+$ ans $\varphi^* \overline{S}^* = \tilde{S} + \tilde{L}$, $\varphi^* \overline{D}^+ = -2\varphi^* \overline{S}^+ - \varphi^* K_{\overline{V}_5^+} = \tilde{D} + 2\tilde{L}$. This shows the claim. (6): Let $\mu : M \longrightarrow \overline{S}^+$ be the minimal resolution with the exceptional set $B := \mu^{-1}(\overline{t}^+)$, where $\overline{t}^+ := \varphi(\tilde{t^*})$. Here since $\overline{S}^+ \overset{\varphi}{\cong} \tilde{S}$, M can be considered the same one as in (3.12.2), via $M \xrightarrow{\mu} \overline{S}^+ \overset{\varphi}{\cong} \tilde{S}$, so we use the same notation as in (3.12.2).

Let $\widehat{F}_1^+$, $\widehat{G}_0^+$, $\widehat{E}^+$, $\widehat{\Gamma}^+$ be the proper transforms of $\overline{F}_1^+$, $\overline{G}_0^+$, $\overline{E}^+, \overline{\Gamma}^+$ in M respectively. Then M is a ruled surface over $\mathbb{P}^1$ with only one singular fiber $\widehat{F}_1^+ \cup B \cup \widehat{G}_0^+$ with $\operatorname{Pic} M \cong \mathbb{Z}[\widehat{E}^+] \oplus \mathbb{Z}[\widehat{G}_0^+] \oplus \mathbb{Z}[\widehat{F}_1^+] \oplus \mathbb{Z}[B]$, where

$$(\widehat{F}_1^+)^2 = (\widehat{G}_0^+)^2 = -1,\ (B)^2 = -2,\ (B \cdot \widehat{F}_1^+) = (B \cdot \widehat{G}_0^+) = 1,\ (\widehat{F}_1^+ \cdot \widehat{G}_0^+) = 0.$$

Taking into an account that $\mu^* \overline{F}_1^+ = \widehat{F}_1^+ + \frac{1}{2}B$, $\mu^* \overline{G}_0^+ = \widehat{G}_0^+ + \frac{1}{2}B$, one has the claim. Since $\tilde{S} \overset{\varphi}{\cong} \overline{S}^+$, we have also $(\overline{E}^+)^2_{\overline{S}^+} = -3$ by (3.14)-(1), hence we have (8). Since $\mu^* K_{\overline{S}^+} = K_M \sim -2\widehat{E}^+ - 5\widehat{G}_0^+ - 4C - 3\widehat{F}_1^+$, one has $K_{\overline{S}^+} \sim -2\widehat{E}^+ - 5\widehat{G}_0^+ - 3\widehat{F}_1^+$. On the other hand $K_{\overline{S}^+} = (K_{\overline{V}_5^+} + \overline{S}^+)|_{\overline{S}^+} = -(\overline{S}^+|_{\overline{S}^+} + \overline{D}^+|_{\overline{S}^+})$ This implies that $\overline{S}^+|_{\overline{S}^+} \sim 3\overline{G}_0^+ + 3\overline{F}_1^+ - \Gamma^+$, since $\overline{D}^+|_{\overline{S}^+} \sim 2\overline{E}^+ + 2\overline{G}_0^+ + \overline{\Gamma}^+$. Since $\widehat{\Gamma}^+ = \widehat{E}^+ + 5\widehat{G}_0^+ + 4B + 3\widehat{F}_1^+$, one gets $\overline{\Gamma}^+ \sim \overline{E}^+ + 5\overline{G}_0 + 3\overline{F}_1^+$. (7): The assertion follows from (5), (3.14)-(4) and the fact that $\overline{S}^+$ is smooth along $\overline{E}^+$. (9): First One sees $(\overline{S}^+ \cdot \overline{F}_1^+) = (\overline{S}^+|_{\overline{S}^+} \cdot \overline{F}_1^+) = -(\overline{E}^+ + 2\overline{G}_0^+) \cdot \overline{F}_1^+ = -2(\overline{G}_0^+ \cdot \overline{F}_1^+) = -1$. Next $(\overline{D}^+ \cdot \overline{F}_1^+) = (\overline{D}^+|_{\overline{S}^+} \cdot \overline{F}_1^+) = (3\overline{E}^+ + 7\overline{G}_0^+ + 3\overline{F}_1^+) \cdot \overline{F}_1^+ = \frac{7}{2} - \frac{3}{2} = 2$.

(3.17) Lemma.

(1) $(\overline{S}^+)^3 = -1$.

(2) $(\overline{D}^+)^2 \cdot \overline{S}^+ = 7$.

(3) $(\overline{S}^+)^2 \cdot \overline{D}^+ = 0$.

(4) $(\overline{D}^+)^3 = -16$.

(5) $(-K_{\overline{V}_5^+})^3 = 18$.

Proof. (1):$(\overline{S}^+)^3 = (\overline{E}^+ + 2\overline{G}_0^+)^2 = -1$ by (3.16)-(8). (2):$(\overline{D}^+)^2 \cdot \overline{S}^+ = (2\overline{E}^+ + 2\overline{G}_0^+ + \overline{\Gamma}^+)^2 = 7$ by (3.16)-(7),-(8). (3):$(\overline{S}^+)^2 \cdot \overline{D}^+ = (\overline{D}^+|_{\overline{S}^+} \cdot \overline{S}^+|_{\overline{S}^+}) = -(3\overline{E}^+ + 7\overline{G}_0^+ + 3\overline{F}_1^+)(\overline{E}^+ + \overline{G}_0^+) = 0$. (4): Since $18 = (-K_{\overline{V}_5^*})^3 = (-K_{\overline{V}_5^+})^3 = 8(\overline{S}^+)^3 + 12(\overline{S}^+)^2 \cdot \overline{D}^+ + 6\overline{S}^+ \cdot (\overline{D}^+)^2 + (\overline{D}^+)^3$, we have $(\overline{D}^+)^3 = -16$. $\square$

By an argument similar to (3.13)-(2), we obtain

(3.18) Lemma. $N_{\overline{F}^+|\overline{V}_5^+} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Therefore we have a birational map $\chi_2 : \overline{V}_5^+ \dashrightarrow \overline{V}_5^{++}$ and a smooth rational curve $\overline{F}_1^{++}$ such that

(1) $\chi_2 : \overline{V}_5^+ - \overline{F}_1^+ \cong \overline{V}_5^{++} - \overline{F}_1^{++}$

(2) $\overline{V}_5^+$ is smooth near the curve $\overline{F}_1^{++}$

The birational map χ_2 is called the flop (or $\overline{D}^+$-flop, [20]) along $\overline{F}_1^+$. Let S^{++}, D^{++}, be the proper transform of $\overline{S}^+$, $\overline{D}^+$ in V_5^{++}. We set $\chi_2(\overline{F}_1^+) = F_1^{++}$, $\chi_2((\overline{E}^+) = E^{++}$. Then one can see that S^{++} is smooth and $S^{++} \cong \mathbb{F}_3$ (see Pagoda in [14], [30]). Taking an account of (3.16) and elementary properties of the flop , we obtain the following

(3.19) Lemma.

(1) $K_{V_5^{++}} = -2S^{++} - D^{++}$.

(2) $D^{++}|_{S^{++}} = 3E^{++} + 7G_0^{++}$.

(3) $S^{++}|_{S^{++}} = -E^{++} - 2G_0^{++}$.

(4) $(S^{++} \cdot F_1^{++}) = 1$.

(5) $(D^{++} \cdot F_1^{++}) = -2$.

(6) $(S^{++} \cdot E^{++}) = 1$.

(7) $(S^{++} \cdot F_1^{++}) = 1$, *in particular,* S^{++} *intersects with* F_1^{++} *transversally at one point.*

Finally we set $\chi := \chi_2 \circ \chi_1$, then we obtain the following

(3.20) Proposition. *There is a projective threefold* V_5^{++} *and disjoint smooth rational curves* E^{++}, F_1^{++} *and a birational map* $\chi : V_5^* \dashrightarrow V_5^{++}$ *such that* $V_5^* - (\overline{F}_1 \cup \overline{E}) \overset{\chi}{\cong} V_5^{++} - (F_1^{++} \cup E^{++})$, *the birational map* χ *is called the* $\overline{D}$*-flop along* $\overline{E}$, $\overline{F}_1$.

(3.21) Corollary. $\mathrm{Bs}\,|D^{++} + 2S^{++}| = \emptyset$.

Proof. By (3.12), $\mathrm{Bs}\,| - K_{\overline{V}_5^*}| = |\overline{D} + 2\overline{S}| = \emptyset$. Since $\chi_*(-K_{\overline{V}_5^*}) = -K_{V_5^{++}} \sim D^{++} + 2S^{++}$. Thus we have the claim. $\square$

(3.22) Lemma.

(1) $(S^{++})^3 = 1$.

(2) $(D^{++})^2 \cdot S^{++} = 15$.

(3) $(S^{++})^2 \cdot D^{++} = -4$.

(4) $(D^{++})^3 = -32$.

(5) $(K_{V_5^{++}})^3 = -18$.

Proof. (1): $(S^{++})^3 = (S^{++}|_{S^{++}})^2 = (-E^{++} - 2G_0^{++})^2 = 1$. (2): $(D^{++})^2 \cdot S^{++} = (D^{++}|_{S^{++}})^2 = (3E^{++}+7G_0^{++})^2 = 15$. (3): $(S^{++})^2 \cdot D^{++} = (S^{++}|_{S^{++}} \cdot D^{++}|_{S^{++}}) = -(E^{++} + 2G_0^{++})(3E^{++} + 7G_0^{++}) = -4$. (4): $(K_{V_5^{++}})^3 = (K_{\overline{V}_5^{+}})^3 = -18$. (5): $18 = (-K_{V_5^{++}})^3 = (D^{++} + 2S^{++})^2 = 50 + (D^{++})^3$. $\square$

(3.33) Proposition. *There is a birational contraction $\Psi : V_5^{++} \longrightarrow V_{22}^*$ with* $\mathrm{Exc}(\Psi) = S^{++}$ *such that*

(1) V_{22}^* *is smooth near* $\ell := \Psi(S^{++}) \cong \mathbb{P}^1$,

(2) $(-K_{V_{22}^*} \cdot \ell) = 1$,

(3) $\mathrm{mult}_\ell\, D^* = 3$,

(4) $-K_{V_{22}^*} = -D^*$, *where* $D^* := \Psi_* D^{++}$,

(5) $(-K_{V_{22}^*})^3 = 22$.

In particular $\Psi : V_5^{++} \longrightarrow V_{22}^*$ *is a blowing up of* V_{22}^* *with the center* ℓ.

Proof. First we recall that E^{++} (resp. G_0^{++}) is the negative section (resp. a fiber) of the Hirzebruch surface $S^{++} \cong \mathbb{F}_3$. By (3.22)-(3), we have $(S^{++} \cdot G_0^{++}) = -1$. Thus we have the contraction $\Psi : V_5^{++} \longrightarrow V_{22}^*$ of S^{++} in a fiber direction. In particular V_{22}^* is smooth neat $\ell := \Phi(S^{++})$. Since $-2S^{++} - D^{++} = K_{V_5^{++}} = \Phi^* K_{V_{22}^*} + S^{++} = -\Phi^* D^* + S^{++}$, one has $\Phi^* D^* = D^{++} + 3S^{++}$, which says $\mathrm{mult}_\ell\, D^* = 3$. $(-K_{V_{22}^*})^3 = (\Phi^* D^*)^3 = (D^{++} + 3S^{++})^3 = 22$. Next $-1 = (D^{++} + 3S^{++}) \cdot (S^{++})^2 = \Phi^* D^* \cdot (S^{++})^2 = (D^* \cdot \ell) \cdot (S^{++} \cdot G_0^{++}) = -(D^* \cdot \ell)$. Thus we have $(-K_{V_{22}^*} \cdot \ell) = -(D^* \cdot \ell) = 1$. $\square$

(3.24) Lemma.

(1) $\dim H^0(V_5^{++}; \mathcal{O}(D^{++} + 3S^{++})) = 14$

(2) $\mathrm{Bs}\,|D^{++} + 3S^{++}| = \emptyset$. *In fact, the birational morphism* $\Psi : V_5^{++} \longrightarrow V_{22}^*$ *is given by the linear system* $|D^{++} + 3S^{++}|$.

Proof. (1): First we will show that $D^{++} + 3S^{++}$ is nef and big. In fact, assume that there is a curve R such that $(D^{++} + 3S^{++}) \cdot R < 0$. Since $D^{++} + 2S^{++}$ is semi-ample, one has $(S^{++} \cdot R) < 0$, that is, $R \subset S^{++}$. On the other hand, since $(D^{++} + 3S^{++})|_{S^{++}} = G_0^{++}$ on $S^{++} \cong \mathbb{F}_3$, $(S^{++} \cdot R) = (G_0^{++} \cdot R) \geq 0$, which is a contradiction. Hence $D^{++} + 3S^{++}$ is nef. Since $(D^{++} + 3S^{++}) = 22$, it is big.

Take a general member $Z^{++} \in |D^{++} + 2S^{++}|$, which is a smooth K-3 surface. Consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}(S^{++}) \longrightarrow \mathcal{O}(D^{++} + 3S^{++}) \longrightarrow \mathcal{O}_{Z^{++}}(D^{++} + 3S^{++}) \longrightarrow 0$$

Since $\mathrm{h}^0(\mathcal{O}_{V_5^{++}}(S^{++})) = 1$, $\mathrm{h}^1(\mathcal{O}_{V_5^{++}}(S^{++})) = 0$, one has

$$\mathrm{h}^0(\mathcal{O}_{V_5^{++}}(D^{++} + 3S^{++})) = 1 + \mathrm{h}^0(\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})).$$

Now since $D^{++} + 3S^{++}$ is nef big on Z^{++}, $\mathrm{h}^i(\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})) = 0$ for $i > 0$. The Riemann-Roch theorem says

$$h^0(\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})) = \frac{1}{2}(D^{++} + 3S^{++})^2(D^{++} + 2S^{++}) + 2 = 13$$

, hence $\mathrm{h}^0(\mathcal{O}_{V_5^{++}}(D^{++} + 3S^{++})) = 14$.

(2): Since $D^{++} + 2S^{++}$ is semi ample, one can take a general member $Z^{++} \in |D^{++} + 2S^{++}|$ such that Z^{++} contains no fixed component of $|D^{++} + 2S^{++}|$. Thus the nef and big line bundle $\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})$ has no fixed component on Z^{++}, hence has no basepoints. A surjection $H^0(\mathcal{O}(D^{++} + 3S^{++})) \longrightarrow H^0(\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})) \longrightarrow 0$ implies the claim. $\square$

Thus we have finally the following

(3.25) Proposition. *V_{22}^* is the Fano threefold of degree 22 in $\mathbb{P}^{13}$ of index $r = 1$ with a hypersurface terminal singularity of A_1-type at the point $p = \Phi(\chi(\sigma^{-1}(p)))$, in particular,* $\mathrm{b}_2(V_{22}^*) = 1$ *and* $\mathrm{b}_2(V_{22}^*) = 2$.

(3.26) Remark.

We set $\Delta_{22} := \Psi(\chi(\sigma^{-1}(\Delta_5)))$ and $W := V_{22}^* - \Delta_{22}$, which is an smooth affine threefold. By construction one sees that $V_5 - \Delta_5 - F_1 \cong V_{22}^* - \Delta_{22} - F_1^{++}$. In particular, $W \supset \mathbb{C}^2 \times \mathbb{C}^*$ and has a decomposition $W = (\mathbb{C}^2 \times \mathbb{C}^*) \dot{\cup} \mathbb{C}^2$ (disjoint union). The author does not know whether $W \cong \mathbb{C}^3$.

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