COMPLEX CROWNS OF RIEMANN SYMMETRIC SPACES AND MATSUKI DUALITY

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Soji Kaneyuki with warmest feelings

It was a very big pleasure for me to participate in the conference honoring my friend and collaborator, the remarkable mathematician Soji Kaneyki. I consider his contribution in the theory of symmetric spaces, especially pseudo Riemannian ones, as absolutely outstanding. On this reason it was natural to talk on this conference about symmetric spaces. I talked about one of my favorite subjects of last 10-15 years - complex crowns of Riemannian symmetric spaces. In the last year there was a substantial progress in this area and therefore I include in this text based on my conference talk also some results of last months.

Let $X = G/K$ be a Riemann symmetric space of noncompact type, G is here a real semisimple Lie group and K - its maximal compact subgroup (the case of compact X can be also included in these considerations but it is relatively simple). The crucial idea is that there is a canonical G-invariant Stein neighborhood of X - complex crown $\operatorname{Crown}(X)$-which must universally appear in geometric and analytic problems on X [G]. There were several reasons to expect the existence of such an object. Let us mention two of them. Spherical functions on X can be holomorphically extend in a complex neighborhood of X since the Laplace-Beltrami operator is elliptic. What is more important that it is not an individual behavior but a "social"phenomenon: there is an joint holomorphy domain for all spherical functions. Such an effect of simultaneous extension was discovered by Aronshain for the Laplacian and was investigated in the case of constant coefficients (Martienau, Kiselman). There were some reasons to expect that similar fact holds also for Riemann symmetric spaces where we have operators with variable coefficients..

Another problem has a geometrical nature. Let us consider flag domains - open G-orbits on flag manifolds $F = G/P$ (P is here a parabolic subgroup) and the family of compact complex submanifolds in them. If to exclude some degenerate situations (which can appear only for groups G of Hermitian type - orbits of holomorphic type [WZ]), then these cycles are parameterized by some points of $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. This parametric set contains X (flag domains admit fiberings over X with compact complex fibers). Let us take the connected component Ξ_c, containing X. Wolf proved that Ξ_c is always Stein [W]. It was a conjecture that in nondegenerate case (in particular, for all groups of non Hermitian type) this domain is the same for all flag domains [G] and coincides with the domain in the previous example.

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There is a serious reason to expect the coincidence of these domain which have the different origin. Let us consider the Penrose transform in a flag domain F_i: the integration of $\bar{\partial}$-cohomology of the appropriate degree along compact cycles in F_i. Functions in their image will be holomorphic on the manifold parameterising the cycles and will satisfy a system of elliptic equations - Schmid equations. The joint holomorphy domain of solutions of this system is naturaly coincides with the parametric manifold of the cycles.

Hence it is natural to build Crown(X) inside of $X_{\mathbb{C}}$, which is a Stein symmetric (pseudo Riemannian) space. It is too big for being the crown, if the group G is not compact. The basic obstruction is the improper action of G on $X_{\mathbb{C}}$. Akhieser and me [AG] suggested the next construction. Let us take the Cartan decomposition in Lie algebras

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

and $\mathfrak{a}$ be a maximal Abelian subspace in $\mathfrak{p}$. Let

$$\Omega = \{X \in \mathfrak{a} : |\alpha(X)| < \frac{\pi}{2}(\forall \alpha \in \Sigma)\},$$

where Σ denotes the restricted root system with respect to $\mathfrak{a}$.

Let

$$\Xi = G \exp(i\Omega) K_{\mathbb{C}}/K_{\mathbb{C}}.$$

We hoped that Ξ be just the canonical complex neighborhood of X. A few words about the motivation of this construction. The domain Ξ is the union of the maximal connected set of G-orbits on $X_{\mathbb{C}}$ with compact isotropy subgroups, which intersect $\tilde{A} = \exp(i\mathfrak{a})$. Probably, Ξ is the union of the maximal set of proper G-orbits (without the condition on the intersection with A), but it does not proved. Let us remark that since the group G is noncompact, the orbit decomposition on $X_{\mathbb{C}}$ is pretty complicate and already on the boundary of Ξ there are orbits which do not intersect $\tilde{A}$. The trouble was that we could not prove that Ξ is Stein. It was proved only at the last year by different ways.

There were a few cases when Ξ was explicitly calculated. The simplest case is when X be a Hermitian symmetric space (then G is called a group of Hermitian type). In this case

$$\Xi = X \times \bar{X}.$$

In the case of $SO(1,n)$ the space X be the hyperboloid

$$\Box(x) = x_0^2 - x_1^2 - \cdots - x_n^2 = 1 \quad x_0 > 0, x \in \mathbb{R}^{n+1},$$

and $X_{\mathbb{C}}$ will be the complex hyperboloid

$$\Box(z) = 1, z \in \mathbb{C}^{n+1}.$$

Then Ξ is defined by conditions

$$\Box(z) = 1, \Box(x) > 0, x_0 > 0.$$

This domain is biholomorphic equivalent to the future tube.

For $SL(n, \mathbb{R})$ the space X consists of positive symmetric matrices of the order n with the determinant 1, $X_{\mathbb{C}}$ of symmetric complex matrices with the determinant

1 and Ξ is the connected component (containing X) of the set of such symmetric matrices that corresponding quadrics in $\mathbb{C}P^{n-1}$ have no real points. In this last example the manifold Ξ is not homogeneous and in the examples considered before Ξ were homogeneous relative to groups bigger then G. For classical groups Ξ has most complicate structure for spaces X connecting with pseudo-orthogonal groups $SO(p,q)$, where $p,q \geq 2$.They are inhomogeneous in these cases.

There is a very natural way to prove that the domain Ξ is Stein. In the reality, it is one more way bringing crowns. It gives a complex neighborhood of X, which is Stein on trivial reasons. Let us consider the Iwasawa decomposition $G = KAN$, where $A = \exp(\mathfrak{a})$, N be a maximal unipotent subgroup. Let us consider holomorphic extension of this decomposition on the complex group $G_{\mathbb{C}}$ and let G_0 be the set $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ where this decomposition exists (but can be non unique). Let G_I be the connecting component of the intersection

$$\bigcap_{g\in K} gG_0,$$

containing G. The set $G_I \subset G_{\mathbb{C}}$ is exactly the maximal set where Iwasawa decompositions can be holomorphically extended for all parabolic subgroups. Let us call

$$\Xi_I = G_I/K_{\mathbb{C}}$$

by the Iwasawa set. It is evident that $\Xi_I \supset X$ and that it is Stein. It was natural suppose that

$$\Xi = \Xi_I.$$

It is not difficult to check this equality if there is an explicit description of Ξ (e.g. for groups of Hermitian type).

At the last year there appeared several preprints about this conjecture. Krötz and Stanton [KS] proved that $\Xi \subset \Xi_I$ for classical groups, Matsuki and Gindikin [GM] that $\Xi = \Xi_I$ for classical groups, Barchini [B] that $\Xi \supset \Xi_I$ in general case. There is also an announcement that Huckelberry proved that $\Xi \subset \Xi_I$ in general case, but I had no chance to see the proof. From other side Burns, Halverscheid and Hind [BHH] found a direct proof of the conjecture that Ξ be Stein proving that Ξ be the maximal Grauert tube for X (without the consideration of Ξ_I).

The consideration of Ξ_I is connected with the conjecture that all zonal spherical functions and solutions of other invariant differential operators admit holomorphic extensions from X to Ξ. The technical reason of such an extension is representations of such solutions by Poisson type integrals where integrands contain some functions type of determinants in the Sylvester conditions - determinant functions. In general case they are characterized as functions on X which multiply under the action of the parabolic subgroup $P = AN$ on characters of A. Determinant functions and the domain Ξ_I in the case of the Grassmanians was consider in [BGW]; the general construction discussed at [G]. The equality of the determinant functions to zero or infinity is just the obstruction to the existence of the Iwasawa decomposition. So they and their G-translations define "barrier"(supporting) submanifolds for Ξ_I. The same conditions restrict the holomorphic extension of Poisson type integrals.This analytic extension was the motivation of the paper of Kröetz-Stanton [KS] which follow an idea of J.Bernstein-Reznikov to apply holomorphic extensions of matrix coefficients to exact estimates of automorphic forms.

The connection of the domain Ξ with the Iwasawa decompositions and holomorphic extensions of spherical functions is only one of properties of the complex crown of X. The way to see the universality of Ξ is the Matsuki duality [M]. I believe that this duality is an extremely deep fact of the geometry of symmetric and flag spaces which nature we do not understand completely yet. Its connection with complex crowns is one more confirmation. Let us remind that we consider on a flag manifold $F = G_{\mathbb{C}}/P$, where P be a parabolic subgroup, G-orbits and $K_{\mathbb{C}}$-orbits. $K_{\mathbb{C}}$-orbit S and G-orbit S' are dual if they have non empty closed intersection. It turns out that it is one-to-one correspondence. In our joint paper with Matsuki [GM] we consider the next problem. Let $G_{\mathbb{C}}(S)$ be the set of such $g \in G_{\mathbb{C}}$ that $gS \bigcap S'$ is closed and non empty (for dual orbits S, S'). Let $\Xi(S) \subset X_{\mathbb{C}}$ be the connecting component of $G_{\mathbb{C}}(S))/K_{\mathbb{C}}$ cotaining X.

We had 2 conjectures. Let

$$\Xi_0 = \bigcap \Xi(S),$$

where we take the intersection for all $K_{\mathbb{C}}$-orbits in all flag manifolds.Our first conjecture is that

$$\Xi_0 = \Xi. \tag{1}$$

More strong conjecture is

$$\Xi(S) = \Xi \tag{2}$$

for all generic orbits S. Degenerate orbits for which (2) is not true can appear only for groups of Hermitian type (orbits of holomorpjic type in the sense of [WZ]).

Different geometrical problems corresponds to different pairs of dual orbits. Let $P = B$ be a Borel subgroup and S be the unique open $K_{\mathbb{C}}$-orbit on $G_{\mathbb{C}}/B$ (correspondingly there is only one closed G-orbit). Then $\Xi(S) = \Xi_I$. The opposite situation, when S is closed, corresponds to the parametrization of cycles gS in the dual open orbit S' - the flag domain (then the domain $\Xi(S)$ is the domain Ξ_c above). The conjecture 2 means that $\Xi_c = \Xi$. It was proved for classical Hermitian spaces in [WZ].

We proved conjecture 1 for all classical groups and for all groups of Hermitian type. We also proved the conjecture (2) in several cases, including the case of open S for $P = B$ and for classical groups, such that in these cases $\Xi_I = \Xi$.

We also proved that for all groups

$$\Xi_0 = \Xi_I. \tag{3}$$

Then, since it proved now that $\Xi = \Xi_I$, we have $\Xi_0 = \Xi$ in general case. It has strong consequences. In particular, Ξ is contained in domains $\Xi(S)$ parameterizing cycles inside flag domains S'. It was proved earlier by Wolf -Zierau for classical Hermitian groups [WZ] and they proved the opposite inclusion $\Xi(S') \subset \Xi$ for all groups of Hermitian type. So the conjecture 2 indeed is true for all groups of Hermitian type.

Usually it is very informative to understand what some results on symmetric spaces mean in special cases when they transform in facts of linear algebra. In the case of $SL(n;\mathbb{R})$ and simplest closed orbit S (quadrics in the compliment of $\mathbb{R}P^n$

to $\mathbb{C}P^n$) we have a remarkable fact of linear algebra: two quadratic forms without joint non trivial zeroes can be simultaneously transform to diagonal forms. This generalization of the classical theorem where one of quadratic forms is positive; it rediscovered by different mathematiiancs time to time. Apparently, this theorem could be in a standard textbook on linear algebra. Other classical groups deliver other interesting examples of new facts of linear algebra. Let us remark that a substantial part of the paper [GM] is just a systematic consideration of these problems. I am sure that this is very informative but now it is important to find a more conceptual proof of the conjecture (2), probabely in the style of the simplectic proof of usual Matsuki duality in [MUV].

The important role of complex crown in geometric and analytic problems is clear now in many points. It has the pretty effective description of G-orbit decomposition, but it does not give an explicit picture of their geometry, including their boundary. It reflects usual troubles in the work with noncompact orbits.

We already mentioned that for some groups the situation is pretty simple. For example, for Hermitian symmetric spaces X we have $\Xi = X \times \bar{X}$. A substantial part of classical Riemann symmetric spaces X can be present as real forms of Hermitian symmetric spaces Y. Then it is natural expect that just these spaces Y will be complex crowns of X. It would be a generalization of the result for Hermitian spaces. It turns out that this coincidence holds only in the the cases of restricted root systems of types C and BC [BHH, KS]. In other cases Ξ are strictly more than Y and inhomogeneous. Boundaries and distinguished boundaries (Shilov boundaries) in these cases are pretty complicate. In joint paper with B.Krötz [GK2] we gave the description of distinguished boundaries of Ξ. The essential part of this description is the result that semisimple symmetric spaces are appear on the boundaries of complex crowns if and only if they are non-compactly causal symmetric spaces. In turn, in a joint work with Krötz and Ólafsson [GKO] we used this geometrical observation for a construction of Hardy type spaces for non-compactly causal symmetric spaces. Earlier Hardy spaces were constructed for compactly causal symmetric spaces, where they are connected with holomorphic discrete series and it was an impression that they do not exist in non-compactly causal cases. It turns out that for non-compactly causal symmetric spaces Hardy spaces are connected with most continuous series of representations.

Together with Krötz [GK1]we showed that the domain $\Xi \subset X_{\mathbb{C}}$ be in a sense the maximal domain where orbits of noncompact groups G conduct as orbits of compact groups. This observation has illustrated by two important facts. Firstly, we proved complex analogue of Kostan's convexity theorem: gave the description of Iwasawa projection on $A_{\mathbb{C}}$ of unions of G-orbits in Ξ. As a result we explicitly present Ξ as the intersection of horospherical tubes - domains invariant relative of fixed parabolic subgroups ($P_{\mathbb{C}} = A_{\mathbb{C}} N_{\mathbb{C}}$ and its K-conjugating). Of course, this presentation as an intersection is more informative for the consideration of the boundary than the representation as the union. It is also a way to consider horospherical transforms in Ξ.

Another fact is the characterization of G-invariant Stein subdomains in Ξ by their intersections with $A_{\mathbb{C}}$, similar to the theorem of Lassale for compact groups G.

These results are a first step in more broad project to connect some complex domains in $X_{\mathbb{C}}$ with all pseudo Riemann symmetric spaces X. These domains will not be Stein and they will canonicaly connect with different series of representations on X [G].

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