The ternary Goldbach problem

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The ternary Goldbach problem: what is it? What was known?

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Every odd number $n \geq 7$ is the sum of three primes.

(Binary Goldbach conjecture:
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Hardy-Littlewood (1923): There is a C such that every odd number $\geq C$ is the sum of three primes, if we assume the generalized Riemann hypothesis (GRH).
Vinogradov (1937): The same result, unconditionally.

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every odd $n > 1$ is the sum of ≤ 5 primes (**Tao, 2012**).

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Ternary Goldbach holds for all n conditionally on the generalized Riemann hypothesis (GRH) (**Deshouillers-Effinger-te Riele-Zinoviev, 1997**)

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Every odd $n \geq C$ is the sum of three primes (Vinogradov)

Bounds for C? $C = 3^{3^{15}}$ (Borodzin, 1939), $C = 3.33 \cdot 10^{43000}$ (Wang-Chen, 1989), $C = 2 \cdot 10^{1346}$ (Liu-Wang, 2002).

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Verification for small n:
every even $n \leq 4 \cdot 10^{18}$ is the sum of two primes (Oliveira e Silva, 2012);
taken together with results by Ramaré-Saouter and Platt, this implies that every odd $5 < n \leq 1.23 \cdot 10^{27}$ is the sum of three primes; alternatively, with some additional computation, it implies that every odd $5 < n \leq 8.875 \cdot 10^{30}$ is the sum of three primes (Helfgott-Platt, 2013).

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$8.875 \cdot 10^{30}$ is much smaller than $2 \cdot 10^{1346}$.

We must diminish C from $2 \cdot 10^{1346}$ to $\sim 10^{30}$.

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where

$\eta(t) = e^{-t}$ (Hardy-Littlewood), $\eta(t) = 1_{[0,1]}$ (Vinogradov),

$\Lambda(n) = \log p$ if $n = p^\alpha$, $\Lambda(n) = 0$ if n is not a prime power (**von Mangoldt** function)

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The crucial identity:

$$\sum_{n_1+n_2+n_3=N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\eta(n_1/x)\eta(n_2/x)\eta(n_3/x)$$
$$= \int_{\mathbb{R}/\mathbb{Z}} (S_\eta(\alpha, x))^3 e(-N\alpha) d\alpha.$$

We must show that this integral is > 0.

Major and minor arcs

We partition $\mathbb{R}/\mathbb{Z}$ into intervals (*"arcs"*) $\mathfrak{m}_{a,q} \subset (a/q - 1/qQ, a/q + 1/qQ)$ around a/q, $q \leq Q$, where $Q \leq x$. (**Farey fractions**)

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Let $\mathfrak{M}$ be the union of major arcs and $\mathfrak{m}$ the union of minor arcs.
We want to estimate $\int_{\mathfrak{M}} (S_\eta(\alpha, x))^3 e(-N\alpha) d\alpha$ and bound $\int_{\mathfrak{m}} |S_\eta(\alpha, x)|^3 d\alpha$ from above.

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We do this studying $L(s, \chi)$ for Dirichlet characters mod q, i.e., characters $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$.

$$L(s, \chi) := \sum_n \chi(n) n^{-s}$$

for $\Re(s) > 1$; this has an analytic continuation to all of $\mathbb{C}$ (with a pole at $s = 1$ if χ is trivial).
We express $S_\eta(\alpha, x)$, $\alpha = a/q + \delta/x$, as a sum of

$$S_{\eta,\chi}(\delta/x, x) = \sum_{n=1}^{\infty} \Lambda(n) \chi(n) e(\delta n/x) \eta(n/x)$$

for χ varying among all Dirichlet characters modulo q.

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(b) ρ runs over the complex numbers ρ with $L(\rho, \chi) = 0$ and $0 < \Re(\rho) \leq 1$ (called "non-trivial zeroes"),
(c) F_δ is the Mellin transform of $\eta(t) \cdot e(\delta t)$.

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Analytic on a strip $x_0 < \Re(s) < x_1$ in $\mathbb{C}$.

It is a Laplace transform (or Fourier transform!) after a change of variables.

Where are the zeroes of $L(s, \chi)$?

Let $\rho = \sigma + it$ be any non-trivial zero of $L(s, \chi)$.

What we believe:
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What we can also know:
for a given χ, we can verify GRH for $L(s, \chi)$ "up to a height T_0". This means: verify that every zero ρ with $|\Im(\rho)| \leq T_0$ satisfies $\sigma = 1/2$.

Verifying GRH up to a given height

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For χ trivial ($\chi(x) = 1$), $L(s,\chi) = \zeta(s)$. The Riemann hypothesis has been verified up to $|t| \leq 2.4 \cdot 10^{11}$ (**Wedeniwski** 2003), $|t| \leq 1.1 \cdot 10^{11}$ (**Platt** 2012, rigourous), $|t| \leq 2.4 \cdot 10^{12}$ (**Gourdon-Demichel** 2004, not duplicated to date).

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This has been extended up to $q \leq 2 \cdot 10^5$, q odd, and $q \leq 4 \cdot 10^5$, q pair ($|t| \leq 200 + 7.5 \cdot 10^7/q$) (**Platt** 2013).

How to use a GRH verification

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For $\eta(t) = e^{-t}$, the Mellin transform of $\eta(t)e(\delta t)$ is

$$F_\delta(s) = \frac{\Gamma(s)}{(1 - 2\pi i \delta)^s}.$$

Decreases as $e^{-\lambda|\tau|}$, $\lambda = \tan^{-1} \frac{1}{2\pi|\delta|}$, for $s = \sigma + i\tau$, $|\tau| \to \infty$. If $\delta \gg 1$, then $\lambda \sim \frac{1}{2\pi|\delta|}$. Problem: $e^{-|\tau|/2\pi\delta}$ does not decay very fast for δ large!

The Gaussian smoothing

Instead, we choose $\eta(t) = e^{-t^2/2}$. The Mellin transform F_δ is then a parabolic cylinder function.

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The main term in $F_\delta(\sigma + i\tau)$ behaves as

$$e^{-\frac{\pi}{4}|\tau|}$$

for δ small, $\tau \to \pm\infty$, and as

$$e^{-\frac{1}{2}\left(\frac{|\tau|}{2\pi\delta}\right)^2}$$

for δ large, $\tau \to \pm\infty$.

Major arcs: conclusions

Thus we obtain estimates for $S_{\eta,\chi}(\delta/x, x)$, where

$$\eta(t) = g(t) e^{-t^2/2},$$

and g is any "band-limited" function:

$$g(t) = \int_{-R}^{R} h(r) t^{-ir} dr$$

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All the rest of the circle must be minor arcs; $m(x)$ must be a constant M. (Writer for *Science*: "Muenster cheese" rather than "Swiss cheese".)
Thus, minor-arc bounds will have to be very strong.

Back to the circle

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$$\int_{\mathfrak{m}} |S_\eta(\alpha, x)|^2 |S_{\eta_*}(\alpha, x)| d\alpha \leq \max_{\alpha \in \mathfrak{m}} |S_{\eta_*}(\alpha, x)| \cdot L_2, \quad (1)$$

where $L_2 = \int_{\mathfrak{m}} |S_\eta(\alpha, x)|_2^2 d\alpha$.

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It is possible to improve (1): **Heath-Brown** replaces $x \log x$ by $2e^\gamma x \log q$. A new approach based on **Ramaré**'s version of the large sieve (cf. *Selberg*) replaces this by $2x \log q$.
The idea is that one can give good bounds for the integral over the arcs with denominator between r_0 and r_1 (say).

What weight η_+?

The main term for the number of (weighted) solutions to $N = p_1 + p_2 + p_3$ will be proportional to

$$\int_0^\infty \int_0^\infty \eta_+(t_1)\eta_+(t_2)\eta_* \left(\frac{N}{x} - t_1 - t_2\right) dt_1 dt_2, \qquad (2)$$

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Solution: since $\eta(t) = g(t)e^{-t^2/2}$, we let g be a band-limited approximation to $e^t \cdot I_{[0,2]}$.

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Vinogradov chose $\eta_* = 1_{[0,1]}$.
We would like: $\eta_+(x) = f *_M f$, where

$$(f *_M f)(t_0) = \int_0^\infty f(t) f\left(\frac{t_0}{t}\right) \frac{dt}{t},$$

f of compact support (e.g. $\eta_2 := f *_M f$, $f = 2 \cdot 1_{[1/2,1]}$, as in Tao).

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Solution: $\eta_* := \eta_0 *_M f *_M f$, where η_0 has a Mellin transform with exponential decay.

What weight η_*?

In order to estimate S_{η_*} on the major arcs, we want a η_* whose Mellin transform decreases exponentially for $\Re(s)$ bounded, $\Im(s) \to \pm\infty$.

To estimate S_{η_*} on the minor arcs, we prefer a η_* with compact support bounded away from 0.

Vinogradov chose $\eta_* = 1_{[0,1]}$.
We would like: $\eta_+(x) = f *_M f$, where

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If we know $S_{f*f}(\alpha, x)$ **or** $S_{\eta_0}(\alpha, x)$, we know $S_{\eta_*}(\alpha, x)$.

The new bound for minor arcs

Theorem (Helfgott, May 2012 – March 2013)

Let $x \geq x_0$, $x_0 = 2.16 \cdot 10^{20}$. Let $2\alpha = a/q + \delta/x$, $\gcd(a, q) = 1$, $|\delta/x| \leq 1/qQ$, where $Q = (3/4)x^{2/3}$. If $q \leq x^{1/3}/6$, then $|S_{\eta_2}(\alpha, x)|/x$ is less than

$$\frac{R_{x,\delta_0 q}(\log \delta_0 q + 0.002) + 0.5}{\sqrt{\delta_0 \phi(q)}} + \frac{2.491}{\sqrt{\delta_0 q}}$$

$$+ \frac{2}{\delta_0 q} \min\left(\frac{q}{\phi(q)}\left(\log \delta_0^{7/4} q^{13/4} + \frac{80}{9}\right), \frac{5}{6}\log x + \frac{50}{9}\right)$$
$$+ \frac{2}{\delta_0 q}\left(\log q^{\frac{80}{9}} \delta_0^{\frac{16}{9}} + \frac{111}{5}\right) + 3.2x^{-1/6},$$

where $\delta_0 = \max(2, |\delta|/4)$,

$$R_{x,t_1,t_2} = 0.4141 + 0.2713 \log\left(1 + \frac{\log 4t_1}{2\log \frac{9x^{1/3}}{2.004 t_2}}\right).$$

The new bound for minor arcs, II

Theorem (Helfgott, May 2012 – March 2013, bound for q large)

If $q > x^{1/3}/6$, then

$$|S_\eta(\alpha, x)| \leq 0.27266 x^{5/6} (\log x)^{3/2} + 1217.35 x^{2/3} \log x.$$

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For $x = 10^{25}$, $q \sim 1.5 \cdot 10^5$, $|\delta| < 8$ (the most delicate case)

$$R_{x,\delta_0 q} = 0.5833\ldots$$

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10^5	0.04521	0.34475
$1.5 \cdot 10^5$	0.03820	0.28836
$2.5 \cdot 10^5$	0.03096	0.23194
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Table: Upper bounds on $x^{-1}|S_\eta(a/2q,x)|$ for $q \geq q_0$, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13|q$, $|\delta| \leq 8$, $x = 10^{25}$. The trivial bound is 1.

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Meaning: GRH verification needed only for $q \leq 1.5 \cdot 10^5$, q odd, and $q \leq 3 \cdot 10^5$, q even.

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- cancellation within Vaughan's identity
- $\delta/x = \alpha - a/q$ is a friend, not an enemy:
  In type I: (a) decrease of $\widehat{\eta}$,
  change in approximations;
  In type II: scattered input to the large sieve
- relation between the circle method and the large sieve – in its version for primes;
- the benefits of a continuous η (also in Tao, Ramaré),

Cancellation within Vaughan's identity

Vaughan's identity:

$$\Lambda = \mu_{\leq U} * \log - \Lambda_{\leq V} * \mu_{\leq U} * 1 + 1 * \mu_{>U} * \Lambda_{>V} + \Lambda_{\leq V},$$

where $f_{\leq V}(n) = f(n)$ if $n \leq V$, $f_{\leq V}(n) = 0$ if $n > V$. (Four summands: type I, type I, type II, negligible.)

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- Advantage: flexibility – we may choose U and V;
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Alternative would have been: use a log-free formula (e.g. Daboussi-Rivat); proceeding as above seems better in practice.

How to recover factors of log

In type I sums:
We use cancellation in $\sum_{n\leq M: d|n} \mu(n)/n$.
This is allowed: we are using only ζ, not L.
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Application of the large sieve for primes.

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In type II:

- $\widehat{\eta}(\delta) \ll 1/\delta^2$ (so that $|\eta''|_1 < \infty$),
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In type II: the angles $m\alpha$ are separated by $\geq \delta/x$ (even when $m \geq q$). We can apply the large sieve to *all* $m\alpha$ in one go. We can even use prime support: double scattering, by δ and by **Montgomery**'s lemma.

Final result

All goes well for $n \geq 10^{30}$ (or well beneath that). As we have seen, the case $n \leq 10^{30}$ is already done (computation).

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Theorem (Helfgott, May 2013)

Every odd number $n \geq 7$ is the sum of three prime numbers.