# PLURIHARMONIC MAPS INTO COMPACT LIE GROUPS AND FACTORIZATION INTO UNITONS

Yoshihiro Ohnita(*) and Giorgio Valli(**)

(*)Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

and

Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya, Tokyo 158

Japan

(**)Mathematics Institute
University of Warwick
Coventry CV4 7AL

United Kingdom

MPI/89-3

Abstract.

We establish the construction of special holomorphic maps into the group $\Omega G$ of based loops, called extended solutions, from a pluriharmonic map of a simply connected complex manifold into a compact Lie group $G$ . As its application many results on a harmonic map of a Riemann surface into a Lie group can be extended to a pluriharmonic map of a complex manifold. Moreover we show the unique factorization theorem for a pluriharmonic map into the unitary group $U(N)$ with the singularity set of codimension at least 2 in the domain complex manifold, by introducing the notion of rational unitons.

Introduction.

Let M be a complex manifold and N be a Riemannian manifold. A smooth map $\varphi : M \longrightarrow N$ is called pluriharmonic if the (0,1)–derivative $\nabla'' \partial\varphi$ of $\partial\varphi$ vanishes identically. The notion of a pluriharmonic map is a natural extension of a harmonic map from a Riemann surface. Though the pluriharmonicity is much stronger than the usual harmonicity, the class of pluriharmonic maps contains so many interesting examples of harmonic maps (cf. [13]). There are many beautiful results on harmonic maps from Riemann surfaces (cf. [4], [5]). It is interesting and important to generalize them to results for a pluriharmonic map from a complex manifold and to develop the theory of pluriharmonic maps. The theory of pluriharmonic maps is closely related to differential, algebraic and analytic geometry of the domain complex manifolds and theory of holomorphic maps and meromorphic maps. In this paper we shall find direct links of pluriharmonic maps with holomorphic maps or meromorphic maps.

In the paper [24], Uhlenbeck gave many remarkable results on the theory of harmonic maps from Riemann surfaces into Lie groups, which are closely related to several works in mathematical physics ([32], [33]). There are many excellent works about this subject (cf. [5]). In this paper we develop such theory for pluriharmonic maps from complex manifolds.

The notion of extended solutions of a harmonic map from a Riemann surface into a Lie group played a central role in the theory of [24]. We shall establish the construction of extended solutions $\Phi_\lambda$, $\lambda \in \mathbb{C}^*$, for a pluriharmonic map $\varphi$ from a complex manifold M into a compact Lie group G. An extended solution can be considered as a special holomorphic map from M into the group $\Omega G$ of based loops in G. In the same way as in [24], we can introduce the notion of unitons and uniton equations for a pluriharmonic map. We shall get the finiteness of the Laurent expansion in $\lambda$ of the based extended solution $\Phi_\lambda$ for a pluriharmonic map from a compact complex manifold, and we shall prove the formula for the difference of energies in adding a uniton, generalizing a previous one due to the second named author [26]. In our theory we need the notion of not only smooth unitons but also rational unitons. By using the method of [24] and results from the theory of rational maps and coherent sheaves, we shall show the unique factorization theorem for pluriharmonic maps from a simply connected compact complex manifold into the unitary group U(N) with the singularity set of complex codimension at least 2 in the domain. Hence we see that any pluriharmonic map from a simply connected compact complex manifold into U(N) can be obtained from a holomorphic map, generally a rational map, into a complex Grassmann manifold. The interesting problems are the removability or resolution of the singularity in the factorization for a pluriharmonic map and the explicit construction of pluriharmonic maps from a specific compact complex manifold into U(N). Moreover by the methods of [32], [33], [24] and [18], [21], [9], we also can make the action of the loop algebra and loop group on the space of pluriharmonic maps into a compact Lie group.

The authors wish to thank Professor J. Eells for kind suggestion. The first named author wishes to thank Dr. S. Udagawa and Dr. M. Bergvelt for useful communication. He was supported by the Max–Planck–Institut für Mathematik in Bonn and partially by the London Mathematical Society during the preparation of this paper. He wishes to thank the Max–Planck–Institut and the Universities of Leeds, Warwick and Durham for the hospitality and especially Miss M. Grau for her typing of this manuscript. The second named author wishes to thank the University of Catania and the International Centre for Theoretical Physics in Trieste for the hospitality and financial support.

## 1. Pluriharmonic maps.

Let M be a connected complex manifold and N be a connected Riemannian manifold with a Riemannian metric $g_N$. Let $\varphi : M \longrightarrow N$ be a smooth map from M to N. The differential $d\varphi : TM \longrightarrow \varphi^{-1}TN$ extends by complex linearity to $d\varphi : TM^{\mathbb{C}} \longrightarrow \varphi^{-1}TN^{\mathbb{C}}$. Relative to the complex structure J of M we have a decomposition $TM^{\mathbb{C}} = TM^{(1,0)} \oplus TM^{(0,1)}$. By restricting $d\varphi$ to each factor we define the bundle maps $\partial\varphi : TM^{(1,0)} \longrightarrow \varphi^{-1}TN^{\mathbb{C}}$ and $\bar{\partial}\varphi : TM^{(0,1)} \longrightarrow \varphi^{-1}TN^{\mathbb{C}}$. Using the induced connection $\nabla^{\varphi}$ and the $\bar{\partial}$–operator of $TM^{(1,0)}$, we define the (0,1)–exterior derivative of $\partial\varphi$ by

$$(\nabla''_{\bar{W}} \partial\varphi)(Z) = \nabla^{\varphi}_{\bar{W}}(\partial\varphi(Z)) - \partial\varphi(d''_{\bar{W}} Z)$$

for each $Z, W \in C^{\infty}(TM^{(1,0)})$. Then $\varphi$ is called <u>pluriharmonic</u> if $\varphi$ satisfies $\nabla'' \partial\varphi = 0$. We see immediately the following.

<u>Proposition 1.1</u>. A smooth map $\varphi$ from a complex manifold M to a Riemannian manifold N is pluriharmonic if and only if for any holomorphic curve $\iota : C \longrightarrow M$, the composite $\varphi \circ \iota$ is always harmonic.

Note that a pluriharmonic map $\varphi : M \longrightarrow N$ is harmonic with respect to any Kähler metric on M (we can always give a Kähler metric in a small neighborhood of M ).

Assume that M is a Kähler manifold. Denote by $g_M$ and $\nabla^M$ its Kähler metric and Riemannian connection. The second fundamental form $\nabla d\varphi$ of the map $\varphi$ is defined by

$$(\nabla d\varphi)(X,Y) = (\nabla_Y d\varphi)(X) = \nabla^\varphi_Y(d\varphi(X)) - d\varphi(\nabla^M_Y X)$$

for each $X,Y \in C^\infty(TM^{\mathbb{C}})$. Since $\nabla^M$ is a Riemannian connection of a Kähler metric $g_M$, the (0,1)–part of $\nabla^M$ is the $\bar\partial$–operator of $TM^{(1,0)}$. Hence the (1,1)–part of the second fundamental form $\nabla d\varphi$ coincides with $\nabla'' \partial\varphi$.

Lemma 1.2. Let $\varphi : M \longrightarrow N$ be a pluriharmonic map from a complex manifold $M$ to a Riemannian manifold $N$. Then we have

$$R^N(d\varphi(Z),d\varphi(V))d\varphi(\overline{W}) = 0$$

for each $Z,V,W \in T_xM^{(1,0)}$ and each $x \in M$, where $R^N$ denotes the curvature tensor field of $N$.

Proof. Choose a Kähler metric $g_M$ defined on some neighborhood $U$ of $x$. We denote by $R^\varphi$ and $R^M$ the curvature forms of $\nabla^\varphi$ and $\nabla^M$ respectively. For any $Z,V,W \in C^\infty_U(TM^{(1,0)})$, by Ricci identity we have

$$\begin{aligned} 0 &= (\nabla^2 d\varphi)(\overline{W},V,Z) \\ &= (\nabla^2 d\varphi)(\overline{W},Z,V) + (\tilde{R}(Z,V)d\varphi)(\overline{W}) \\ &= R^\varphi(Z,V)d\varphi(\overline{W}) - d\varphi(R^M(Z,V)\overline{W}) \\ &= R^N(d\varphi(Z),d\varphi(V))d\varphi(\overline{W}) . \end{aligned}$$

q.e.d.

Remark. (1) Let $(M,g,J)$ be a general Hermitian manifold and $\omega$ be its fundamental 2–form, $\omega(X,Y) = g(JX,Y)$. $(M,g,J)$ is called cosymplectic if $\omega$ is coclosed, $d\omega^{m-1} = 0$. Then $(M,g,J)$ is cosymplectic (resp. Kähler) if and only if any pluriharmonic map from $(M,g,J)$ is harmonic (resp. (1,1)–geodesic).

(2) In the case when $(M,g,J)$ is complex 2–dimensional, M is cosymplectic if and only if M is Kähler. Let $M = G/C(T_1) = G^{\mathbb{C}}/P$ be a generalized flag manifold with a fixed homogeneous complex manifold structure. If g is a G–invariant Riemannian metric on M, then g is a cosymplectic Hermitian metric on M.

Lemma 1.3. If the curvature operator $\mathcal{R}^N$ of N is nonnegative or nonpositive, then the curvature form $R^{\varphi}$ of $\nabla^{\varphi}$ in $\varphi^{-1}TN^{\mathbb{C}}$ is of type (1,1).

Proof. By Lemma 1.2, for any $Z,V \in T_xM^{(1,0)}$ we have

$$g_N(R^N(d\varphi(Z),d\varphi(V))d\varphi(\bar{V}),d\varphi(\bar{Z}))$$

$$= g_N(\mathcal{R}^N(d\varphi(Z)\wedge d\varphi(V)), \overline{d\varphi(Z)\wedge d\varphi(V)}) = 0 .$$

Since $\mathcal{R}^N$ is positive semi–definite or negative semi–definite, we get $\mathcal{R}^N(d\varphi(Z)\wedge d\varphi(V)) = 0$. Hence $R^{\varphi}(Z,V) = R^N(d\varphi(Z),d\varphi(V)) = 0$. Similarly $R^{\varphi}(\bar{Z},\bar{V}) = 0$.

q.e.d.

Proposition 1.4. Let $\varphi : M \longrightarrow N$ be a pluriharmonic map from a complex manifold to a Riemannian manifold whose curvature operator is nonnegative or nonpositive. Set $E = \varphi^{-1}TN^{\mathbb{C}}$ and denote by h the Hermitian metric of E induced from $g_N$ through $\varphi$. Then there exists a unique holomorphic vector bundle structure in E such that the connection $\nabla^{\varphi}$ coincides with the Hermitian connection of $(E,h)$ with respect to this holomorphic structure.

Proposition 1.4 follows from Lemma 1.3 and a well–known result of [11] (cf. [10]).

Remark. It is well–known that if N is an n–dimensional simply connected compact irreducible Riemannian manifold with nonnegative curvature operator, then

(i) N is homeomorphic to a sphere (in case $n \leq 6$, diffeomorphic to a sphere),

(ii) N is diffeomorphic to a complex projective space, or

(iii) N is isometric to a symmetric space of compact type.

Let $\varphi : M \longrightarrow N$ be a smooth map from a complex m–dimensional complex manifold $(M,J)$ to a Riemannian manifold $(N,g_N)$. Set $\beta(X,Y) = (\varphi^* g_N)(JX,Y)$ for $X,Y \in T_xM$. Then the (1,1)–part $\beta^{(1,1)}$ of $\beta$ is a nonnegative real (1,1)–form on M. We define the energy form $\mathcal{E}(\varphi)$ as $\mathcal{E}(\varphi) = \beta^{(1,1)}$;

$$\mathcal{E}(\varphi) = \sqrt{-1}\sum_{i,j=1}^{m} g_N(\varphi_i,\varphi_{\bar{j}})dz^i \wedge d\bar{z}^j ,$$

where $\{z^i\}$ is a local complex coordinate system in M and $\{\varphi_i,\varphi_{\bar{i}}\}$ denote the components of $d\varphi$ with respect to $\{z^i\}$. The energy of $\varphi$ relative to a Hermitian metric g on M of fundamental 2–form $\omega$ is given by

$$E_\omega(\varphi) = (1/(m-1)!)\int_M \mathcal{E}(\varphi)\wedge\omega^{m-1} \tag{1.1}$$

if M is compact. Because $e(\varphi)*1 = (1/(m-1)!)\,\mathcal{E}(\varphi)\wedge\omega^{m-1}$.

Lemma 1.5. (1) If $\varphi$ is a pluriharmonic map, then the (1,1)–form $\mathcal{E}(\varphi)$ is closed

([12], [13]). Moreover if M is compact and M has a cosymplectic Hermitian metric of fundamental 2–form $\omega$, then the energy $E_\omega(\varphi)$ depends only on the cohomology class $[\omega^{m-1}] \in H^{2m-2}(M,\mathbb{R})$.

(2) Let g be a cosymplectic Hermitian metric on M of fundamental 2–form $\omega$. If $\varphi : (M,g) \longrightarrow (N,g_N)$ is harmonic, then we have

$$(1.2) \qquad \sqrt{-1}(d'd''\mathscr{E}(\varphi))\wedge\omega^{m-2}$$

$$= (1/m(m-1))(|\nabla''\partial\varphi|^2 - \sum_{i,j=1}^{m} g_N(R^N(\varphi_i\wedge\varphi_j),\overline{\varphi_i\wedge\varphi_j}))\omega^m ,$$

where $\{\varphi_i\}$ are the components of $\partial\varphi$ with respect to a local unitary frame field of $TM^{(1,0)}$ relative to g.

Proof. (1) follows from the closedness of $\mathscr{E}(\varphi)$ and (1.1). By using the Ricci identity, simple computations show

$$(1.3) \qquad \sqrt{-1}(d'd''\mathscr{E}(\varphi))\wedge\omega^{m-2}$$

$$= (1/m(m-1))(|\nabla''\partial\varphi|^2 - |\mathrm{tr}\nabla''\partial\varphi|^2 - \sum_{i,j=1}^{m} g_N(R^N(\varphi_i\wedge\varphi_j),\overline{\varphi_i\wedge\varphi_j}))\omega^m ,$$

where $\mathrm{tr}_g\,\nabla''\partial\varphi = \sum_{i=1}^{m}(\nabla''_{\bar{i}}\partial\varphi)_i$. Since $\omega$ is coclosed, we have $\mathrm{tr}_g\,\nabla''\partial\varphi = (1/2)\tau_\varphi$, where $\tau_\varphi$ is a tension field of $\varphi$. Hence (1.3) reduces to (1.2).

q.e.d.

Remark. (1.2) is a slight extension of Bochner type identity of [20]. From this we see the following. Let $\varphi : M \longrightarrow N$ be a harmonic map from a compact Kähler manifold $M$ to a Riemannian manifold $N$ with nonnegative curvature operator. Then the following statements are equivalent each other:

(i) $\varphi$ is a pluriharmonic map.

(ii) The curvature form of the induced connection $\nabla^{\varphi}$ in $\varphi^{-1}TN^{\mathbb{C}}$ is of type (1,1).

(iii) There exists a holomorphic vector bundle structure in $\varphi^{-1}TN^{\mathbb{C}}$ with the $\bar{\partial}$–operator $(\nabla^{\varphi})''$, where $(\nabla^{\varphi})''$ denotes the (0,1)–part of the connection $\nabla^{\varphi}$.

## 2. Pluriharmonic maps into Lie groups.

Let M be a complex m–dimensional connected complex manifold and G be a Lie group with the Lie algebra $\mathfrak{g}$ . Denote by $\mu_G$ the Maurer–Cartan form of G , which is a left invariant $\mathfrak{g}$–valued 1–form on G . Let $\varphi : M \longrightarrow G$ be a smooth map. Set $\alpha = \varphi^* \mu_G$ , which is a $\mathfrak{g}$–valued 1–form on M . Then we have the decomposition $\alpha = \alpha' + \alpha''$ of $\alpha$ into (1,0)–and (0,1)–parts relative to the complex structure of M . By the Maurer–Cartan equation, the $\mathfrak{g}$–valued 1–form $\alpha$ satisfies

$$(2.1) \qquad d\alpha + (1/2)[\alpha \wedge \alpha] = 0 .$$

This is equivalent to the flatness of the connection $d+\alpha$ , which means an integrability condition for a smooth map into G . (2.1) is equivalent to the following system of equations:

$$(2.2) \qquad d''\alpha' + d'\alpha'' + [\alpha' \wedge \alpha''] = 0 ,$$

$$(2.3) \qquad d'\alpha' + (1/2)[\alpha' \wedge \alpha'] = 0 ,$$

$$(2.4) \qquad d''\alpha'' + (1/2)[\alpha'' \wedge \alpha''] = 0 .$$

We recall that in the case when M is a Riemann surface, the harmonic map equation of $\varphi$ is $d''\alpha' - d'\alpha'' = 0$ (cf. [24], [5]).

<u>Lemma 2.1</u>. $\varphi : M \longrightarrow G$ is a pluriharmonic map if and only if the 1–form $\alpha$ satisfies

(2.5) $$d''\alpha' - d'\alpha'' = 0 .$$

Proof. By Proposition 1.1 the pluriharmonicity of $\varphi$ is equivalent to that, for each holomorphic curve $i : C \longrightarrow M$ , $\varphi \circ i : C \longrightarrow M$ is harmonic. Since the harmonic map equation of $\varphi \circ i$ is $d''(i^*\alpha)' - d'(i^*\alpha)'' = i^*(d''\alpha' - d'\alpha'') = 0$ , this is equivalent to that $(d''\alpha' - d'\alpha'')(Z,\bar{Z}) = 0$ for each $Z \in TM^{(1,0)}$ . Hence (2.5) is the pluriharmonic map equation for $\varphi$ .

q.e.d.

Note that the pair of (2.2) and (2.5) is equivalent to the pair of

(2.6) $$d''\alpha' + (1/2)[\alpha' \wedge \alpha''] = 0 \text{ and}$$

(2.7) $$d'\alpha'' + (1/2)[\alpha' \wedge \alpha''] = 0 .$$

Lemma 2.2. Assume that $G$ is a compact Lie group equipped with a biinvariant Riemannian metric $g_G$ induced by an AdG–invariant inner product $( \, , \, )$ of $\mathfrak{g}$ . Let $\varphi : M \longrightarrow G$ be a pluriharmonic map from a complex manifold $M$ to $G$ . Then (1) we have

(2.8) $$[\alpha' \wedge \alpha'] = [\alpha'' \wedge \alpha''] = 0 \text{ and}$$

(2.9) $$d'\alpha' = d''\alpha'' = 0 .$$

Moreover (2), setting $A = (1/2)\alpha$ , the curvature form of the G–connection $d_A = d+A$ in the trivial vector bundle $\underline{V} = M \times V$ is of type $(1,1)$ , where $V$ is an arbitrary

G–module.

Proof. (1) Since the curvature tensor $R^G$ of the Riemannian manifold $(G,g_G)$ at the identity e is given by $R^G(X,Y)_e = -(1/4)\mathrm{ad}[X,Y]$ for $X,Y \in \mathfrak{g}$, the curvature operator of $(G,g_G)$ is nonnegative, and hence by Lemma 1.3 we have $R^G(d\varphi(Z),d\varphi(W)) = R^G(d\varphi(\bar{Z}),d\varphi(\bar{W})) = 0$ for each $Z,W \in T_x^{(1,0)}M$. Thus by the left translation and the AdG–invariance of ( , ) we get $|[\alpha(Z),\alpha(W)]|^2 = |[\alpha(\bar{Z}),\alpha(\bar{W})]|^2 = 0$ for each $Z,W \in T_xM^{(1,0)}$, which is equivalent to (2.8). (2.9) follows from (2.3), (2.4) and (2.8). (2) By (2.1) the curvature form of the connection $d_A$ is $dA + (1/2)[A \wedge A] = -(1/8)[\alpha \wedge \alpha]$. By (1) we see that the curvature form is of type (1,1).

q.e.d.

Remark. (1) In case $m = 1$ the statements of Lemma 2.2 are trivial. In case $m \geq 2$ the compactness of G is essential to the statements. (2) When $\underline{V} = \mathfrak{g}$, the connection $d_A$ coincides with the connection induced from the Riemannian connection of $(G,g_G)$.

Set $d'_A = d'+A'$ and $d''_A = d''+A''$, where $A' = (1/2)\alpha'$ and $A'' = (1/2)\alpha''$.

It follows from (2.2), (2.5), (2.6) and (2.8) that a smooth map $\varphi : M \longrightarrow G$ is pluriharmonic if and only if $A = (1/2)\varphi^*\mu_G$ satisfies

$$d''_A \circ d''_A = 0 \quad \text{and} \quad d''_A A' = 0 \ . \tag{2.10}$$

The condition $d''_A \circ d''_A = 0$ is the integrability condition for the $\bar{\partial}$–operator $d''_A$; by [11] it produces a holomorphic vector bundle structure in $\underline{V}$. The condition $d''_A A' = 0$ means that $A'$ is a $d''_A$–holomorphic section of $T^*M^{(1,0)} \otimes \mathrm{End}(\underline{V})$. So $A'$ can be considered as a Higgs field. In [19], Simpson called a pair $(E,\Psi)$ of a holomorphic vector bundle E and a holomorphic 1–form $\Psi$ with coefficients in End(E) a Higgs bundle, and investigated such pairs satisfying $[\Psi \wedge \Psi] = 0$. These objects had previously been investigated by Hitchin [8] in the case of Riemann surfaces. We conclude with the following.

Corollary 2.3. If $\varphi : M \longrightarrow G$ is a pluriharmonic map into a compact Lie group $G$, then for any G–module $\underline{V}$ the pair $(E,\Psi)$ of $E = \underline{V}$ with the $d''_A$–holomorphic structure and $\Psi = A'$ is a Higgs bundle over $M$ satisfying $[\Psi \wedge \Psi] = 0$.

The complex structure J of M induces endomorphisms of $T^*M^{\mathbb{C}}$ and $T^*M^{\mathbb{C}} \otimes g^{\mathbb{C}}$. We define a $g^{\mathbb{C}}$–valued 1–form $J\alpha$ on M as $(J\alpha)(X) = \alpha(J(X))$. Then the pluriharmonic map equation (2.5) is written as $\{d(J\alpha)\}^{(1,1)} = 0$.
For any smooth map $\varphi : M \longrightarrow G$, the energy form is given by

$$\mathcal{E}(\varphi) = \sqrt{-1}(\alpha' \wedge \alpha'') \tag{2.11}$$

which is real (1,1)–form on M. Here ( , ) is the AdG–invariant inner product of g which induces the biinvariant Riemannian metric $g_G$ on G, and $(\alpha' \wedge \alpha'')(X,Y) = (\alpha'(X),\alpha''(Y)) - (\alpha'(Y),\alpha''(X))$. Let g be a cosymplectic Hermitian metric on M of fundamental 2–form $\omega$. Then $\varphi : (M,g) \longrightarrow G$ is harmonic if and only if

$$d(J\alpha) \wedge \omega^{m-1} = 0 . \tag{2.12}$$

The energy of $\varphi$ relative to a Hermitian metric g of fundamental 2–form $\omega$ is given by

$$E_\omega(\varphi) = (m/m!)\int_M \mathscr{E}(\varphi)\wedge\omega^{m-1} \tag{2.13}$$

if M is compact. Moreover the following lemma follows from Lemma 1.5 directly.

Lemma 2.4. Let $\varphi : M \longrightarrow G$ be a smooth from a complex manifold $M$.

(1) If $\varphi$ is pluriharmonic, then $\mathscr{E}(\varphi)$ is a closed real (1,1)–form. Moreover if $M$ is compact and $M$ has a cosymplectic Hermitian metric of fundamental 2–form $\omega$, then the energy $E_\omega(\varphi)$ depends only on the cohomology class $[\mathscr{E}(\varphi)] \in H^2(M,\mathbb{R})$ and $[\omega^{m-1}] \in H^{2m-2}(M,\mathbb{R})$.

(2) For any fundamental 2–form $\omega$ of a cosymplectic Hermitian metric $g$ in $M$, if $\varphi : (M,g) \longrightarrow G$ is harmonic, then we have

$$\sqrt{-1}(d'd''\mathscr{E}(\varphi))\wedge\omega^{m-2} = (4/m(m-1))(|d''_A A'|^2 - |[A'\wedge A']|^2)\omega^m . \tag{2.14}$$

Remark. (1) In general, for any (1,1)–form $\omega$ on $M$, we also define $E_\omega(\varphi)$ as (2.13). (2) The formula (2.14) is useful. (2.8) follows also from the formula (2.14).

## 3. Extended solutions of a pluriharmonic map.

Let M be an m–dimensional connected complex manifold. First we prepare the following lemma.

Lemma 3.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Let $\alpha$ be a $\mathfrak{g}$–valued 1–form on M and $\alpha = \alpha' + \alpha''$ be the decomposition of $\alpha$ into (1,0)–and (0,1)–parts. For $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, set $\alpha_\lambda = (1/2)(1-\lambda^{-1})\alpha' + (1/2)(1-\lambda)\alpha''$, which is a $\mathfrak{g}^{\mathbb{C}}$–valued 1–form on M. Then

$$d\alpha_\lambda + (1/2)[\alpha_\lambda \wedge \alpha_\lambda] = 0 \tag{3.1}$$

holds for each $\lambda \in \mathbb{C}^*$ if and only if the 1–form $\alpha$ satisfies (2.2), (2.5), (2.8) and (2.9).

Proof. We compute

$$\begin{aligned}
&d\alpha_\lambda + (1/2)[\alpha_\lambda \wedge \alpha_\lambda] \\
&= (1/2)(1-\lambda^{-1})d'\alpha' + (1/2)(1-\lambda^{-1})d''\alpha' + (1/2)(1-\lambda)d'\alpha'' + (1/2)(1-\lambda)d''\alpha'' \\
&+ (1/8)(1-\lambda^{-1})^2[\alpha' \wedge \alpha'] + (1/4)(1-\lambda^{-1})(1-\lambda)[\alpha' \wedge \alpha''] + (1/8)(1-\lambda)^2[\alpha'' \wedge \alpha''] \\
&= (1/2)(1-\lambda^{-1})\{d''\alpha' + (1/2)[\alpha' \wedge \alpha'']\} + (1/2)(1-\lambda)\{d'\alpha'' + (1/2)[\alpha' \wedge \alpha'']\} \\
&+ (1/2)(1-\lambda^{-1})\{d'\alpha' + (1/4)(1-\lambda^{-1})[\alpha' \wedge \alpha']\} \\
&+ (1/2)(1-\lambda)\{d''\alpha'' + (1/4)(1-\lambda)[\alpha'' \wedge \alpha'']\}.
\end{aligned}$$

Lemma 3.1 follows from this equation. q.e.d.

Let G be a compact Lie group with a biinvariant Riemannian metric and

$\varphi : M \longrightarrow G$ be a smooth map. Set $\alpha = \varphi^* \mu_G = \alpha' + \alpha''$ , where $\alpha'$ and $\alpha''$ denotes the (1,0)–part and (0,1)–part of $\alpha$ respectively. Set for each $\lambda \in \mathbb{C}^*$ ,

$$\alpha_\lambda = (1/2)(1-\lambda^{-1})\alpha' + (1/2)(1-\lambda)\alpha'' . \tag{3.2}$$

Let $G^{\mathbb{C}}$ be the complexification of $G$ . We consider linear differential equations

$$\Phi_\lambda^* \mu_{G^{\mathbb{C}}} = \alpha_\lambda \tag{3.3}$$

of smooth maps $\Phi_\lambda : M \longrightarrow G^{\mathbb{C}}$ for each $\lambda \in \mathbb{C}^*$ . If we fix a realization of $G^{\mathbb{C}}$ in a general linear group $GL(N,\mathbb{C})$ such that $G = G^{\mathbb{C}} \cap U(N)$ , then (3.3) can be written as

$$\begin{cases} d'\Phi_\lambda = (1/2)(1-\lambda^{-1})\Phi_\lambda\alpha' \\ d''\Phi_\lambda = (1/2)(1-\lambda)\Phi_\lambda\alpha'' \end{cases} , \tag{3.4}$$

for each $\lambda \in \mathbb{C}^*$ . Since the integrability condition of the linear equations (3.3) or (3.4) is (3.1) for each $\lambda \in \mathbb{C}^*$ , hence it follows from Lemmas 2.1, 2.2 and 3.1 that "the pluriharmonicity of a smooth map $\varphi$ from a complex manifold $M$ to a compact Lie group $G$ is equivalent to the integrability condition of the linear equations (3.3) or (3.4) for all $\lambda \in \mathbb{C}^*$ ". Thus we get the following.

Theorem 3.2. (1) Let $M$ be a connected complex manifold with the fixed base point $x_0 \in M$ and $G$ be a compact Lie group. Assume that $M$ is simply connected, more generally, $\mathrm{Hom}(\pi_1(M),G) = \{1\}$ . If $\varphi : M \longrightarrow G$ is a pluriharmonic map, then for any map $h : \mathbb{C}^* \longrightarrow G^{\mathbb{C}}$ there exists a unique map $\Phi : \mathbb{C}^* \times M \longrightarrow G^{\mathbb{C}}$ such that $\Phi_\lambda(x_0) = \Phi(\lambda,x_0) = h(\lambda)$ and $\Phi_\lambda^* \mu_{G^{\mathbb{C}}} = \alpha_\lambda$ for each $\lambda \in \mathbb{C}^*$ .

(2) Conversely, if a map $\Phi : \mathbb{C}^* \times M \longrightarrow G^{\mathbb{C}}$ satisfies $\Phi_\lambda^* \mu_{G^{\mathbb{C}}} = (1/2)(1-\lambda^{-1})\alpha' + (1/2)(1-\lambda)\alpha''$ for each $\lambda \in \mathbb{C}^*$, where $\alpha'$ and $\alpha''$ are $\mathfrak{g}^{\mathbb{C}}$–valued (1,0)– and (0,1)–forms on $M$ independent of $\lambda \in \mathbb{C}^*$, then $\Phi_{-1} : M \longrightarrow G^{\mathbb{C}}$ satisfies (2.5), (2.8) and (2.9).

The map $\Phi : \mathbb{C}^* \times M \longrightarrow G^{\mathbb{C}}$ is called an extended solution of a pluriharmonic map $\varphi = \Phi_{-1}$ (or extended pluriharmonic map).

Remark. (1) $\Phi_1$ is always a constant map. (2) $\Phi_{-1} = a\,\varphi$ for some $a \in G^{\mathbb{C}}$. (3) If $h : \mathbb{C}^* \longrightarrow G^{\mathbb{C}}$ is holomorphic, then $\Phi_\lambda(x)$ is holomorphic in $\lambda \in \mathbb{C}^*$ for each fixed $x \in M$. (4) If $h(\lambda) \in G$ for each $\lambda \in S^1$, then $\Phi_\lambda : M \longrightarrow G$ for each $\lambda \in S^1$.

From now on we consider only extended solutions satisfying $\Phi_1 = e$. An extended solution $\Phi$ is called real if $\Phi_\lambda : M \longrightarrow G$ for each $\lambda \in S^1$. An extended solution $\Phi$ is called based if $\Phi_\lambda(x_0) = e$ for each $\lambda \in \mathbb{C}^*$. Note that a based extended solution is always real.

Let $\Omega G$ be the group of all based smooth loops in $G$, that is, $\Omega G = \{\gamma : S^1 \longrightarrow G$ smooth, $\gamma(1) = e\}$. Let $\pi : \Omega G \longrightarrow G$ denote the natural projection defined by $\pi(\gamma) = \gamma(-1)$ for $\gamma \in \Omega G$. Any real extended solution $\Phi : \mathbb{C}^* \times M \longrightarrow G^{\mathbb{C}}$ can be regarded as a smooth map $\Phi : M \longrightarrow \Omega G$ by $(\Phi(x))(\lambda) = \Phi_\lambda(x)$ for $x \in M$ and $\lambda \in S^1$. We can give some observations on extended solutions as maps into $\Omega G$ similar to [2] and [5]. It is well–known that $\Omega G$ has the standard infinite dimensional complex manifold structure $J_1$ which makes, together with $L^2_{1/2}$–metric, $\Omega G$ into a Kähler manifold of Kähler form $-S$ (cf. [16]). Moreover $\Omega G$ has an interesting nonintegrable almost complex structure $J_2$ ([2], [5]). Then we see that any real extended solution $\Phi : M \longrightarrow \Omega G$ of a pluriharmonic map $\varphi : M \longrightarrow G$ is a $J_1$– and $J_2$–holomorphic map.

Hence we obtain the following.

Theorem 3.3. Assume that a complex manifold M satisfies $\mathrm{Hom}(\pi_1(M),G) = \{1\}$. If $\varphi : M \longrightarrow G$ is a pluriharmonic map, then there exists a $J_1$–holomorphic map $\Phi : M \longrightarrow \Omega G$ such that the diagram

$$\begin{array}{ccc} & & \Omega G \\ & \overset{\Phi}{\nearrow} & \big\downarrow \pi \\ M & \xrightarrow[\varphi]{} & G \end{array}$$

commutes.

Remark. By results of Atiyah–Donaldson [1], there exists a bijective correspondence between the space of based $J_1$–holomorphic maps of a Riemann sphere $S^2$ into G and the space of framed G–instantons over $\mathbb{R}^4$.

We recall the following fact on $\Omega G$ (cf. [6]):

Fact. (1) The left invariant symplectic form on $\Omega G$ is given by

$$S(X,Y) = \int_0^1 (\dot{X}(t), Y(t))dt$$

for each $X,Y \in \Omega\mathfrak{g}$. Here $\lambda = e^{2\pi\sqrt{-1}\,t}$.

(2) If $H^3(G,\mathbb{Z}) \cong H^2(\Omega G,\mathbb{Z}) \cong \mathbb{Z}$, then the positive generator $[\gamma_{\Omega G}] \in H^2(\Omega G,\mathbb{Z})$ is represented by $\gamma_{\Omega G} = -(|\delta|^2/8\pi^2)S$, where $\delta$ denotes the highest root of G.

Theorem 3.4. Let M be an m–dimensional connected complex manifold. Assume that a pluriharmonic map $\varphi : M \longrightarrow G$ has a real extended solution $\Phi : M \longrightarrow \Omega G$ with $\Phi_{-1} = \varphi$. Then the following statements hold:

(1) The cohomology class $[(|\delta|^2/16\pi)\, \mathcal{E}(\varphi)]$ is integral.

(2) The energy of $\varphi$ relative to a (1,1)–form $\omega$ on M is given by

$$E_\omega(\varphi) = (16m\pi/m!\,|\delta|^2) \int_M \Phi^* \gamma_{\Omega G} \wedge \omega^{m-1} \tag{3.5}$$

if M is compact.

(3) Moreover if $\omega$ is a Hodge metric on M so that M is a projective algebraic manifold, or more generally $\omega$ is a fundamental 2–form of a cosymplectic Hermitian metric on M such that the cohomology class $[\omega^{m-1}]$ is integral, then we have

$$E_\omega(\varphi) = (16m\pi/m!\,|\delta|^2)\deg_\omega(\varphi) \ , \tag{3.6}$$

where $\deg_\omega(\varphi) = \int_M \Phi^* [\gamma_{\Omega G}] \wedge [\omega^{m-1}]$ is a nonnegative integer.

<u>Proof</u>. By a simple computation we have

$$\Phi^* S = -(\pi/2)\, \mathcal{E}(\varphi) \ . \tag{3.7}$$

Hence by Fact (2) and (3.7) we get

$$(|\delta|^2/16\pi)\ \mathcal{E}(\varphi) = \Phi^* \gamma_{\Omega G} \ . \tag{3.8}$$

Thus we get (1). Integrating (3.8) over M relative to $\omega^{m-1}$, we obtain (2). (3) follows from (2). q.e.d.

This theorem provides <u>a priori quantization</u> of the energy form of pluriharmonic maps $\varphi : M \longrightarrow G$, or in the language of physicists, a <u>topological charge</u>.

Assume that $\varphi : M \longrightarrow G$ is a nonconstant pluriharmonic map from a compact complex manifold and $\varphi$ has a real extended solution $\Phi : M \longrightarrow \Omega G$ with $\Phi_{-1} = \varphi$. Let g be any Hermitian metric on M of fundamental 2–form $\omega$. For $\lambda = e^{\sqrt{-1}t} \in S^1$, the energy of $\Phi_\lambda : M \longrightarrow G$ reduces to

$$E_\omega(\Phi_\lambda) = (1/2)\int_M \|\alpha_\lambda\|^2 * 1 = \int_M (\alpha'_\lambda, \alpha''_\lambda) * 1$$

$$= (1/2)(1-\cos t)E_\omega(\varphi) \ .$$

Therefore we have $(d^2/dt^2)E_\omega(\Phi_\lambda)\Big|_{t=\pi} = -(1/2)E_\omega(\varphi) < 0$.

<u>Theorem 3.5</u>. Let M be a simply connected compact cosymplectic Hermitian manifold and G be a compact Lie group with a biinvariant Riemannian metric. Then any nonconstant pluriharmonic map $\varphi : M \longrightarrow G$ is unstable as a harmonic map for any cosymplectic Hermitian metric on M.

<u>Corollary 3.6</u>. Let $\mathbb{C}P^m$ be a complex projective space with the Fubini–Study metric. Then any nonconstant harmonic map $\varphi : \mathbb{C}P^m \longrightarrow G$ is unstable.

<u>Proof</u>. It follows from Theorem 3.5 and a result of [12].

q.e.d.

<u>Remark</u>. Every argument and result in [27], [28] and this paper remains valid in

the case of pluriharmonic gauges in the spirit of [28], i.e. solutions $(A,\Phi)$ of

$$\begin{cases} F(A) + (1/2)[\Phi,\Phi] = -2\pi\sqrt{-1}\,\mu(V) , \\ \overline{\partial}_A \Phi_z = 0 , \quad [\Phi_z,\Phi_z] = 0 , \\ \overline{\partial}_A \circ \overline{\partial}_A = 0 , \end{cases}$$

on a Hermitian vector bundle $V \longrightarrow M$ ; in the notation of the paper.

4. Uniton and minimal uniton number for a pluriharmonic map.

Assume that G is the unitary group U(N). Set $Gr(\mathbb{C}^N) = \{a \in U(N); a^2 = I\}$. Each connected component of $Gr(\mathbb{C}^N)$ is a complex Grassmann manifold $G_\ell(\mathbb{C}^N)$ for $0 \leq \ell \leq N$. Each $G_\ell(\mathbb{C}^N)$ has the Hermitian symmetric space structure induced from biinvariant Riemannian metric of U(N).

Let M be an m–dimensional complex manifold. We call that a pluriharmonic map $\varphi : M \longrightarrow U(N)$ has <u>at most uniton number</u> n if there exists a real extended solution $\Phi$ such that

(i) $\Phi$ has the Laurent expansion in $\lambda \in \mathbb{C}^*$ of the form $\Phi_\lambda = \sum_{i=0}^{n} T_i \lambda^i$, $T_n \not\equiv 0$,

(ii) $\Phi_1 = I$, and

(iii) $\Phi_{-1} = a\varphi$ for some $a \in U(N)$. Here $T_i : M \longrightarrow \mathfrak{gl}(N,\mathbb{C})$.

A pluriharmonic map $\varphi$ is called an <u>n–uniton</u> if $\varphi$ has minimal uniton number n. A 0–uniton is a constant map to the identity.

We can show the following fundamental results about unitons in the same way as in [24].

<u>Theorem 4.1</u>. A pluriharmonic map $\varphi : M \longrightarrow U(N)$ is a 1–uniton if and only if $\varphi = a\,h$ for some $a \in U(N)$ and a holomorphic map $h : M \longrightarrow G_\ell(\mathbb{C}^N) \subset U(N)$.

<u>Proof</u>. Consider maps $\Phi_\lambda = P + \lambda Q : M \longrightarrow GL(N,\mathbb{C})$ for $\lambda \in \mathbb{C}^*$. By simple computations we observe that $\Phi$ satisfies (3.4), the reality condition and $\Phi_1 = I$ if and only if $P^2 = P^* = P$, $P^\perp d''P = 0$ and $Q = I-P = P^\perp$. Note that the condition

$P^{\perp}d''P = 0$ is equivalent to the holomorphicity of the vector subbundle $\underline{P}$ of the trivial holomorphic vector bundle $\underline{\mathbb{C}}^N = M \times \mathbb{C}^N$ corresponding to the Hermitian projections $P$. Theorem 4.1 follows from this observations. q.e.d.

Theorem 4.2. Assume that $M$ is a compact complex manifold. If $\Phi$ is a based extended solution of a pluriharmonic map $\Phi_{-1} = \varphi : M \longrightarrow U(N)$ with $\varphi(x_0) = I$, then $\Phi$ has finite Laurent expansion in $\lambda \in \mathbb{C}^*$, that is, $\Phi_\lambda = \sum_{i=-p}^{q} T_i\lambda^i$ for some $p,q \geq 0$. In particular any pluriharmonic map $\varphi : M \longrightarrow U(N)$ from a simply connected compact complex manifold $M$ always has finite uniton number.

Proof. By (3.4), (2.5), (2.6) and (2.7) a simple computation shows

$$\begin{aligned}(4.1)\qquad & d'(d''\Phi_\lambda - \Phi_\lambda\alpha'') - d''(d'\Phi_\lambda - \Phi_\lambda\alpha') \\ &= -(1/2)\Phi_\lambda\{\lambda(d'\alpha'' + (1/2)[\alpha' \wedge \alpha'']) + (d'\alpha'' - d''\alpha') \\ &\quad - \lambda^{-1}(d''\alpha' + (1/2)[\alpha' \wedge \alpha''])\} = 0 \ .\end{aligned}$$

Choose a Hermitian metric $g$ on $M$. By (2.5), (4.1) becomes

$$2d'd''\Phi_\lambda - d'\Phi_\lambda \wedge \alpha'' + d''\Phi_\lambda \wedge \alpha' = 0 \ .$$

In particular each coefficient $T_i$ of $\Phi_\lambda$ in $\lambda$ is a solution of a linear elliptic equation

$$(4.2)\qquad L(f) = 2\,\mathrm{tr}_g d'd''f - \mathrm{tr}_g d'f\wedge\alpha'' + \mathrm{tr}_g d''f\wedge\alpha' = 0 \ ,$$

for a $\mathfrak{gl}(N,\mathbb{C})$-valued smooth function $f$ on $M$. By the compactness of $M$, the solution space of $L(f) = 0$ is finite dimensional. Now we can apply the argument in proof of [24, Theorem 11.5] to our situation by using (3.4). Hence we get the first statement of Theorem 4.2. The second statement follows from it and Theorem 3.2 (1).

q.e.d.

By applying the argument of [24, Section 13] to extended solutions of a pluriharmonic map, we get the following.

Theorem 4.3. Let $\varphi : M \longrightarrow U(N)$ be a pluriharmonic map from a complex manifold $M$ with finite uniton number. Then there exists a unique real extended solution $\Phi$ such that $\Phi_1 = I$, $\Phi_{-1} = a\varphi$ for some $a \in U(N)$, $\Phi_\lambda = \sum_{i=0}^{n} T_i \lambda^i$ $(\lambda \in \mathbb{C}^*)$, $T_n \not\equiv 0$, and $V_0(\Phi) = \mathbb{C}^N$, where $V_0(\Phi)$ denotes the vector subspace of $\mathbb{C}^N$ spanned by $\{(T_0)_x v \,;\, x \in M,\ v \in \mathbb{C}^N\}$. Moreover $n$ is equal to the minimal uniton number of $\varphi$.

5. Uniton equations and rational unitons.

Let $\varphi : M \longrightarrow U(N)$ be a pluriharmonic map from a complex manifold $M$. We recall the bijective correspondence between complex subbundles $\eta = \underline{P}$ of the vector bundle $\underline{\mathbb{C}}^N = M \times \mathbb{C}^N$ with rank $\ell$ and smooth maps $\Pi_\eta - \Pi_\eta^\perp = P - P^\perp : M \longrightarrow G_\ell(\mathbb{C}^N) \subset U(N)$, where $\Pi_\eta$ (resp. $\Pi_\eta^\perp$) denotes Hermitian projections onto $\eta$ (resp. its orthogonal complement $\eta^\perp$ in $\underline{\mathbb{C}}^N$) and $\underline{P}$ denotes the image subbundle of the Hermitian projections $P$ (i.e. $P^2 = P^* = P$) in $\underline{\mathbb{C}}^N$. We assume that $\Phi : \mathbb{C}^* \times M \longrightarrow GL(N,\mathbb{C})$ is a real extended solution of the pluriharmonic map $\Phi_{-1} = \varphi$. For a smooth map $P - P^\perp : M \longrightarrow G_\ell(\mathbb{C}^N)$, set $\tilde{\Phi}_\lambda = \Phi_\lambda(P + \lambda P^\perp) : M \longrightarrow GL(N,\mathbb{C})$ for each $\lambda \in \mathbb{C}^*$. Note that $\tilde{\Phi}_\lambda : M \longrightarrow U(N)$ for each $\lambda \in S^1$. Set $\tilde{\varphi} = \tilde{\Phi}_{-1} : M \longrightarrow U(N)$.

Theorem 5.1. $\tilde{\Phi}$ is also an extended solution, i.e. a solution to (3.4), if and only if the Hermitian projections $P$ satisfy

(5.1) $$P^\perp \alpha' P = 0 \quad \text{and}$$

(5.2) $$P^\perp(d''P + (1/2)\alpha''P) = 0 \ .$$

(5.1) and (5.2) are equivalent to

(5.3) $$P\alpha'' P^\perp = 0 \quad \text{and}$$

(5.4) $$P(d'P^\perp + (1/2)\alpha' P^\perp) = 0 \ .$$

Moreover in this case $\tilde{\varphi} : M \longrightarrow U(N)$ is pluriharmonic and $\tilde{\alpha} = \tilde{\varphi}^{-1} d\tilde{\varphi} = \tilde{\alpha}' + \tilde{\alpha}''$ is given by

$$(5.6) \qquad \tilde{\alpha}' = \alpha' - 2d'P \ ,$$

$$(5.7) \qquad \tilde{\alpha}'' = \alpha'' + 2d''P \ .$$

Proof. By direct computations we have

$$(5.8) \qquad \tilde{\Phi}_\lambda d'\tilde{\Phi}_\lambda = (1/2)(1-\lambda^{-1})\{\lambda^{-1}P^\perp\alpha'P + (\alpha'-2d'P) - P^\perp\alpha'P - P(2d'P^\perp + \alpha'P^\perp) + \lambda P(2d'P^\perp + \alpha'P^\perp)\} \text{ , and}$$

$$(5.9) \qquad \tilde{\Phi}_\lambda d''\tilde{\Phi}_\lambda = (1/2)(1-\lambda)\{\lambda^{-1}P^\perp(2d''P + \alpha''P) + (\alpha'' + 2d''P) - P\alpha''P^\perp - P^\perp(2d''P + \alpha''P) + \lambda P\alpha''P^\perp\} \ .$$

Theorem 5.1 follows from (5.8), (5.9) and Theorem 3.2 (2). q.e.d.

The equations (5.1) and (5.2) are called uniton equations for a pluriharmonic map $\varphi$. A solution of the uniton equations is called a (smooth) uniton for $\varphi$.

Following Lemma 2.2 (2) and (2.10), we endow the trivial bundle $\underline{\mathbb{C}}^N$ over M with the holomorphic vector bundle structure of the $\bar{\partial}$–operator $d''_A = d''+A''$ , and $A'$ is a $d''_A$–holomorphic 1–form with values in $\mathrm{End}(\underline{\mathbb{C}}^N,\underline{\mathbb{C}}^N)$ . We can restate Theorem 5.1 as follows.

Theorem 5.2. $\tilde{\Phi}$ is an extended solution if and only if the subbundle $\underline{P}$ of $\underline{\mathbb{C}}^N$ is (i)

$A'$–stable, i.e. $(A'(Z))C^\infty(\underline{P}) \subset C^\infty(\underline{P})$ for any $Z \in C^\infty(TM^{(1,0)})$, and (ii) a $d''_A$–holomorphic subbundle, i.e. $(d''_A)_Z C^\infty(\underline{P}) \subset C^\infty(\underline{P})$ for any $Z \in C^\infty(TM^{(1,0)})$.

This procedure of making a new pluriharmonic map $\tilde{\varphi}$ from a given pluriharmonic map $\varphi$ is called addition of a union. We show a formula for the difference of energies in adding a uniton. In case $m = 1$, the formula was shown by the second named author ([26]).

We define a complex bilinear form ( , ) and a Hermitian inner product ⟨ , ⟩ of $\mathfrak{gl}(N,\mathbb{C})$ as

$$(A,B) = -\operatorname{tr}(AB),$$
$$\langle A,B\rangle = -(A,B^*) = \operatorname{tr}(AB^*).$$

Theorem 5.3. Let $M$ be an $m$–dimensional compact complex manifold and $\varphi : M \longrightarrow U(n)$ be a pluriharmonic map. If $\tilde{\varphi} = \varphi(P-P^\perp) : M \longrightarrow U(N)$ is a pluriharmonic map obtained from $\varphi$ by addition of a uniton $\underline{P}$, then for any real (1,1)–form $\omega$ on $M$ with $d\omega^{m-1} = 0$ we have

$$E_\omega(\tilde{\varphi}) - E_\omega(\varphi) = -(8m\pi/m!)\deg_\omega(\underline{P}), \tag{5.10}$$

where $\deg_\omega(\underline{P}) = \int_M c_1(\underline{P})\wedge\omega^{m-1}$ is the $\omega$–degree of the complex vector bundle $\underline{P}$, and $\deg_\omega(\underline{P})$ is an integer if the cohomology class $[\omega^{m-1}]$ is integral.

We prove the formula (5.10) by showing the following lemmas.

Those are local results. We do not need the compactness of $M$. First it is standard to observe the following.

Lemma 5.4. Let $\nabla_A$ be the connection of a complex subbundle $\underline{P}$ induced from a connection $d_A$ in $\underline{\mathbb{C}}^N$; for each section $s$ of $\underline{P}$,

$$\nabla_A s = Pd_A s = d_A s - (d_A P)s \ .$$

Then the curvature from $F(\nabla_A)$ of the connection $\nabla_A$ is given by the End($\underline{P}$,$\underline{P}$)–valued 2–form

$$F(\nabla_A) = P(F(A) + d_A P \wedge d_A P)P \ ,$$

where $F(A)$ denotes the curvature form of the connection $d_A$ .

Set

$$c_1(\underline{P},\nabla_A) = (\sqrt{-1}/2\pi)\mathrm{tr}\, F(\nabla_A) \tag{5.12}$$

$$= (\sqrt{-1}/2\pi)\mathrm{tr}(F(A) + d_A P \wedge d_A P)P \ ,$$

which is a closed real 2–form on $M$ . By the Chern–Weil theory, $c_1(\underline{P},\nabla_A)$ represents the first Chern class $c_1(\underline{P})$ of the complex vector bundle $\underline{P}$ .

Lemma 5.5. If $P$ is a uniton for a pluriharmonic map $\varphi : M \longrightarrow U(N)$ and we set $\tilde{\varphi} = \varphi(P - P^{\perp}) : M \longrightarrow U(N)$ , then we have

$$\mathcal{E}(\tilde{\varphi}) - \mathcal{E}(\varphi) = -8\pi\, c_1(\underline{P},\nabla_A) \ . \tag{5.13}$$

In particular we have

$$[(1/8\pi)\,\mathscr{E}(\tilde{\varphi})]-[(1/8\pi)\,\mathscr{E}(\varphi)] = -c_1(\underline{P}) \tag{5.14}$$

as integral cohomology classes.

Proof. By (5.6), (5.7) and the uniton equations we have

$$\tilde{\alpha}' = P\alpha' P + P^{\perp}\alpha' P^{\perp} - 2P^{\perp}d_A' P \ , \tag{5.15}$$

$$\tilde{\alpha}'' = P\alpha'' P + P^{\perp}\alpha'' P^{\perp} + 2Pd_A'' P \ . \tag{5.16}$$

Hence using uniton equations we compute

$$\begin{aligned}\mathscr{E}(\tilde{\varphi}) &= \sqrt{-1}(\tilde{\alpha}'\wedge\tilde{\alpha}'') \\ &= \sqrt{-1}\{(\alpha' P\wedge\alpha'') + (P^{\perp}\alpha'\wedge\alpha'') - 4(P^{\perp}d_A' P\wedge d_A'' P)\} \ .\end{aligned}$$

Thus we get

$$\mathscr{E}(\tilde{\varphi})-\mathscr{E}(\varphi) = \sqrt{-1}\{(\alpha' P\wedge\alpha'')-(P\alpha'\wedge\alpha'')-4(d_A' P\wedge d_A'' P)\} \ .$$

Since $P(d_A P\wedge d_A P)P = -d_A' P\wedge d_A'' P$ by $d_A''$–holomorphicity of $\underline{P}$ and $F(A) = F(A)^{(1,1)} = -(1/4)\,[\alpha'\wedge\alpha'']$ , we have

$$\begin{aligned}\mathscr{E}(\tilde{\varphi})-\mathscr{E}(\varphi) &= \sqrt{-1}\{-([\alpha'\wedge\alpha''],P)-4(d_A' P\wedge d_A'' P)\} \\ &= -4\sqrt{-1}\ \mathrm{tr}(F(A) + d_A P\wedge d_A P)P \ .\end{aligned} \tag{5.17}$$

By (5.12) and (5.17) we get (5.13). q.e.d.

Proof of Theorem 5.3. Since $M$ is compact, by contracting (5.13) with $\omega^{m-1}$ and integrating it over $M$, we obtain (5.10). q.e.d.

In our theory it is important to introduce the notion of unitons with the singularity since we work over higher dimensional complex manifolds.

Let $\varphi : M \longrightarrow U(N)$ be a pluriharmonic map from an m–dimensional connected complex manifold $M$.

Definition. A complex subbundle $\eta$ of $\underline{\mathbb{C}}^N | W$ with rank $\ell$ defined on a dense open set $W$ of $M$ is called a rational uniton in $M$ for $\varphi$ if the following conditions are satisfied:

(i) $\eta$ is a smooth uniton for $\varphi$ defined on $W$.

(ii) For any local trivialization $\{(U_i,\sigma_i)\}$ of the holomorphic vector bundle $(\underline{\mathbb{C}}^N, d''_A)$ with holomorphic transition functions $\sigma_i \circ \sigma_j^{-1} : U_i \cap U_j \longrightarrow GL(N,\mathbb{C})$, each holomorphic map $\Pi_i - \Pi_i^{\perp} : U_i \cap W \longrightarrow G_\ell(\mathbb{C}^N)$ is rational in $U_i$, where $\zeta_i = \sigma_i(\eta | U_i \cap W)$ is the image bundle of $\eta | U_i \cap W$ under $\sigma_i : \underline{\mathbb{C}}^N|_{U_i} \longrightarrow U_i \times \mathbb{C}^N$ and $\Pi_i = \Pi_{\zeta_i}$ is the Hermitian projections onto $\zeta_i$ in $(U_i \cap W) \times \mathbb{C}^N$.

The following lemma is fundamental for rational unitons.

Lemma 5.6. Let $\eta$ be a rational uniton in $M$ for $\varphi$ defined on a dense open set $W$ of $M$. Set

$$G(\eta) = \{(x,(\Pi_\eta - \Pi_\eta^\perp)(x)) \in W \times G_\ell(\mathbb{C}^N);\ x \in M\} ,$$

the graph of the map $\Pi_\eta - \Pi_\eta^\perp : W \longrightarrow G_\ell(\mathbb{C}^N)$. Let p and q denote the Hermitian projection of the product manifold $M \times G_\ell(\mathbb{C}^N)$ onto M and $G_\ell(\mathbb{C}^N)$, respectively. Then the following statements hold:

(1) The closure $\overline{G(\eta)}$ of $G(\eta)$ in $M \times G_\ell(\mathbb{C}^N)$ has a structure of a complex analytic space (in general it is not an analytic subset of $M \times G_\ell(\mathbb{C}^N)$) such that $p : \overline{G(\eta)} \longrightarrow M$ is a proper surjective holomorphic map and $p : G(\eta) \longrightarrow W$ is a biholomorphic diffeomorphism.

(2) There exists an analytic subset $S(\eta)$ of M with $\dim_{\mathbb{C}} S(\eta) \leq m-2$ and $W \subset M - S(\eta)$ such that $p : G(\eta) - p^{-1}S(\eta) \longrightarrow M - S(\eta)$ is a biholomorphic diffeomorphism. Hence $q \circ (p \,|\, G(\eta) - p^{-1}S(\eta))^{-1} : M - S(\eta) \longrightarrow G_\ell(\mathbb{C}^N)$ induces a smooth uniton on $M - S(\eta)$, which extends $\eta$, and it is also a rational uniton in M for $\varphi$.

(3) There exists an m–dimensional connnected complex manifold $\hat{M}$ and a proper surjective holomorphic map $\nu : \hat{M} \longrightarrow M$ with a biholomorphic diffeomorphism $\nu : \hat{M} - \nu^{-1}S(\eta) \longrightarrow M - S(\eta)$ such that the smooth uniton $\nu^{-1}\eta$ on $\nu^{-1}W$ is extended to a smooth uniton $\hat{\eta}$ for a pluriharmonic map $\hat{\varphi} = \varphi \circ \nu : \hat{M} \longrightarrow U(N)$ defined globally on $\hat{M}$.

<u>Proof</u>. Applying fundamental facts on meromorphic maps ([17] and c.f. [7]) to each $\Pi_i - \Pi_i^\perp : U_i \cap W \longrightarrow G_\ell(\mathbb{C}^N)$, we obtain (1) and (2). We show (3). By Hironaka's resolution of singularity, there exists an m–dimensional connected complex manifold $\hat{M}$ and a proper surjective holomorphic map $\hat{\nu} : \hat{M} \longrightarrow \overline{G(\eta)}$ such that $\hat{\nu} : \hat{M} - \hat{\nu}^{-1}(\overline{G(\eta)} - p^{-1}S(\eta)) \longrightarrow \overline{G(\eta)} - p^{-1}S(\eta)$ is a biholomorphic diffeomorphism. Set $\nu = p \circ \hat{\nu} : \hat{M} \longrightarrow M$, $\Pi_{\hat{\eta}} - \Pi_{\hat{\eta}}^\perp = q \circ \hat{\nu} : \hat{M} \longrightarrow G_\ell(\mathbb{C}^N)$ and $\hat{\varphi} = \varphi \circ \nu : \hat{M} \longrightarrow U(N)$.

Here $\hat{\eta}$ denotes the complex subbundle of $\hat{M} \times \mathbb{C}^N$ corresponding to the map $q \circ \hat{\nu}$. Since $\nu$ is a holomorphic map, $\hat{\varphi}$ is also a pluriharmonic map. Then $\hat{\eta}$ is a smooth uniton for $\hat{\varphi}$ defined globally on $\hat{M}$. q.e.d.

Let $\varphi : M \longrightarrow U(N)$ be a pluriharmonic map and $\eta$ be a rational uniton in $M$ for $\varphi$ defined on a dense open set $W$ of $M$. Then we have a pluriharmonic map $\tilde{\varphi} = \varphi(\Pi_\eta - \Pi_\eta^\perp) : W \longrightarrow U(N)$. Following Lemma 5.6, we take a resolution $(\hat{M},\nu,\hat{\eta})$ of the singularity for the rational uniton $\eta$. Since $\hat{\eta}$ is a smooth uniton for $\hat{\varphi} = \varphi \circ \nu : \hat{M} \longrightarrow U(N)$, we get a pluriharmonic map $\psi = \hat{\varphi}(\Pi_{\hat{\eta}} - \Pi_{\hat{\eta}}^\perp) : \hat{M} \longrightarrow U(N)$ such that $\tilde{\varphi} \circ \nu = \psi | \nu^{-1}(W)$.

Assume that $M$ is compact. Since $\nu$ is proper, $\hat{M}$ is also compact. Let $\omega$ be a fundamental 2–form of a Hermitian metric on $M$. Obviously we have $E_\omega(\varphi) < \infty$. Then by the compactness of $\hat{M}$ we have

$$(5.19) \qquad \infty > \int_{\hat{M}} \mathcal{E}(\psi)\Lambda(\nu^*\omega)^{m-1} = \int_{\hat{M}-\nu^{-1}S(\eta)} \mathcal{E}(\psi)\Lambda(\nu^*\omega)^{m-1}$$

$$= \int_{M-S(\eta)} \mathcal{E}(\tilde{\varphi})\Lambda\omega^{m-1} = (m-1)!E_\omega(\tilde{\varphi}) \ .$$

Therefore $\tilde{\varphi} : W \longrightarrow U(N)$ also has finite energy. Moreover if $\omega$ is a real (1,1)–form on $M$ with $d\omega^{m-1} = 0$, then by Theorem 5.3 we get

$$(5.20) \qquad E_\omega(\tilde{\varphi}) - E_\omega(\varphi) = -(8m\pi/m!) \int_{\hat{M}} c_1(\hat{\eta})\Lambda(\nu^*\omega)^{m-1} \ .$$

Let $\mathcal{O} = \mathcal{O}_M$ be the structure sheaf of $M$, i.e. the sheaf of germs of holomorphic functions on $M$, and let $\mathcal{O}(\underline{\mathbb{C}}^N, d_A'')$ be the sheaf of germs of $d_A''$–holomorphic sections of

$\underline{\mathbb{C}}^N$ . $\mathcal{O}_M$ and $\mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$ are locally free coherent analytic sheaves. Then any rational uniton $\eta$ in M for $\varphi$ induces a coherent subsheaf $\mathscr{A}(\eta)$ of $\mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$ satisfying

(5.21) $$A'(Z)\Gamma(U, \mathscr{A}) \subset \Gamma(U, \mathscr{A})$$

for each open set U of M and each $Z \in \Gamma(U, TM^{(1,0)})$, such that $\mathcal{O}(\eta \mid M - S(\eta)) = \mathscr{A}(\eta) \mid M - S(\eta)$.

Indeed, let $X_i$ be the closure of the graph of each rational map $\Pi_i - \Pi_i^\perp : U_i \longrightarrow G_\ell(\mathbb{C}^N)$, and let $p_i$ and $q_i$ denote the projections of $U_i \times G_\ell(\mathbb{C}^N)$ onto $U_i$ and $G_\ell(\mathbb{C}^N)$, respectively. Let $T_i$ be the tautological bundle over $G_\ell(\mathbb{C}^N)$. Then by Grauert direct image theorem $(p_i)_* \mathcal{O}(q_i^{-1} T_i)$ is a coherent sheaf on $U_i$, and $\{(U_i, (p_i)_* \mathcal{O}(q_i^{-1} T_i))\}$ define a coherent subsheaf $\mathscr{A}(\eta)$ of $\mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$. Obviously it satisfies the condition (5.21).

Conversely, let $\mathscr{A}$ be a coherent subsheaf of $\mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$ satisfying the condition (5.21). Set $S(\mathscr{A}) = \{x \in M$ ; the stalk $\mathscr{A}_x$ at x is not free$\}$. Then $S(\mathscr{A})$ is an analytic subset of M and $\mathscr{A}$ is locally free on $M - S(\mathscr{A})$. Let $\ell = \operatorname{rank} \mathscr{A} = \operatorname{rank} \mathscr{A}_x$ for $x \in M - S(\mathscr{A})$. Since $\mathscr{A}$ is a subsheaf of a torsion–free sheaf, $\mathscr{A}$ is also torsion–free, and hence we have $\dim_{\mathbb{C}} S(\mathscr{A}) \leq m-2$ (cf. [10]). Let $\det \mathscr{A} = (\Lambda^\ell \mathscr{A})^{**}$ be the determinant line bundle of $\mathscr{A}$. The inclusion map $j : \mathscr{A} \longrightarrow \mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$ induces a sheaf homomorphism

$$\tilde{j} : \det \mathscr{A} \longrightarrow \Lambda^\ell \mathcal{O}(\underline{\mathbb{C}}^N, d''_A) .$$

Then $\tilde{j}$ is injective. Let $\hat{j}$ be the holomorphic section of the bundle $\Lambda^\ell(\underline{\mathbb{C}}^N, d''_A) \otimes (\det \mathscr{A})^*$ which corresponds to $\tilde{j}$, and let B be the zero set of $\hat{j}$, which is an analytic subset of M. Let $D_i$, $i = 1,\dots,k$, be the irreducible components of B of codimension 1. Let Y denote the union of all irreducible components of B of codimension at least 2 so that

$$B = \bigcup_{i=1}^{k} D_i \cup Y .$$

For each $D_i$, define its multiplicity $n_i > 0$ as follows. If $x \in D_i - \bigcup_{j \neq i} D_j \cup Y$ and if $D_i$ is defined by $w = 0$ in a neighborhood of $x$, then $n_i$ is the largest integer $n$ such that $\hat{j}/w^n$ is holomorphic. Then $\hat{j}/w^{n_i}$ is a local holomorphic section of the bundle $\Lambda^{\ell}(\underline{\mathbb{C}}^N, d_A'') \otimes (\det \mathscr{S})^*$ not vanishing at $x$. We set

$$D = \sum_{i=1}^{k} n_i D_i$$

and $[D]$ denotes the holomorphic line bundle defined by the divisor $D$. Let $\delta$ be the natural holomorphic section of $[D]$. Let

$$j' : \det \mathscr{S} \otimes [D] \longrightarrow \Lambda^{\ell}(\underline{\mathbb{C}}^N, d_A'')$$

be the bundle homomorphism defined by $j' = \hat{j} \otimes (1/\delta)$. Set $M^* = M - Y$. Then $j'$ is injective over $M^*$. Hence there exists a holomorphic subbundle $\eta(\mathscr{S})$ of $(\underline{\mathbb{C}}^N, d_A'') | M^*$ such that

$$\det \eta(\mathscr{S}) = j'(\det \mathscr{S} \otimes [D] \, | M^*) .$$

In particular we have $\mathscr{S} \, |(M-X) = \mathcal{O}(\eta(\mathscr{S}) \, |(M-X))$, where $X = S(\mathscr{S}) \cup B$. Note that by the argument of [10, (V.8.5)] we have

(5.22) $$\int_{M^*} c_1(\eta(\mathcal{S}),\nabla_A)\wedge\Omega = \int_M c_1(\det \mathcal{S} \otimes [D])\wedge\Omega$$

for any closed real (m–1,m–1)–form $\Omega$ on M with compact support, and if $\Omega$ is non-negative, then we have

(5.23) $$\int_M c_1(\det \mathcal{S} \otimes [D])\wedge\Omega \geq \int_M c_1(\mathcal{S})\wedge\Omega \ .$$

Here the first Chern class $c_1(\mathcal{S})$ of a coherent sheaf $\mathcal{S}$ is defined by $c_1(\mathcal{S}) = c_1(\det \mathcal{S})$. Then $\eta(\mathcal{S})$ is a smooth uniton for $\varphi$ defined on $M^*$. Hence, by applying Levi extension theorem (cf. [7]) to each holomorphic map $\Pi_i - \Pi_i^\perp : U_i \cap M^* \longrightarrow G_\ell(\mathbb{C}^N)$, we get the following.

Proposition 5.7. The complex subbundle $\eta(\mathcal{S})$ of $\underline{\mathbb{C}}^N$ is a rational uniton in M for $\varphi$. In general, a smooth uniton $\eta$ defined on the complement of an analytic subset of codimension at least 2 in M is always a rational uniton in M.

Remark. By using a result of [25] on the regularity of weakly holomorphic sub-bundles we see that a section $P \in L_1^2(\mathrm{End}(\underline{\mathbb{C}}^N))$ satisfies

$$P^2 = P^* = P\ , \ (I-P)d_A''P = 0 \quad \text{and} \quad (I-P)A'P = 0$$

if and only if P defines a rational uniton in M for $\varphi$.

Here we show the formula for the difference of energies in adding a rational uniton.

Theorem 5.8. Let M be an m–dimensional compact complex manifold and $\varphi : M \longrightarrow U(N)$ be a pluriharmonic map. Assume that $\tilde{\varphi} = \varphi(\Pi_\eta - \Pi_\eta^\perp) : M-S \longrightarrow U(N)$ is a pluriharmonic map obtained from $\varphi$ by addition of a rational uniton $\eta$. Here S denotes the singularity set of the rational uniton $\eta$, which is an analytic subset of $\dim_{\mathbb{C}} S \le m-2$. Let $\omega$ be any real (1,1)–form on M with $d\omega^{m-1} = 0$. If we let $\mathscr{S}(\eta)$ the coherent subsheaf $\mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$ induced by $\eta$, then we have

$$E_\omega(\tilde{\varphi}) - E_\omega(\varphi) = -(8m\pi/m!)\deg_\omega(\mathscr{S}(\eta)) \quad , \tag{5.24}$$

where $\deg_\omega(\mathscr{S}(\eta)) = \int_M c_1(\mathscr{S}(\eta))\wedge\omega^{m-1}$. If $\eta = \eta(\mathscr{S})$ is induced from a coherent subsheaf $\mathscr{S}$ of $\mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$ in the above manner, then we have

$$E_\omega(\tilde{\varphi}) - E_\omega(\varphi) = -(8m\pi/m!)\int_M c_1(\det \mathscr{S} \otimes [D])\wedge\omega^{m-1} \quad . \tag{5.25}$$

Proof. The inclusion $j : \mathscr{S}(\eta) \longrightarrow \mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$ induces a bundle homomorphism

$$\det(j) : \det \mathscr{S}(\eta) \longrightarrow \Lambda^\ell \underline{\mathbb{C}}^N \quad ,$$

where $\ell = \operatorname{rank}\eta = \operatorname{rank}\mathscr{S}(n)$. Since $\det(j)$ is injective on M–S with $\dim_{\mathbb{C}} S \le m-2$, by (5.22) for $D = 0$ we have

$$\int_M c_1(\mathscr{S}(\eta))\wedge\omega^{m-1} = \int_{M-S} c_1(\eta, \nabla_A)\wedge\omega^{m-1} \quad . \tag{5.26}$$

Hence we have

$$\int_{M} c_1(\mathscr{A}(\eta))\wedge\omega^{m-1} = \int_{\hat{M}-\nu^{-1}S} c_1(\hat{\eta},\nabla_A)\wedge(\nu^*\omega^{m-1}) \tag{5.27}$$

$$= \int_{\hat{M}} c_1(\hat{\eta})\wedge(\nu^*\omega^{m-1}) \ .$$

Thus (5.24) follows from (5.20) and (5.27). Similarly, we get (5.25) by (5.22).

q.e.d.

We define the degree $\deg_\omega\eta$ of a rational uniton $\eta$ relative to a real (1,1)–form $\omega$ on M with $d\omega^{m-1} = 0$ as

$$\deg_\omega\eta = \int_{M-S} c_1(\eta,\nabla_A)\wedge\omega^{m-1} \ . \tag{5.28}$$

6. Factorization theorem for a pluriharmonic map from a simply connected complex manifold.

Let M be an m–dimensional connected complex manifold. Assume that $\varphi : M \longrightarrow U(N)$ is a nonconstant pluriharmonic map with minimal uniton number n . Then by Theorem 4.3, $\varphi$ has a unique real extended solution $\Phi : \mathbb{C}^* \times M \longrightarrow GL(N,\mathbb{C})$ such that $\Phi_1 = I$, $\Phi_{-1} = a\,\varphi$ for some $a \in U(N)$, $V_0(\Phi) = \mathbb{C}^N$ and

(6.1) $$\Phi_\lambda = \sum_{i=0}^{n} T_i \lambda^i \quad \text{for } \lambda \in \mathbb{C}^* .$$

The linear equations (3.4) become

(6.2) $$\begin{cases} d' T_i = (T_i - T_{i+1}) A' \ , \\ d'' T_i = (T_i - T_{i-1}) A'' \ . \end{cases}$$

The reality condition of $\Phi_\lambda$ becomes

(6.3) $$\sum_{i=0}^{n} T_i T_{j+i}^* = \delta_{j0} I \ .$$

Here $T_i \equiv 0$ for $i < 0$ or $i > n$ .

Lemma 6.1. Given any Hermitian projection $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ , then $PT_0$ is a holomorphic section of $\mathrm{Hom}((\underline{\mathbb{C}}^N, d''_A), (\underline{\mathbb{C}}^N, d''))$ .

Proof. By (6.2) we have $d''(PT_0) - (PT_0)A'' = 0$ . This means just the statement of

Lemma 6.1. q.e.d.

Let $\mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$ be an analytic sheaf of germs of $d''_A$–holomorphic sections of $\underline{\mathbb{C}}^N$. Now we give a Hermitian projection $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$. Let $\mathcal{K}_P$ be the kernel subsheaf of $PT_0$, that is, the sheaf of germs of $d''_A$–holomorphic sections of $\underline{\mathbb{C}}^N$ such that $(PT_0)s = 0$. Then $\mathcal{K}_P$ is a reflexive coherent subsheaf of $\mathcal{O}(\underline{\mathbb{C}}^N, d''_A)$, and hence the singularity set $S(\mathcal{K}_P)$ of the coherent sheaf $\mathcal{K}_P$ satisfies $\dim_{\mathbb{C}} S(\mathcal{K}_P) \leq m-3$ (cf. [10]). As the argument in Section 5, the coherent subsheaf $\mathcal{K}_P$ induces a $d''_A$–holomorphic vector subbundle $\eta_P = \eta(\mathcal{K}_P)$ of $\underline{\mathbb{C}}^N$ on $M^* = M-S$, where $S$ is an analytic subset of $M$ of $\dim_{\mathbb{C}} S \leq m-2$. Let $\Pi_P$ (resp. $\Pi_P^{\perp} = I - \Pi_P$) $: M^* \longrightarrow \mathfrak{gl}(N,\mathbb{C})$ be Hermitian projections onto $\eta_P$ (resp. $\eta_P^{\perp}$).

Lemma 6.2. The subbundle $\eta_P$ is a rational uniton in $M$ for $\varphi$ defined on $M^*$.

Proof. Since $(PT_0)A' = 0$ by (6.2), we see that $\eta_P$ is a smooth uniton on $M^*$. Hence it follows from Proposition 5.7 that $\eta_P$ is a rational uniton in $M$.

q.e.d.

We consider the case $P = I$. Then we get the following in the same way as [24, Section 14]. We give the proof for the sake of completeness.

Lemma 6.3. (1) $\tilde{\Phi}_\lambda = \lambda^{-1}\Phi_\lambda(\Pi_I + \lambda\Pi_I^{\perp}) : M^* \longrightarrow GL(N,\mathbb{C})$ $(\lambda \in \mathbb{C}^*)$ is a real extended solution defined on $M^*$ and $\tilde{\Phi}$ has the Laurent expansion $\tilde{\Phi}_\lambda = \sum_{i=0}^{n-1} \tilde{T}_i\lambda^i$, where $\tilde{T}_i = T_{i+1}\Pi_I + T_i\Pi_I^{\perp}$ for $i = 0,\dots,n-1$ and $V_0(\tilde{\Phi}) = \mathbb{C}^N$.

(2) rank $\tilde{T}_0 >$ rank $T_0$ at each point of a dense open subset of $M^*$.

Proof. (1) From Theorem 5.2 and Lemma 6.2 we already know that $\tilde{\Phi}$ is a real extended solution defined over $M^*$ . We compute
$\tilde{\Phi} = T_0\Pi_I\,\lambda^{-1} + \sum_{i=0}^{n-1}(T_{i+1}\Pi_I + T_i\Pi_I^{\perp})\lambda^i + T_n\Pi_I^{\perp}\lambda^n$ . Set $M' = \{x \in M;\ \mathrm{rank}\,(T_0)_x$ is maximal$\}$ , which is a connected open dense subset of $M^*$ . Note that $\eta_I = \mathrm{Ker}\,T_0$ on $M'$ . Therefore we have $T_0\Pi_I = 0$ on $M'$ and hence on $M^*$ . Since $T_nT_0^* = 0$ by (6.3) and $\eta_I^{\perp} = \mathrm{Im}\,T_0^*$ on $M'$ , we get $T_n\Pi_I^{\perp} = 0$ on $M'$ and hence on $M^*$ . Since
$\tilde{T}_0 = T_0\Pi_I^{\perp} + T_1\Pi_I = T_0 + T_1\Pi_I$ on $M^*$ , we have $\tilde{T}_0\eta_I = \tilde{T}_1\eta_I$ and $\tilde{T}_0\eta_I^{\perp} = T_0\eta_I^{\perp}$ on $M^*$ . Hence $\mathrm{Im}\,T_0 \subset \mathrm{Im}\,\tilde{T}_0$ at each point of $M^*$ . Thus we get
$V_0(\tilde{\Phi}) = V_0(\Phi|_{M^*}) = V_0(\Phi) = \mathbb{C}^N$ . (2) Set $M'' = \{x \in M';\ \mathrm{rank}(T_0)_x$ and $\mathrm{rank}(\tilde{T}_0)_x$ are maximal$\}$, which is a connected dense open subset of $M$ . Assume that
$\mathrm{rank}\,T_0 = \mathrm{rank}\,\tilde{T}_0$ on $M''$ . Since $\mathrm{Im}\,T_0 = \mathrm{Im}\,\tilde{T}_0$ on $M''$ , we have $\mathrm{Im}(T_1\Pi_I) \subset \mathrm{Im}\,T_0$ and hence $\mathrm{Im}(T_1\alpha') \subset \mathrm{Im}\,T_0$ on $M''$ . Therefore, by it and (6.2) for $i = 0$ , $\mathrm{Im}\,T_0|M''$ is $d'$– and $d''$–stable, and hence $\mathrm{Im}\,T_0|M'' = M'' \times V$ is a trivial subbundle of $\underline{\mathbb{C}}^N|M'' = M'' \times \mathbb{C}^N$ for some vector subspace $V$ of $\underline{\mathbb{C}}^N$ . Thus we see $\mathrm{Im}\,T_0 \subset M \times V$ . Since $\mathbb{C}^N = V_0(\Phi) = V$ , we have $\mathrm{Im}\,T_0|M'' = M'' \times \mathbb{C}^N$ , i.e. $\mathrm{rank}\,T_0 = N$ on $M''$ . By (6.3) for $j = n$ , we get $T_n \equiv 0$ , a contradiction. Therefore we conclude
$\mathrm{rank}\,T_0 < \mathrm{rank}\,\tilde{T}_0$ on $M''$ . q.e.d.

Theorem 6.4. If $\varphi : M \longrightarrow U(N)$ is a pluriharmonic map with minimal number $n < \infty$ , then $n \leq N$ .

This theorem follows from Lemma 6.3. In case $m = 1$ , this is a result in [24].

We obtained a new pluriharmonic map $\tilde{\varphi} = \tilde{\Phi}_{-1} : M^* = M - S \longrightarrow U(N)$ with minimal uniton number $n-1$ from a given pluriharmonic map $\varphi : M \longrightarrow U(N)$ with minimal uniton number $n$ . Again we can apply the same process to $\tilde{\varphi} : M^* \longrightarrow U(N)$ .

Repeating this process, we obtain a sequence of pluriharmonic maps $\varphi^{(n)} = \varphi$, $\varphi^{(n-1)} = \tilde{\varphi}, \varphi^{(n-2)},\ldots,\varphi^{(1)}, \varphi^{(0)} = a \in U(N)$. Naturally a singularity set arises at each step. By modify the domain complex manifold $M$, we can take a resolution of the singularity set for a pluriharmonic map $\tilde{\varphi}$ as in Lemma 5.6.

Repeating these processes, we get the following sequences of pluriharmonic maps and complex manifolds.

$$\begin{array}{ccccccccc}
M_0 & \xrightarrow{\nu_1} & M_1 & \longrightarrow - - - - - \longrightarrow & M_{n-2} & \xrightarrow{\nu_{n-1}} & M_{n-1} = \hat{M} & \xrightarrow{\nu_n = \nu} & M_n = M \\
\big\downarrow \varphi_0 & & \big\downarrow \varphi_1 & & \big\downarrow \varphi_{n-2} & & \big\downarrow \varphi_{n-1} = \hat{\varphi} & & \big\downarrow \varphi_n = \varphi \\
U(N) & & U(N) & & U(N) & & U(N) & & U(N)
\end{array}$$

Here $\varphi_k : M_k \longrightarrow U(N)$ $(k = 0,\ldots,n)$ is a pluriharmonic map with minimal uniton number $k$ and each $\nu_k : M_k \longrightarrow M_{k+1}$ is a proper surjective holomorphic map such that $\nu_k : M_k - \nu_k^{-1}S \longrightarrow M_{k+1} - S$ is a biholomorphic diffeomorphism for some analytic subset $S$ of $M_{k+1}$ of codimension at least 2.

Now, combining Theorems 3.2, 4.2, 4.3 and results of this section, we obtain the following factorization theorem for a pluriharmonic map into $U(N)$.

Theorem 6.5. Let $M$ be an m–dimensional simply connected compact complex manifold and $\varphi : M \longrightarrow U(N)$ be a pluriharmonic map. Then $\varphi$ has a factorization

$$\varphi = a(\Pi_1 - \Pi_1^{\perp}) \,\ldots\ldots\, (\Pi_n - \Pi_n^{\perp})$$

over $M - S$ into $a \in U(N)$ and $\Pi_k - \Pi_k^{\perp} : M - S \longrightarrow Gr(\mathbb{C}^N)$ $(k = 1,\ldots,n)$, where

(1) $S$ is an analytic subset of $M$ with $\dim_{\mathbb{C}} S \leq m-2$,

(2) each $\varphi^{(k)} = a(\Pi_1 - \Pi_1^{\perp}) \,\ldots\ldots\, (\Pi_k - \Pi_k^{\perp}) : M - S \longrightarrow U(N)$ $(k = 1,\ldots,n)$ is a pluriharmonic map,

(3) $\Pi_k$ is a rational uniton in M for $\varphi^{(k)}$ defined on M–S for each $k = 1,\ldots,n$ ,

(4) $\Pi_1 - \Pi_1^{\perp} : M - S \longrightarrow Gr(\mathbb{C}^N)$ is a holomorphic map, and moreover it induces a meromorphic map $M \longrightarrow Gr(\mathbb{C}^N)$ ,

(5) $n(\leq N)$ is equal to the minimal uniton number of $\varphi$ .

Furthermore, for any Hermitian metric on M of fundamental 2–form $\omega$ , each energy $E_\omega(\varphi^{(k)})$ is finite.

Remark. (1) The uniqueness of the factorization also holds under the same conditions as in [24].

(2) In case $m = 1$ , the singularity set S is empty. This is a result in [24].

(3) The unique factorization theorem for a pluriharmonic map into a complex Grassmann manifold $G_\ell(\mathbb{C}^N) \subset U(N)$ also holds in the same way as in [24]. By using the method of [3], [29] and [30], the first named author and Udagawa investigated the construction of pluriharmonic maps into complex Grassmann manifolds ([14]).

Furthermore, in the case when M has a cosymplectic Hermitian metric, we show the factorization $\varphi^{(n)} = \varphi, \varphi^{(n-1)},\ldots,\varphi^{(1)}$ in Theorem 6.5 is energy–decreasing.

Lemma 6.6. Let $\varphi : M \longrightarrow U(N)$ be a nonconstant pluriharmonic map with a real extended solution $\Phi_\lambda = \sum_{i=0}^{n} T_i \lambda^i$ and $\mathscr{K}_P$ be the kernel subsheaf of $PT_0$ . Assume that M is compact and $\omega$ is a nonnegative real (1,1)–form on M with $d\omega^{m-1} = 0$ such that $\omega$ is positive at some point of M . Then we have

$$\deg_\omega(\eta_P) \geq \deg_\omega(\mathscr{K}_P) \geq 0 \ . \tag{6.4}$$

If $P = I$ and $V_0(\Phi) = \mathbb{C}^N$ , then

(6.5) $$\deg_\omega(\eta_I) \geq \deg_\omega(\mathscr{K}_I) > 0 \ .$$

Proof. By Lemma 6.1, $PT_0$ induces a sheaf homomorphism

$$PT_0 : \mathcal{O}(\underline{\mathbb{C}}^N, d''_A) \longrightarrow \mathcal{O}(\underline{\mathbb{C}}^N, d'') \ .$$

Then we have an exact sequence of the kernel subsheaf $\mathscr{K}_P$ of $PT_0$ and the image subsheaf $\mathscr{I}_P$ of $PT_0$:

(6.6) $$0 \longrightarrow \mathscr{K}_P \longrightarrow \mathcal{O}(\underline{\mathbb{C}}^N, d''_A) \longrightarrow \mathscr{I}_P \longrightarrow 0 \ .$$

Since $(\underline{\mathbb{C}}^N, d'')$ is the trivial holomorphic vector bundle, by (5.22) and (5.23) we see

$$\deg_\omega(\mathscr{I}_P) \leq 0 \ ,$$

and $\deg_\omega(\mathscr{I}_P) = 0$ if and only if $\mathscr{I}_P$ induces a trivial subbundle $M \times \mathbb{C}^p$ of $\underline{\mathbb{C}}^N$ for some complex subspace $\mathbb{C}^p$ of $\mathbb{C}^N$. On the other hand, from (6.6) we have

$$\deg_\omega(\mathscr{K}_P) + \deg_\omega(\mathscr{I}_P) = \deg_\omega(\underline{\mathbb{C}}^N) = 0 \ .$$

Hence we get (6.4). Assume that $P = I$. If $V_0(\Phi) = \mathbb{C}^N$, then we see $\deg_\omega(\mathscr{I}_P) < 0$, and hence $\deg_\omega(\mathscr{K}_P) > 0$. Since by (5.22) and (5.28)

$$\deg_\omega(\eta_P) = \deg_\omega(\mathscr{K}_P) + \deg_\omega([D])$$

for some effective divisor D on M , hence we get $\deg_\omega(\eta_P) \geq \deg_\omega(\mathscr{K}_P)$ .

q.e.d.

Theorem 6.7. In the factorization of Theorem 6.5, if we suppose that M has a co-symplectic Hermitian metric of fundamental 2–form $\omega$ , then we have

$$E_\omega(\varphi^{(k)})-E_\omega(\varphi^{(k-1)}) > 0 \quad \text{for each} \quad k = 1,\dots,n \ .$$

Proof. It follows from Theorem 5.8 and Lemma 6.6, by using the sequence $\{\varphi_n,\dots,\varphi_0\}$ of pluriharmonic maps. q.e.d.

7. Pluriconformal maps.

Definition. A smooth map $\varphi : M \longrightarrow N$ from a complex manifold $M$ to a Riemannian manifold $(N, g_N)$ is called pluriconformal if $(\varphi^* g_N)^{(2,0)} = \overline{(\varphi^* g_N)}^{(0,2)} = 0$, or equivariantly, if for any holomorphic curve $i : C \longrightarrow M$, $\varphi \circ i$ is conformal.

Assume that $\varphi : M \longrightarrow N$ is a pluriharmonic map. Then $(\varphi^* g_N)^{(2,0)}$ is a holomorphic section of $\otimes^2 T^* M^{(1,0)}$ and hence if $M$ is a compact complex manifold with $c_1(M) > 0$, then $\varphi$ is pluriconformal ([12], [13]). We can show a slight extension of this fact in the case of $N = U(N)$ as follows.

Proposition 7.1. Let $M$ be a simply connected compact complex manifold and $\varphi : M \longrightarrow U(N)$ be a pluriharmonic map. Then $\varphi$ is pluriconformal.

Proof. This proposition is a simple consequence of Theorem 6.5, the fact that $\varphi$ is pluriconformal iff $tr(A' \otimes A') = tr(A')^2 = 0$, where $\otimes$ denotes the tensor product, and the following lemma for $k = 2$. q.e.d.

Lemma 7.2. Let $\varphi : M \longrightarrow U(N)$ be a pluriharmonic map and $\tilde{\varphi} = \varphi(P - P^\perp)$ be a pluriharmonic map obtained by addition of a uniton $P$. Then we have, for each $k > 0$,

$$tr(A')^k = tr(\tilde{A}')^k \tag{7.1}$$

and they are both holomorphic sections of $\otimes^k T^* M^{(1,0)}$.

Proof. Since $\tilde{A}' = PA'P + P^{\perp}A'P^{\perp} - P^{\perp}d'P$ by (5.14), we have $\mathrm{tr}(\tilde{A}')^k = \mathrm{tr}(PA'P)^k + \mathrm{tr}(P^{\perp}A'P^{\perp})^k$. From $P^{\perp}A'P = 0$, we easily get $\mathrm{tr}(PA'P)^k = \mathrm{tr}(P(A')^k)$ and $\mathrm{tr}(P^{\perp}A'P^{\perp})^k = \mathrm{tr}(P^{\perp}(A')^k)$. Hence we obtain (7.1).

q.e.d.

## 8. Action of infinite dimensional Lie algebra and Lie group on the space of pluriharmonic maps.

In Section 3 we established the construction of extended solutions from a pluriharmonic map into a compact Lie group. As one of its applications we can construct actions of certain infinite dimensional Lie algebra and Lie group of loop algebra and loop group type on the moduli space of pluriharmonic maps into a compact Lie group. There seem to be two methods for the construction of the Lie algebra and Lie group actions. One is the method of Riemann–Hilbert transform due to Zakharov–Mikhailov–Shabat [32], [33] and Uhlenbeck [24]. Another is the infinite dimensional Grassmann method due to Sato [18] and Takasaki [21] (cf. [9] for chiral model). The actions of the infinite dimensional Lie algebra and Lie group preserve the minimal uniton number and the actions on the moduli space of pluriharmonic maps with the fixed minimal uniton number reduce to the actions of suitable finite dimensional quotient Lie algebra and Lie group. In particular the actions on 1–unitons are essentially equal to the actions of holomorphic transformations of complex Grassmann manifolds on the space of holomorphic maps. We shall discuss them in detail elsewhere.

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