ON THE FEDOSOV DEFORMATION QUANTIZATION BEYOND THE REGULAR POISSON MANIFOLDS

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Abstract

A simple iterative procedure is suggested for the deformation quantization of (irregular) Poisson brackets associated to the classical Yang-Baxter equation. The construction is shown to admit a pure algebraic reformulation giving the Universal Deformation Formula (UDF) for any triangular Lie bialgebra. A simple proof of classification theorem for inequivalent UDF's is given. As an example the explicit quantization formula is presented for the quasi-homogeneous Poisson brackets on two-plane.

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1 Introduction

The Deformation Quantization, as it was originally formulated in [1, 2], has undergone an explosive development during the last two decades [3] with a number of important achievements including Kontsevich's $*$-product construction for a general Poisson manifold [4]. Fedosov [5, 6] gave a simple and manifestly covariant quantization method working well for any symplectic or regular Poisson manifold and yet giving a simple classification for inequivalent quantizations. Meanwhile, this method can not be directly applied to the irregular Poisson brackets because the main ingredient is lacking for the Fedosov construction: no affine connection can exist respecting an irregular Poisson structure.

In this paper we suggest a simple method extending the Fedosov approach to a broad class of manifolds with irregular Poisson brackets, that gives an explicitly covariant and geometrically transparent algorithm for constructing the $*$-products in the irregular case. To make the method working, these Poisson manifolds are assumed to be equipped with certain additional algebraic structures. We also expect that the method will allow for generalizations essentially relaxing conditions imposed on the class of Poisson manifolds. Although the brackets of considered class, being more general than the regular ones, are of interest in their own rights, our approach admits also a purely algebraic reformulation, which allows one to simply derive the Universal Deformation Formula for any triangular Lie bialgebra (see Sec. 5).

To explain the class of the irregular brackets we are going to study, let us give a simple example of this type considered in [7, 8].

Consider a collection of pair-wise commuting vector fields X_i, $i = 1, ..., n$ defined on a smooth real manifold M. These fields may be viewed as derivatives of the commutative algebra of smooth functions $C^\infty(M)$. A constant skew-symmetric matrix r assigns M with a Poisson bracket of the form

$$\{f, g\} = r^{ij}(X_i f)(X_j g)\,, \qquad \forall\, f, g \in C^\infty(M)\,. \tag{1}$$

In general, this bracket is irregular since the rank of the distribution X_i can vary from point to point. Nevertheless, it can be easily quantized by the Weyl-Moyal like formula

$$f * g = f \cdot g + \sum_{k=1}^{\infty} \left(-\frac{i\hbar}{2}\right)^k \frac{1}{k!} r^{i_1 j_1} \ldots r^{i_k j_k} (X_{i_1} \ldots X_{i_k} f)(X_{j_1} \ldots X_{j_k} g)\,, \tag{2}$$

$\hbar$ being a formal deformation parameter. The associativity of the $*$ -product follows trivially from the commutativity of vector fields X_i. Note that (2) may be regarded as the *universal deformation formula*, in the sense that it works for the action of *any* commutative Lie algebra.

A natural generalization of the above construction, we are going to study in this paper, consists in allowing the vector fields X_i to form a noncommutative Lie algebra, say

$$[X_i, X_j] = f_{ij}^k X_k\,, \tag{3}$$

f_{ij}^k being structure constants. The Jacobi identity for the Poisson bracket (1) implies that

$$\{f, \{g, h\}\} + cycle(f, g, h) = \Lambda^{ijk}(X_i f)(X_j g)(X_k h) = 0\,, \tag{4}$$

where

$$\Lambda^{ijk} = f_{mn}^i r^{mj} r^{nk} + cycle(i, j, k)\,. \tag{5}$$

For the commutative algebra we have $\Lambda \equiv 0$, and the Jacobi identity is automatically satisfied. In general case, equation $\Lambda = 0$ is rather nontrivial and known in mathematical literature as the *classical Yang-Baxter equation* (CYBE). For reasons, which we will not explain here, a skew-symmetric matrix r satisfying this equation is called the *classical triangular r-matrix* [9]. If all the vector fields X_i are linearly independent (at least at one point on M), the CYBE provides both necessary and sufficient conditions for the Jacobi identity to hold. In the opposite case, the equation $\Lambda = 0$ gives only sufficient condition. The important observation concerning r-bracket (1) is that one may always assume the r-matrix to be non-degenerate without restricting generality. Indeed, as for any skew-symmetric matrix r, one may always find such a basis of the generators $X_i = (X_A, X_\alpha)$ in which $r^{i\alpha} = 0$ and $\det(r^{AB}) \neq 0$. Thus, only the vector fields X_A actually contribute to the Poisson bracket (1). From the CYBE it then follows that

$$\Lambda^{\alpha AB} = f_{MN}^\alpha r^{MA} r^{NB} = 0 \quad \Rightarrow \quad f_{MN}^\alpha = 0$$

and hence, the vector fields X_A form a Lie subalgebra, for which r^{AB} is a non-degenerate triangular r-matrix. For non-degenerate r -matrices the CYBE reduces to the ordinary cocycle condition

$$f_{ij}^n r_{nk} + cycle(i, j, k) = 0\,, \tag{6}$$

where $r_{ik}r^{kj} = \delta_i^j$. The solutions to the equation (6), forming a linear space of 2-cocycles, are known to be in one-to-one correspondence with the central extensions of the Lie algebra. In other words, equation (6) arises as a part of the Jacobi identity for the Lie algebra

$$[y_i, y_j] = f_{ij}^k y_k + r_{ij} c\,, \qquad [c, y_i] = 0 \tag{7}$$

The Lie algebra admitting a central extension defined by a non-degenerate cocycle r_{ij} is called *quasi-Frobenius*, and *Frobenius* if at least one of such cocycles is trivial (representable in the form $r_{ij} = f_{ij}^k \xi_k$ for some vector ξ_k)[10]. An extended list of examples of (quasi-)Frobenius Lie algebras (from here on QFL algebras, for short) can be found in recent papers [11], [12], [13].

In this paper we give a simple quantization formula for the Poisson bracket (1) associated to the action of an arbitrary QFL algebra (3). Our method, being conceptually similar to that of Fedosov, does not require regularity of the bracket (1, 3). The main distinction is a possibility to work with an auxiliary quantum bundle associated to the enveloping of the underlying Lie algebra, instead of the usual quantum Weyl algebra exploited in the original Fedosov construction and its various adaptations and reinterpretations [14], [15], [16], [17], [18]. Since the output $*$-product involves only initial algebraic data (much like its commutative version (2)) one may regard it as the Universal Deformation Formula for the respective algebra.

Mention that a covariant quantization of general Poisson manifolds has been recently suggested in the paper [19]. We develop a different explicitly covariant method to quantize a particular class of irregular Poisson brackets (1), (3) which seems much more simple than the general scheme [19].

Finally, we should remark that the quantization problem for the r-brackets is deeply rooted to the theory of quantum groups [20], dating back to the seminal Drinfeld's paper [9], where the existence of quantization was proved for an arbitrary triangular r-matrix. Actually, the computations required for obtaining explicit expressions by the Drinfeld method are appeared to be impracticable except for the Abelian case. That is why the most of known examples of the deformation has been constructed by means of *a twisting* transformation technique rather than by the quantization [11], [12], [13], [21]. In the recent paper [22], the Fedosov method has been used to quantize a non-degenerate triangular *dynamical* r-matrices.

In the formal algebraic settings the application of the Fedosov deformation quantization to a certain class of irregular Poisson brackets, including r-brackets (1), (3) has been also discussed in [23]. In principle, the method of the work [22], as well as it's predecessor [9], gives rise to the quantization of the r-bracket (1), and even more general Poisson structures. At the technical level, however, this method appears to be unduly cumbersome for the application to such a simple class of Poisson brackets as we are going to consider. In this respect, our approach, being elementary in essence, will hopefully be found useful both from theoretical and practical viewpoints.

The structure of the paper is as follows: The deformation quantization of general r-brackets is exposed in Section 2. Section 3 is devoted to the classification of such quantizations and some possible modifications of our method. In Section 4, we consider the case of quasi-homogeneous Poisson brackets on two-plane which is shown to provide an interesting class of r-brackets associated with the two-dimensional (and therefore quasi-Frobenius) Lie algebras. In Section 5, we show how the construction of the Universal Deformation Formula for a triangular r-matrix results in a quantization of the respective Lie bialgebra. In concluding Section, we summarize the results and sketch proposals for further generalizations of our construction within the framework of BRST quantization.

2 Construction of the $*$-product

Let us start with the action $\rho : L \to \mathrm{Vect}(M)$ of an abstract QFL algebra L on a smooth real manifold M. Denote by L_c the central extension of L associated with a non-degenerate 2-cocycle r. We assume that upon choosing a basis the structure of the algebra L_c is described by the Lie brackets (7), so that $L = L_c/\mathbb{R}c$ and the vector fields $\rho(y_i) = X_i$ commute according to (3).

Introduce the associative algebra $\mathcal{A} = C^\infty(M) \otimes U(L_c)$, being a tensor product of the commutative algebra of smooth functions on M and the universal enveloping of the Lie algebra L_c over $\mathbb{C}$ (or more precisely, its formal completion by infinite series). The elements of $\mathcal{A}$ may be viewed as the sections of the trivial vector bundle $p : U(L_c) \times M \to M$, with the fiber-wise associative algebra structure induced by the associative multiplication in $U(L_c)$. Choosing the basis of monomials being symmetric in the Lie algebra generators, we identify the elements of $U(L_c)$ with their Weyl symbols and set $c = 1 \in \mathbb{C}$. Then the generic element $a \in \mathcal{A}$ reads

$$a(y) = \sum_{k=0}^{\infty} a^{i_1 \cdots i_k} y_{i_1} \cdots y_{i_k} ,$$

where y_i are formal *commuting* variables and the coefficients $a^{i_1\cdots i_k} \in C^\infty(M)$ are symmetric in the indices $i_1, \dots, i_k$. The product of two symbols $a, b \in \mathcal{A}$, which we will denote by $\circ$, is given by [1]

$$(a \circ b)(y) = a(\hat{L}) b(y) , \qquad \hat{L}_i = \left(y_j + \frac{1}{2} r_{jn} \frac{\partial}{\partial y_n} \right) \mathcal{R}_i^j \left(\frac{\partial}{\partial y_k} \right) , \tag{8}$$

where

$$\mathcal{R}(x) = \sum_{m=0}^{\infty} \frac{b_m}{m!} \Lambda^m(x) = \left(\frac{e^{\Lambda(x)} - 1}{\Lambda(x)} \right)^{-1} , \qquad \Lambda_j^i(x) = x^k f_{kj}^i , \tag{9}$$

b_m being Bernoulli numbers. The formula involves well-known group-theoretical constructions: matrices $\Lambda(x)$ realize the adjoint representation of the Lie algebra L; $\mathcal{R}(x)$ are the matrices of the right shifts on the group $G = \mathrm{Exp}(L)$, written in the first kind coordinates x^i; finally, operators $\hat{L}_i$ define the left regular representation of the algebra L_c corresponding to the Weyl ordering [24],

$$[\hat{L}_i, \hat{L}_j] = f_{ij}^k \hat{L}_k + r_{ij} . \tag{10}$$

In what follows the explicit form (8) of the $\circ$-product will be inessential for our considerations.

The space of smooth functions $C^\infty(M)$ is embedded into $\mathcal{A}$ as a central subalgebra. Denote by $\pi : \mathcal{A} \to C^\infty(M)$ the canonical projection $(\pi a)(y) = a(0)$. To assign $C^\infty(M)$ with a $*$-product compatible with the Poisson brackets (1) we construct another, less trivial embedding $\sigma : C^\infty(M) \to \mathcal{A}$, allowing one to induce the desired $*$-product via pull-back of $\circ$-product. To this end, introduce the following Lie algebra of external differentiations:

$$\begin{aligned} & D_i a = X_i a + [y_i, a] , \\ & D_i(a \circ b) = (D_i a) \circ b + a \circ (D_i b) , \quad \forall\, a, b \in \mathcal{A} , \\ & [D_i, D_j] = f_{ij}^k D_k . \end{aligned} \tag{11}$$

Hereafter we set $[a, b] = a \circ b - b \circ a$. Denote by $\mathcal{A}_D$ the subalgebra of D-constant elements

$$\mathcal{A}_D = \{ a \in \mathcal{A} |\ D_i a = 0 \} . \tag{12}$$

[1] See [24] for a detailed survey of the operator calculus, in particular, for the Weyl calculus on Lie algebras.

The following theorem establishes isomorphism between the linear spaces $C^\infty(M)$ and $\mathcal{A}_D$.

Theorem 1. *Any D-constant element $a \in \mathcal{A}_D$ is uniquely determined by its projection πa onto subspace $C^\infty(M)$ of y-independent elements and vice versa.*

Proof. Let

$$a(y) = \sum_{n=0}^{\infty} a_n\,, \qquad a_n = a^{i_1\dots i_n} y_{i_1}\dots y_{i_n}\,, \tag{13}$$

be an arbitrary element from $\mathcal{A}$ with a given $a_0 = a(0) \in C^\infty(M)$. Expanding the condition $D_i a = 0$ via the basis of homogeneous monomials we get a chain of equations

$$\partial_i a_{n+1} = V_i a_n\,, \tag{14}$$

where we have denoted

$$\partial_i = r_{ji}\frac{\partial}{\partial y_j}\,, \qquad V_i = X_i + y_k f_{ij}^k \frac{\partial}{\partial y_j}\,. \tag{15}$$

Besides, the system (14) should be supplemented by a set of compatibility conditions resulted from the commutativity of the partial derivatives ∂_i,

$$\partial_i(V_j a_n) - \partial_j(V_i a_n) = 0\,. \tag{16}$$

To construct a solution to equations (14), (16) we use the induction over the degree n in the expansion (13). For $n = 0$ the equations are obviously consistent and we get

$$a_1 = y_i r^{ji} X_j a_0\,, \quad a_0 \in C^\infty(M)\,. \tag{17}$$

Suppose that we have found all a_k with $k = 1, 2, ..., n$. Then equations (14) for a_{n+1} are compatible provided the conditions (16) are satisfied. Using identities

$$[\partial_i, V_j] - [\partial_j, V_i] = f_{ij}^k \partial_k\,, \qquad [V_i, V_j] = f_{ij}^k V_k\,, \tag{18}$$

we can rewrite (16) as

$$\partial_i(V_j a_n) - \partial_j(V_i a_n) = f_{ij}^k \partial_k a_n - (V_i\partial_j - V_j\partial_i)a_n\,, \tag{19}$$

and since by induction hypothesis $\partial_i a_n = V_i a_{n-1}$ we finally get

$$f_{ij}^k \partial_k a_n - [V_i, V_j]a_{n-1} = f_{ij}^k(\partial_k a_n - V_k a_{n-1}) = 0\,. \tag{20}$$

Upon the compatibility conditions have satisfied the straightforward integration of (14) leads to the explicit recurrent formula

$$a_{n+1}(y) = \int_0^1 y_j r^{ij} \left(X_i a_n(ty) + y_k f^k_{im} \frac{\partial a_n(ty)}{\partial y_m} \right) dt \tag{21}$$

for the unique D-constant element $a \in \mathcal{A}_D$ with a prescribed $a_0 = a(0) \in C^\infty(M)$. q.e.d.

Up to the second order in y the lift of an arbitrary function $a \in C^\infty(M)$ to a D-constant element of $\mathcal{A}$, denoted by $\sigma(a)$, reads

$$\sigma(a) = a + y^i X_i a + \frac{1}{2}(y^i y^j X_i X_j a + y_j y^i f^j_{in} X^n a) + \cdots \tag{22}$$

$$y^i = r^{ji} y_j\,, \quad X^i = r^{ji} X_j\,.$$

As the next step we introduce the deformation parameter $\hbar$ and extend the space of smooth functions, i.e. the space of classical observables, to that of quantum observables $C^\infty(M)[[\hbar]]$. The latter consists of elements

$$a(\hbar) = \sum_{k=0}^\infty \hbar^k a_k\,, \qquad a_k \in C^\infty(M)\,,$$

being formal power series in $\hbar$ with coefficients in smooth functions. The above definitions of $\circ$-product and derivatives (11) are extended by linearity to the algebra $\mathcal{A}[[\hbar]] = C^\infty(M)[[\hbar]] \otimes U(L_c)$; in so doing, we rescale the structure constants of the algebra L_c by multiplying on $i\hbar$,

$$f^k_{ij} \to i\hbar f^k_{ij}\,, \qquad r_{ij} \to i\hbar r_{ij}\,, \tag{23}$$

and redefine the action of the derivatives

$$D_i \to D_i = X_i + \frac{1}{i\hbar}[\, y_i, \cdot\,]\,, \tag{24}$$

so that the lifting map (21) remains intact. The relation of the above theorem to the quantization of a general r-bracket (1) is established by the following

Theorem 2. *The pull-back of $\circ$-product on $\mathcal{A}[[\hbar]]$ via $\mathbb{C}[[\hbar]]$-linear isomorphism $\sigma : C^\infty(M)[[\hbar]] \to \mathcal{A}[[\hbar]]$ induces an associative $*$-product on $C^\infty(M)[[\hbar]]$,*

$$a * b = \pi(\sigma(a) \circ \sigma(b)) = \sum_{k=o}^\infty \hbar^k D_k(a, b) \tag{25}$$

satisfying the properties

i) *Locality :* $D_k(a,b)$ *are bi-differential operators of order* k*;* (25.a)

ii) Correspondence principle: $a*b = a\cdot b - \frac{i\hbar}{2}\{a,b\} \pmod{\hbar^2}$; (25.$b$)

iii) Normalization condition: $1*a = a*1 = a$. (25.c)

Proof. The associativity of $*$-product directly follows from associativity of $\circ$-product and the identity $\sigma\pi|_{\mathcal{A}_D} = \mathrm{id}$. To prove the locality condition it is convenient to assign the algebra $\mathcal{A}[[\hbar]]$ with a filtration

$$\mathcal{A}[[\hbar]] = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots$$

associated to the sequence of subspaces $\mathcal{A}_k$ consisting of elements

$$a = \sum_{2m+n\geq k} \hbar^m a_m^{i_1\ldots i_n} y_{i_1}\ldots y_{i_n} \in \mathcal{A}_k\,.$$

This filtration defines the topology and convergence in the space of formal power series in $\hbar$ and y's. From the recurrent formula (21) we see that the element a_n belongs to $\mathcal{A}_n$ and involves *finite* linear combinations of a_0 and its derivatives along X_i up to order n. In view of the obvious inclusions (keep in mind redefinitions (23))

$$\mathcal{A}_m \circ \mathcal{A}_n \subset \mathcal{A}_{n+m}\,, \qquad \pi(\mathcal{A}_n) \subset \mathcal{A}_n$$

we can write

$$a*b = \sum_{n+m\leq 2k} \pi(a_n \circ b_m) \pmod{\hbar^{k+1}}$$

and hence, each power of $\hbar$ in the product expansion is given by a finite order bi-differential operator. More accurate consideration of the above formula shows that

$$D_k(a,b) = \frac{(-i)^k}{2^k k!} r^{i_1 j_1}\ldots r^{i_k j_k}(X_{i_1}\ldots X_{i_k}a)(X_{j_1}\ldots X_{j_k}b) + \cdots\,,$$

where dots stand for the bi-differential operators of order less than k. For the commutative Lie algebra L the rest terms vanish and we come to the Weyl-Moyal $*$-product (2).

The correspondence principle (25.b) is proved by straightforward verification with the help of eq.(22). Finally, to prove the normalization condition (25.c) it suffices to note that $\sigma(1) = 1$. q.e.d.

Remark 1. The constructed $*$-product has a rather special structure: the bi-diffrential operators at each power of $\hbar$ are determined by repeated differentiations along the vector fields X_i,

$$D_n(a,b) = \sum_{k,l\leq n} D_n^{i_1\cdots i_k j_1\cdots j_l}(X_{i_1}\cdots X_{i_k}a)(X_{j_1}\cdots X_{j_l}b)\,, \tag{26}$$

where the coefficients $D_n^{i_1\cdots i_k j_1\cdots j_l}\in\mathbb{C}$ are universally expressed via the structure constants f_{ij}^k and r_{ij}, irrespective of the concrete representation $\rho : X_i \to \mathrm{Vect}(M)$. For this reason one may refer to this $*$-product as the universal quantization formula. It would be very interesting to compare this result with the Kontsevich quantization formula [4] and/or give its "sigma-model interpretation" [25].

Remark 2. The above quantization admits a natural deformation of the algebraic data entering to the $*$-product construction. Namely, the cocycle r may be replaced by any formal series

$$r'_{ij} = r_{ij} + \hbar r_{ij}^{(1)} + \hbar^2 r_{ij}^{(2)} + \cdots\,, \tag{27}$$

where $r^{(k)}$ are arbitrary 2-cocycles of the algebra L. Note that the matrix r' is formally invertible since r is so. As the result we get another star-product having the same quasi-classical limit (25.b). The question arises as to whether the deformation (27) leads to essentially different quantization or an equivalence transform can be found to establish an isomorphism between two algebras of quantum observables. In the next section we formulate the necessary and sufficient conditions for two star-products, obtained in such a way, to be equivalent.

3 Two theorems of equivalence

Given a QFL algebra L, consider two star-products $*$ and $*'$ associated to non-degenerate 2-cocycles r and r' of the form (27). Following [9], we call these star-products to be *equivalent* if there exists an invertible operator

$$B : C^\infty(M)[[\hbar]] \mapsto C^\infty(M)[[\hbar]], \tag{28}$$

establishing isomorphism of two star-algebras, i.e.

$$B(a *' b) = (Ba * Bb) \tag{29}$$

The most general expression for such an operator B preserving the peculiar structure of $*$-product (see Remark 1 of the previous section) looks like

$$B = 1 + \hbar B_1 + \hbar^2 B_2 + \cdots , \tag{30}$$

where B_k are finite order differential operators of the from

$$B_k = \sum_{n=1}^{N} C^{i_1\cdots i_n} X_{i_1}\cdots X_{i_n} \qquad C^{i_1\cdots i_n} \in \mathbb{C}. \tag{31}$$

In the paper by Drinfeld [9] the theorem was stated that the inequivalent Universal Deformation Formulas are classified by the formal series in $\hbar$ with values in the second cohomologies of the Lie algebra. However the proof of the theorem was given neither in this paper [9] nor in any of consequent papers. Here we present a simple proof of the classification theorem for the inequivalent Universal Deformation Formulas, obtained by our procedure.

Theorem 3. *Two star-products $*$ and $*'$ associated to formal 2-cocycles r and r' are equivalent if and only if*

i) $r = r' \pmod{\hbar}$;

ii) $r - r'$ *is a coboundary .*

Proof. The necessity of the above conditions can be proved by standard deformation quantization arguments based on the consideration of Hochschild cohomologies. So we omit this part of the proof and turn to the sufficiency. By assumption of the theorem we have $r_{ij} - r'_{ij} = f_{ij}^k \xi_k$, where $\xi_i = \hbar\xi_i^{(1)} + \hbar^2\xi_i^{(2)} + \ \ldots$ is a formal vector. Then the Lie algebra isomorphism $Q : L_c \to L'_c$, defined on the basis elements as

$$Q(y_i) = y_i + \xi_i c\,, \qquad Q(c) = c \tag{32}$$

induces isomorphism of the universal enveloping algebras $U(L_c)$ and $U(L'_c)$, which then extends to isomorphism of associative algebras $(\mathcal{A}[[\hbar]], \circ)$ and $(\mathcal{A}[[\hbar]], \circ')$. This means that the operator

$$Qa(y, \hbar) = a(y + \xi, \hbar) \tag{33}$$

intertwines the circle-products $\circ$ and $\circ'$ constructed by r and r' respectively. Since

$$D'_i a = X_i a + \frac{1}{i\hbar}[y_i\,, a]' = QDQ^{-1}\,, \qquad [a, b]' = a \circ' b - b \circ' a\,, \tag{34}$$

Q maps D-constant elements to D'-constant ones. Then it is easy to see that operator

$$Ba = \pi Q \sigma(a) \tag{35}$$

has the form (30,31) and intertwines the star-products $*$ and $*'$. Indeed,

$$(Ba) *' (Bb) = (\pi Q\sigma(a)) *' (\pi Q\sigma(b)) = \pi(Q\sigma(a) \circ' Q\sigma(b)) =$$

$$= \pi Q(\sigma(a) \circ \sigma(b)) = B\pi(\sigma(a) \circ \sigma(b)) = B(a * b)$$

q.e.d.

So far the role of QFL algebra underlying the quantization was twofold. First, it had appeared as the Lie algebra of vector fields entering to the r-bracket construction (1). Second, it had defined the fiber-wise $\circ$-product on the auxiliary bundle of the formal universal enveloping algebra $U(L_c)[[\hbar]]$. Now we are going to separate these functions to obtain a more flexible formalism which, as we hope, admits a generalization beyond the case of Lie algebras (see Sec. 6). This may also be viewed as a further deformation of the fiber-wise $\circ$-product giving an equivalent $*$-product on the base manifold M.

Theorem 4. *Let (L, r) and (L', r') be two QFL algebras with non-degenerate 2-cocycles. If $\dim L = \dim L'$ then the associative algebras $U(L_c)[[\hbar]]$ and $U(L'_c)[[\hbar]]$ are isomorphic.*

Remark. Without loss of generality we may assume that $r = r'$ (if not, choose another basis in the Lie algebra). Then it is sufficient to construct such an isomorphism $\phi : U(L_c)[[\hbar]] \to U(L'_c)[[\hbar]]$ for the case when algebra L is fixed. For definiteness, we take L_c to be the Heisenberg-Weyl algebra:

$$[z_i, z_j] = i\hbar r_{ij} c\,, \tag{36}$$

whose formal universal enveloping algebra, denoted by W, consists of formal power series in z_i and $\hbar$ with complex coefficients

$$a(z, \hbar) = \sum_{k,n\geq 0} \hbar^k a_k^{i_1\cdots i_n} z_{i_1} \cdots z_{i_n}\,. \tag{37}$$

Since the underlying QFL algebra L is commutative the matrix $\mathcal{R}$ entering to the definition (8) of $\circ$-product is equal to unit and we arrive at the usual Weyl-Moyal formula

$$a \circ b = a\left(z_i + \left(\frac{i\hbar}{2}\right) r_{ij} \frac{\partial}{\partial z_j}\right) b(z) = \exp\left(\frac{i\hbar}{2} r_{ij} \frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}\right) a(z)b(w)|_{z=w}\,. \tag{38}$$

In what follows we will refer to W as the Weyl algebra.

Now let L and L' be arbitrary QFL algebras of equal dimension, then the stated isomorphism $\phi : U(L_c)[[\hbar]] \to U(L'_c)[[\hbar]]$ can be written as the composition $\phi = \rho\rho'^{-1}$ of isomorphisms $\rho : W \to U(L_c)[[\hbar]]$ and $\rho' : W \to U(L'_c)[[\hbar]]$.

Proof. Prescribing the degrees to the formal variables: deg $y_i = 1$ and deg $\hbar = 2$, we turn W to a graded associative algebra assigned by the natural filtration $W \supset W_1 \supset W_2 \dots$ with respect to the total degree $2k + n$ of the series terms (37). The desired isomorphism $\rho : W \to U(L_c)[[\hbar]]$ is then completely determined by its action on the generators z_i of the Heisenberg-Weyl algebra. Having in mind the invertibility of the map ρ, we are looking for ρ of the form

$$y_i = \rho(z_i) = z_i + Y_i(z, \hbar)\,, \tag{39}$$

where $Y_i \in W_2$. Substituting this anzatz to the commutation relation of the algebra L_c we get

$$[\rho(z_i), \rho(z_j)] = i\hbar(f_{ij}^k \rho(z_k) + r_{ij})\,,$$

or explicitly

$$\partial_i Y_j - \partial_j Y_i = f_{ij}^k (z_k + Y_k) - \frac{1}{i\hbar}[Y_i, Y_j]\,, \tag{40}$$

where

$$\partial_i = r_{ij} \frac{\partial}{\partial z_j}\,.$$

Let us show that the last equation has a unique solution for $Y_i \in W_2$ subject to additional condition

$$z_i r^{ij} Y_j(z, \hbar) = 0 \tag{41}$$

This in particular implies $Y_i(0, \hbar) = 0$. Consider an expansion of Y_i in the homogeneous components

$$Y_i = \sum_{k \geq 2} Y_i^{(k)}\,, \qquad \deg\ Y_i^{(k)} = k\,.$$

Substituting this expansion back to the equation (40) we obtain a chain of equations

$$\begin{aligned} &\partial_i Y_j^{(2)} - \partial_j Y_i^{(2)} = f_{ij}^k z_k, \\ &\partial_i Y_j^{(n+1)} - \partial_j Y_i^{(n+1)} = f_{ij}^k Y_k^{(n)} - \frac{1}{i\hbar} \sum_{k=2}^{n} [Y_i^{(n+2-k)}, Y_j^{(k)}]\,, \qquad n > 2\,. \end{aligned} \tag{42}$$

Treating now $Y_i^{(n)}$ as 1-forms on a linear space with coordinates $z^i = r^{ij} z_j$ we see that above equations have a structure $dY^{(n)} = F^{(n)}$, where 2-forms $F^{(n)}$ are known. A necessary and sufficient condition for these equations to be solvable is $dF^{(n)} = 0$ (the Poincare' lemma). The last condition can be proved by induction over degree n. For $n = 2$ the r.h.s. of the first line in (42) is obviously closed in view of the cocycle condition $f_{ij}^l r_{lk} + cycle(i,j,k) = 0$. Suppose now that $dF^{(n)} = 0$ and $Y^{(n)}$ is a solution to equation $dY^{(n)} = F^{(n)}$. Then it is not hard to check that

$$(dF^{(n+1)})_{ijk} = f_{ij}^l (dY^{(n)} - F^{(n)})_{kl} + cycle(i,j,k) = 0\,. \tag{43}$$

Thus, using the induction, we obtain that $dF^{(n)} = 0$, at any n. By the Poincare' lemma there exists a unique 1-form $Y^{(n)} = d^{-1}F^{(n)}$ vanishing on the linear vector field $v = z^i \partial/\partial z^i$ (condition (41)). The explicit expression reads

$$Y_i^{(n)}(z) = \int_0^1 dt F_{ij}^{(n)}(tz) z^j\,. \tag{44}$$

The inverse map $\rho^{-1} : U(L_c)[[\hbar]] \to W$ is obtained by iterating equation

$$z_i = y_i - Y_i(z,\hbar) \tag{45}$$

with respect to z_i. Since $\deg Y_i \geq 2$ these iterations converge to the unique solution $z_i = \rho^{-1}(y_i) \in W$. q.e.d.

Using this theorem we can reformulate the quantization procedure of Sec. 2 in terms of Weyl algebra bundle for any QFL algebra L of vector fields $X_i \in \mathrm{Vect}(M)$. Namely, starting with the associative algebra $A[[\hbar]] = C^\infty(M) \otimes W$ we introduce a set of its derivatives $D_i : A[[\hbar]] \to A[[\hbar]]$,

$$D_i a = X_i a + \frac{1}{i\hbar}[y_i,\, a]\,, \tag{46}$$

In view of the Theorem 4, one can choose the element $y_i = z_i + Y_i(z)$ to satisfy the commutation relations for the Lie algebra L_c. Upon this choice for y's the derivatives D_i form the Lie algebra L with respect to the commutator. Now, repeating the proof of the Theorem 1 one can show the existence of an isomorphism between the subspace of D-constant elements in $\mathcal{A}[[\hbar]]$ and the space of quantum observables $C^\infty(M)[[\hbar]]$. The pull-back of $\circ$-product on $\mathcal{A}[[\hbar]]$ via this isomorphism induces $*$-product on $C^\infty(M)[[\hbar]]$. The latter is obviously equivalent to the $*$-product from Sec. 2.

4 Nonabelian example

The simplest example of the Frobenius Lie algebra is provided by two-dimensional Borel algebra B with the Lie bracket

$$[H, E] = E\,. \tag{47}$$

The coadjoint action of the algebra B on its dual space B^* is generated by the pair of linear vector fields[2]

$$H = y\partial_y\,, \qquad E = y\partial_x\,, \tag{48}$$

where (x, y) are coordinates on $B^* \sim \mathbb{R}^2$. This yields a quadratic Poisson brackets on $\mathbb{R}^2$

$$\omega = E \wedge H = y^2 \partial_x \wedge \partial_y\,. \tag{49}$$

In principle, this bracket can be quantized by means of the general iterative procedure described in the previous sections. In this simple case, however, there is a more direct way to obtain the respective $*$-product. Namely, observe that the above Poisson bivector can also be written as the wedge product of the following vector fields:

$$X = \partial_x\,, \qquad Y = y^2\partial_y\,, \tag{50}$$

so that

$$\omega = X \wedge Y\,, \qquad [X, Y] = 0\,.$$

As the vector fields commute, the bracket can easily be quantized by the Weyl-Moyal like formula

$$f * g = \sum_{n=0}^{\infty} \left(-\frac{i\hbar}{2}\right)^n \frac{1}{n!}(X^n f)(Y^n g) \tag{51}$$

On the other hand, it is easy to prove by induction that

$$(y^2\partial)^n = y^n \prod_{k=0}^{n-1}(y\partial_y + k) \tag{52}$$

[2]As is seen, the carrier space B^* stratifies on two coadjoint orbits of dimension 0 and 2. The former coincides with the origin $0 \in B^*$, while the latter is given by $B^* - \{0\} \sim S^1 \times \mathbb{R}^1$. The existence of a coadjoint orbit of dimension equal to the dimension of the corresponding Lie algebra is a general characteristic property of the Frobenius algebras [10].

Using this identity we can rewrite the above $*$-product in terms of noncommuting vector fields (48) as

$$f * g = \sum_{n=0}^{\infty} \left(-\frac{i\hbar}{2}\right)^n \frac{1}{n!}(E^n f)(H^{(n)} g)\,, \tag{53}$$

where we put

$$H^{(n)} = H(H+1)\cdots(H+n-1)\,.$$

Actually, expression (53) is the universal quantization formula as it gives an associative $*$-product for any pair of vector fields E and H realizing the Borel algebra (47). This may be seen from the following line of reasons. The vector fields (48) induce a special representation $\rho : U(B) \to D(\mathbb{R}^2)$ of the universal enveloping algebra $U(B)$ in the algebra of differential operators on plane $D(\mathbb{R}^2)$. Since H and E are linearly independent with functional coefficients this representation is exact, i.e. $\ker \rho = 0$ [3]. In other words, there are no algebraic relations among the first-order differential operators E and H other than (47) and its algebraic consequences. On the other hand, the associativity condition for the $*$-product (53), being written explicitly, has a form of algebraic equations for the generators E and H and, as we have just argued, they should be satisfied as consequence of the commutation relations of the Borel algebra only. Thus the associativity holds for any representation $\rho : B \to \mathrm{Vect}(\mathbb{R}^2)$.

Originally, the Universal Deformation Formula (53) was obtained in the paper [26]. The skew-symmetric form of the above quasi-exponential formula (that corresponds to the Weyl ordering) was derived in [21].

The formula (53) allows one to quantize an interesting class of irregular Poisson brackets on $\mathbb{R}^2$. Any bracket on plane is given by a single function

$$\omega = f(x,y)\partial_x \wedge \partial_y \tag{54}$$

The function f is said to be *quasi-homogeneous of weight* λ upon the weights of x and y equal to α and β respectively if it is an eigen-vector to the Euler operator v with the eigen-value λ,

$$vf = \lambda f, \qquad v = \alpha x \partial_x + \beta y \partial_y\,. \tag{55}$$

[3] Obviously, this property is valid for the coadjoint action of any Frobenius Lie algebra L since there is an open set of points (the orbit of maximal dimension) at which the corresponding vector fields are linearly independent.

In what follows we assume $\lambda \neq 0$. Introduce a Hamiltonian vector field u with respect to the canonical Poisson bracket on $\mathbb{R}^2$,

$$u = \partial_x f \partial_y - \partial_y f \partial_x \tag{56}$$

Then it is easy to check that

$$\omega = \frac{v \wedge u}{\lambda}, \qquad [v, u] = (\lambda - \alpha - \beta) u \tag{57}$$

If $\alpha + \beta = \lambda$ the bivector (54) is representable as exterior product of the commuting vector fields u, v, otherwise the vector fields form (after an obvious redefinition) the Borel algebra (47). Thus, every quasi-homogeneous Poisson bracket on plane can be quantized in a pure algebraic manner by formulas (51) or (53).

For example, let (m, n, k, l) be a set of four integers, then function

$$f = x^m y^n + x^k y^l \tag{58}$$

is quasi-homogeneous of weight $\lambda = nk - lm$ with $\alpha = n - l$ and $\beta = k - m$. Conversely, given the integer weights λ, α, β, then the most general quasi-homogeneous, analytical at zero, function has the form

$$f(x, y) = \sum_k a_k x^{n_k} y^{m_k}\,, \tag{59}$$

a_k being arbitrary constants and the sum goes over all non-negative integers solutions (n_k, m_k) to the linear Diophantus equation

$$\alpha n + \beta m = \lambda\,. \tag{60}$$

Depending on α, β and λ this equation may have finite or infinite number of non-negative solutions, or may have none.

5 Quantization of triangular Lie bialgebras

In this section we briefly discuss how the output algebraic structures entering the quantization formula for r-bracket (26) can be interpreted in the formal language of the Lie bialgebra quantization. Here, we do not want to go into detailed definition of all the related mathematical constructions but only recall some basic points, useful for our purposes. More details can be found, for example, in [20], [21].

As it was first shown by Drinfeld [9] the deformation problem for the triangular Lie bialgebra (L,r) is equivalent to constructing the Universal Deformation Formula. The latter can be immediately read off from the explicit expression for the $*$-product (25, 26) if one replace the vector fields X_i by the Lie algebra generators y_i. This leads to the following *universal twisting element*:

$$F = I\otimes I + \sum_{n,m,k=1}^{\infty} \hbar^n D_n^{i_1\dots i_m j_1\dots j_k} y_{i_1}\dots y_{i_m}\otimes y_{j_1}\dots y_{j_k} \in U(L)\otimes U(L)[[\hbar]]\,, \tag{61}$$

I being the unit in $U(L)$. The element F possesses special algebraic properties (to be specified further) resulted from the associativity and normalization conditions for the respective $*$-product, which, in turn, are based essentially on the Leibnitz rule for the derivatives $X_i(ab) = X_i(a)b + aX_i(b)$ and the "null on constant condition" $X_i(1) = 0$. The algebraic formalization for the last two relations is naturally achieved by introducing the co-algebra structure on the universal enveloping algebra $U(L)$, i.e. the co-multiplication $\triangle : U(L)\to U(L)\otimes U(L)$ and co-unit $\varepsilon : U(L)\to\mathbb{C}$ mappings. Being homomorphisms of the associative algebra $U(L)$ with unit I, these operations are completely determined by their action on the generators y_i and I:

$$\begin{aligned} &\triangle y_i = y_i\otimes I + I\otimes y_i\,, \qquad &&\triangle I = I\otimes I\,,\\ &\varepsilon(y_i) = 0\,, &&\varepsilon(I) = 1\,. \end{aligned} \tag{62}$$

Now the associativity $(a*b)*c = a*(b*c)$ and normalization condition $(25.c)$ are encoded in the following relations for the universal twisting element F:

$$\begin{aligned} &((\triangle\otimes I)\,F)\;(F\otimes I) = ((I\otimes\triangle)\,F)\;(I\otimes F)\,,\\ &(\varepsilon\otimes I)F = (I\otimes\varepsilon)F = I\,, \end{aligned} \tag{63}$$

while the correspondence principle $(25.b)$ looks like

$$F = I\otimes I - \frac{i\hbar}{2} r^{ij} e_i\otimes e_j \mod (\hbar^2)\,. \tag{64}$$

Another natural operation $S : U(L)\to U(L)$, called antipode, is induced by the involutive anti-homomorphism of the Lie algebra L:

$$S(y_i) = -y_i\,,\quad S(I) = I\,,\quad S^2 = \mathrm{id}\,. \tag{65}$$

The collection of the operations $(\triangle,\varepsilon,S)$ endows $U(L)$ with a structure of the Hopf algebra defined by three axioms:

i) $(\triangle \otimes I)\triangle = (I \otimes \triangle)\triangle$ - *co-associativity*;

ii) $(\varepsilon \otimes I)\triangle = I = (I \otimes \varepsilon)\triangle$;

iii) $m(S \otimes I)\triangle = u \circ \varepsilon = m(I \otimes S)\triangle$ ($\circ$ *stands for the composition of maps*).

Here m denotes the standard multiplication in $U(L)$ and the map u sends $c \in \mathbb{C}$ to $c \cdot I \in U(L)$. If in addition $\triangle = \tau\triangle$, where automorphism $\tau : a \otimes b \to b \otimes a$ permutes the multipliers in the tensor square $U(L) \otimes U(L)$, then the Hopf algebra is called co-commutative[4].

Thus we arrive at the one-to-one correspondence between the variety of $*$-products of the form (25) and solutions to the equations (63) subject to the "boundary condition" (64); in doing so, the invariance of equations (63) under the transformation

$$F \longrightarrow F' = (\triangle\, B)\, F\, (B^{-1} \otimes B^{-1})\,, \quad B \in U(L)[[\hbar]]\,, \quad \varepsilon B = I\,,$$

corresponds the to equivalence of two such $*$-products [9]. As another example of invariance we present the following discrete symmetry:

$$F \to \widetilde{F} = [(S \otimes S)\, \tau F \tau]^{-1} \tag{66}$$

Using the new solution $\widetilde{F}$ to the equations (63), having the same quasi-classical limit (64), one can construct another $*$-product which is not in general equivalent to the initial one (61) in the sense of (29). It would be interesting to explicitly compute the characteristic class of the respective $*$-product.

In order to illustrate how the universal deformation formula gives rise to the deformation of the Lie bialgebra structure, we remind the construction of the quantum universal enveloping algebra $U_\hbar(L)$. As an associative algebra, $U_\hbar(L)$ coincides with $U(L)[[\hbar]]$. The respective co-unit ε remains the same as in (62), while the co-multiplication and the antipode for $U_\hbar(L)$ are obtained from those of $U(L)$ by the twisting transformation

$$\begin{aligned} &\triangle_\hbar a = F^{-1}(\triangle a)F\,, && S_\hbar a = U^{-1} S a U\,, \\ &U = m((S \otimes I)F)\,, && a \in U_\hbar(L)\,. \end{aligned} \tag{67}$$

The validity of the Hopf algebra axioms for the deformed co-algebra structure can be verified with the help of conditions (63).

[4] Omitting the antipode operation S and the related axiom (iii) one arrives at the notion of an associative bialgebra.

Given the associative bialgebra $U_\hbar(L)$, one can twist the multiplication of any $U_\hbar(L)$-module algebra or co-algebra, in particular, to write an explicit symbol realization for the deformed coordinate rings of various algebraic varieties [21]. Among the most important examples of such a kind it is worth to mention the Takhtajan approach to the quantum group construction [20]. Let G be the Lie group corresponding to the Lie algebra L, and denote by $G_\hbar$ corresponding quantum group. The latter is defined in terms of an associative algebra of functions $\mathrm{Fun}(G_\hbar)$. As a linear space, the algebra $\mathrm{Fun}(G_\hbar)$ is identified with $C^\infty(G)[[\hbar]]$ and equipped with the following star-product:

$$a \star b = M(F^L (F^{-1})^R (a \otimes b)) \,, \tag{68}$$

where M stands for the ordinary commutative multiplication in $C^\infty(G)$ and F^L, $(F^{-1})^R$ denote the twisting elements F, F^{-1} in the representation of the left $\{X_i^L\}$ and anti-representation of the right $\{X_i^R\}$ invariant vector fields on the Lie group G, respectively. The associativity of $\star$-product follows from the first equation in (63) and the fact that $X_i^R \mapsto X_i^L$ is involutive anti-homomorphism of the Lie algebras of vector fields. Evaluation of the quasi-classical limit for $\star$-product (68) leads to the following Poisson bracket:

$$\{a,\, b\} = r^{ij}\, (X_i^L a\; X_j^L b - X_i^R a\; X_j^R b) \,.$$

As is seen this bracket is a particular example of r-brackets considered in the previous sections.

Note that one can use the transformed twisting element (66) or even a combination of two different elements (61) and (66) in the definition (68) to obtain a different inequivalent quantizations. However the special "adjoint" structure of the deformation (68) (the "right" twisting element is taken to be inverse to the "left" one) implies also that the standard group co-multiplication $\triangle : \mathrm{Fun}(G_\hbar) \to \mathrm{Fun}(G_\hbar) \otimes \mathrm{Fun}(G_\hbar)$ defined by the relation

$$(\triangle_G f)(g_1, g_2) = f(g_1 g_2) \tag{69}$$

is indeed a homomorphism of the algebra $(C^\infty(G))[[\hbar]], \star)$ to $(C^\infty(G)[[\hbar]], \star) \otimes (C^\infty(G)[[\hbar]], \star)$.

Our explicit formulas, being combined with the Takhtajan construction [20], provide a transparent and explicit description for the quantum algebra of functions $\mathrm{Fun}(G_\hbar)$, that can be viewed as an alternative to the standard FRT method [27].

6 Discussion and Conclusion

To summarize, in this paper we have proposed a simple quantization procedure for the Poisson brackets associated to the classical triangular r-matrix and giving, as a byproduct, the explicit quantization formula for Lie bialgebras. Abstracting from the peculiar form of the r-brackets (1), (3) one can find a noticeable similarity between our quantization algorithm and Fedosov's construction for symplectic manifolds. Indeed, in both the cases one begins with an auxiliary quantum bundle assigned by a fiber-wise $\circ$-product; in so doing, the space of quantum observables is identified to the center of this algebra. At the next step a new nontrivial embedding is constructed to induce the $*$-product on the Poisson manifold as a pull-back of the $\circ$-one. This is achieved by introducing an appropriate Lie algebra of external derivatives and identifying the image of the embedding map with their kernel subspace (in fact $\circ$-subalgebra). Moreover, the Theorem 4 of Sec. 3 suggests that, similar to the Fedosov approach, we can always start with the quantum bundle of Weyl algebras and construct the embedding map in an explicit recurrent manner with any accuracy in the deformation parameter. In our special algebraic situation, however, the use of the universal enveloping algebra bundle seems to be more adequate to the problem, considerably simplifying the whole construction. Another novel point of our approach, looking from the pure algebraic perspective, is the non-commutativity of the aforementioned derivatives for any choice of fiber-wise $\circ$-product. This is a distinction from the Fedosov deformation quantization which is essentially based on the notion of the *Abelian* connection corresponding, in this language, to a certain pair-wise commuting set of derivatives associated with any symplectic connection.

Let us briefly discuss a possibility to extend our star product construction to a more general class of Poisson manifolds than considered in this paper. It seems useful to seek for the generalizations making use of a natural relationship between the Fedosov deformation quantization and BRST theory [15] [5]. The key idea behind the identification of both the quantization schemes was a realization of the symplectic manifold M as a second class constrained surface embedded into cotangent bundle T^*M. Let us explain how this approach,

[5] On the other hand, the BRST language could be useful in attempts to apply the present $*$-product construction to the field theory problems, as it provides a uniform description for both deformation quantization of the phase space observable algebra and gauge symmetry or/and the Hamiltonian constraint algebra, see [16].

originally found for the symplectic manifold, can be first extended to the case of r-brackets (1), (3). Consider a trivial vector bundle $\mathcal{M} = M \times V$ over a smooth manifold M, assigned by the following Poisson brackets

$$\{x^\mu, x^\nu\} = 0\,, \quad \{p_i, p_j\} = f_{ij}^k p_k + r_{ij}\,, \quad \{p_i, x^\mu\} = X_i^\mu(x)\,. \tag{70}$$

Here x^μ are local coordinates on M and p_i are linear coordinates on V. It is easy to check that these brackets satisfy the Jacobi identity iff (i) the vector fields X_i generate a Lie algebra (3) and (ii) matrix r satisfies cocycle condition (6). If in addition $\det(r_{ij}) \neq 0$, then the second class constraints $T_i = p_i \approx 0$ define an embedding of M into $\mathcal{M}$ as a zero section. The corresponding Dirac bracket on $\mathcal{M}$ induces Poisson bracket on M,

$$\{x^\mu, x^\nu\}_{Dirac} = r^{ij} X_i^\mu(x) X_j^\nu(x)\,. \tag{71}$$

This relation identifies the Hamiltonian theory on M to the second class constrained theory on $\mathcal{M}$. Note that in general both the brackets (70) and (71) are irregular, that does not, however, obstruct the mentioned identification. The common method of BRST treatment of the second class constrained systems implies the conversion of the second class constraints T_i to the first class ones by means of a proper extension of the phase manifold $\mathcal{M}$ (see [28], and references therein). The minimal possibility to do this is to duplicate the vector bundle V, that is to consider the direct product $\mathcal{M}' = \mathcal{M} \times V$, where V is a linear symplectic space assigned by canonical Poisson bracket

$$\{z_i, z_j\} = r_{ij}\,, \tag{72}$$

z_i being linear coordinates on the second copy V. Together Rels. (70) and (72) define the Poisson structure on $\mathcal{M}'$ (we mean that bracket of z are vanishing to other variables). Now it is possible to extend the constraints T_i from $\mathcal{M}$ to $\mathcal{M}'$ getting an equivalent first class constrained theory $T_i' = T_i + z_i + o(z^2)$, where the higher orders in the formal variables z_i are determined from the requirement of involution of the effective first class constraints T_i'

$$\{T_i', T_j'\} = f_{ij}^k T_k'\,. \tag{73}$$

The Poisson brackets (70), (72) on $\mathcal{M}'$ can be easily quantized if one restricts the space of observables to the functions on $\mathcal{M}'$ at most linear in p_i. The connection with constructions of previous sections is established by the relation $D_i a = [T_i, a]$, where $a = a(x, z, \hbar)$. The construction of nilpotent BRST charge and identification of physical observable algebra with the special BRST cohomology imply a further enlargement of the phase space $\mathcal{M}'$ by the ghost variables. For the case of M being a symplectic manifold, this BRST quantization programme has been implemented in [15], and it seems no special modification is required in the case of the Poisson manifold (1), (3).

In this form the above quantization procedure may be generalized to the case when all the structure constants f_{ij}^k, r_{ij} entering to the fundamental Poisson brackets (70), (72) are allowed to depend on the point $x \in M$ [29]. As a particular case this includes the quantization of dynamical r-matrix [22] with the vector fields X_i generating a Lie bialgebroid.

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