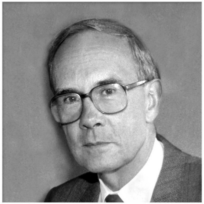
# Bonn Arbeitstagung

# The Hirzebruch Signature Theorem and Branched Coverings

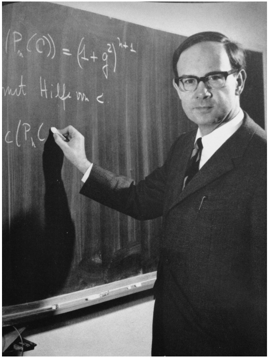


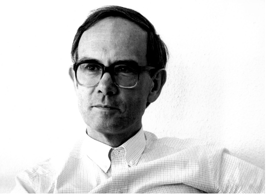
Michael Atiyah (Edinburgh)

Dedicated to Fritz Hirzebruch

26 May, 2013

# Fritz

## Princeton 1953

- **Chern classes** $c_j$ of complex vector bundle
- $1 + c_1 + c_2 ... + c_n = \prod_{i=1}^{n}(1 + x_i)$
- For complex manifold $X$

$$c_i(X) = i\text{th Chern class of tangent bundle}$$

- **Pontryagin classes** $p_j$ of real vector bundle

$$1 + p_1 + ..., p_k = \prod_{i=1}^{2k}(1 + x_i^2)$$

- For real manifold $X$

$$p_i(X) = i\text{th Pontryagin class of tangent bundle}$$

## Borel and Chern

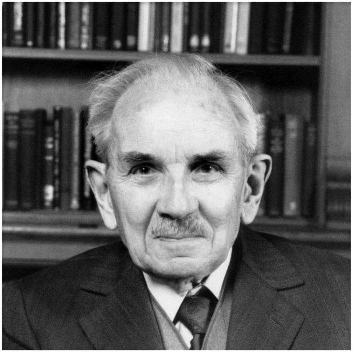
## Todd genus

$$T = \sum_n T_n(c_1, c_2, \dots) = \prod_i \frac{x_i}{1 - e^{-x_i}}$$

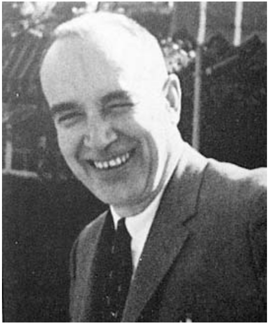
$$T_1 = \frac{c_1}{2}, \quad T_2 = \frac{c_1^2 + c_2}{12}$$

$$T_3 = \frac{c_1 c_2}{24}, \quad T_4 = \frac{1}{720}(-c_4 + c_3 c_1 + 3c_2^2 + 4c_2 c_1^2 - c_1^4)$$

# Leray and Cartan

## Spencer, Serre, Kodaira, Weyl



## L-genus

- $$L = \sum L_k(p_1, p_2, \ldots) = \prod \left( \frac{x_i}{\tanh x_i} \right)$$
- $$\begin{aligned} L_1 &= \frac{p_1}{3} \\ L_2 &= \frac{7p_2 - p_1^2}{45} \\ L_3 &= \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_1 p_2 + 2p_1^3) \end{aligned}$$
- Relation between $T$ and $L$

$$\frac{x}{\tanh x} + x = \frac{2x}{1 - e^{-2x}}$$

## Riemann-Roch

- $X$ compact complex manifold $\dim_{\mathbb{C}} X = n$
- $\mathscr{O}$ sheaf of holomorphic functions on $X$
- $H^q(X, \mathscr{O})$ cohomology groups
- $\chi(X, \mathscr{O}) \;=\; \sum_{q=0}^{n} (-1)^q \dim H^q(X, \mathscr{O})$ **Arithmetic Genus**
- **Theorem 1** (Hirzebruch Riemann-Roch)

$$\chi(X, \mathscr{O}) \;=\; T_n(X)$$

- $n = 1$, $X$ Riemann surface

$$\chi \;=\; \frac{c_1}{2} \;=\; 1 - g \;.$$

## Signature

- $X$ compact oriented manifold of dimension $4k$
- $H^{2k}(X;\mathbb{R})$ has a non-degenerate quadratic form, with $p+q = \dim H^{2k}(X;\mathbb{R})$ non-zero eigenvalues, $p$ positive signs, $q$ negative signs
- The signature of $X$ is the signature of the form

$$\mathrm{Sign}(X) \;=\; p - q \in \mathbb{Z}\ .$$

- **Theorem 2** (Hirzebruch Signature Theorem)

$$\mathrm{Sign}(X) \;=\; L_k(X)$$

- $k=1$, $\dim X = 4$, $\mathrm{Sign}(X) = p_1/3$.

# Mexico, 1956

**Bonn, 1977**

## $\widehat{A}$-genus

- $\widehat{A}(p_1, p_2, \dots) \ = \ \prod_i \dfrac{x_i/2}{\sinh x_i/2}$
- $T = e^{-c_1/2}\widehat{A}$ (involves only $c_1$ and $p_j$)
- **Theorem 3 (A-S 1963)**

$$\widehat{A}(X) \ = \ \operatorname{index} D$$

- $D$ Dirac operator

## Bernoulli numbers

- $$\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} x^{2k}$$
- Define
$$B_k = (-1)^{k-1} b_{2k}$$
- $$B_1 = \frac{1}{6}$$
$$B_2 = \frac{1}{30}$$
$$\vdots$$
$$B_8 = \frac{3617}{510}$$

## Cauchy Residues

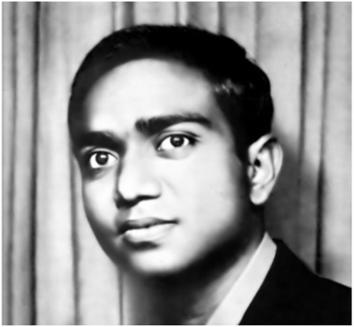
- HRR for $P_n(\mathbb{C})$ gives $T(P_n(\mathbb{C})) = 1$
- total Chern class of $P_n(\mathbb{C}) = (1+x)^{n+1}$
- $$T(P_n(\mathbb{C})) \;=\; \text{coefficient of } x^n \text{ in } \left(\frac{x}{1-e^{-x}}\right)^{n+1}$$
- shown to be 1 by **Cauchy residue formula**
- $$\frac{1}{2\pi i}\int \frac{dx}{(1-e^{-x})^{n+1}} \;=\; \frac{1}{2\pi i}\int \frac{dy}{y^{n+1}(1-y)} \;=\; 1$$

  (where $y = 1 - e^{-x}$)

## Defects (of singularities)

- If $X$ has Riemannian metric (Hermitian in complex case) then the $p_j$ and $c_j$ are represented by differential forms and Theorems 1 and 2 express $\chi$ and Sign as **integrals** over $X$.
- If $X$ has a singular set $\Sigma$, but $\chi$ or Sign are still defined, then the difference between this invariant and the integral is called the **defect** due to $\Sigma$.
- Three cases where this happens are:
  1. $X$ is a rational homology manifold (e.g. an orbifold), so signature still defined.
  2. $X$ is a complex variety with singular set $\Sigma$, but $\chi$ is still defined by sheaf cohomology.
  3. $X$ is a manifold but the metric has singularities along $\Sigma$.

# Zagier and Patodi

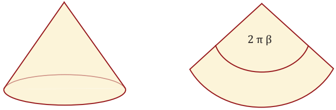


## Special Cases

1. Hirzebruch (Zagier) studied orbifold singularities using the G-signature theorem (equivariant version of Theorem 2) and found interesting relations with number theory (Dedekind sums).
2. Hirzebruch also studied cusp singularities of Hilbert modular surfaces and this motivated extension of Theorems 1 and 2 to manifolds with boundary and introduction of $\eta$-invariant (A-Patodi-Singer, 1973)
3. If $\Sigma \subset X$ is real codimension 2 sub-manifold (e.g. complex codimension 1) then we can have metrics on $X$ with conical singularities (of fixed angle $\beta$) along $\Sigma$. (A. 2013)

## Cones

- **Local model - dimension 2**
- $\mathbb{C} = \mathbb{R}^2$ vertex at origin




Cone

Slit cone

- metric $= dr^2 + \beta^2 r^2 d\theta^2$
- flat except at origin (vertex) where curvature $\kappa$ is multiple of delta function: $2\pi(1-\beta)\delta$

  $\beta < 1 \quad \kappa > 0$ positive curvature
- $\beta = 1 \quad \kappa = 0$ flat

  $\beta > 1 \quad \kappa < 0$ negative curvature
- Smooth out metric near vertex preserving rotational symmetry. Then Gauss-Bonnet relates curvature integral to geodesic curvature along boundary.
- Note: For $\beta > 1$ picture cannot be drawn in $\mathbb{R}^3$.

## Integer Angles

- When $\beta = \frac{1}{q}$ with $q$ integer, cone is just quotient of $\mathbb{R}^2$ by $\mathbb{Z}_q$ cyclic group of order $q$. In complex coordinates

$$z = u^q$$

  and the $u$-plane is $q$-fold branched covering of $z$-plane.
- The standard flat metric on $u$-plane pushes down to a conical metric with $\beta = \frac{1}{q}$ on $z$-plane.
- But we can reverse the process and lift up the flat metric on $z$-plane to give a conical metric on $u$-plane with $\beta = q$.
- Note. $\mathbb{C} \cong \mathbb{C}/\mathbb{Z}_q$ either in **topology** or in **complex** analysis (invariant functions are functions on quotient). But **not in real differential geometry, which is where cones appear.**

## Rational Angles I.

- Cones with $\beta = \frac{p}{q}$ rational occur for the correspondence between $u$-plane and $v$-plane where

$$u^q \ = \ v^p \ ( \ = \ z)$$

- The flat $u$-metric **pushed down** to $z$-plane and then **lifted up** to the $v$-plane becomes conical with

$$\beta \ = \ \frac{p}{q}$$

In polar coordinates if $u = e^{i\theta}$, $v = e^{i\phi}$

$$q\theta \ = \ p\phi \ .$$

## Rational Angles II.

- All maps are compatible with rotation. Formally they are $U(1)$-equivariant where $U(1)$ is the phase group of the $w$-plane, with $z = w^{pq}$
- $$\begin{tikzcd} & w\text{-plane} \arrow[dl] \arrow[dr] & \\ u\text{-plane} \arrow[dr] & & v\text{-plane} \arrow[dl] \\ & z\text{-plane} & \end{tikzcd}$$

## Todd genus defect I.

- $\Sigma \subset X$ codimension 2 with metric on $X$ with (constant) angle $2\pi\beta$. Define the defect

$$\delta_T(\beta) \;=\; \int_X T_n - \int_{X-\Sigma} T_n(\beta)$$

  where $T_n$ is the Todd form of a smooth metric on $X$ and $T_n(\beta)$ is the Todd form of the conical metric.
- **Theorem 1** $(\beta)$

$$\delta_T(\beta) \;=\; \left[\frac{T(\Sigma)}{x}\left\{\frac{x}{1-e^{-x}} - \frac{\beta x}{1-e^{-\beta x}}\right\}\right][\Sigma]$$

  where $x \in H^2(\Sigma)$ is $c_1$ of normal bundle.

## Todd genus defect II.

- Expanding in terms of Bernoulli numbers we get

$$\delta_T(\beta) \;=\;$$

$$\left\{ \frac{1-\beta}{2} T_{n-1}(\Sigma) + \sum_{k\geqslant 1} (-1)^{k-1} \frac{T_{n-2k}(\Sigma) B_k (1-\beta^{2k})}{2k!} x^{2k-1} \right\} [\Sigma]$$

- **Example**. $\dim X = 4$

$$\delta \;=\; \frac{1-\beta}{2}(1-g) + \frac{(1-\beta^2)}{12}\Sigma^2$$

## L-genus defect

- **Theorem 2 ($\beta$)**
- Similar formula to Theorem 1($\beta$) but with $L$-genus instead of $T$-genus and using the formula

$$\frac{1}{\tanh x} = \frac{2}{1 - e^{-x}} - 1$$

- We get essentially same extra terms involving Bernoulli numbers but with the constant term dropped.
- **Example** $\dim X = 4$ we get no dependence on the genus of $\Sigma$ only a term $\dfrac{1-\beta^2}{3}\Sigma^2$.
- There is also Theorem 3($\beta$) dealing with the Dirac index of a spin-manifold and more generally the Dirac index of a Spin$^c$-manifold where the formula is just that of the Todd-genus.

## Euler characteristic defect

- Theorems $1(\beta)$, $2(\beta)$ and $3(\beta)$ should be compared with the more elementary formula for the ordinary Euler characteristic $E$ where the defect is just

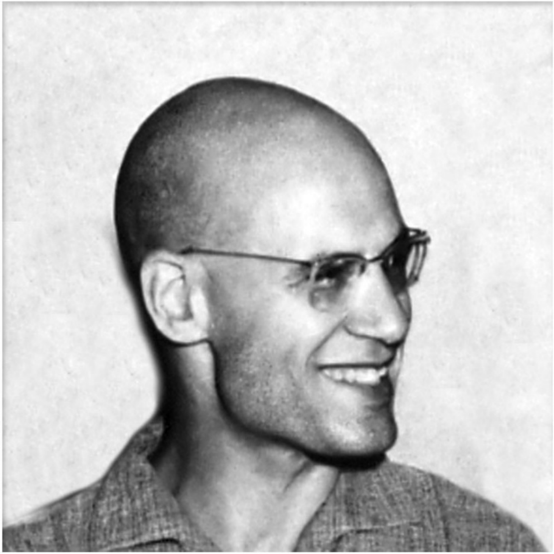
$$(1-\beta)E(\Sigma)$$

- This just comes from the one odd Bernoulli number $b_1$ and is the obvious extension of the formula for dimension 2.

## First Arbeitstagung 1957

- **Grothendieck-Riemann Roch**
- **Algebraic Geometry**
- $K$-theory of coherent sheaves, vector bundles and resolutions.
- Key components:
  1. $K$-theory of vector bundles via exact sequences.
  2. $K$-theory of coherent sheaves isomorphic (for non-singular $X$) to $K$-theory of vector bundles: use projective resolutions.
  3. Definition of $f_! : K(X) \to K(Y)$ for a map $f : X \to Y$, reducing to $\chi(X)$ when $Y = \{\text{point}\}$.
  4. Functoriality of $f_!$.
  5. $K(X \times P_1) \cong K(X) \otimes K(P_1)$, $K(P_1) = \mathbb{Z} \oplus \mathbb{Z}$

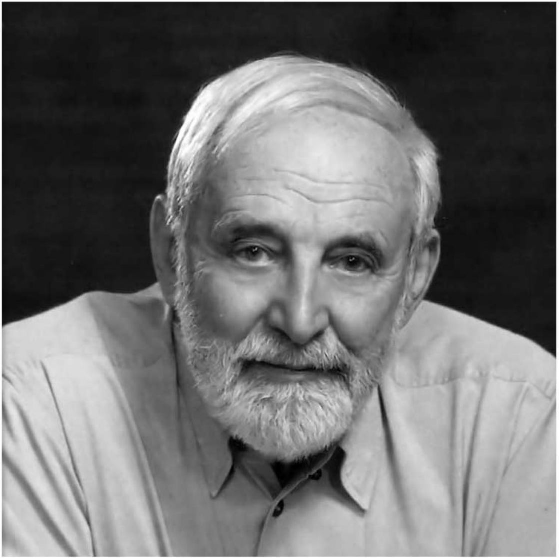
**Grothendieck**



## First Decade of AT

- Bott Periodicity 1957
- Topological $K$-Theory (AH) 1959
- Index Theory (AS) 1963
- Equivariant $K_G$-theory (Segal) 1968
- **Key Point** for topological $K$-theory:
  Bott periodicity is essentially equivalent to
  (A) $K(X \times P_1) \cong K(X) \otimes K(P_1)$
  or
  (B) $K_G(\mathbb{C}) \cong K_G(\text{point}) = R(G) = \mathbb{Z}[\eta, \eta^{-1}]$
  where $G = U(1)24$

**Bott**



## Basic Exact Sequence

- $A =$ origin in $\mathbb{C}$, $\mathscr{O}$ holomorphic functions on $\mathbb{C}$, or on $P_1(\mathbb{C})$, exact sequence

$$0 \longrightarrow \mathscr{O}(-1) \xrightarrow{z} \mathscr{O} \longrightarrow \mathscr{O}_A \longrightarrow 0$$

- View this equivariantly for $G = U(1)$ in $K$-theory (Grothendieck or Bott)

$$i_* \;:\; K_G(A) \xrightarrow{\cong} \widetilde{K}_G(\mathbb{C}) \;\; \text{(compact support)}$$

$$i_*(1) \;=\; 1 - \eta^{-1}$$

where $\eta$ is line bundle $\mathscr{O}(1)$

## Localization

- Pass from ring $R(G)$ to field $\mathbb{C}(\eta)$ of rational functions. Torsion modules drop out and compact support can be ignored, so can consider element 1 and write

$$i_*^{-1}(1) \;=\; \frac{1}{1-\eta^{-1}}$$

- Passing to equivariant cohomology of $G$, via Chern character, we get

$$\frac{1}{1-e^{-x}} \;=\; \frac{1}{x}+\frac{1}{2}+\ldots$$

- Clearly the polar term $\dfrac{1}{x}$ has to be dealt with!

## Cancelling the pole

- Consider the $q$-fold branched cover $u \mapsto z^q$. The polar terms in the difference

$$\left(\frac{1}{1-e^{-x}} - \frac{q}{1-e^{-qx}}\right)$$

  cancel, and this gives the formula appearing in Theorem $1(\beta)$ for $\beta = \dfrac{1}{q}$.
- Doing the same for an integer $p$ and using the correspondence $u^q = v^p$ we get the formula for $\beta = \dfrac{p}{q}$.
- Continuity gives it for all $\beta$.

## Proof of Theorem 1 ($\beta$) : Outline

1. First we note that the difference of integrals can be localized near the subspace $\Sigma$, since the two metrics can be chosen to agree elsewhere.
2. This gives us $U(1)$ symmetry and means that the contribution of the normal bundle is a universal calculation for $U(1)$ acting on $\mathbb{C}$. The formulae involve equivariant cohomology of $U(1)$ but using the Weil model we get equivariant differential forms with basic 2-form $\omega$ representing the Chern class $x$.
3. The local calculation has been sketched above.

# Weil

## Final comments I.

1. For function $f$ on circle, the "distribution property" is that, for all $q$,
$$\frac{1}{q}\sum_{\gamma} f(z\gamma) \;=\; f(z^q)\ (|z|=1)$$
$\gamma$ in the finite cyclic group of $q$-th roots of 1.
2. Holds for $f(z) = \dfrac{1}{1-z^{-1}}, \dfrac{1}{1-z}, 1$
3. In the space of Schwartz distributions on the circle
$$f(z) \;=\; \sum_{-\infty}^{\infty} a_n z^n \ (a_n \text{ polynomial growth})$$
the only ones with "distribution property" are those in (2) (expanded as power series) and linear combinations.
4. The three functions in 2. correspond (essentially) to the $L$-genus, Todd-genus and Euler characteristic.

## Final comments II.

5. Can extend Theorems to include the Hirzebruch $\chi_y$ genus

$$\chi_y \ = \ \sum_p y^p \chi(\Omega^p)$$

6. Distributional characters occur in index theory for transversally elliptic operators. Interprets the pole of $f(z)$ at $z = 1$, and its appearance in the expansion of $(1 - e^{-x})^{-1}$ at $x = 0$. Example: Holomorphic functions on $\mathbb{C}$, graded by degree.
7. The limit case of $\beta = 0$ in Theorems 1 ($\beta$) and 2 ($\beta$) is of interest, and was studied (with Lebrun) in dimension 4.

# Edinburgh 2009

# Edinburgh 2010