On Thurston's boundary of Teichmüller space and the extension of earthquakes

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1. Introduction

Consider a closed surface M of genus ≥ 2, equipped with a hyperbolic metric, and let μ be a geodesic lamination on M, which has the property that its complementary regions are all isometric to ideal triangles.

Thurston has shown in [10] how we can associate to μ a global parametrization for the Teichmüller space, **T**, of M. The parameter space is a subspace of the space **MF** of equivalence classes of measured foliations on M, which is defined as the set of equivalence classes which can be represented by measured foliations transverse to μ.

There are two main results that we prove in this paper. The first one is that the parametrization above extends to Thurston's boundary, **PMF**, of Teichmüller space. More precisely, we prove that if a sequence of hyperbolic metrics tends to infinity, then it converges to a point on the boundary if and only if the sequence of projective classes of measured foliations associated to these metrics converges in **PMF**, and we prove that in this case the limits of the two sequences are the same (see the precise statement in section (4.1)). This generalizes a result we have already obtained in a simpler case: the case where the surface M has cusps and where the lamination has only a finite number of leaves, each tending to a cusp (cf.[7]).

The second main result is proven as a consequence of the first; it is about the extension of the earthquake flow to the boundary of Teichmüller space. To state this result, we need first to recall that there are two distinct ways of parametrizing the earthquake flow, which are both natural. To see how these two parametrizations are defined, recall first that we can define the (parametrized) earthquake flow associated to a measured geodesic lamination ν by taking a sequence of weighted simple closed geodesics $x_i.C_i$ converging to ν in the topology of **MF**, and taking the limit of the sequence of (parametrized) Fenchel-Nielsen flows (twist flows) along the geodesics C_i, weighted by the sequence of real numbers x_i.

There are two natural ways of parametrizing the Fenchel-Nielsen flow along a geodesic. The first one (used for example by Kerckhoff in [2]) consists, at time t, of twisting by an amount equal to t, along the geodesic C_i, w.r.t. the metric on C_i induced by the hyperbolic metric on the surface. With this parametrization, it is easy to see that the

Fenchel-Nielsen flow extends continuously by the identity to the boundary of Teichmüller space. Now it is natural to ask whether this flow can be reparametrized so that it induces continuously a nontrivial flow on the boundary; in other words, we can ask whether the foliation of Teichmüller space induced by the flow admits an extension as a foliation tangent to the boundary of that space.

Indeed, there is a second way of parametrizing the Fenchel-Nielsen flow along a geodesic C_i. This is defined by twisting, at time t, by an amount equal to $t.l_g(C_i)$, where $l_g(C_i)$ is the length of C_i w.r.t. the hyperbolic metric g which we are twisting. It is easy to see that the time-1 map associated to this flow induces on Teichmüller space the same action as that of the mapping class defined by the Dehn twist along C_i. So this parametrization is also quite natural. And in fact it turns out that the Fenchel-Nielsen flow with this parametrization extends continuously to Thurston´s boundary, where it induces a flow which is the quotient flow of a flow defined on the space **MF**, which can be described as "twisting" the measured foliations which have nonzero intersection number with C_i. The zero set of the flow on **PMF** is equal to the subset defined by the equation $i(C_i,.) = 0$. This is explained in [7], and it uses local parameters near a point on the boundary of Teichmüller space which are adapted to the curve C_i.

We shall refer to the Fenchel-Nielsen flow with this second parametrization as the "normalized" Fenchel-Nielsen flow. It defines, by taking limits, a normalized earthquake flow on **T**.

In this paper, we consider the normalized earthquake flow associated to a measured geodesic lamination μ which is maximal (i.e. for which every complementary component is isometric to an ideal triangle). We use Thurston´s parameters of Teichmüller space associated to μ that we referred to above, together with our result on the behaviour of these parameters when a sequence of hyperbolic metrics converges to a point on the boundary, to prove that the normalized earthquake flow associated to μ extends continuously to the boundary of Teichmüller space, on which it induces a nontrivial flow.

The plan of this paper is as follows:

In section 2, we describe the parameters of Teichmüller space that we will be working with. This parametrization has been defined by Thurston and is contained in his paper [10]. For the convenience of the reader, we recall the necessary definitions together with the main construction, which is that of the horocyclic foliation associated to a maximal geodesic lamination on a hyperbolic surface.

We state as theorem (2.1) Thurston´s result that we shall be using, which says that a certain map from Teichmüller space to a subset of **MF** is a homeomorphism. This is the parametrization of Teichmüller space associated to the maximal geodesic lamination μ.

Section 3 contains some material about lengths of measured foliations and related facts on a hyperbolic surface. This notion of length is a generalization of the notion of length for simple closed curves, and is defined in the same way Thurston has defined the length of a measured geodesic lamination. We prove somme facts about lengths and intersection functions which are used in the next section.

Section 4 contains the proof of the main result on the convergence of sequences of hyperbolic metrics to points on the boundary of Teichmüller space. This is stated as theorem (4.1).

Section 5 is independent of the remaining part of the paper. We discuss in it some simple facts concerning the geodesics of a new metric on Teichmüller space, which is defined by Thurston in his paper [10]. These geodesics are called "stretch lines" and are defined in terms of the parameters of Teichmüller space that are associated to a maximal geodesic lamination μ, which we were using before. An immediate consequence of our work is that any stretch line converges to a definite point on the boundary of Teichmüller space. We consider also "anti-stretch" lines (i.e. stretch lines equipped with the opposite orientation); these are not geodesics for the metric. (The metric is nonsymmetric.) We discuss their convergence to the boundary.

Section 6 contains the proof of the result about the extension of the normalized earthquake flow. To prove this result, we use a description of the earthquake flow that Thurston gives in his paper [10], and this description makes use of shear coordinates for measured foliations. For the convenience of the reader, we have included in this section a description of these coordinates.

We conclude this introduction by fixing the notations for the rest of the paper.

In all this paper, M is a closed surface of genus $g \geq 2$. We begin by recalling a few definitions.The details about all the notions that are used are contained in [1] and [9].

The Teichmüller space of M is denoted by **T** and is viewed as the space of hyperbolic metrics on M up to homotopy. More precisely, it is the space of couples (f,S) where S is a hyperbolic surface and $f : M \to S$ is a homeomorphism defined up to homotopy, with the equivalence relation that identifies two couples $(f_1,S_1)$and (f_2,S_2) if there exists an isometry $g: S_1 \to S_2$ s.t. the homotopy classes $f_1 \circ g$ and f_2 are equal.

We shall denote our surface by S or M, depending on whether or not it is equipped with a hyperbolic structure.

MF denotes the space of measured foliations on M up to isotopy and Whitehead moves, and **PMF** is the quotient space of **MF** w.r.t. the action of the set $\mathbf{R}_+$ of positive real numbers.

S denotes the set of isotopy classes of simple closed curves on M which are not homotopic to a point. There is a natural injection from the set $\mathbf{R}_+.\mathbf{S}$ into **MF**. We shall denote by i (.,.) the intersection function defined on the product **MF** x **MF**, which extends continuously the geometric intersection function defined on couples of weighted simple closed curves (cf. [8]).

For any element g in **T**, and any α element in **S**, we denote by $l_g(\alpha)$ the length of the unique geodesic in the class α, measured with the hyperbolic metric g.

We recall that the topologies of the spaces **MF** and **T** are defined by the inclusion of these spaces in the space $R_+^{\mathbf{S}}$ of positive functions on **S**, *via* the functions i(F,.) and $l_g(.)$ respectively.

Finally, if F is either a measured foliation or an elememt of the space **MF**, we denote by [F] its image in **PMF**.

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2. A parametrization of Teichmüller space (following Thurston)

This section contains some terminology and a construction of Thurston which provides the global parameters of Teichmüller space that we shall be working with.

For every maximal geodesic lamination μ on the hyperbolic surface S, Thurston constructs in [10] a measured foliation, denoted by $F_\mu(g)$, and which we shall call the *horocyclic foliation* (associated to μ and to the hyperbolic structure g on S).

The construction is as follows:

The complementary components of μ are all isometric to ideal triangles, and we can consider in each component a partial foliation (i.e. a foliation whose support is a subsurface) whose leaves are made up of pieces of horocycles, subject to the following property: Each segment joins 2 boundary components of the triangle in a perpendicular way, and the support of the foliation is equal to the whole triangle except for a little triangle bounded by 3 pieces of horocycles which meet tangentially (see figure 1). These conditions uniquely determine the foliation.

These partial foliations in the ideal triangles fit together on the surface and define a partial foliation on this surface, which has a well-defined invariant transverse measure, which is uniquely specified by the fact that on the leaves of μ, this transverse measure coincides with hyperbolic distance. We obtain in this way a measured foliation, $F_\mu(g)$, which has a well-defined class in **MF**. Note that by construction, this class has the property that it can be represented by a measured foliation that is transverse to μ.

Conversely, Thurston proves in [10] that the elements of **MF** which possess this last property are classes of horocyclic foliations which arise as above, for the same μ and for some hyperbolic metric.

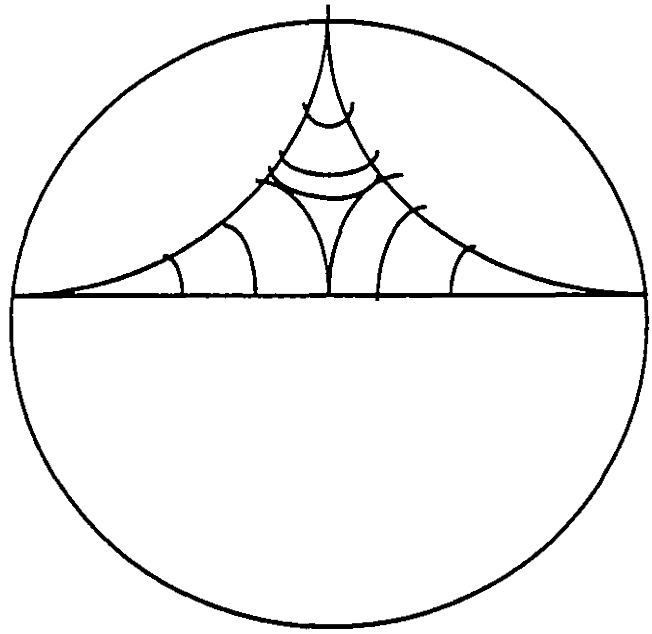


Figure 1

More precisely, let **MF**(μ) denote the subset of **MF** which consists of equivalence classes which admit representatives transverse to μ. Then we have the following

(2.1) **Theorem** (Thurston, [10],§ 9) *For any maximal geodesic lamination μ, the map ϕ_μ which associates to each hyperbolic metric the equivalence class of its horocyclic foliation is a homeomorphism from Teichmüller space to the subset $PMF(\mu)$ of MF.*

Remark Although Thurston's Theorem is valid when μ is any maximal geodesic lamination, in this paper we shall always suppose that μ is a measured geodesic lamination in the usual sense, i.e. that it admits a transverse measure of full support. However, the result in section 5 about the behaviour of stretch lines is valid also in the case where μ does not admit necessarily a transverse measure, as we remark it in that section.

3. The length of a foliation and of a lamination, and the geometric intersection function

(3.1) If μ is a measured geodesic lamination on the hyperbolic surface S, Thurston has defined the *length* of μ, as the total mass on the surface w.r.t. the product measure dt x dl where dl is the 1-dimensional Lebesgue measure on the leaves of μ, and dt the 1-dimensional transverse measure of μ. We shall denote the length of μ w.r.t. the metric g by $l_g(\mu)$ (see [9] and [3]).

So there is a function $l_{(.)}(.)$ (which we shall also denote as $l(.,.)$) defined on the product space **T** x **MF** , where for $g \in$ **T** and $F \in$ **MF**, in order to compute $l(g,F)$, we have to replace F by the measured geodesic lamination (w.r.t. the hyperbolic structure g) which represents it.We know that this function is continuous in the two variables (see [3]). If $x_i.C_i$ is a sequence of weighted simple closed geodesics converging to μ in the topology of **MF**, the quantity $l_g(\mu)$ is therefore equal to the limit of the sequence of real numbers $x_i.l_g\,(C_i)$.

We need to generalize this notion of length to any measured foliation or lamination (which is not necessarily geodesic) on the surface S. So we make the following definition.

(3.2) **Definition** Let F be a measured foliation on S, or a partial measured foliation (i.e. a measured foliation supported on a subsurface of S, like for example the horocyclic foliation associated to a maximal geodesic lamination), or a measured lamination (we can stick to laminations isotopic to measured geodesic laminations). We define the *length* of F with respect to the hyperbolic metric g, which we denote by $L(g,F)$, as the total mass on S of the product measure dt x dl , where dt is the transverse measure of F and dl is the Lebesgue measure along the leaves of F.

One way of making the above definition more explicit is the following:

By compactness, we can cover the support of the foliation or the lamination with a finite number of rectangles (flow boxes) of disjoint interiors, where if such a rectangle is parametrized by IxI (where I is an interval), the induced foliation (or lamination) is the horizontal product foliation on IxB, where B is a closed subset of the interval I. (In the case where F is a foliation, B is equal to the whole interval I.) Now, computing $L(g,F)$ is just a matter of integrating a product measure on the rectangles and adding the results. It is easy to see that the definition does not depend on the choice of the cover.

Recall that for every F as above, we denote by $l(g,F)$ the length of the unique measured geodesic lamination representing F.

If C is a simple closed curve which is not homotopic to a point, let γ be the closed geodesic on S which represents it. It is a classical result that the length of C is bounded below by the length of γ. We shall need a generalization of this result to the case of a measured foliation F; and this is the following:

(3.3) **Proposition** *Let F be a measured foliation, or a partial measured foliation, or a measured lamination on the surface S. Then we have:*

$$l(g,F) \leq L(g,F).$$

Proof. For the proof, we shall suppose that F is a measured foliation. The cases of a partial measured foliation and of a lamination can be handled in the same way. We shall prove first the inequality in a special case and then deduce the general case by a density argument.

Suppose to begin with that all the leaves of F are closed leaves, so that F is the union of a finite number of cylinders $D_1, \ldots, D_n$ foliated by parallel circles, the interiors of the cylinders being disjoint. Let $C_1, \ldots, C_n$ be the corresponding homotopy classes, and $\gamma_1, \ldots, \gamma_n$ be the geodesics representating them.

For each of the cylinders D_i, we know that the length of a closed leaf is bounded below by the length of the geodesic γ_i. Now if h_i is the hight of D_i (i.e. the transverse measure of an arc joining the two boundary components of the cylinder and transverse to the foliation), the length of this foliated cylinder (w.r.t. the product measure of the Lebesgue measure along the leaves with the transverse measure) is therefore bounded below by the quantity $h_i.l(\gamma_i)$. Therefore, L(g,F) is bounded below by the sum $\Sigma_i h_i.l(\gamma_i)$, which is equal precisely to the length of the measured geodesic lamination representing F.

Now that we are done with the particular case, let F be any measured foliation on S.

We claim that there exists a sequence F_n of measured foliations which has the following 3 properties:

- For every n, the foliation F_n has all its leaves closed

-When $n \to \infty$, F_n converges to F for the topology of **MF**

-When $n \to \infty$, $L(g,F_n)$ converges to $L(g,F)$.

There are many ways of proving the existence of such a sequence F_n; one of them uses the machinery of train tracks (explained by Thurston in [9]), and we can describe it as follows :

We can see the foliation F as supported on the fibred neighborhood of a train track, and where each complementary region of that neighborhood has been collapsed onto a spine. Thus, the foliation F appears as a union of rectangles, each rectangle foliated by (say) horizontal leaves, the interiors of the rectangles being disjoint. These rectangles are in natural one-to-one correspondence with the edges of the train track, and there is a system of positive weights on the train track which is induced by F, where the height of the foliation induced on each rectangle is equal to the weight on the corresponding edge of the train track. Now we can approximate this system of weights on the train track by a sequence of rational systems of weights, which represents a sequence F_n of measured foliations each of which has all its leaves closed, and which converges to F in the topology of **MF**. Furthermore, we can choose the sequence of representatives to converge geometrically to F in the train track neighborhood, so that each foliation in the sequence is a union of foliated rectangles (the same rectangles as for F), with the length of each rectangle w.r.t. the foliation induced by F_n converging to the length of the rectangle w.r.t.

the foliation induced by F. From this last property, we deduce that $L(g,F_n)$ converges to $L(g,F)$ when n goes to infinity.

On the other hand, we know by the continuity of the geodesic length function on the product **TxMF** (see (3.1)) that $l(g,F_n)$ converges to $l(g,F)$ when n tends to infinity.

Finally, by the particular case proven above, we have $l(g,F_n) \leq L(g,F_n)$.

By making n tend to infinity in the last inequality, we obtain that $l(g,F) \leq L(g,F)$, which proves the proposition.

We shall also make use of the following few facts about the geometric intersection function, which are easy to prove. First, we make a definition:

(3.4) For $i = 1$ and 2, let F_i be either a measured foliation, a partial measured foliation or a measured lamination on the surface M, and suppose that F_1 is transverse to F_2 (transverse at each point where they intersect; we do not suppose that the supports are the same). We define the quantity $I(F_1,F_2)$ as the total mass on the surface of the product measure $dx_1 x\, dx_2$, where for $i = 1$ and 2, dx_i denotes the transverse measure of F_i. Then we have the following

(3.5) **Lemma** $I(F_1,F_2) \geq i(F_1,F_2)$.

(3.6) **Lemma** Suppose furthermore that there is no *Whitney disk* for the couple (F_1,F_2), that is, a disk on the surface whose boundary is the union of a segment in F_1 and a segment in F_2. Then, we have $I(F_1,F_2) = i(F_1,F_2)$.

For the proof of lemmas (3.5) and (3.6), notice first that in the case where F_1 or F_2 is a simple closed curve, a proof is contained in [1], exposé 5. The general case can be deduced by a density argument analogous to the one we have made during the proof of (3.3).

We make now the following two remarks:

(3.7) If F_1 and F_2 are transverse measured foliations with the support of each one being equal to the whole surface, then there do not exist Whitney disks.This is a consequence of the fact that there does not exist a measured foliation (with allowed singularities) on the closed disk. Therefore we have $I(F_1,F_2) = i(F_1,F_2)$.

(3.8) If μ is a maximal geodesic lamination on S and $F_\mu(g)$ an associated horocyclic foliation (for some metric g), then this couple satisfies the condition of lemma (3.6).

We prove now some more facts which will be useful in the sequel:

(3.9) **Lemma** For any hyperbolic metric g, and for any maximal geodesic lamination μ, the quantity $L(g,F_\mu(g))$ is equal to -6K, where K is the Euler characteristic of the surface.

Proof The transverse measure of the foliation $F_\mu(g)$ being a diffuse measure (with no atoms), and the lamination μ being transversely of measure zero (see Thurston [9], ch.8), we can compute the quantity $L(g,F_\mu(g))$ as the total mass of the product measure (Lebesgue measure along the leaves of $F_\mu(g)$ x transverse measure of this foliation) *in the complement of* μ.

By the Gauss-Bonnet formula, there are -2K ideal triangles in the complement of μ. All the ideal triangles being isometric, we can make the calculation in any one of them, and we take the one in the upper half-plane with vertices at the points 0, 1 and ∞. Now for the cusp of this triangle which corresponds to the point ∞, the pieces of horocycles are segments parallel to the x-axis; they start at the ordinate 1, and each piece of horocycle which has ordinate $t > 1$ has length equal to $e^{-(t-1)}$. The length of the horocyclic foliation in this region is therefore equal to $\int e^{-(t-1)}dt = 1$.

Therefore, the length of the foliation in each idal triangle is equal to 3.

We conclude that $L(g,F_\mu(g))$is equal to -6K, which proves the lemma.

In fact, we shall only use the following

(3.10) **Corollary** the quantity $l(g,F_\mu(g))$ is bounded above by a quantity which is independ of the hyperbolic metric g and of the lamination μ.

(3.11) **Definition** Given a geodesic lamination μ, with a measured foliation F transverse to it (F can be a partial measured foliation, as in the case of an $F_\mu(g)$), we shall make use now and later on of the notion of a *rectangular cover adapted to the couple* (F,μ).

By definition, this is a finite set $\{B_i\}$ of rectangles on the surface, s.t. their union contains the supports of F and of μ, s.t. for every distinct indices i and j, B_i and B_j have disjoint interiors, and for each i, F induces a "vertical" foliation on B_i, and μ induces a "horizontal" lamination on this rectangle.

Given F and μ as above, we can always find such a cover; first locally by using the product structure, and then globally, using compactness.

Note that with this definition, the "vertical" sides of each rectangles are contained in the leaves of F, whereas the "horizontal" sides are not necessarily contained in μ (they can be disjoint from μ).

(3.12) **Lemma** For any maximal measured geodesic lamination μ on the hyperbolic surface S, we have $l_g(\mu) = i(\mu,F_\mu(g))$.

Proof Let dt denote the transverse measure of the lamination μ, and let dx denote that of $F_\mu(g)$.

Consider a rectangular cover adapted to $(\mu, F_\mu(g))$.

Recall that dx coincides with the Lebesgue measure along the leaves of μ. Therefore, the *length* of the lamination induced on a given rectangle (in the sense of (3.1)) is equal to its total mass w.r.t. the product measure dt x dx. We deduce that $l_g(\mu)$ is equal to the intersection $I(\mu, F_\mu(g))$. By (3.8), this quantity is equal to $i(\mu, F_\mu(g))$, which proves (3.12).

4. Converging to Thurston's boundary

Recall that we defined the set **PMF**(μ) to be the set of all projective classes of measured foliations which can be represented by measured foliations transverse to μ. Because μ is a *maximal* geodesic lamination, this is also the set of classes of measured foliations having nonzero geometric intersection with μ.

The aim of this section is to prove the following theorem:

(4.1) **Theorem** *1. Let g_n be a sequence of elements in Teichmüller space which converges to a point in PMF(μ). Then the sequence $[F_\mu(g_n)]$ of projective classes of the associated horocyclic foliations converges to that same point.*

2. Suppose that the sequence g_n tends to infinity in Teichmüller space, with the sequence $F_\mu(g_n)$ of horocyclic measured foliations tending also to infinity (in MF). If the sequence of associated projective classes $[F_\mu(g_n)]$ converges in **PMF**, *then the sequence g_n converges also and the two limits are the same.*

The proof of the theorem is divided into several steps. First, we prove some lemmas and we state separately a few facts which are used in the proof.

(4.2) Given a maximal measured lamination μ and a real number ε, we define the set $V(\mu,\varepsilon)$ as the subset of Teichmüller space consisting of the hyperbolic metrics g for which we have $l_g(\mu) > \varepsilon$.

Let ε be a given real number, and [F] a given element of **PMF**(μ). We have the following

Lemma *The subset $PMF(\mu) \cup V(\mu,\varepsilon)$ of the compactified Teichmüller space is an open neighborhood of $[F]$ in that space.*

Proof Suppose for contradiction that g_n is a sequence of points in Teichmüller space which converges to $[F]$ and s.t. for every index n, g_n is contained in the complement of $V(\mu,\varepsilon)$. We have therefore $i(g_n, \mu) \leq \varepsilon$.

By ([1], exposé 8, corollary (2.3)), there exists a representative F of the class [F], and a sequence x_n of real numbers, with

$x_n \to 0$ and $x_n.g_n \to F$.

Therefore, we have $i(F,\mu) = \lim x_n.l(g_n,\mu) = 0$, which is a contradiction.

The lemma follows easily.

Assumption By the above lemma, in order to prove part 1 of theorem (4.1), we can make the following assumption without loss of generality:

There exists an $\varepsilon > 0$ s.t. the sequence g_n belongs to the set $V(\mu,\varepsilon)$.

(4.3) **Lemma** *Suppose that the sequence g_n converges to a point $[G]$ in $PMF(\mu)$, and let $[F_\mu(g_n)]$ be the associated sequence of projective measured foliations. Consider a converging subsequence of the sequence $[F_\mu(g_n)]$ and let $[F]$ denote its limit. Then we have $i(F,G) = 0$ (i.e. the intersection number is zero for any choice of representatives, F of [F] and G of [G]).*

Proof of (4.3) Denote by $[F_n]$ the converging subsequence.

As $g_n \to [G]$, there exists a sequence x_n of real numbers, with

$x_n \to 0$ and $x_n.g_n \to G$

As $[F_n] \to F$, there is a sequence y_n of real numbers s.t. $y_n.F_n \to F$.

Note now that $x_n.\ l(g_n,\mu) \to i(F,\mu) \neq 0$, which implies that the sequence of lengths $l(g_n,\mu)$ tends to infinity. By lemma (3.12), $l(g_n,\mu) = i(F_n,\mu)$. We deduce that the sequence y_n tends to 0.

Now we have $i(G,F) = \lim_{n \to \infty} x_n.y_n.l(g_n,F_n)$.

By corollary (3.10), the sequence $l(g_n,F_n)$ is bounded.

Therefore, $i(F,G) = 0$, which proves lemma (4.3).

The following fact is well-known, and we state it for future reference.

(4.4) Suppose that $[F_n]$ is a sequence in $\mathbf{PMF}(\mu)$ which converges to an element $[F]$ in that space. Then, we can represent $[F_n]$ and $[F]$ by measured foliations F_n and F on the surface (i.e. not only up to equivalence) s.t. the convergence $F_n \to F$ is *geometric*, which means here that the foliations have the same singular points, and around each point on the surface, the sequence of foliations F_n converges to the foliation F in the topology of line fields.

There are several ways of proving this statement (for example by using the theory of normal forms of measured foliations w.r.t. a pants decomposition of the surface). In the present context, we can deduce this statement from the fact that elements in $\mathbf{MF}(\mu)$ have geometric representatives, which are the horocyclic foliations, which have the desired property.

Actually, for a converging sequence $[F_n]$ as above, it will be convenient for us to use these horocyclic foliations as representatives which converge geometrically.

(4.5) We note now the fact that the measured geodesic lamination μ being maximal (the complementary components are all triangles) implies that every leaf of μ is bi-infinite and dense in the support of μ. This can be deduced from the corresponding (may be better-known) fact that if a measured foliation has no leaves connecting singularities, then every leaf is dense.

Given such a μ and a hyperbolic metric g, let β be a rectangular cover adapted to (F,μ). We have the following:

Lemma There exists an integer N s.t. if L is any segment in a leaf of μ which has the property that it intersects at least N times the union of (the vertical) sides of the rectangles in the cover, then L crosses at least one time each of the rectangles in the cover.

The proof can be done easily by taking a cross-section of μ and using the density of the leaves.

(4.6) Let F_n be now a sequence of elements in $\mathbf{PMF}(\mu)$ which converges to the element $[F]$ in that space, and let F_n be a sequence of representatives of $[F_n]$ by horocyclic foliations, converging geometrically to a horocyclic foliation F representing $[F]$.

Suppose that we are given a rectangular cover β adapted to (F,μ) in the sense defined in (3.11). Then the following holds:

For each F_n, we can find a sequence β_n of rectangular covers adapted to (F_n,μ) s.t. the sequence of covers β_n converges geometrically to the cover β. More precisely, we need a property of the following kind to hold:

For all integers i and j, there exists a one-to-one correspondence between the rectangles of the cover β_i and the rectangles of the cover β_j, s.t. each sequence of corresponding rectangles converges (in the Hausdorff topology on closed subsets of the surface) to a rectangle of β.

With this condition on the sequence β_n, the following fact is true as a consequence of the lemma in (4.5):

Lemma There exists an integer N s.t. for every $i = 1,2,\ldots$, and for every segment L in a leaf of μ which intersects at least N times the set of vertical sides of the rectangles of β_i, the segment L passes at least one time through each rectangle of β_i.

(4.7) We continue now the discussion begun in (4.1), with the assumption made in (4.2).

Let g_n be a sequence of elements in $V(\mu,\varepsilon)$ converging in $\mathbf{R}_+^{\mathbf{S}}$ to a point [F] in $\mathbf{PMF}(\mu)$. Consider the sequence $[F_\mu(g_n)]$ of projective classes of the associated horocyclicmeasured foliations, and let [F] be a cluster point of this sequence. We wish to prove that [F] = [G]. This will prove that the sequence $[F_\mu(g_n)]$converges to [G].

Before going on, we need to make a definition:

(4.8) **Definition** A subset S′ of S is said to be *complete* if any element of **T** or of **PMF** is completely determined by its intersection number with **S′**.

By the classification of measured foliations and of hyperbolic structures on surfaces, we know that there exists complete subsets of **S** which are finite.

We shall need also the following:

Lemma If V is any nonempty open subset of **PMF**, we can find a complete subset of S which is contained in the subsetV′ of **MF**, defined as the R_+-cone over V.

Proof One proof consists in taking a pseudo-Anosov mapping class of the surface, whose projective class of unstable foliation is contained in V. Then, by the dynamics of a pseudo-Anosov on **PMF** (see [1], exposé 12), if S′ is any finite complete subset of S, for all n large enough, $f^n(S')$ (which is also a complete subset), is contained in V′. This proves the lemma.

With the notations of (4.7), consider a subsequence of $[F_n]$ converging to [F], and to simplify notations, suppose that the subsequence is equal to the sequence itself.

In order to prove that [F] = [G], it suffices to prove the following lemma, which is an analogue of the " fundamental lemma" of [1], exposé 8.

(4.9) **Lemma** *There exists a complete subset S′ of S which has the following property:*

For every element α of S′, there exists a constant C s.t. the following is true:

For every $n = 1,2,\ldots$, we have $i(F_n,\alpha) \leq l(g_n,\alpha) \leq i(F_n,\alpha) + C$

The proof of lemma (4.9) is given below in sections (4.11) through (4.16). Let′s prove first the following:

(4.10) *Claim : lemma (4.9) implies that [F] = [G].*

Proof of the Claim As g_n converges to [G] in $\mathbf{PR_+}^S$, there exists a sequence x_n of real numbers, converging to 0, s.t. $x_n.g_n$ converges to G in $\mathbf{R_+}^S$.

By (4.9), we have, for every α in S′,

$i(x_n.F_n,\alpha) \leq x_n.l(g_n,\alpha) \leq i(x_n.F_n,\alpha) + x_n.C.$

Therefore, $| i(x_n.F_n,\alpha) - x_n.l(g_n,\alpha) | \to 0$ when $n \to \infty$, which implies that $x_n.F_n \to F$, and $[F_n] \to [F]$.

(4.11) *Proof of the first inequality of (4.9)*

The proof can be done in the same way as for the case treated in [7], where the lamination μ is finite (i.e. where the surface has cusps and the lamination is the 1-skeleton of an ideal triangulation). For the convenience of the reader, we reproduce here the main steps.

The proof applies to any element α in S.

Given the hyperbolic metric g_n on the surface, let α^* denote the closed geodesic representing the class α. (To save notations, we do not put an index to α.)

α^* is transverse to μ, since μ has no closed leaves.

Define K to be the subset of the surface S equal to the complement of the lamination μ in the support of F_n.

K is equal to a union of foliated parts of ideal triangles (where we do not include the boundary of the triangle).

Consider a connected component of the intersection of α^* with K, and let k denote its closure. k is a segment which can be of one of the following 3 types, represented respectively in figure 2 (a), (b) and (c).

-type 1: the two endpoints of k are on μ

-type 2: there is exactly one endpoint of k is on μ

-type 3: no endpoint of k is on μ.

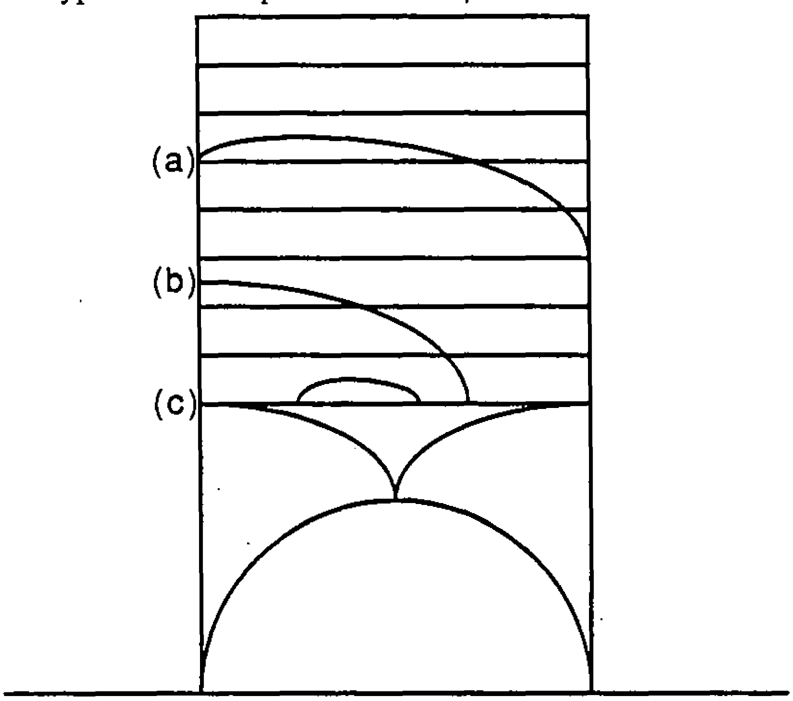


figure 2

By elementary hyperbolic geometry, the segment k can have at most one point of tangency with the foliation F_n. We can therefore modify k in the following manner:

If k is of type 1 or 2, we replace it by a segment k' having the same endpoints, and which is transverse to F_n. If it is of type 3, we push it in the nonfoliated region of the surface.

Note now that in the cases where k is of type 1 or 2, the transverse measure $I(F_n,k')$ of the segment k' is equal to the length of the projection of the segment k on one side of the ideal triangle in which it is contained (the projection being done along the leaves of the horocyclic foliation).

Again, by elementary hyperbolic geometry, the length of the projection is not greater than that of k.

Therefore, we have:

(4.11.1) $I(F_n,k') \leq l(g_n,k)$.

Let α^{**} denote the closed curve obtained out of α^* by applying the above modification to each connected component of the intersection of α^* with K.

By (4.11.1), we have $I(\alpha^{**},F_n) \leq l(g_n,\alpha)$.

Therefore, $I(\alpha, F_n) \leq l(g_n,\alpha)$, which is the desired inequality.

In sections (4.12) through (4.16) we prove the second inequality of (4.9).

(4.12) The second inequality of (4.9) is also true for any isotopy class α, but to avoid further technicalities in the proof, we shall prove it only for the following complete family: This is the family of classes α which have the property that they can be represented by simple closed curves on the surface which have minimum intersection number with F and which are transverse to this foliation. (Recall that in the general case, i.e. for a general isotopy class α, curves with minimum intersection number are made up of segments which are either transverse to F or contained in a leaf of F and joining singular points; this is discussed in [1], exposé 5).

It is easy to find a nonempty open subset of **PMF** in which the isotopy classes of curves satisfy the property we are requiring (for example, by taking a foliation which is transverse to F, and taking a sufficiently small neighborhood of its class in **PMF**). Therefore, by the lemma in (4.8), the collection of such isotopy classes constitutes a complete subset of **S**. We call this subset **S**´.

(4.13) Let α be an element of **S**´, represented by a curve α' which has minimum intersection number with F and which is transverse to that foliation.

Let β be a rectangular cover adapted to (F,μ) (in the sense of (3.11)). The curve α' is transverse to the vertical sides of the rectangles, since these are in F. By general position, we may also assume that it is transverse to the horizontal sides.

Consider now a segment k in α', which is equal to the closure of a connected component of the intersection of α with the interior of a rectangle in β. By transversality to F, the intersection pattern of the segment k with the sides of the rectangle in which it is contained is of one of the types represented in figure 3 (in which the vertical sides of the rectangles are of course, those in F).

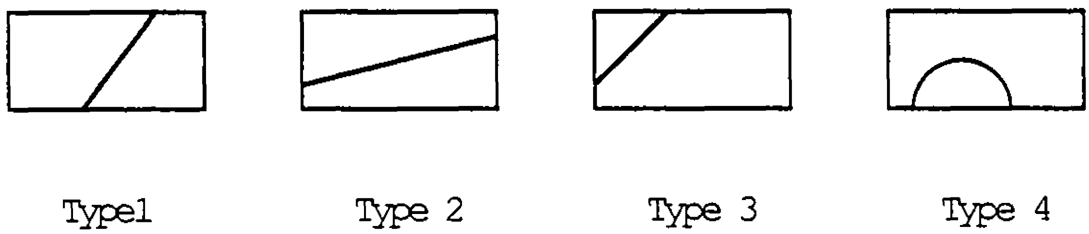


Figure 3

We perform now the following operations on k:

If k is of one of the types 1, 2 or 3, then we replace it by a segment in the same homotopy class with endpoints fixed, which is transverse to F and which is either contained in a leaf of μ or transverse to μ.

If k is of type 4, we push it along the leaves of F into the neighboring rectangle, as indicated in figure 4.

In each case, we perform the above operation without changing the intersection number of the curve with the foliation F.

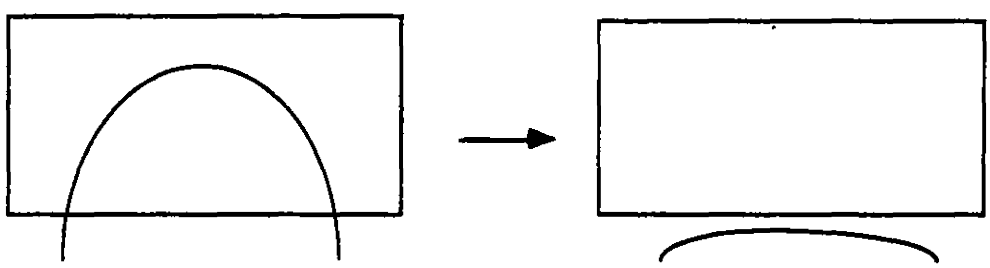


Figure 4

Note that in the case of type 4, the operation we perform reduces strictly the number of intersection points of the curve with the sides of the rectangle, so that after a finite number of steps, we can suppose that the connected components of the intersection of α' with the rectangular cover are made up of segments which are of type 1, 2 or 3, and which are either transverse to μ or contained in a leaf of μ.

(4.14) Now, following (4.4) and (4.6), as $[F_n]$ converges to $[F]$, we can consider representatives F_n which are horocyclic foliations converging geometrically to F, together with a sequence of rectangular covers β_n adapted to (F_n,μ) converging geometrically to the cover β (in the sense of (4.6)).

Given the element α in the set S′ defined in (4.12), we can suppose (by the geometric convergence of F_n to F) that for n large enough, we have a representative α_n' of α which is transverse to F_n,which has minimum intersection number with that measured foliation, that the patterns of intersection of α_n' with the rectangles of β_n are of the types 1, 2 or 3 of figure 3, and that in each rectangle, α_n' is either transverse to μ, or is a segment in μ joining the two opposite vertical sides of the rectangle. We can also suppose that for n large enough, the following two properties hold:

(4.14.1) The number of connected components of the intersection of α_n' with each of the rectangles in β_n is independent of n (we are using here the fact that the rectangles in any two covers in the sequence β_n are in one-to-one correspondence).

(4.14.2) the numbers of connected components in α_n'contained in the nonfoliated region of the surface is independent of n.

(4.15) For each n, we replace α_n' by a curve α_n'' constructed in the following way:

Let s_1 be a connected component of the intersection of α_n' with the interior of a rectangle in β_n. We replace s_1 by two consecutive segments s_2 and s_3 which are projections of s_1 on μ and F_n respectively, as indicated in figure 5. (If s_1 is contained in μ, we let $s_2 = s_1$, and s_3 is reduced to a point.)

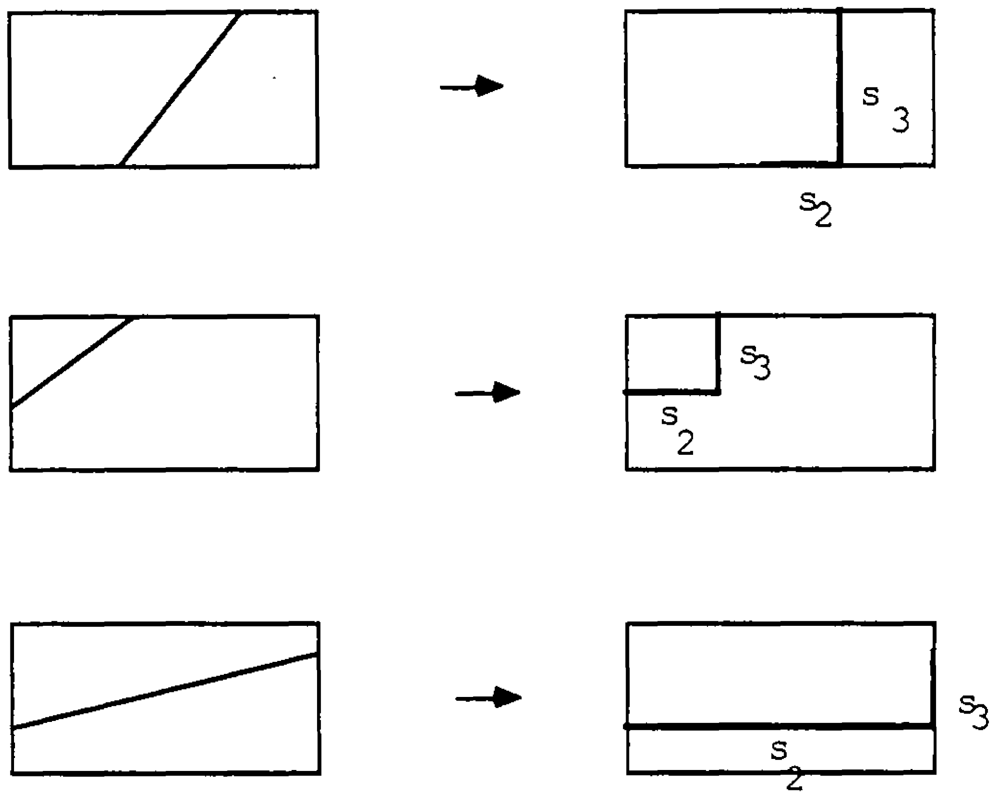


Figure 5

We have now the following

Lemma *The length of each of the segments s_3 is bounded above by a quantity which is independent of n .*

Proof Recall that by assumption, all the metrics g_n in the sequence satisfy $l(g_n,\mu) > \varepsilon$. By (3.12), for each n, this length is equal to the sum, over all rectangles in β_n, of the area of each rectangle, where the area is measured w.r.t. the product of the transverse measures of F_n and μ.

The transverse measure (w.r.t. μ) of the vertical side of a rectangle is bounded above independently of μ, since the sequences of sides converge geometrically to sides of F, and the transverse measure of μ is independent of n.

Therefore, we have the following:

(4.15.1) The sum of the F_n-measures of the horizontal sides of the rectangles in β_n is bounded below by a constant h which is independent of n.

By the lemma in (4.6), there is an integer N, which is independent of n, s.t. if a segment L in μ has at least N points of intersection with the union of the vertical sides of the cover β_n, then it crosses at least one time each rectangle in the cover F_n. By (4.15.1), L has therefore length > h.

Consider now one of the segments b_i referred to above.

The set of intersection points of s_3 with the lamination μ has measure zero (w.r.t. the Lebesgue measure on s_3). Therefore, we can calculate $l(g_n,s_3)$ as the infinite sum of the lengths of the components of the intersection of s_3 with the complement of μ, i.e. with the interiors of the ideal triangles.

There are finitely many ideal triangles, so it suffices to show that the sum of the lengths of the segments of intersection of s_3 with any one of them is bounded above independently of n.

Consider one of the ideal triangles, and look at one of the 3 foliated regions in this triangle.

The intersection of s_3 with this foliated region is an infinite union of segments of the horocyclic foliation, and we have proven that if two of these segments in a cusp of μ are separated by at least N other segments in that cusp, then they are separated by a distance at least equal to h.

By looking (as in (3.9)), in the ideal triangle in the upper half-plane with vertices at 0, 1 and ∞, to the foliated region corresponding to the point ∞, we see that if a horocyclic segment is at distance t from the nonfoliated region, its length is equal to e^{-t}.

This proves that the infinite sum converges and is bounded above by a quantity which is independent of n.

This finishes the proof of the lemma.

(4.16) We can finish now the proof of the second inequality of (4.9):

We have $l(g_n,\alpha) \leq L(g_n,\alpha_n'')$.

We write the quantity $L(g_n,\alpha_n'')$ as the sum of the following 3 quantities(using the notations of (4.15)):

-the sum of the lengths of segments of the form s_2

-the sum of the lengths of segments of the form s_3

-the sum of the lengths of the connected components of the intersection of α_n'' with the nonfoliated regions of the surface.

Note now that the number of segments involved in each of the 3 quantities above is finite and is independent of n (by (4.14.1) and (4.14.2)).

The first quantity is equal, by construction, to $i(F_n,\alpha)$.

The second quantity is bounded above independently of n, by the lemma in (4.15).

Finally, we can change each connected component of the intersection of α_n'' with the nonfoliated part of the surface, so that it has length ≤ 1 (since the diameter of such a region is bounded by 1).

Therefore, there exists a constant C, independent of n, s.t. $l(g_n,\alpha) \leq i(F_n,\alpha) + C$.

The proof of lemma (4.9) is now complete.

(4.17) We finish now the proof of theorem (4.1).

We have already completed the proof of part 1 of that theorem (see (4.7)).

To prove part 2, suppose that the sequence g_n tends to infinity in **T**, and suppose that the sequence $F_\mu(g_n)$ tends to infinity in **MF**, with $[F_\mu(g_n)]$ converging to a point [F] in **PMF**(μ).

Let [G] be a cluster point of the sequence g_n.

By the proof of lemma (4.3), we have also $i(G,F) = 0$, so that G is also in **PMF**(μ). By the part of the theorem we have already proven, we have [F] = [G].

Therefore, the whole sequence g_n converges to [F].

The proof of theorem (4.1) is now complete.

5. Remarks on stretch lines in Teichmüller space

There is a family of lines in Teichmüller space which arises naturally in terms of the parameters of that space associated to the lamination μ, and this family is particularly interesting.

One of the properties of the lines in this family is that they are geodesic lines for a metric on Teichmüller space which is defined by Thurston in his paper [10].

We will not recall the definition of the metric since we do not need it; we refer the interested reader to the paper [10]. WE shall prove a property of the behaviour at infinity of these lines.

To see what these lines are, recall that the set $\mathbf{MF}(\mu)$ of equivalence classes of measured foliations transverse to μ has a natural cone structure, and define a *positive ray* in that space to be a one-parameter family of measured foliation classes of the form $(x.F)_{x\geq 0}$, where F is an element of $\mathbf{MF}(\mu)$, and where this ray is equipped with the positive orientation induced by the real numbers. The image of such a ray by the map ϕ_μ (defined in theorem 2.1), is called a *stretch line* in Teichmüller space, and is a geodesic for Thurston's metric that we referred to above.

Note It is important to specify the orientation in the definition of a stretch line, because this line equipped with the opposite orientation is not a stretch line, and is not a geodesic (the metric is nonsymmetric).

When μ is a maximal measured geodesic lamination, we have, as an immediate consequence of theorem (4.1), the following result on stretch lines:

(5.1) **Theorem** Any stretch line which is the image by the map ϕ_μ of a ray of the form $(x.F)_{x\geq 0}$ converges to the point [F] on the boundary of Teichmüller space.

Remark Thurston's theorem (2.1) is valid when μ is any maximal geodesic lamination (not necessarily equipped with a transverse measure of full support). But to prove theorem (4.1), we had to suppose that μ was a measured geodesic lamination; we made an assumption about the length of μ not being too small, and we used the transverse measure of full support. However, theorem (5.1) is true for any maximal geodesic lamination μ, and to prove this, we can prove the double inequality in lemma (4.9) in the case where the sequence of hyperbolic metrics goes to infinity on a stretch line, without the hypothesis on the existence of a transverse measure for μ; we can just follow step by step the proof done in sections (4.11) to (4.16), and what makes things easier here is that the sequence of foliations F_n is the same as F, except that its transverse measure is being multiplied by a sequence of real numbers going to infinity.

(5.2) Since we have been talking about stretch lines, it is natural to ask what is their limiting behaviour when we endow them with the opposite orientation.

Again, these lines are defined for any maximal geodesic lamination μ, and their definition does not use a transverse measure on the lamination.

Recall that without a transverse measure, μ does not define an element on the boundary of T. It can, for example, admit more than one proportionality class of transverse measures (this is the case of a nonuniquely ergodic lamination), or no transverse measure at all (like in the case of a punctured surface, with the lamination μ having a finite number of leaves, all converging to cusps; recall here that in the case of punctured surfaces, in order for a transverse measure for μ to define a point on the boundary of Teichmüller space, it has to be of compact support.) But in all these cases, the mapping ϕ_μ still makes sense.

There is a case where we can assure that these "anti-stretch" lines have a limit point on the boundary of Teichmüller space, and this is the case where μ is a uniquely ergodic measured lamination. In that case, we have the following

(5.2.1) **Proposition** *For a uniquely ergodic μ, every anti-stretch line converges to the point on the boundary of Teichmüller space defined by the class $[\mu]$ of μ.*

Before proving this proposition, we prove the following lemma

(5.2.2) **Lemma** Let μ be uniquely ergodic maximal geodesic lamination, and let g_n be a sequence of hyperbolic metrics going to infinity. If the sequence of lengths of μ w.r.t. g_n is bounded, then g_n converges to $[\mu]$ in the topology of $\mathbf{PR_+S}$.

Proof of the lemma Let [F] be a cluster point of the sequence g_n in **PMF**. Then there exists a representative F of this class, with a sequence x_n of real numbers converging to 0 s.t. the sequence $x_n.g_n$ converges to F in the topology of $\mathbf{R_+S}$.

We have $i(F,\mu) = \lim x_n.l(g_n,\mu) = 0$. The fact that μ is uniquely ergodic implies now that $[F] = [\mu]$ (see Masur [4]), and therefore g_n converges to [F].

Proof of proposition (5.3.1) Let g_n be a sequence of elements of T, associated by the map ϕ_μ to a sequence of elements F_n in MF(μ), where $F_n = 1/y_n.F$, with y_n a sequence of positive reals converging to infinity.

The sequence g_n goes to infinity in T, for by lemma (3.12), we have

$l(g_n,\mu) = i(\mu,1/y_n.F) = 1/y_n.i(\mu,F)$ which tends to 0.

The fact that $l(g_n,\mu)$ is bounded above implies also, by lemma (5.3.1), that g_n converges to μ.

(5.3.3) *Remarks*

1. For the proof of proposition (5.3.1), we could not appeal to theorem (4.1), because, although the sequence of metrics goes to infinity, the sequence of horocyclic foliations does not go to infinity in **MF**, but to 0.

2. In the case treated in [7], where the surface has cusps and where the lamination μ is finite, the behaviour of the anti-stretch lines is completely different. In this case (where μ does not have compactly supported transverse measures), the lines do not converge to infinity but to a point in Teichmüller space which corresponds to a "symmetric gluing of the ideal triangles", using the terminology of Thurston in [10],§ 9.4. As Thurston points out, in that case the cone MF(μ) is naturally a vector space, with the origin corresponding to this point of symmetric gluing.

6. The extension of the earthquake flow

(6.1) Throughout the rest of this paper, $\mathcal{E}_\mu$ will denote the *normalized* earthquake flow associated to the maximal measured lamination μ, in the sense defined in the introduction.

In [10], Thurston gives a description of this flow in terms of the parameters of Teichmüller space associated to μ which we have been working with. This description is simple, in the sense that it does not involve taking limits of earthquake flows along simple closed curves. On the other hand, using our theorem (4.1), we shall show that the flow on **MF** which appears naturally in this description gives rise to a flow on **PMF** which extends continuously the earthquake flow $\mathcal{E}_\mu$.

For the convenience of the reader, we shall recall Thurston´s description of the earthquake flow that we are referriong to; this involves the notion of *shear coordinates* for the space **MF**(μ) of equivalence classes of measured foliations transverse to μ, on which we make some digresssion.

(6.2) Let F be a measured foliation transverse to μ, and τ a train track on the surface S, which is an ε-approximation of μ (in the sense of [9], ch. 8).

If the approximation is fine-enough, we can represent τ by a train track which is transverse to F, since μ is itself transverse to F.

Following Thurston [10], § 9, the measured foliation F induces a system of weights τ on , which are called the *shear coordinates* of F, and which are defined in the following manner:

Let e be an edge of τ. To each one of the two sides of e on the surface, there is naturally associated a component of S-τ. This component is a triangle, and by the transversality of F to τ, it is easy to see that each such triangle contains exactly one singular point of F, which is a 3-prong singularity, and furthermore, if we consider the foliation induced by F on this region, for each one of the 3 sides of this region, there is a leaf issuing from the singular point that hits this side.

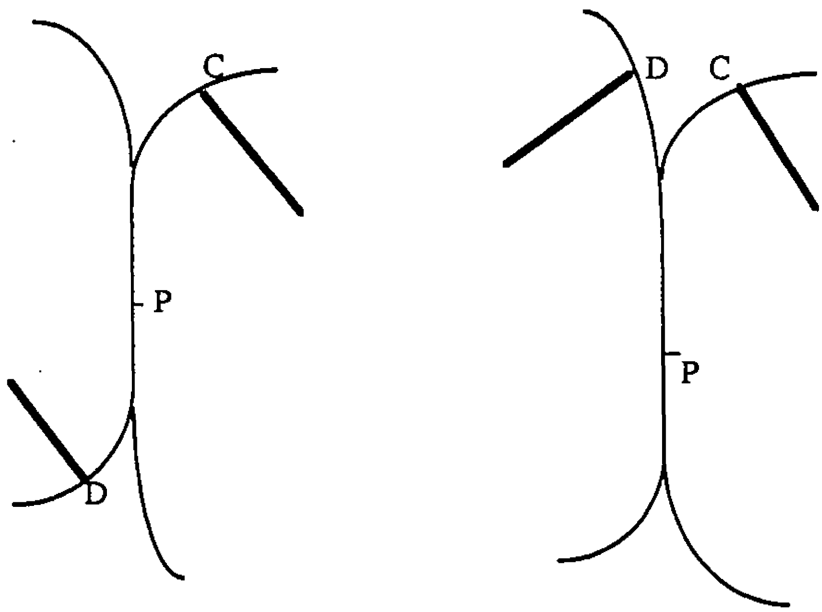
Let V_1 and V_2 be the complementary regions of τ associated to the edge e. (Note that V_1 may be equal to V_2.) The edge e is contained in a *side* of each of these components (a side being a union of edges joining two cusps), and on such a side, there is a distinguished point , which is the point where the leaf issuing from the singularity of F contained in that component hits this side.

Now for the edge e, there is a real number s(e) which we can associate to it, which is the algebraic distance between the hitting points on each side of this edge, with the rule that left shears are counted positively, and right shears negatively. Because the two hitting points are not neccesarily joined by a smooth arc in the train track, we have to be more precise in the definition. For this, we choose a reference point P on the side e, and we measure the shear coordinates on each side of e w.r.t. that reference point. The sign convention is the one that is used usually to define left earthquakes or left Fenchel-Nielsen deformations. Note that the definition does not involve the choice of an orientation on the edge e. For the sign convention, we refer to figure 6.

Up to symmetry, there are 2 cases, and these 2 cases are represented in figures 6 (a) and 6(b).

In figure 6, P is the reference point on the edge e, and C and D are the hitting points on the corresponding sides (the leaves of F are represented in fat lines).

In figure 6 (a), the shear coordinate induced by F on the edge e is equal to the sum x + y , where x and y are respectively the transverse measures of the arcs PC and PD. In figure 6 (b), the shear coordinate is equal to the difference x - y.



(a) (b)

Figure 6

(6.3) According to Thurston ([10], § 9), we can define the earthquake flow $\mathcal{E}_\mu$ on Teichmüller space by transporting, *via* the homeomorphism ϕ_μ defined in section (2.1), a flow h_μ defined on the space $\mathbf{MF}(\mu)$. In other words, the deformation of hyperbolic structures is defined *via* a deformation of the associated horocyclic foliations.

To define the flow h_μ, let F be an element of **MF**(μ). Then as before, we consider a train track τ which is an ε-approximation of μ, and which is transverse to F. F induces a system of shear coordinates on the edges of τ.

The measured lamination μ also induces a system of weights on the edges of τ, which we shall call the *transverse coordinates* of μ, to distinguish them from shear coordinates, and which are the set of weights that the lamination μ induces on the fibres of a regular neighborhood of μ. (So these are the usual train track coordinates, defined in [9].)

We can now define he flowline $h_\mu{}^t\,(F)_{t\in \mathbb{R}}$ passing through the point F:

(6.3.1) The element $F^t = h_\mu{}^t\,(F)$ in **MF**(μ) is defined by its shear coordinates on τ. For any edge e of this train track, if s(a) is the shear coordinate of F on e, and x(e) the transverse coordinate of μ, then the shear coordinate of F^t on e is equal to the quantity $s(e) + t.x(e).i(\mu,F)$. This defines the element F^t of **MF**(μ).

We note the following 2 facts:

(6.3.2) The quantity $i(\mu,F)$ is equal (by (3.12)) to the length of μ w.r.t. the initial metric (at time $t = 0$). It is also equal to the length of μ w.r.t. the metric at any time t, since the length of μ is invariant under the earthquake flow. Therefore, it is also equal to $i(\mu,F^t)$.

(6.3.3) In the desription of the earthquake flow given by Thurston in ([10], § 9), the term $i(\mu,F)$ does not appear in the formula giving the new shear coordinates. This is because he is considering the non-normalized earthquake flow.

Now we prove an extension property of the flow h_μ on measured foliations space:

(6.4) **Proposition** *The flow h_μ extends to a continuous flow defined on the whole space MF, which is the identity flow on the complement of the space MF(μ).*

Proof From the description in local parameters, it is clear that on the space **MF**(μ), the flow is continuous. So what we have to prove is the continuous extension to the complement of this set.

Consider the *non-normalized* flow defined on the set **MF**(μ), i.e. the one where at time t, the new shear coordinate (using the notations of ((6.3.1)) is equal to $s(e) + t.x(e)$. This non-normalized flow has constant derivative in the whole coordinate system associated to the train track τ. This shows in particular that the flow h_μ is a Lipschitz flow on the whole space **MF**(μ).

On the other hand, the intersection function $i(\mu,.)$ defined on the whole space **MF** is continuous and takes the value zero on the complement of the space **MF**(μ). From this, we deduce that the normalized flow h_μ extends continuously by the identity flow to the complement of $\mathbf{MF}(\mu)$.

Remark This flow h_μ which is a generalized twist flow on **MF**, was studied in [6].

The extended flow on **MF** is homogeneous, and defines therefore a quotient flow on the space **PMF** which is continuous, and which we shall denote by $\mathcal{H}_\mu{}^t$.

The zero set of $\mathcal{H}_\mu{}^t$ is equal to the complement in **PMF** of the set $\mathbf{PMF}(\mu)$.

We can now prove the following

(6.5) **Theorem** *The normalized earthquake flow $\mathcal{E}_\mu$ admits a continuous extension to Thurston's boundary of Teichmüller space, and the induced flow on the boundary is the flow $\mathcal{H}_\mu{}^t$.*

Proof Let g_n be a sequence of points in **T** converging to the point [F] in **PMF**.

Suppose first that [F] is in $\mathbf{PMF}(\mu)$.

By the first part of theorem (4.1), we know that the sequence $[F_\mu(g_n)]$ of projective classes of the associated horocyclic foliations converges also to [F].

Let t be an arbitrary real number.

By the continuity of the flow $\mathcal{H}_\mu$ on **PMF** (see (6.4)), the sequence $\mathcal{H}_\mu{}^t[F_\mu(g_n)]$ converges to the point $\mathcal{H}_\mu{}^t[F]$.

Therefore, $[h_\mu{}^t(F_\mu(g_n))]$ converges to $\mathcal{H}_\mu{}^t[F]$.

Note now that by (6.3), for every n, $h_\mu{}^t(F_\mu(g_n))$ is the horocyclic foliation of the hyperbolic structure $\mathcal{E}_\mu{}^t(g_n)$.

Note also that as the sequence g_n converges to the class [F], with $i(F,\mu) \neq 0$, the sequence $l(g_n,\mu)$ tends to infinity. But $l(g_n,\mu) = i(F_\mu(g_n),\mu)$, by lemma (3.12).

Therefore, the sequence $F_\mu(g_n)$ tends to infinity in **MF**.

By the continuity of the flow h_μ on **MF**, the sequence $h_\mu{}^t(F_\mu(g_n))$ also tends to infinity in **MF**.

Therefore, we can use the second part of theorem (4.1) to conclude that the sequence $\mathcal{E}_\mu{}^t(g_n)$ converges to the point $\mathcal{H}_\mu{}^t[F]$.

We deal now with the case where the point [F] is not in $\mathbf{PMF}(\mu)$.

We have a sequence x_n of positive real numbers that converges to 0, with

(6.5.1) $\quad x_n.g_n \to F$ (in the topology of $\mathbf{R}_+{}^{\mathbf{S}}$).

In particular, we have

(6.5.2) $\quad x_n.l(g_n,\mu) \to i(F,\mu) = 0$, when $n \to \infty$.

Now note that for any real number t, and any element α in the set S, we have the following inequality:

$$| l(\mathcal{E}_\mu{}^t(g_n), \alpha) - l(g_n, \alpha) | \leq | t | . l(g_n,\mu). i(\mu,\alpha).$$

To see that the above inequality is true, note that it is true, by [5], if μ, instead of being a minimal measured geodesic lamination, was a simple closed geodesic (and so the flow would be a Fenchel-Nielsen twist flow). The case where μ is a measured lamination follows then by the density of weighted simple closed geodesics in measured laminations space and the fact that the earthquake flow is equal to the limit of the twist flows along a sequence of weighted geodesics approximating the lamination.

Therefore, we have also

$$| x_n. l(\mathcal{E}_\mu{}^t(g_n), \alpha) - x_n.l(g_n,\alpha) | \leq x_n.| t | . l(g_n,\mu). i(\mu,\alpha).$$

Now using (6.5.1) and (6.5.2), we have

$$| x_n.l(\mathcal{E}_\mu{}^t(g_n), \alpha) - i(F,\alpha) | \to 0 \text{ as } n \to \infty.$$

Therefore, $x_n.\mathcal{E}_\mu{}^t(g_n) \to F$ in $\mathbf{R}_+{}^{\mathbf{S}}$,and the sequence $\mathcal{E}_\mu{}^t(g_n)$ converges to [F] in $\mathbf{PR}_+{}^{\mathbf{S}}$.

This completes the proof of theorem (6.5)

References

[1] A. Fathi, F. Laudenbach and V. Poénaru, *ed.* Travaux de Thurston sur les Surfaces, Astérisque 66-67, S.M.F. 1979

[2] S. P. Kerckhoff The asymptotic geometry of Teichmüller space, Topology 19 (1980) p. 23-41

[3] S. P. Kerckhoff, Earthquakes are analytic, Comment. Math. Helvetici 60 (1985) p. 17-30

[4] H. Masur, Two boundaries of Teichmüller space, Duke Math. Jour. 49 (1982) p.183-190

[5] A. Papadopoulos, L´extension du flot de Fenchel-Nielsen au bord de Thurston de l´espace de Teichmüller, Comptes Rendus Acad. Sc. Paris t. 302, série 1 (1986), p. 325-327

[6] A. Papadopoulos, Geometric intersection functions and hamiltonian flows on the space of measured foliations on a surface . Pac. Jour. of Math. Vol 124 No. 2 (1986) p.375-402

[7] A. Papadopoulos, Sur le bord de Thurston de l'espace de Teichmüller d'une surface non compacte, Mathematische Annalen ,1988

[8] M. Rees, An alternative approach to the ergodic theory of measured foliations on surfaces, Erg. Theory and Dyn. Syst. 1 (1981) p.461-488

[9] W. P. Thurston, The geometry and Topology of 3-manifolds, Lecture notes, Princeton University,1976-77

[10] W. P. Thurston, Minimal stretch maps between hyperbolic surfaces, preprint 1985 (to appear)

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