Representations of Algebraic Groups I

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Preliminary Foreword

In August and September 1984 I delivered a series of lectures on the representation theory of reductive groups at the East China Normal University in Shanghai. I had been asked before to prepare some lecture notes. This is the first part of a revised and extended version of the notes I had written for those lectures.

When writing down the first version I had not yet decided about the precise contents of my lectures. Therefore I included more than I could hope to cover in the lectures (and than I did cover). On the other hand, some parts of the theory about which I lectured (the relationship with the representations of finite groups) were not covered in the written notes as I had not had enough time before leaving for Shanghai.

In this revised version of my notes I intend to deal also with these matters missing in the original version. The first part written up so far contains the general theory of group schemes and their representations upto the amoung that appears to be necessary for the more concrete representation theory of reductive groups and of their most important subgroups (like Borel subgroups, Frobenius kernels, finite groups of Lie type).

This first part contains an introduction to some fundamental concepts in the theory of algebraic groups as schemes, group schemes, quotients, factor groups, algebras of distributions, Frobenius kernels. The main source in these matters is [SGA 3] to which one should add [T] in the case of algebras of distributions.

The books [DG] and [W] contain more accessible approaches to the theory. I have tried to be understandable to someone who has not read these books but is familiar with linear algebraic groups (in the sense of [Bo],[Hu],[Sp]) and fundamental notions of commutative algebra (especially flatness) and of algebraic geometry. So I give all the definitions and indicate the proofs where they are not too involved but give a feeling what standard arguments look like. In the case of deeper results I usually refer to [DG] from where I usually take also the terminology.

Furthermore this first part contains an introduction to the main tools in the representation theory of algebraic group schemes like induction, injective modules, cohomology, associated sheaves, reduction mod p and to some special aspects of the representation theory of finite group schemes. Most things done here are generalizations of constructions in the representation theory of (finite) abstract groups and of Lie groups. I have therefore usually not tried very hard to trace all sources and to attribute priorities.

The list of references is divided into three parts. The first one contains text books on algebraic groups and related topics to which is referred by a letter code like [Bo]. The second part contains research articles on representations of algebraic groups and related topics to which is referred by the family name of the author(s) like [Curtis 2]. The last part contains references from other areas of mathematics to which is referred to by numbers like [3].

I should like to thank Henning Haahr Andersen for his useful

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Contents

Part I : General Theory

I General Theory

1. Schemes

The reader is assumed to be familiar with the theory of algebraic groups as exposed in the text books by Borel, Humphreys and Springer (i.e. in [Bo],[Hu],[Sp]). He also ought to know the fundamental properties of varieties as to be found in these books. Though we are interested mainly in representations of such groups it will be necessary to look at more general objects, called group schemes, and similarly at the corresponding objects in algebraic geometry, called schemes.

It is the purpose of this first chapter to give the necessary introduction to schemes following the functorial approach of [DG]. This approach appears to be most suitable when dealing with group schemes later on. After trying to motivate the definitions in 1.1 we discuss affine schemes in 1.2 - 1.6. What is done there is fundamental for the understanding of everything to follow. As far as arbitrary schemes are concerned it is most of the time enough to know that they are functors with some properties so that all affine schemes are functors and so that over an algebraically closed field any variety gives rise to a scheme in a canonical way. Sometimes, e.g. when dealing with quotients, it is useful to know more. So we give the appropriate definitions in 1.7 - 1.9 and mention the comparison with other approaches to schemes and with varieties in 1.11. The elementary discussion of a base change in 1.10 is again necessary for many parts later on.

A ring or an associative algebra will always be assumed to have a 1 and homomorphisms are assumed to respect this 1. Let k be a fixed commutative ring. Notations of linear algebra (like Hom, $\otimes$) without special reference to a ground ring always refer to structures as k-modules. A k-algebra is always assumed to be commutative and associative. (For non-commutative algebras we shall use the terminology: algebras over k.)

1.1 Before giving the definitions I want to point out how functors arise naturally in algebraic geometry. Assume for the moment that k is an algebraically closed field.

Consider a Zariski closed subset X of some k^n and denote by I the ideal of all polynomials $f \in k[T_1,T_2,\ldots,T_n]$ with $f(X) = 0$. Instead of looking at the zeroes of I only over k we can look also at the zeroes over any k-algebra A, i.e. at $\underline{\underline{X}}(A) = \{x \in A^n | f(x) = 0 \text{ for all } f \in I\}$. The map $A \mapsto \underline{\underline{X}}(A)$ from {k-algebras} to {sets} is a functor: Any homomorphism $\varphi\colon A \to A'$ of k-algebras induces a map $\varphi^n\colon A^n \to (A')^n, (a_1,a_2,\ldots,a_n) \mapsto (\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n))$ with $f(\varphi^n(x)) = \varphi(f(x))$ for all $x \in A^n$ and $f \in k[T_1,\ldots,T_n]$. Therefore φ^n maps $\underline{\underline{X}}(A)$ to $\underline{\underline{X}}(A')$. Denote its restriction by $\underline{\underline{X}}(\varphi)\colon \underline{\underline{X}}(A) \to \underline{\underline{X}}(A')$. For another homomorphism $\varphi'\colon A' \to A''$ of k-algebras one has obviously $\underline{\underline{X}}(\varphi')\underline{\underline{X}}(\varphi) = \underline{\underline{X}}(\varphi' \circ \varphi)$ proving that $\underline{\underline{X}}$ is indeed a functor.

A regular map from X to a Zariski closed subset Y of some k^m is given by m polynomials $f_1,f_2,\ldots,f_m \in k[T_1,T_2,\ldots,T_n]$ as $f\colon X \to Y, x \mapsto (f_1(x),f_2(x),\ldots,f_m(x))$. The f_i define for each k-algebra A a map $f(A)\colon A^n \to A^m, x \mapsto (f_1(x),\ldots,f_m(x))$. The comorphism $f(k)^*\colon k[T_1,\ldots,T_m] \to k[T_1,\ldots,T_n]$ maps the ideal defining Y into the ideal I defining X. This implies that any $f(A)$ maps $\underline{\underline{X}}(A)$ into $\underline{\underline{Y}}(A)$. The family of all $f(A)$ defines a morphism $\underline{\underline{f}}: \underline{\underline{X}} \to \underline{\underline{Y}}$ of functors, i.e. a natural transformation. The more general discussion in 1.3 (cf. 1.3(2)) shows that the map $f \mapsto \underline{\underline{f}}$ is bijective (from {regular maps $X \to Y$} to {natural transformations $\underline{\underline{X}} \to \underline{\underline{Y}}$}).

Taking this for granted we have embedded the category of all affine algebraic varieties over k into the category of all functors from {k-algebras} to {sets} as a full subcategory. This embedding can be extended to the category of all algebraic varieties, see 1.11.

One advantage of working with functors instead of varieties (i.e. of working with $\underline{\underline{X}}$ instead of X) will be that it gives a natural way to work with "varieties" over other fields and also over rings. Furthermore we get also over k (algebraically closed) new objects in a natural way. Instead of working with I we might have taken any ideal $I' \subset k[T_1,\ldots,T_n]$ defining X, i.e. with $X = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in I'\}$ or, equivalently by Hilbert's Nullstellensatz, with $\sqrt{I'} = I$. Replacing I by I' in the definition of $\underline{\underline{X}}$ we get a functor, say $\underline{\underline{X}}'$, with $\underline{\underline{X}}'(A) = \underline{\underline{X}}(A)$ for each field extension $A \supset k$ (or even each integral domain), but with $\underline{\underline{X}}(A') \neq \underline{\underline{X}}(A)$ for some A if $I \neq I'$. Such functors arise in a natural way even when we deal with varieties and they play an important role in representation theory.

Before giving the proper definitions let us describe the functor $\underline{\underline{X}}$ without using the embedding of X into k^n. For each k-algebra A we have a bijection $\mathrm{Hom}_{k\text{-alg}}(k[T_1,T_2,\ldots,T_n],A) \to A^n$, sending any α to $(\alpha(T_1),\alpha(T_2),\ldots,\alpha(T_n))$. The inverse image of $\underline{\underline{X}}(A)$ consists of those α with $0 = f(\alpha(T_1),\ldots,\alpha(T_n)) = \alpha(f)$ for all $f \in I$, hence can be identified with $\mathrm{Hom}_{k\text{-alg}}(k[T_1,T_2,\ldots,T_n]/I,A)$. As $k[T_1,T_2,\ldots,T_n]/I$ is the algebra $k[X]$ of regular functions on X we have thus a bijection $\underline{\underline{X}}(A) \simeq \mathrm{Hom}_{k\text{-alg}}(k[X],A)$. If $\varphi\colon A \to A'$ is a homomorphism of k-algebras, then $\underline{\underline{X}}(\varphi)$ corresponds to the map $\mathrm{Hom}_{k\text{-alg}}(k[X],A) \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k[X],A')$ with $\alpha \to \varphi\circ\alpha$. A morphism $f\colon X \to Y$ is given by its comorphism $f^*\colon k[Y] \to k[X]$. Then the morphism $\underline{\underline{f}}\colon \underline{\underline{X}} \to \underline{\underline{Y}}$ is given by $\underline{\underline{f}}(A)\colon \mathrm{Hom}_{k\text{-alg}}(k[X],A) \to \mathrm{Hom}_{k\text{-alg}}(k[Y],A)$, $\alpha \mapsto \alpha\circ f^*$ for any k-algebra A.

<u>1.2</u> (k-<u>functors</u>). Let us assume k to be arbitrary again. In the definitions to follow we shall be rather careless about the foundations of mathematics. Instead of working with "all" k-algebras we should (as in [DG]) take only those in some universe. We leave the appropriate modifications to the interested reader.

A k-<u>functor</u> is a functor from the category of k-algebras to the category of sets.

Let X be a k-functor. A <u>subfunctor</u> of X is a k-functor Y with $Y(A) \subset X(A)$ and $Y(\varphi) = X(\varphi)|_{Y(A)}$ for all k-algebras A,A' and all $\varphi \in \mathrm{Hom}_{k\text{-alg}}(A,A')$.

Obviously a map Y which associates to each k-algebra A a subset $Y(A) \subset X(A)$ is a subfunctor if and only if $X(\varphi)Y(A) \subset Y(A')$ for each homomorphism $\varphi\colon A \to A'$ of k-algebras.

For any family $(Y_i)_{i\in I}$ of subfunctors of X we define their intersection $\bigcap_{i\in I} Y_i$ through $(\bigcap_{i\in I} Y_i)(A) = \bigcap_{i\in I} (Y_i(A)$ for each k-algebra A. This is again a subfunctor. The obvious definition of a union is not the useful one, so we shall not denote it by $\bigcup_{i\in I} Y_i$.

For any two k-functors X, X' we denote by $\mathrm{Mor}(X,X')$ the set of all morphisms (i.e. natural transformations) from X to X'. For any $f \in \mathrm{Mor}(X,X')$ and any subfunctor Y' of X' we define the inverse image $f^{-1}(Y')$ of Y' under f through $f^{-1}(Y')(A) = f(A)^{-1}(Y'(A))$ for each k-algebra A. Clearly $f^{-1}(Y')$ is a subfunctor of X. (The obvious definition of an image of a subfunctor is not the useful one.) Obviously f^{-1} commutes with intersections.

For two k-functors X_1, X_2 the direct product $X_1 \times X_2$ is defined through $(X_1 \times X_2)(A) = X_1(A) \times X_2(A)$ for all A. The projections $p_i\colon X_1 \times X_2 \to X_i$ are morphisms and $(X_1 \times X_2, p_1, p_2)$ has the usual universal property of a direct product.

For three k-functors X_1, X_2, S and two morphisms $f_1\colon X_1 \to S$, $f_2\colon X_2 \to S$ the fibre product $X_1 \times_S X_2$ (relative f_1, f_2) is defined through

$$(X_1 \times_S X_2)(A) = X_1(A) \times_{S(A)} X_2(A) = \{(x_1, x_2) \mid x_1 \in X_1(A), x_2 \in X_2(A),$$
$$f_1(A)(x_1) = f_2(A)(x_2)\}.$$

The projections from $X_1 \times_S X_2$ to X_1 and X_2 are morphisms and $X_1 \times_S X_2$ together with these projections has the usual universal property of a fibre product. Of course we may also regard $X_1 \times_S X_2$ as the inverse image of the diagonal subfunctor $D_S \subset S \times S$ (with $D_S(A) = \{(s,s) \mid s \in S(A)\}$ for all A) under the (obvious) morphism $(f_1, f_2)\colon X_1 \times X_2 \to S \times S$. (On the other hand inverse images and intersections can also be regarded as special cases of fibre products.)

1.3 (Affine Schemes). For any $n \in \mathbb{N}$ the functor $\mathbb{A}^n$ with $\mathbb{A}^n(A) = A^n$ for all A and $\mathbb{A}^n(\varphi) = \varphi^n\colon (a_1,\ldots,a_n) \mapsto (\varphi(a_1),\ldots,\varphi(a_n))$ for all $\varphi\colon A \to A'$ is called the affine n-space over k. (We use sometimes also the notation $\mathbb{A}^n_k$ when it may be doubtful which k we consider.) Note that $\mathbb{A}^0$ is the functor with $\mathbb{A}^0(A) = \{0\}$ for all A. Hence there is for each k-functor X exactly one morphism from X to $\mathbb{A}^0$ (i.e. $\mathbb{A}^0$ is a final object in the category of k-functors) and that we can regard any direct product as a fibre product over $\mathbb{A}^0$.

For any k-algebra R we can define a k-functor $\mathrm{Sp}_k R$ through $(\mathrm{Sp}_k R)(A) = \mathrm{Hom}_{k\text{-alg}}(R,A)$ for all A and $(\mathrm{Sp}_k R)(\varphi)\colon \mathrm{Hom}_{k\text{-alg}}(R,A) \to \mathrm{Hom}_{k\text{-alg}}(R,A')$, $\alpha \mapsto \varphi \circ \alpha$ for all homomorphisms $\varphi\colon A \to A'$. We call $\mathrm{Sp}_k R$ the spectrum of R. Any k-functor isomorphic to some $\mathrm{Sp}_k R$ is called an affine scheme over k. (Note that the $\mathrm{Sp}_k R$ generalize the functors $\underline{\underline{X}}$ considered in 1.1.) For example the affine n-space $\mathbb{A}^n$ is isomorphic to $\mathrm{Sp}_k k[T_1,\ldots,T_n]$ (and will usually be identified with it), where $k[T_1,\ldots,T_n]$ is the polynomial ring over k in n variables $T_1,\ldots,T_n$.

We can recover R from $\mathrm{Sp}_k R$. This follows from:

Yoneda's Lemma: For any k-algebra R and any k-functor X the map $f \mapsto f(R)(\mathrm{id}_k)$ is a bijection

$$\mathrm{Mor}(\mathrm{Sp}_k R, X) \tilde{\to} X(R).$$

Indeed, take any k-algebra A and any $\alpha \in \mathrm{Hom}_{k\text{-alg}}(R,A) = (\mathrm{Sp}_k R)(A)$. As f is a natural transformation, we have $X(\alpha)\circ f(R) = f(A)\circ(\mathrm{Sp}_k R)(\alpha)$. Let us abbreviate $x_f = f(R)(\mathrm{id}_R)$. As $(\mathrm{Sp}_k R)(\alpha)(\mathrm{id}_R) = \alpha\circ\mathrm{id}_k = \alpha$, we get

(1) $$f(A)(\alpha) = X(\alpha)(x_f).$$

This shows that f is uniquely determined by x_f and indicates how to construct an inverse map. For each $x \in X(R)$ and any k-algebra A let $f_x(A)\colon (\mathrm{Sp}_k R) \to X(A)$ be the map $\alpha \mapsto X(\alpha)(x)$. Then one may check that $f_x \in \mathrm{Mor}(\mathrm{Sp}_k R, X)$ and that $x \mapsto f_x$ is inverse to $f \mapsto x_f$.

An immediate consequence of Yoneda's lemma is

(2) $$\mathrm{Mor}(\mathrm{Sp}_k R, \mathrm{Sp}_k R') \tilde{\to} \mathrm{Hom}_{k\text{-alg}}(R', R)$$

for any k-algebras R, R'. We denote this bijection by $f \mapsto f^*$ and call f^* the <u>comorphism</u> corresponding to f. As $\mathrm{Hom}_{k\text{-alg}}(k[T_1], R) \tilde{\to} R$ under $\alpha \mapsto \alpha(T_1)$ we get especially

(3) $$\mathrm{Mor}(\mathrm{Sp}_k R, \mathbb{A}^1) \tilde{\to} R.$$

For any k-functor X we denote $\mathrm{Mor}(X, \mathbb{A}^1)$ by $k[X]$. This set has a natural structure as a k-algebra and (3) is an isomorphism $k[\mathrm{Sp}_k R] \tilde{\to} R$ of k-algebras. (For $f_1, f_2 \in k[X]$ define f_1+f_2 through $(f_1+f_2)(A)(x) = f_1(A)(x)+f_2(A)(x)$ for all A and all

$x \in X(A)$. Similarly f_1f_2 and af_1 for $a \in k$ are defined). We shall usually write $f(x) = f(A)(x)$ for $x \in X(A)$ and $f \in k[X]$. Note that for $X = Sp_k R$ and $f \in R \simeq k[X]$ we have $f(x) = x(f)$ for $x \in (Sp_k R)(A) = Hom_{k-alg}(R,A)$.

The universal property of the tensor product implies immediately that a direct product $X_1 \times X_2$ of affine schemes over k is again an affine scheme over k with $k[X_1 \times X_2] \simeq k[X_1] \otimes k[X_2]$. More generally a fibre product $X_1 \times_S X_2$ with X_1, X_2, S affine schemes is an affine scheme with

(4) $k[X_1 \times_S X_2] \simeq k[X_1] \otimes_{k[S]} k[X_2]$.

<u>1.4</u> (<u>Closed Subfunctors of Affine Schemes</u>). Let X be an affine scheme over k.

For any subset $I \subset k[X]$ we define a subfunctor $V(I)$ of X through

(1) $$V(I)(A) = \{x \in X(A) \mid f(x) = 0 \text{ for all } f \in I\}$$
$$\simeq \{\alpha \in Hom_{k-alg}(k[X],A) \mid \alpha(I) = 0\}$$

for all A. (It is immediate to check that this is indeed a subfunctor, i.e. that $X(\varphi)V(I)(A) \subset V(I)(A')$ for any homomorphism $\varphi\colon A \to A'$.)

Of course $V(I)$ depends only on the ideal generated by I in $k[X]$. We claim:

(2) <u>The map</u> $I \mapsto V(I)$ <u>from</u> $\{$<u>ideals in</u> $k[X]\}$ <u>to</u> $\{$<u>subfunctors of</u> $X\}$ <u>is injective</u>.

More precisely we claim for two ideals I, I' of $k[X]$:

(3) $\quad I \subset I' \Longleftrightarrow V(I) \supset V(I')$.

Of course the direction "$\Longrightarrow$" is trivial. On the other hand, consider the canonical map $\alpha\colon k[X] \to k[X]/I'$ which we regard as an element of $X(k[X]/I')$. As $\alpha(I') = 0$ it belongs to $V(I')(k[X]/I')$. If $V(I') \subset V(I)$, then $\alpha \in V(I)(k[X]/I')$ and $\alpha(I) = 0$, hence $I \subset I'$.

We call a subfunctor Y of X closed, if it is of the form $Y = V(I)$ for some ideal $I \subset k[X]$. Obviously any closed subfunctor is again an affine scheme over k as

(4) $\quad V(I) \simeq Sp_k(k[X]/I)$.

For any family $(I_j)_{j \in J}$ of ideals in $k[X]$ one checks easily

(5) $\quad \bigcap_{j \in J} V(I_j) = V(\sum_{j \in J} I_j)$.

Thus the intersection of closed subfunctors is closed again.

For each subfunctor Y of X there is a smallest closed subfunctor $\bar{Y}$ of X with $Y(A) \subset \bar{Y}(A)$ for all A. (Take the intersection of all closed subfunctors with the last property.) This subfunctor $\bar{Y}$ is called the <u>closure</u> of Y. We really do not have to assume here that Y is a subfunctor: Any map Y will do which associates to each A a subset $Y(A) \subset X(A)$. We can for example fix an A and consider a subset $M \subset X(A)$. Then the closure $\bar{M}$ of M is the smallest closed subfunctor of X with $M \subset \bar{M}(A)$.

Let I_1, I_2 be ideals in $k[X]$. Because of (3) the closure of the subfunctor $A \mapsto V(I_1)(A) \cup V(I_2)(A)$ is equal to $V(I_1 \cap I_2)$. If A is an integral domain, then one checks easily that $V(I_1)(A) \cup V(I_2)(A) = V(I_1 \cap I_2)(A)$. For arbitrary A this equality can be false. Still we <u>define</u> the union as $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$.

Let $f: X' \to X$ be a morphism of affine schemes over k. One easily checks for any ideal I of $k[X]$ that

$$(6) \qquad f^{-1}V(I) = V(k[X']f^*(I)).$$

Thus the inverse image of a closed subfunctor is again a closed subfunctor. For any ideal $I' \subset k[X']$ the closure of the subfunctor $A \mapsto f(A)(V(I')(A))$ is $V((f^*)^{-1}I')$. This functor is also denoted as $\overline{f(V(I'))}$, but we do not want to define $f(V(I'))$ here.

For two affine schemes X_1, X_2 over k and ideals $I_1 \subset k[X_1], I_2 \subset k[X_2]$ one checks easily

$$(7) \quad V(I_1) \times V(I_2) \cong V(I_1 \otimes k[X_2] + k[X_1] \otimes I_2).$$

If S is another affine scheme and if morphisms $X_1 \to S$, $X_2 \to S$ are fixed, then one gets

$$(8) \quad V(I_1) \times_S V(I_2) \cong V(I_1 \otimes_{k[S]} k[X_2] + k[X_1] \otimes_{k[S]} I_2).$$

(Use e.g. that $V(I_1) \times_S V(I_2) = p_1^{-1}V(I_1) \cap p_2^{-1}V(I_2)$ together with (5),(6) where $p_i: X_1 \times_S X_2 \to X_i$ for $i = 1,2$ are the canonical projections.)

1.5 (Open Subfunctors of Affine Schemes) Let X be an affine scheme over k.

A subfunctor Y of X is called open, if there is a subset $I \subset k[X]$ with $Y = D(I)$ where we set for all k-algebras A:

$$(1) \quad D(I)(A) = \{x \in X(A) \mid \sum_{f \in I} Af(x) = A\}$$

$$= \{\alpha \in \mathrm{Hom}_{k\text{-alg}}(k[X],A) \mid A\alpha(I) = A\}.$$

Note that (1) defines for each ideal I a subfunctor: For each $\varphi \in \mathrm{Hom}_{k\text{-alg}}(A,A')$ and each $x \in D(I)(A)$ one has $\sum_{f \in I} A'f(X(\varphi)x) = \sum_{f \in I} A'\varphi(f(x)) = A'\varphi(\sum_{f \in I} Af(x)) = A'\varphi(A) = A'$. Obviously:

(2) If A is a field, then $D(I)(A) = \bigcup_{f \in I} \{x \in X(A) \mid f(x) \neq 0\}$.

Of course, the right hand side in (2) would be the obvious choice for something open. But it does not define a subfunctor as homomorphisms between k-algebras are not injective in general. Therefore we have to take (1) as the appropriate generalization to rings.

For I of the form $I = \{f\}$ for some $f \in k[X]$ one writes $X_f = D(f) = D(\{f\})$ and gets

$$(3) \quad X_f(A) = \{\alpha \in \mathrm{Hom}_{k\text{-alg}}(k[X],A) \mid \alpha(f) \in A^{\times}\},$$

hence

$$(4) \quad X_f \cong \mathrm{Sp}_k(k[X]_f)$$

where $k[X]_f = k[X][f^{-1}]$ is the localization of $k[X]$ at f. So the open subfunctors of the form X_f are again affine schemes.

For arbitrary I, however, $D(I)$ may be no longer an affine scheme.

Obviously $D(I)$ depends only on the ideal of $k[X]$ generated by I. As any proper ideal in any ring is contained in a maximal ideal, we have for any A

$$\begin{aligned} D(I)(A) &= \{\alpha \in \mathrm{Hom}_{k\text{-alg}}(k[X],A) \mid \alpha(I) \not\subset \underline{m} \text{ for any } \underline{m} \in \mathrm{Max}(A)\} \\ &= \{\alpha \in \mathrm{Hom}_{k\text{-alg}}(k[X],A) \mid \alpha_{\underline{m}} \in D(I)(A/\underline{m}) \text{ for any } \underline{m} \in \mathrm{Max}(A)\} \end{aligned}$$

where $\mathrm{Max}(A)$ is the set of all maximal ideals of A and $\alpha_{\underline{m}}$ is the composed map $k[X] \xrightarrow{\alpha} A \xrightarrow{\text{can}} A/\underline{m}$. This shows that $D(I)$ is uniquely determined by its values over fields and especially that $D(I) = D(\sqrt{I})$ for any ideal $I \subset k[X]$. Denote for each prime ideal $P \subset k[X]$ the quotient field of $k[X]/P$ by Q_P and the canonical homomorphism $k[X] \to k[X]/P \to Q_P$ by α_P. Then

$$\alpha_P \notin D(I)(Q_P) \Longleftrightarrow \alpha_P(I) = 0 \Longleftrightarrow P \supset I.$$

As $\sqrt{I}$ is the intersection of all prime ideals $P \supset I$ of $k[X]$, we get for any two ideals I, I' of $k[X]$

(5) $\quad D(I) \subset D(I') \Longleftrightarrow \sqrt{I} \subset \sqrt{I'}.$

Thus $I \mapsto D(I)$ is a bijection $\{$ideals I of $k[X]$ with $I = \sqrt{I}\}$ $\to$ $\{$open subfunctors of $X\}$.

For two ideals I, I' in $k[X]$ one checks easily

(6) $\quad D(I) \cap D(I') = D(I \cap I') = D(I \cdot I')$

and gets especially for any $f, f' \in k[X]$

(7) $X_f \cap X_{f'} = X_{ff'}$.

For any ideal I in $k[X]$ one has:

(8) <u>If</u> A <u>is a field, then</u> $X(A)$ <u>is the disjoint union of</u> $D(I)(A)$ <u>and</u> $V(I)(A)$.

For arbitrary A the union may be smaller. Also the next statement may be false for arbitrary A: Consider a family $(I_j)_{j \in J}$ of ideals in $k[X]$. Then obviously

(9) <u>If</u> A <u>is a field, then</u> $\bigcup_{j \in J} D(I_j)(A) = D(\sum_{j \in J} I_j)(A)$.

For any morphism $f: X' \to X$ of affine schemes over k one has

(10) $f^{-1}D(I) = D(k[X']f^*(I))$

for any ideal $I \subset k[X]$ as one may check easily. We get especially for any $f' \in k[X]$

(11) $f^{-1}(X_{f'}) = X'_{f^*(f')}$.

For any fibre product $X_1 \times_S X_2$ of affine schemes over k (with respect to suitable morphisms) and any ideals $I_1 \subset k[X_1]$, $I_2 \subset k[X_2]$ one has

(12) $D(I_1) \times_S D(I_2) = D(I_1 \otimes_{k[S]} I_2)$.

(Argue as for 1.4(8).)

<u>1.6</u> (<u>Affine Varieties and Affine Schemes</u>) An affine scheme

X is called <u>algebraic</u>, if $k[X]$ is isomorphic to a k-algebra of the form $k[T_1,\dots,T_n]/I$ for some $n \in \mathbb{N}$ and a finitely generated ideal I in the polynomial ring $k[T_1,\dots,T_n]$. It is called <u>reduced</u>, if $k[X]$ does not contain any nilpotent element other than 0.

Assume until the end of this section 1.6 that k is an algebraically closed field. Any affine variety X over k defines as in 1.1 a k-functor $\underline{\underline{X}}$ which we may identify with $Sp_k[X]$. One gets in this way exactly all reduced algebraic affine schemes over k. For two affine varieties X,X' one has $Mor(X,X') \simeq Hom_{k-alg}(k[X'],k[X]) \simeq Mor(\underline{\underline{X}},\underline{\underline{X}}')$. So we have indeed embedded the category of affine varieties as a full subcategory into the category of affine schemes.

When doing this, one has to be aware of several points. Any closed subset Y of an affine variety X is itself an affine variety. The functor $\underline{\underline{Y}}$ is the closed subfunctor $V(I) \subset \underline{\underline{X}}$ where $I = \{f \in k[X] \mid f(Y) = 0\}$. In this way one gets an embedding $\{$closed subsets of $X\} \to \{$closed subfunctors of $\underline{\underline{X}}\}$. On the level of ideals (cf. 1.4(2)) it corresponds to the inclusion $\{$ideals I of $k[X]$ with $I = \sqrt{I}\} \subset \{$ideals of $k[X]\}$. The embedding is certainly compatible with inclusions (i.e. $Y \subset Y' \Longleftrightarrow \underline{\underline{Y}} \subset \underline{\underline{Y}}'$), but in general not with intersections: It may happen that $\underline{\underline{Y}} \cap \underline{\underline{Y}}'$ is strictly larger than $\underline{\underline{Y \cap Y'}}$. Take for example in $X = k^2$ (where $k[X] = k[T_1,T_2]$) the line $Y = \{(a,0) \mid a \in k\}$ and the parabola $Y' = \{(a,a^2) \mid a \in k\}$. Then $Y \cap Y' = \{(0,0)\}$. The ideals I,I' of Y,Y' are $I = (T_2)$ and $I' = (T_1^2-T_2)$, hence $I+I' = (T_1^2, T_2) \neq (T_1, T_2)$ and $\underline{\underline{Y}} \cap \underline{\underline{Y}}' = V(I) \cap V(I') = V(I+I') \supsetneqq V(T_1, T_2) = \underline{\underline{Y \cap Y'}}$.

So, when regarding affine varieties as (special) affine schemes, we have to be careful, whether intersections are taken as varieties or as schemes. The same is true for inverse images and (more generally) for fibre products.

Similar problems do not arise with open subsets. To any open $Y \subset X$ we can associate the ideal $I = \{f \in k[X] \mid f(X-Y) = 0\}$ and then the open subfunctor $D(I)$ which we denote by $\underline{\underline{Y}}$. Because of 1.5(5) the map $Y \mapsto \underline{\underline{Y}}$ is a bijection from {open subsets of X} to {open subfunctors of $\underline{\underline{X}}$} which is compatible with intersections. It follows from 1.5(10),(12) that this bijection is also compatible with inverse images and fibre products. (In case Y is affine the notation $\underline{\underline{Y}}$ is compatible with the earlier one.)

<u>1.7</u> (<u>Open Subfunctors</u>) (Let k again be arbitrary.)

Let X be a k-functor. A subfunctor $Y \subset X$ is called <u>open</u> if for any affine scheme X' over k and any morphism $f: X' \to X$ there is an ideal $I \subset k[X']$ with $f^{-1}(Y) = D(I)$.

Note that this definition is compatible with the one at the beginning of 1.5 because of 1.5(10). From 1.5(6) one gets

(1) <u>If</u> Y, Y' <u>are open subfunctors of</u> X, <u>then so is</u> $Y \cap Y'$.

Let $f: X' \to X$ be a morphism of k-functors. Then one has obviously:

(2) <u>If</u> Y <u>is an open subfunctor of</u> X, <u>then</u> $f^{-1}(Y)$ <u>is an open subfunctor of</u> X'.

Let X_1, X_2, S be k-functors and suppose $X_1 \times_S X_2$ is defined with respect to some morphisms. Then one gets (using $Y_1 \times_S Y_2 = p_1^{-1}(Y_1) \cap p_2^{-1}(Y_2)$).

(3) <u>If</u> $Y_1 \subset X_1$ <u>and</u> $Y_2 \subset X_2$ <u>are open subfunctors, then</u> $Y_1 \times_S Y_2$ <u>is an open subfunctor of</u> $X_1 \times_S X_2$.

Let Y, Y' be open subfunctors of X. Then:

(4) $Y = Y' \Longleftrightarrow Y(A) = Y'(A)$ <u>for each</u> k-<u>algebra</u> A <u>which is a field</u>.

(Of course "$\Longrightarrow$" is trivial. In order to show "$\Longleftarrow$" suppose $Y \neq Y'$. Then there is some k-algebra A with $Y(A) \neq Y'(A)$. Assume that there is $x \in Y(A)$ with $x \notin Y'(A)$. Via $Y(A) \simeq Mor(Sp_k A, Y) \subset Mor(Sp_k A, X)$ regard x as a morphism $Sp_k A \to X$. Then $id_A \in x^{-1}(A)(A), \notin x^{-1}(Y')(A)$, hence $x^{-1}(Y) \neq x^{-1}(Y')$. Now the result follows from the discussion preceding 1.5(5).)

A family $(Y_j)_{j \in J}$ of open subfunctors of X is called an <u>open covering</u> of X, if $X(A) = \bigcup_{j \in J} Y_j(A)$ for each k-algebra A <u>which is a field</u>.

If X is affine and if $Y_j = D(I_j)$ for some ideal $I_j \subset k[X]$, then formula 1.5(9) implies that the $D(I_j)$ form an open covering of X if and only if $k[X] = \sum_{j \in J} I_j$. A comparison with the case of an affine variety shows that this is the appropriate generalization of the notion of an affine cover. Note that especially

(5) <u>Let</u> X <u>be affine and</u> $f_1, f_2, \ldots, f_r \in k[X]$. <u>Then the</u> X_{f_i}

<u>form an open covering of</u> X <u>if and only if</u> $k[X] = \sum_{i=1}^{r} k[X]f_i$.

<u>1.8</u> <u>(Local Functors)</u> As the notion of an affine scheme generalizes the notion of an affine variety we want to define the notion of a scheme generalizing the notion of a varieties. Certainly a scheme should (by analogy) be a k-functor admitting an open covering by affine schemes. This is however not enough.

Consider two k-functors X,Y and an open covering $(Y_j)_{j\in J}$ of Y. If X,Y correspond to geometric objects then a morphism $f: Y \to X$ ought to be determined by its restrictions $f_{|Y_j}$ to all Y_j. Furthermore, given for each j a morphism $f_j: Y_j \to X$ such that $f_{j|Y_j\cap Y_{j'}} = f_{j'|Y_j\cap Y_{j'}}$ for all $j,j' \in J$, then there ought to be a (unique) morphism $f: Y \to X$ with $f_{|Y_j} = f_j$ for all j. In other words, the sequence

$$(1)\quad \mathrm{Mor}(Y,X) \xrightarrow{\alpha} \prod_{j\in J} \mathrm{Mor}(Y_j,X) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{j,j'\in J} \mathrm{Mor}(Y_j\cap Y_{j'},X)$$

ought to be exact where $\alpha(f) = (f_{|Y_j})_{j\in J}$ and $\beta((f_j)_{j\in J})$ resp. $\gamma((f_j)_{j\in J})$ has (j,j')-component $f_{j|Y_j\cap Y_{j'}}$ resp. $f_{j'|Y_j\cap Y_{j'}}$.

For arbitrary $X,Y,(Y_j)$ the sequence (1) will not be exact. So we define a k-functor X to be <u>local</u> if the sequence (1) is exact for all k-functors Y and all open coverings $(Y_j)_{j\in J}$. (One can express this as saying that the functor $\mathrm{Mor}(?,X)$ is a sheaf in a suitable sense.)

For any k-algebra R and any $f_1, \ldots, f_r \in R$ with $\sum_{i=1}^{r} Rf_i = R$ the $Sp_k(R_{f_i})$ form an open covering of the affine scheme $Sp_k R$. In this case the sequence (1) takes (because of Yoneda's lemma) the form

$$(2) \quad X(R) \to \sum_{i=1}^{r} X(R_{f_i}) \rightrightarrows \prod_{1 \leq i,j \leq r} X(R_{f_i f_j})$$

where the maps have components of the form $X(\alpha)$ with α one of the canonical maps $R \to R_{f_i}$ or $R_{f_i} \to R_{f_i f_j}$. Now one can prove (cf. [DG], I, §1, 4.13)

Proposition: A k-functor X is local if and only if för any k-algebra R and any $f_1, \ldots, f_r \in R$ with $\sum_{i=1}^{r} R_{f_i} = R$ the sequence (2) is exact.

(Note that in [DG] the second property is taken as the definition of "local".)

For R and $f_1, \ldots, f_r$ as in (2) the sequence

$$(3) \quad R \to \prod_{i=1}^{r} R_{f_i} \rightrightarrows \prod_{1 \leq i,j \leq r} R_{f_i f_j}$$

(induced by the natural maps $R \to R_{f_i}$ and $R_{f_i} \to R_{f_i f_j}$) is exact. (This is really the descriptión of the structural sheaf on Spec R e.g. in [Ha] , II, 2.2.) For an affine scheme X over k the exactness property of $Hom_{k-alg}(k[X],?) = X(?)$ shows that the exactness of (3) implies the exactness of (2). Thus we get

(4) Any affine scheme over k is a local k-functor.

Consider k-algebras $A_1, A_2, \dots, A_n$ and the projections $p_j: \prod_{i=1}^{n} A_i \to A_j$. If we apply (2) to $R = \prod_{i=1}^{n} A_i$ and the $f_i = (0,\dots,0,1,0,\dots,0)$, then we get

(5) <u>If</u> X <u>is a local functor, then</u> $X(\prod_{i=1}^{n} A_i) \xrightarrow{\sim} \prod_{i=1}^{n} X(A_i)$ <u>for all</u> k-<u>algebras</u> $A_1, A_2, \dots, A_n$.

(The bijection maps any x to $(X(p_i)x)_{1\le i\le n}$.)

<u>1.9</u> (<u>Schemes</u>) A k-functor is called a <u>scheme</u> (over k), if it is local and if it admits an open covering by affine schemes.

Obviously 1.8(4) implies

(1) <u>Any affine scheme over</u> k <u>is a scheme over</u> k.

The category of schemes over k (a full subcategory of {k-functors}) is closed under important operations:

(2) <u>If</u> X <u>is a local</u> k-<u>functor (resp. a scheme over</u> k) <u>and if</u> X' <u>is an open subfunctor of</u> X, <u>then</u> X' <u>is local (resp. a scheme</u>).

In the situation of 1.8(1) the injectivity of α for X implies its injectivity for X'. In order to show the exactness for X one has to show then for any $f \in Mor(Y,X)$ such that each $f_{|Y_j}$ factors through X', that also f factors through X'. The assumption implies $Y_j \subset f^{-1}(X')$ for each j', hence by the definition of an open covering that $f^{-1}(X')(A) = Y(A)$ for each k-algebra A which is a field. Then 1.7(4) implies $Y \subset f^{-1}(X')$ and f factors through X'. In order to get the affine covering of X' in case X is a scheme one can restrict to the case where X is affine, hence $X' = D(I)$ for some ideal. Then the $(X_f)_{f\in I}$ form an open affine covering.

Let X_1, X_2, S be k-functors and form $X_1 \times_S X_2$ with respect to suitable morphisms. Then:

(3) If X_1, X_2, S are local (resp. schemes), then so is $X_1 \times_S X_2$.

The proof may be left to the reader.

1.10 (Base Change) Let k' be a k-algebra. Any k'-algebra A is in a natural way also a k-algebra, just by combining the structural homomorphisms $k \to k'$ and $k' \to A$. We can therefore associate to each k-functor X a k'-functor $X_{k'}$ by $X_{k'}(A) = X(A)$ for any k'-algebra A. For any morphism $f: X \to X'$ of k-functors we get a morphism $f_{k'}: X_{k'} \to X'_{k'}$ of k'-functors simply by $f_{k'}(A) = f(A)$ for any k'-algebra A. In this way we get a functor $X \mapsto X_{k'}$, $f \mapsto f_{k'}$ from {k-functors} to {k'-functors} which we call base change from k to k'.

For any subfunctor Y of a k-functor X the k'-functor $Y_{k'}$ is a subfunctor of $X_{k'}$. Furthermore the base change commutes with taking inverse images under morphisms, with taking intersections of subfunctors and with forming fibre products.

The universal property of the tensor product implies that $(Sp_k R)_{k'} \simeq Sp_{k'}(R \otimes k')$ for any k-algebra R. In other words, if X is an affine scheme over k, then $X_{k'}$ is an affine scheme over k' with $k'[X_{k'}] \simeq k[X] \otimes k'$. For any ideal I of $k[X]$ one gets then $V(I)_{k'} = V(I \otimes k')$ and $D(I)_{k'} = D(I \otimes k')$.

(Well, we really ought to replace $I \otimes k'$ in these formulas by its canonical image in $k[X] \otimes k'$, but for once we shall indulge in some abuse of notation.)

For any k'-algebras A, R one has

$$(Sp_{k'}R)(A) = \mathrm{Hom}_{k'\text{-alg}}(R,A) \subset \mathrm{Hom}_{k\text{-alg}}(R,A) = (Sp_k R)_{k'}(A).$$

Thus we have embedded $Sp_{k'}R$ as a subfunctor into $(Sp_k R)_{k'}$. For any ideal I of R denote the corresponding subfunctors as in 1.4/5 by $V(I), D(I) \subset Sp_k R$ and $V_{k'}(I), D_{k'}(I) \subset Sp_{k'}R$. Then one sees immediately $D_{k'}(I) = (Sp_{k'}R) \cap D(I)_{k'}$ and $V_{k'}(I) = (Sp_{k'}R) \cap V(I)_{k'}$.

Using the last results it is easy to show for any open subfunctor Y or a k-functor X that $Y_{k'}$ is open in $X_{k'}$. If X is a local k-functor, then obviously $X_{k'}$ is a local k'-functor. Now it is easy to show that $X_{k'}$ is a scheme over k' if X is one over k.

1.11 ("Schemes") In text books on algebraic geometry (like that by Hartshorne to which I shall usually refer in such matters) another notion of scheme is introduced which I shall denote by "schemes" in case a distinction is useful. A "scheme" is a topological space together with a sheaf of k-algebras and an open covering by "affine schemes". The "affine schemes" are the prime spectra $\mathrm{Spec}(R)$ of the k-algebras R endowed with the Zariski topology and a sheaf having sections R_f on each $\mathrm{Spec}(R_f) \subset \mathrm{Spec}(R)$ for all $f \in R$. To each such "scheme" X one can associate a k-functor $\underline{\underline{X}}$ via $\underline{\underline{X}}(A) = \mathrm{Mor}(\mathrm{Spec}\ A, X)$ for all A.

On the other hand one can associate in a functorial way to each k-functor X a topological space $|X|$ together with a sheaf such that $|\mathrm{Sp}_k R| = \mathrm{Spec}(R)$ for each k-algebra R. It turns out that $|X|$ is a "scheme" if and only if X is a scheme and that $X \mapsto |X|$ and $X' \mapsto \underline{\underline{X}}'$ are quasi-inverse equivalences of categories. (This is the content of the comparison theorem [DG], I, §1, 4.4.)

One property of this construction is that the open subfunctors of any k-functor X correspond bijectively to the open subsets of $|X|$, cf. [DG], I, §1, 4.12. More precisely, if Y is an open subfunctor of X, then $|Y|$ can be identified with an open subset of $|X|$ and the k-algebra of sections in $|Y|$ of the structural sheaf of $|X|$ is isomorphic to $\mathrm{Mor}(Y, \mathbb{A}^1)$, ibid. 4. 14/15.

Suppose that k is an algebraically closed field. Consider a scheme X over k which has an open covering by algebraic affine schemes. We can define on $X(k)$ a topology such that the open subsets are the $Y(k)$ for open subfunctors $Y \subset X$. The map $Y \mapsto Y(k)$ turns out to be injective ([DG], I, §3, 6.8). We can define a sheaf $\mathcal{O}_{X(k)}$ on $X(k)$ through $\mathcal{O}_{X(k)}(Y(k)) = \mathrm{Mor}(Y, \mathbb{A}^1)$. Then $X \mapsto (X(k), \mathcal{O}_{X(k)})$ is a faithful functor and its image contains all varieties over k in the usual sense.

2. Group Schemes and Representations

In this section we define group schemes and modules over these objects and discuss their fundamental properties. As in chapter 1 we follow more or less [DG].

After making the definitions of k-group functors and k-group schemes in 2.1 we describe some examples in 2.2. The relationship between algebraic groups and Hopf algebras generalizes to group schemes. This is done in 2.3/4. (We always assume our group schemes to be affine.) We then discuss the class of diagonalizable group schemes in 2.5 and group operations in 2.6.

We then go on to define representations (2.7) and to discuss the relationship between G-modules and k[G]-comodules (2.8). We generalize standard notions of representation theory to G-modules: submodules (2.9), fixed points (2.10), centralizers and stabilizers (2.12), and simple modules (2.14). The definition of a submodule has some unpleasant aspect which disappears only when G is a flat group scheme (i.e. a group scheme such that $k[G]$ is a flat k-module). This is the reason why we shall restrict ourselves later on to such groups. We also discuss one special property of representations of group schemes: They are locally finite (2.13). Furthermore we describe representations of diagonalizable group schemes (2.11) and mention results about modules for trigonalizable and unipotent groups over fields (in 2.14). Here we refer to [DG] for the proofs (which require the notion of factor groups). Otherwise all proofs are rather elementary.

2.1 (Definitions) A k-group functor is a functor from the category of all k-algebras to the category of groups. We can regard any k-group functor also as a k-functor by composing it with the forgetful functor from {groups} to {sets}. In this way we can and shall apply all ideas and notions from section 1 also to k-group functors. For two group functors G,H we shall denote by $\mathrm{Mor}(G,H)$ the set of all morphisms (= natural transformations) from G to H considered as k-functors, and by $\mathrm{Hom}(G,H)$ the set of all morphisms from G to H considered as k-group functors. So $\mathrm{Hom}(G,H)$ consists of those $f \in \mathrm{Mor}(G,H)$ with $f(A)$ a group homomorphism for each k-algebra A. These elements are called homomorphisms from G to H.

A k-group scheme is a k-group functor which is an affine scheme over k when considered as a k-functor. (Of course, we really ought to call such an object an affine k-group scheme and drop the word "affine" in the definition of a k-group scheme. But we shall consider only affine group schemes and then it is more economical to call them group schemes.) An algebraic k-group is a k-group scheme which is algebraic as an affine scheme. A k-group scheme is called reduced if it is so as an affine scheme. Over an algebraically closed field the category of algebraic groups as in [Hu] or [Sp] can be identified with the subcategory of all reduced algebraic k-groups.

Let G be a k-group functor. A subgroup functor of G is a subfunctor H of G such that each $H(A)$ is a subgroup of $G(A)$. The intersection of subgroup functors is again a subgroup functor. The inverse image of a subgroup functor under a homomorphism is again one. A direct product of k-group functor is again a k-group functor, so is a fibre product if the morphisms used in its definition are homomorphisms of k-group functors.

A subgroup functor H of G is called <u>normal</u> (resp. <u>central</u>) if each $H(A)$ is a normal (resp. <u>central</u>) subgroup of $G(A)$. Again, normality is preserved under taking intersections and inverse images under homomorphisms. The kernel $\ker \varphi$ of a homomorphism $\varphi: G \to G'$ is always a normal subgroup scheme.

A <u>closed subgroup scheme</u> of a k-group scheme G is a subgroup functor H which is closed if considered as a subfunctor of the affine scheme G over k. If G and H are algebraic k-groups we simply call H a <u>closed subgroup</u> of G.

A k-group functor G is called <u>commutative</u>, if all $G(A)$ are commutative.

<u>2.2</u> (<u>Examples</u>) The notations introduced here for special group functors G and their algebras $k[G]$ will be used always. The <u>additive group</u> over k is the k-group functor G_a with $G_a(A) = (A,+)$ for all k-algebras A. It is an algebraic k-group with $k[G_a]$ isomorphic to (and usually identified with) the polynomial ring $k[T]$ in one variable.

Any k-module M defines a k-group functor M_a with $M_a(A) = (M \otimes A,+)$ for all A. (So we have $G_a = k_a$). If M is finitely generated and projective as a k-module, then M_a is an algebraic k-group with $k[M_a] = S(M^*)$, the symmetric algebra over the dual k-module M^*. In case $M = k^n$ for some $n \in \mathbb{N}$ we may identify M_a with $G_a \times G_a \times \ldots \times G_a$ (n factors) and $k[M_a]$ with the polynomial ring $k[T_1, T_2, \ldots, T_n]$.

The <u>multiplicative group</u> over k is the k-group functor G_m with $G_m(A) = A^\times = \{\text{units of } A\}$ for all A. It is an algebraic k-group with $k[G_m] \cong k[T, T^{-1}]$.

Any k-module M defines a k-group functor $GL(M)$ with $GL(M)(A) = (\mathrm{End}_A(M \otimes A))^\times$ called the <u>general linear group of</u> M. In case $M = k^n$ we may identify $GL(M)$ with GL_n where $GL_n(A)$ is the group of all invertible $(n \times n)$-matrices over A. Obviously GL_n is an algebraic k-group with $k[GL_n]$ isomorphic to the localization of the polynomial ring $k[T_{ij}, 1 \leq i,j \leq n]$ with respect to $\{(\det)^n | n \in \mathbb{N}\}$. More generally, if M is a finitely generated and projective k-module, then the k-functor $A \mapsto \mathrm{End}_A(M \otimes A)$ can be identified with the affine scheme $(M^* \otimes M)_a$ from above and $GL(M)$ with the open subfunctor $D(\det)$. For such M (projective of finite rank) the determinant defines a homomorphism of algebraic k-groups $GL(M) \to G_m$. Its kernel is denoted by $SL(M)$ and called the <u>special linear group</u> of M. It is an algebraic k-group. Similarly we define $SL_n \subset GL_n$. Note that $GL_1 = G_m$ and $SL_n = 1$ = the group functor associating to each A the trivial group $\{1\}$.

For each $n \in \mathbb{N}$ let T_n be the algebraic k-group such that $T_n(A)$ is the group of all invertible upper-triangular $(n \times n)$-matrices of A, i.e. of all upper-triangular matrices such that all diagonal entries belong to $A^{\times}$. One may identify $k[T_n] \cong k[T_{ij} | 1 \le i \le j \le n, T_{ii}^{-1} | 1 \le i \le n]$. Furthermore let U_n be the algebraic k-group such that each $U_n(A)$ consists of all $g \in T_n(A)$ having all diagonal entries equal to 1. We may identify $k[U_n] \cong k[T_{ij} | 1 \le i < j \le n]$.

For any $n \in \mathbb{N}$ we denote by $\mu_{(n)}$ the group functor with $\mu_{(n)}(A) = \{a \in A | a^n = 1\}$ for all A. It is an algebraic k-functor with $k[\mu_{(n)}] = k[T]/(T^n - 1)$ and a closed subgroup of G_m.

Let p be a prime number and assume $p1 = 0$ in k. Then we can define for each $r \in \mathbb{N}$ a closed subgroup $G_{a,r}$ of G_a through $G_{a,r}(A) = \{a \in A | a^{p^r} = 0\}$.

<u>2.3</u> (<u>Group Schemes and Hopf Algebras</u>) Let G be a k-group functor. The group structures on the G(A) define morphisms of k-functors $m_G\colon G \times G \to G$ (such that each $m_G(A)\colon G(A) \times G(A) \to G(A)$ is the multiplication), and $1_G\colon Sp_k k \to G$ (such that $1_G(A)$ maps the unique element of $(Sp_k k)(A)$ to the 1 of G(A)), and $i_G\colon G \to G$ (inducing on each G(A) the map $g \to g^{-1}$).

Now assume G to be a k-group scheme. Then these morphisms correspond uniquely to their comorphisms $\Delta_G = m_G^*\colon k[G] \to k[G] \otimes k[G]$ (called <u>comultiplication</u>), and $\varepsilon_G = 1_G^*\colon k[G] \to k$ (called <u>counit</u> or <u>augmentation</u>), and $\sigma_G = i_G^*\colon k[G] \to k[G]$ (called <u>coinverse</u> or <u>antipode</u>). So, if $\Delta_G(f) = \sum_{i=1}^{r} f_i \otimes f_i'$ for some $f \in k[G]$, then $f(g_1 g_2) = \sum_{i=1}^{r} f_i(g_1) f_i'(g_2)$ for each $g_1, g_2 \in G(A)$ and any A. Furthermore we have $\varepsilon_G(f) = f(1)$ and $\sigma_G(f)(g) = f(g^{-1})$ for any $g \in G(A)$ and any A.

We shall drop in our notations the index G, whenever no confusion is possible.

As in the case of algebraic groups (cf. [Bo], 1.5 or [Hu], 7.6 or [Sp], 2.1.2) the group axioms imply that $\Delta, \varepsilon, \sigma$ satisfy

(1) $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$,

(2) $(\varepsilon \mathbin{\bar{\otimes}} \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \mathbin{\bar{\otimes}} \varepsilon) \circ \Delta$,

(3) $(\sigma \mathbin{\bar{\otimes}} \mathrm{id}) \circ \Delta = \bar{\varepsilon} = (\mathrm{id} \mathbin{\bar{\otimes}} \sigma) \circ \Delta$.

(Here we denote by $\varphi \mathbin{\bar{\otimes}} \psi$ the map $a \otimes a' \mapsto \varphi(a)\psi(a')$ in contrast to $\varphi \otimes \psi$: $a \otimes a' \longmapsto \varphi(a) \otimes \psi(a')$ and by $\bar{\varepsilon}$ the endomorphism $a \mapsto \bar{\varepsilon}(a)1$ of $k[G]$.)

A morphism $\varphi: G \to G'$ between two k-group schemes is a homomorphism if and only if its comorphism $\varphi^*: k[G'] \to k[G]$ satisfies

(4) $\Delta_G \circ \varphi^* = (\varphi^* \otimes \varphi^*) \circ \Delta_{G'}$.

If so, then one has automatically

(5) $\varepsilon_G \circ \varphi^* = \varepsilon_{G'}$

and

(6) $\sigma_G \circ \varphi^* = \varphi^* \circ \sigma_{G'}$.

A <u>Hopf algebra</u> over k is an associative (not necessarily commutative) algebra R over k together with homomorphisms of algebras $\Delta: R \to R \otimes R$, $\varepsilon: R \to k$, and $\sigma: R \to R$ satisfying (1)-(3). A homomorphism between two Hopf algebras is a homomorphism of algebras satisfying additionally (4)-(6) (with the appropriate changes in the notation.) We call R commutative, if it is so as an algebra, and <u>cocommutative</u>, if $s \circ \Delta = \Delta$, where $s: R \otimes R \to R \otimes R$ is the map $a \otimes b \mapsto b \otimes a$.

Let R be a commutative Hopf algebra over k. Then we can define on each $(Sp_k R)(A) = Hom_k(R,A)$ a multiplication via $\alpha\beta = (\alpha \tilde{\otimes} \beta) \circ \Delta$. In this way we get on $Sp_k R$ a structure as a k-group scheme. It is elementary to see that we get in this way a functor {commutative Hopf algebras over k} $\to$ { k-group schemes} which is quasi-inverse to $G \mapsto k[G]$. Thus these categories are anti-equivalent.

Note that G is commutative, if and only if $k[G]$ is cocommutative.

<u>2.4</u> (<u>Continuation</u>) Let us look at the Hopf algebra structures on $k[G]$ on our examples in 2.2. In the case of G_a one has $\Delta(T) = 1 \otimes T + T \otimes 1$, $\varepsilon(T) = 0$, and $\sigma(T) = -T$. Similar formulas hold for the $G_{a,r}$. In the case of G_m one has $\Delta(T) = T \otimes T$, $\varepsilon(T) = 1$, and $\sigma(T) = T^{-1}$. In GL_n one has $\Delta(T_{ij}) = \sum_{m=1}^{n} T_{im} \otimes T_{mj}$ and $\varepsilon(T_{ij}) = \delta_{ij}$ (the Kronecker delta). The formula for $\sigma(T_{ij})$ is more complicated. Furthermore one has $\Delta(\det) = \det \otimes \det$, $\varepsilon(\det) = 1$, and $\sigma(\det) = \det^{-1}$.

Let G be a k-group scheme and set $I_1 = \ker \varepsilon$, the <u>augmentation ideal</u> in $k[G]$. One has $k[G] = k1 \oplus I_1$ and $a \mapsto a1, k \to k1$ is bijective. This implies $k[G] \otimes k[G] = k(1 \otimes 1) \oplus (k \otimes I_1) \oplus (I_1 \otimes k) \oplus (I_1 \otimes I_1)$. The formula 2.3(2) implies

(1) $\Delta(f) \in f \otimes 1 + 1 \otimes f + I_1 \otimes I_1$ for all $f \in I_1$

and then the formula 2.3(3) implies

(2) $\sigma(f) \in -f+I_1^2$ for all $f \in I_1$.

Set

(3) $X(G) = \mathrm{Hom}(G,G_m)$.

This is a commutative group in a natural way. The embedding of affine schemes $G_m \subset G_a = \mathbb{A}^1$ yields an embedding

$$X(G) \subset \mathrm{Mor}(G,G_m) \subset \mathrm{Mor}(G,G_a) \simeq k[G]$$

which is compatible with the multiplication. Take $f \in k[G]$. One has $f^*(T) = f$. Therefore 2.3(4) implies easily

(4) $X(G) \simeq \{f \in k[G] \mid f(1) = 1, \Delta_G(f) = f \otimes f\}$.

Of course $\Delta_G(f) = f \otimes f$ implies $f(1)^2 = f(1)$. If $f(1) = 0$, then $f(g) = f(g \cdot 1) = f(g)f(1) = 0$ for all $g \in G(A)$ and all A, hence

(4') <u>If</u> k <u>is an integral domain, then</u> $X(G) \simeq \{f \in k[G] \mid \Delta_G(f) = f \otimes f, f \neq 0\}$.

Let me refer to [DG], II, §1, 2.9 for the proof of

(5) <u>If</u> k <u>is a field, then</u> $X(G)$ <u>is linearly independent</u>.

(This is just another variation on the theme "linear independence of characters".) <u>Usually</u> we shall write the group law in $X(G)$ <u>additively</u>.

Let I be an ideal in $k[G]$. Using 1.4(6),(7) one checks easily that $V(I)$ is a subgroup functor if and only if

(6) $\Delta(I) \subset I \otimes k[G] + k[G] \otimes I, \ \varepsilon(I) = 0, \ \sigma(I) \subset I.$

If so, it will be a normal subgroup if and only if

(7) $c^*(I) \subset k[G] \otimes I,$

where c^* is the comorphism of the conjugation map $c\colon G \times G \to G$ with $c(A)(g_1, g_2) = g_1 g_2 g_1^{-1}$ for all A and $g_1, g_2 \in G(A)$. One may check that

(8) $c^* = t \circ (\Delta \otimes \mathrm{id}) \circ \Delta$

where $t(f_1 \otimes f_2 \otimes f_3) = f_1 \sigma(f_3) \otimes f_2$.

2.5 (Diagonalizable Groups) Let Λ be a commutative group (written multiplicatively) and let us identify Λ with the canonical basis of the group algebra $k[\Lambda]$. We make $k[\Lambda]$ into a commutative and cocomutative Hopf algebra via $\Delta(\lambda) = \lambda \otimes \lambda$ and $\varepsilon(\lambda) = 1$ and $\sigma(\lambda) = \lambda^{-1}$ for all $\lambda \in \Lambda$. In this way we associate to Λ a k-group scheme which we denote by $\mathrm{Diag}(\Lambda)$. If Λ is finitely generated, then $\mathrm{Diag}(\Lambda)$ is an algebraic k-group.

We call a k-group scheme diagonalizable, if it is isomorphic to $\mathrm{Diag}(\Lambda)$ for some commutative group Λ. For example $G_m \cong \mathrm{Diag}(\mathbb{Z})$ and $\mu_{(n)} \cong \mathrm{Diag}(\mathbb{Z}/(n))$ are diagonalizable. We get also direct products of these groups as $\mathrm{Diag}(\Lambda_1 \times \Lambda_2) \cong \mathrm{Diag}(\Lambda_1) \times \mathrm{Diag}(\Lambda_2)$ for all commutative groups Λ_1, Λ_2.

Any group homomorphism $\alpha\colon \Lambda_1 \to \Lambda_2$ induces a homomorphism of group algebras $\alpha^*: k[\Lambda_1] \to k[\Lambda_2]$ which is a homomorphism of Hopf algebras, hence we get a homomorphism $\mathrm{Diag}(\alpha): \mathrm{Diag}(\Lambda_1) \to \mathrm{Diag}(\Lambda_2)$ of k-group schemes. Thus $\Lambda \mapsto \mathrm{Diag}(\Lambda)$ is a functor from {commutative groups} to {k-group schemes} which maps {finitely generated commutative groups} into {algebraic k-groups}.

Suppose that k is an integral domain. Then an easy computation shows (cf. [DG], II, §1, 2.11) for all Λ, Λ'

(1) $X(\mathrm{Diag}(\Lambda)) \simeq \Lambda$ (k integral)

and

(2) $\mathrm{Hom}_{gp}(\Lambda, \Lambda') \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Diag}(\Lambda'), \mathrm{Diag}(\Lambda))$ (k integral)

Thus in this case $\mathrm{Diag}(?)$ is an anti-equivalence of categories from {commutative groups} to {diagonalizable k-group schemes}. Furthermore Λ is finitely generated if and only if $\mathrm{Diag}(\Lambda)$ is an algebraic k-group. We get from (1) that a k-group scheme G is diagonalizable if and only if $X(G)$ is a basis of $k[G]$ (for k integral).

<u>2.6</u> (<u>Operations</u>) Let G be a k-group functor. A left <u>operation</u> of G on a k-functor X is a morphism $\alpha: G \times X \to X$ such that for each k-algebra A the map $\alpha(A): G(A) \times X(A) \to X(A)$ is a (left) operation of the group $G(A)$ on the set $X(A)$. We usually write gx instead of $\alpha(A)(g,x)$ for $g \in G(A)$ and $x \in X(A)$. We can similarly define right operations.

For example the conjugation map c in 2.4 is an operation of G on itself. Other operations of G on itself are by left $(\alpha(A)(g,g') = gg')$ and right $(\alpha(A)(g,g') = g'g^{-1})$

multiplication.

Let k' be a k-algebra. Then any operation of G on a k-functor X defines in a natural way an operation of $G_{k'}$ on $X_{k'}$.

For any operation α as above we set

(1) $X^G(k) = \{x \in X(k) | gx = x$ for all $g \in G(A)$ and all $A\}$.

(This is done by some abuse of notation. The x in $gx = x$ is really the image of x under the map $X(k) \to X(A)$ corresponding to the structural morphism $k \to A$. We shall stick to this abuse.) We can define a subfunctor X^G of X, the <u>fixed point functor</u> via

(2) $$X^G(A) = (X_A)^{G_A}(A)$$
$$= \{x \in X(A) |\ gx = x \text{ for all } g \in G(A') \text{ and all } A\text{-algebras } A'\}$$

See [DG], II, §1, n° 3 for elementary properties of X^G and of normalizers and centralizers, also defined there.

Suppose G acts on another k-group functor H such that each $G(A)$ acts on $H(A)$ through group automorphisms. Then we can define the <u>semi-direct product</u> $G \ltimes H$ where each $(G \ltimes H)(A)$ is the usual semi-direct product $G(A) \ltimes H(A)$. As a k-functor $G \ltimes H$ is of course the direct product of G and H.

Let H, N be subgroup functors of G such that H normalizes N, i.e. that each $H(A)$ normalizes $N(A)$. We can then construct $H \ltimes N$ as above and get a homomorphism $\varphi: H \ltimes N \to G$ via $(h,n) \mapsto$

hn for all $h \in H(A)$, $n \in N(A)$ and all A. Its kernel is isomorphic to $H \cap N$ under $h \mapsto (h,h^{-1})$ for all $h \in H(A) \cap N(A)$ and all A. If φ is an isomorphism, then we say that G is the semi-direct product of H and N and write $G = H \ltimes N$. (If G is a k-group scheme and $G = H \ltimes N$, then necessarily H and N are closed subgroup schemes.)

2.7 (Representations) Let G be a k-group factor and M a k-module. A representation of G on M (or: a G-module structure on M) is an operation of G on the k-functor M_a (as in 2.2) such that each $G(A)$ operates on $M_a(A) = M \otimes A$ through A-linear maps. Such a representation gives for each A a group homomorphism $G(A) \to \mathrm{End}_A(M \otimes A)^{\times}$, leading to a homomorphism $G \to GL(M)$ of group functors. Vice versa, any such homomorphism defines a representation of G on M. There is an obvious notion of a G-module homomorphism (or G-equivariant map) between two G-modules M and M'. The k-module of all such homomorphisms is denoted by $\mathrm{Hom}_G(M,M')$.

The representations of G on the k-module k, for example, correspond bijectively to the group homomorphisms from G to $GL_1 = G_m$, i.e. to the elements of $X(G)$. For each $\lambda \in X(G)$ we denote k considered as a G-module via λ by k_λ. In case $\lambda = 1$ we simply write k.

Given one or several G-modules we can construct in a natural way other G-modules. For example

(1) Any direct sum of G-modules is a G-module in a natural way.

(2) The tensor product of two G-modules is a G-module in a natural way.

(3) Any symmetric and exterior power of a G-module is a G-module in a natural way.

In (3) for example we consider for each commutative ring R the functor F_R from R-modules to itself with $F_R(M) = S^n M$. We have for each R-algebra R' canonical isomorphisms $F_R(M) \otimes_R R' \simeq F_{R'}(M \otimes_R R')$ for all R-module M, i.e. the functors $M \mapsto F_R(M) \otimes_R R'$ and $M \mapsto F_{R'}(M \otimes_R R')$ are isomorphic. If M is a G-module, then G operates on the functor $A \mapsto F_A(M \otimes A)$, each $g \in G(A)$ via $F_A(g)$. By our assumption this functor is isomorphic to $F_k(M)_a$, hence we get a G-module structure on $F_k(M)$. The functor $M \mapsto \Lambda^n M$ has the same property, hence we can argue as above. Our reasoning can easily be extended to functors in several variables and then yields (1),(2).

If we deal with contravariant functors $(F_R)_R$ in our situation above, we ought to let $g \in G(A)$ act via $F_A(g^{-1})$. This applies to the functor $M \mapsto M^*$ which will however "commute with ring extensions" only when restricted to finitely generated and projective modules. Thus we get

(4) Let M be a G-module which is finitely generated and projective over k. Then M^* is a G-module in a natural way.

For M as in (4) one has canonically $M^* \otimes M' \simeq \mathrm{Hom}(M,M')$ for any k-module M'. So we get combining (2) and (4)

(5) Let M,M' be G-modules with M finitely generated and

<u>and projective over</u> k. <u>Then</u> $Hom(M,M')$ <u>is a</u> G-<u>module in a natural way</u>.

The following result is obvious from the definitions:

(6) <u>Let</u> k' <u>be a</u> k-<u>algebra and</u> M <u>a</u> G-<u>module. Then</u> $M \otimes k'$ <u>is a</u> $G_{k'}$-<u>module in a natural way</u>.

Another way, how representations arise, is from an operation of G on an affine scheme X. Then we get a G-module structure on $k[X]$: If $g \in G(A)$ and $f \in k[X] \otimes A = A[X_A]$ for some k-algebra A, then $gf \in A[X_A]$ is defined through $(gf)(x) = f(g^{-1}x)$ resp. $= f(xg)$ (for a left resp. right operation) for all $x \in X(A') = X_A(A')$ and all A-algebras A'. (Again, the g in $g^{-1}x$ or xg is really the image of g under $G(A) \to G(A')$...).

In case G is a k-group scheme we get thus the <u>left</u> and <u>right regular representations</u> of G on $k[G]$ derived from the action of G on itself by left and right multiplications. We shall always denote the corresponding homomorphisms $G \to GL(k[G])$ by ρ_ℓ and ρ_r. The coinverse σ_G is an isomorphism of G-modules from $k[G]$ with ρ_r to $k[G]$ with ρ_ℓ. Furthermore the conjugation action of G on itself gives rise to the <u>conjugation representation</u> of G on $k[G]$.

<u>2.8</u> (<u>The Comodule Map</u>) Let G be a k-group scheme. If M is a G-module then $id_{k[G]} \in G(k[G]) = End_{k\text{-}alg}(k[G])$ acts on $M \otimes k[G]$, so we get a k-linear map $\Delta_M: M \to M \otimes k[G]$ with $\Delta_M(m) = id_{k[G]}(m \otimes 1)$ for all $m \in M$. We call Δ_M the comodule map of the

G-module M. It determines the representation of G on M completely: For any k-algebra A and any $g \in G(A) = \mathrm{Hom}_{k\text{-alg}}(K[G],A)$ we have a commutative diagram

$$\begin{array}{ccc} G(k[G]) \times (M \otimes k[G]) & \longrightarrow & M \otimes k[G] \\ \Big\downarrow {\scriptstyle G(g) \times (id_M \otimes g)} & & \Big\downarrow {\scriptstyle id_M \otimes g} \\ G(A) \times (M \otimes A) & \longrightarrow & M \otimes A \end{array}$$

by the functorial property of an operation. As $G(g)\varphi = g \circ \varphi$ for any $\varphi \in G(k[G])$, we have $g = G(g)id_{k[G]}$, hence $g(m \otimes 1) = (id_M \otimes g) \circ \Delta_M(m)$ for all $m \in M$. More explicitly, if $\Delta_M(m) = \sum_{i=1}^{r} m_i \otimes f_i$, then

(1) $$g(m \otimes 1) = \sum_{i=1}^{r} m_i \otimes f_i(g).$$

The fact that each $G(A)$ operates on $M \otimes A$ (i.e. $g(g'm) = (gg')m$ and $1m = m$) yields easily the following formulas:

(2) $$(\Delta_M \otimes id_{k[G]}) \circ \Delta_M = (id_M \otimes \Delta_G) \circ \Delta_M$$

and

(3) $$(id_M \otimes \varepsilon_G) \circ \Delta_M = id_M.$$

If M' is another G-module and if $\varphi: M \to M'$ is a linear map, then φ is a homomorphism of G-modules if and only if

(4) $$\Delta_{M'} \circ \varphi = (\varphi \otimes id_{k[G]}) \circ \Delta_M.$$

A <u>comodule</u> over the Hopf algebra $k[G]$ is a k-module M

together with a linear map $\Delta_M\colon M \to M \otimes k[G]$ such that (2),(3) are satisfied. A homomorphism between two comodules is a linear map satisfying (4). So we have defined a faithful functor from {G-modules} to {k[G]-comodules}. On the other hand, any $k[G]$-comodule gives rise to a G-module: Just take (1) as a definition. In this way we can see that the two categories of G-modules and of k[G]-comodules are equivalent.

Let $\alpha\colon X\times G \to X$ be an action of G on an affine scheme X over k. Then $k[X]$ is a G-module in a natural way (see 2.7) and the comodule map $\Delta_{k[X]}\colon k[X] \to k[X] \otimes k[G]$ is easily checked to be the comorphism α^*. If we take $X = G$ and the action by right multiplication, we get thus

(5) $\quad \Delta_{\rho_r} = \Delta_G.$

(We write Δ_{ρ_r} and also Δ_{ρ_ℓ} below instead of $\Delta_{k[G]}$ in order to indicate which representation is considered.) For the left regular action one gets

(6) $\quad \Delta_{\rho_\ell} = s\circ(\sigma_G \otimes \mathrm{id}_{k[G]})\circ\Delta_G$

with $s(f\otimes f') = f'\otimes f$ for all f,f'. For the conjugation representation on $k[G]$ the comodule map is equal to

(7) $\quad t'\circ(\mathrm{id}_{k[G]}\otimes\Delta_G)\circ\Delta_G$

where $t'(f_1\otimes f_2\otimes f_3) = f_2\otimes\sigma_G(f_1)f_3$

<u>Remark</u>: Suppose for the moment that k is an algebraically closed field and that G is a reduced algebraic k-group. There is

a natural notion of representations of $G(k)$ as an algebraic group (or of a rational $G(k)$-module), cf. [Hu], p. 60. One can show as above that the category of $G(k)$-modules is equivalent to the category of comodules over $k[G(k)] = k[G]$, hence to that of G-modules. (To a G-module M we associate the operation of $G(k)$ on M given by the definition of a G-module.) Similarly one can show that the notions of G-submodules (to be defined in 2.9) and of $G(k)$-submodules coincide, using 2.9(1), and that $M^{G(k)} = M^G$ (to be defined in 2.10), using 2.10(2). Furthermore, one has $\mathrm{Hom}_G(M,M') = \mathrm{Hom}_{G(k)}(M,M')$ for any two G-modules M,M' (using (4) above).

2.9 (Submodules) Let G be a k-group functor. If k is a field, we can define a submodule of a G-module M as a subspace $N \subset M$ such that $N \otimes A$ is a $G(A)$-stable submodule of $M \otimes A$ for each k-algebra A. Then N itself is a G-module in a natural way. For arbitrary k this works out well as long as the natural map $N \otimes A \to M \otimes A$ is injective for each A, e.g. if N is a direct summand of M. Taking only such "pure" submodules (as in [DG], II, 1.3/4) will be too restrictive and not allow kernels and images of all homomorphisms.

So let us define a submodule of a G-module M to be a k-submodule N of M which has itself a G-module structure such that the inclusion of N into M is a homomorphism of G-modules. If so, then M/N has a natural structure as G-module: We have for each A an exact sequence of $G(A)$-modules $N \otimes A \to M \otimes A \to (M/N) \otimes A \longrightarrow 0$. We call M/N the factor module of M by N. It has the usual property of a factor module.

Still, our definition of a submodule has one disadvantage. A given k-submodule N of M may conceivably carry more than one structure as a G-module. In order to prevent this we shall prefer to make special assumptions on our group and not on the modules.

An affine scheme X over k is called <u>flat</u> if $k[X]$ is a flat k-module. A k-group scheme is called flat, if it is so as an affine scheme. This property is obviously preserved under base change.

Assume now that G is a flat k-group scheme. If N is a submodule of a G-module M, then $N \otimes k[G]$ is a $G(k[G])$-stable submodule of $M \otimes k[G]$ (by our assumption of flatness). Then we get obviously

(1) $\Delta_M(N) \subset N \otimes k[G]$

and

(2) $\Delta_N = (\Delta_M)_{|N}$.

The second equality implies together with 2.8 that the G-module structure on N is unique. On the other hand, if N is a k-submodule of M satisfying (1), then (2) defines a G-module structure on N and N is a G-submodule of M. So the G-submodules of M are exactly the k-submodules N satisfying (1).

Using 2.8(4) one checks now easily:

(3) <u>Let</u> G <u>be a flat</u> k-<u>group scheme. For each homomorphism</u> $\varphi\colon M \to M'$ <u>of</u> G-<u>modules its kernel</u> $\ker(\varphi)$ <u>and its image</u> $\operatorname{im}(\varphi)$

are G-submodules of M resp. M'.

We get from this that the G-modules

form an abelian category (for G flat). Under the same assumption intersections and sums of submodules are again submodules. Note that inductive limits exist in the category of G-modules (for G-flat): Just take the inductive limit as k-modules: This is a factor module of the direct sum (which is O.K. by 2.7(1)) where we divide by a sum of images of homomorphisms.

2.10 (Fixed Points) Let G be a k-group scheme and M a G-module. Set

(1) $M^G = \{m \in M \mid g(m \otimes 1) = m \otimes 1$ for all $g \in G(A)$ and all $A\}$.

This is a k-submodule of M and its elements are called the fixed points of G on M. We call M a trivial G-module, if $M = M^G$. In the notations of 2.6 one has $M^G = (M_a)^G(k)$. If we take $g = \mathrm{id}_{k[G]} \in G(k[G])$ in (1), then we get

(2) $M^G = \{m \in M \mid \Delta_M(m) = m \otimes 1\}$.

This description of M^G as kernel of $\Delta_M - \mathrm{id}_M \otimes 1$ yields

(3) Let k' be a k-algebra which is flat as a k-module. Then $(M \otimes k')^{G_{k'}} = M^G \otimes k'$.

In case k is a field, this implies of course $(M_a)^G = (M^G)_a$. (See [DG], II, §2, 1.6 for a generalization to k-group functors.)

If $\varphi: M \to M'$ is a homomorphism of G-modules, then obviously $\varphi(M^G) \subset (M')^G$. In this way $M \to M^G$ is a functor from {G-modules} to {k-modules} which we call fixed point functor (relative to G).

It is certainly additive. We get from (2):

(4) *If* G *is flat, then the fixed point functor is left exact.*

Furthermore, it commutes with taking direct sums, intersections of submodules, and direct limits (but in general not with arbitrary inductive limits).

If we consider $k[G]$ as a G-module via the left or right regular representation, then the definition immediately yields:

(5) $k[G]^G = k1$ (for ρ_ℓ and ρ_r).

Let M' be another G-module and suppose that M is finitely generated and projective over k. We can then regard $\mathrm{Hom}(M,M')$ as a G-module and get easily

(6) $\mathrm{Hom}(M,M')^G = \mathrm{Hom}_G(M,M')$.

Therefore (3) implies

(7) *Let* k' *be a flat* k-*algebra and let* M *be finitely generated and projective as a* k-*module. Then the canonical map*

$$\mathrm{Hom}_G(M,M') \otimes k' \to \mathrm{Hom}_{G_{k'}}(M\otimes k', M'\otimes k')$$

is an isomorphism for all G-*modules* M'.

This generalizes to the case where M is a direct limit of such modules, hence to all M when k is a field, and to all flat M when k is a Dedekind ring.

We can generalize (1)-(4) as follows. For each $\lambda \in X(G)$ set

(1') $M_\lambda = \{m \in M | g(m \otimes 1)| = m \otimes \lambda(g)$ for all $g \in G(A)$ and all $A\}$.

Then:

(2') $M_\lambda = \{m \in M | \Delta_M(m) = m \otimes \lambda\}$.

(3') For k' as in (3) we have $(M \otimes k')_{\lambda \otimes 1} = M_\lambda \otimes k'$.

(4') If G is flat, then the functor $M \longmapsto M_\lambda$ is left exact.

Furthermore, we have

(8) If k is a field, then the sum of all M_λ is direct.

(If $\sum_\lambda a_\lambda m_\lambda = 0$, where $a_\lambda \in k, m_\lambda \in M_\lambda$, then $0 = \Delta_M(\sum_\lambda a_\lambda m_\lambda)$ $= \sum_\lambda a_\lambda m_\lambda \otimes \lambda$. Now apply 2.4(5).)

2.11 (Representations of Diagonalizable Group Schemes) Let Λ be a commutative group and take $G = \mathrm{Diag}(\Lambda)$ as in 2.5. As $k[G]$ is a free k-module with basis Λ we can write the comodule map Δ_M for any G-module M as

(1) $\Delta_M(m) = \sum_{\lambda \in \Lambda} p_\lambda(m) \otimes \lambda$

for suitable $p_\lambda \in \mathrm{End}(M)$. Using the description of Δ_G, ε_G in 2.5 and the formulas 2.8(2),(3) one checks easily (cf. [DG], II, §2, 2.5) that $\sum_{\lambda \in \Lambda} p_\lambda = \mathrm{id}_M$ and $p_\lambda p_{\lambda'} = 0$ for $\lambda \neq \lambda'$ and $p_\lambda^2 = p_\lambda$ for all λ. This implies that M is the direct sum of all $p_\lambda(M)$, that

(2) $p_\lambda(M) = \{m \in M | \Delta_M(m) = m \otimes \lambda\} = M_\lambda$

(using 2.10(6)) and

(3) $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$.

It follows easily that for all G-modules M,M'

(4) $\mathrm{Hom}_G(M,M') \simeq \prod_{\lambda\in\Lambda} \mathrm{Hom}(M_\lambda, M'_\lambda)$

and that the functor $M \mapsto M_\lambda$ is exact for all λ.

If we consider for example $k[G]$ und ρ_r then we get

(5) $k[G]_\lambda = k\lambda$ for all $\lambda \in \Lambda$.

Let $(e(\lambda))_{\lambda\in\Lambda}$ be the canonical basis of the group ring $\mathbb{Z}[\Lambda]$ over $\mathbb{Z}$. So $e(\lambda)e(\lambda') = e(\lambda+\lambda')$, if we agree to write Λ additively. If M is a G-module such that each M_λ is a finitely generated projective k-module, then we define its formal character

(6) $\mathrm{ch}\, M = \sum_{\lambda\in\Lambda} \mathrm{rk}(M_\lambda)e(\lambda)$.

For an exact sequence $0 \to M' \to M \to M'' \to 0$ of G-modules of this type one has

(7) $\mathrm{ch}\, M = \mathrm{ch}\, M' + \mathrm{ch}\, M''$.

For two G-modules M_1, M_2 of this type, also $M_1 \otimes M_2$ has this property and one has

(8) $\mathrm{ch}(M_1 \otimes M_2) = (\mathrm{ch}\, M_1)(\mathrm{ch}\, M_2)$.

One uses for (8) that for any M_1, M_2 and all $\lambda, \lambda' \in \Lambda$

(9) $(M_1)_\lambda \otimes (M_2)_{\lambda'} \subset (M_1 \otimes M_2)_{\lambda+\lambda'}$

(One can generalize (6) to the case where the M_λ are only assumed to be finitely generated over k and where we replace $\mathbb{Z}$ by the Grothendieck group of these k-modules.)

If k' is a k-algebra, then one has obviously for all λ:

(10) $\quad (M_{k'})_\lambda = (M_\lambda) \otimes k'.$

If $ch(M)$ is defined, then so is $ch(M_{k'})$ and it is equal to $ch(M)$.

<u>2.12</u> (<u>Centralizers and Stabilizers</u>) Let G be a k-group scheme and M a G-module.

For any subset $S \subset M$ we define its <u>centralizer</u> $Z_G(S)$ as the subgroup functor of G with

(1) $\quad Z_G(S)(A) = \{g \in G(A) \mid g(m \otimes 1) = m \otimes 1 \quad \text{for all} \quad m \in S\}.$

Obviously $Z_G(S)$ depends only on the k-module generated by S. It is equal to the intersection of all $Z_G(m)$ with $m \in S$.

For any k-submodule $N \subset M$ we define its <u>stabilizer</u> $\mathrm{Stab}_G(N)$ in G as the subgroup functor of G with

(2) $\quad \mathrm{Stab}_G(N)(A) = \{g \in G(A) \mid g(n \otimes 1) \in \overline{N \otimes A} \quad \text{for all} \quad n \in N\}.$

Here $\overline{N \otimes A}$ is the canonical image of $N \otimes A$ in $M \otimes A$.

For two k-submodules $N' \subset N$ of M we define another subgroup functor $G_{N',N}$ of G through

(3) $\quad G_{N',N}(A) = \{g \in G(A) \mid g(n \otimes 1) - n \otimes 1 \in \overline{N' \otimes A} \quad \text{for all} \quad n \in N\}.$

Obviously $G_{O,N} = Z_G(N)$ and $G_{N,N} = \mathrm{Stab}_G(N)$.

Suppose that N' and M are free k-modules and that N' is a direct summand. Choose a basis $(e_i)_{i\in I}$ of M containing a basis $(e_j)_{j\in J}$ for some $J\subset I$ of N'. For any $n \in N$ there are $a_i(n) \in k$ and $f_{i,n} \in k[G]$ (almost all 0 in both cases) with $n = \sum_{i\in I} a_i(n)e_i$ and $\Delta_M(n) = \sum_{i\in I} e_i\otimes f_{i,n}$. Then $g(n\otimes 1) - n\otimes 1 = \sum_{i\in I}(f_{i,n}(g)-a_i(n))e_i$. Then $G_{N',N}$ is the closed subgroup scheme defined by the ideal generated by all $f_{j,n} - a_j(n)1$ with $j \in J$ for all possible n. We get:

(4) _If_ M _is a free_ k-_module, then each_ $Z_G(S)$ _is a closed subgroup scheme of_ G.

(5) _If_ M _and_ N _are free_ k-_modules and if_ N _is a direct summand of_ M, _then_ $\mathrm{Stab}_G(N)$ _is a closed subgroup scheme of_ G.

(6) _If_ M _and_ N' _are free_ k-_modules and if_ N' _is a direct summand of_ M, _then_ $G_{N',N}$ _is a closed subgroup scheme of_ G.

The assumptions on M and N,N' are always satisfied over a field. In general one can replace "free" by "projective of finite rank", see [DG], II, §2, 1.4.

_2.13_ (_Local Finiteness_) Let G be a _flat_ k-group scheme and M a G-module.

We know that any intersection of G-submodules of M is again a G-submodule. So for each subset S of M there is a smallest G-submodule of M containing S. It is called the G-submodule

<u>generated</u> by S and usually denoted by kGS. (Note that in general $kGS \neq kG(k)S$, the k-$G(k)$-submodule of M generated by S.)

Now take $m \in M$ and write $\Delta_M(m) = \sum_{i=1}^{r} m_i \otimes f_i$ with $m_i \in M$ and $f_i \in k[G]$. We claim

(1) $\quad kGm \subset \sum_{i=1}^{r} km_i.$

Let us write $M' = \sum_{i=1}^{r} km_i$. As $1m = m$ we have $m = \sum_{i=1}^{r} f_i(m)m_i \in M$. The same argument proves $N \subset M'$ where we set $N = \{m_1 \in M \mid \Delta_M(m_1) \in M' \otimes k[G]\}$. Obviously $m \in N$. So it will be enough to show that N is a G-submodule of M, i.e. that $\Delta_M(N) \subset N \otimes k[G]$. By definition $N = \Delta_M^{-1}(M' \otimes k[G])$. Using the flatness of $k[G]$ we get $N \otimes k[G] = (\Delta_M \otimes \mathrm{id}_{k[G]})^{-1}(M' \otimes k[G] \otimes k[G])$. Therefore it is enough to show $(\Delta_M \otimes \mathrm{id}_{k[G]})\Delta_M(N) \subset M' \otimes k[G] \otimes k[G]$. By 2.8(2) the left hand side is equal to $(\mathrm{id}_M \otimes \Delta_G)\Delta_M(N) \subset (\mathrm{id}_M \otimes \Delta_G)(M' \otimes k[G]) \subset M' \otimes k[G] \otimes k[G]$.

As kGm is a G-submodule we have $\Delta_M(m) \in (kGm) \otimes k[G]$. We therefore may choose the m_i above all in kGm. Then $kGm = \sum_{i=1}^{r} km_i$. This shows:

(2) <u>Each</u> kGm <u>with</u> $m \in M$ <u>is a finitely generated</u> k-<u>module</u>

and:

(3) <u>Each finitely generated</u> k-<u>submodule in</u> M <u>is contained in a</u> G-<u>submodule of</u> M <u>which is finitely generated over</u> k.

This property is usually expressed as "any G-module is locally finite".

In the case of a field one can show:

(4) If k is a field and if $\Delta_M(m) = \sum_{i=1}^{r} m_i \otimes f_i$ with $(f_i)_{1 \le i \le r}$ linearly independent, then $kGm = \sum_{i=1}^{r} km_i$

(We may assume that also the m_j are linearly independent. If $(m_j')_{1 \le j \le s}$ is a basis of kGm then there are $a_{ji} \in k$ with $m_j' = \sum_{i=1}^{r} a_{ji} m_i$ for all j (by (1)) and there are $f_j' \in k[G]$ with $\Delta_M(m) = \sum_{j=1}^{s} m_j' \otimes f_j' = \sum_{i=1}^{r} m_i \otimes (\sum_{i=1}^{r} a_{ji} f_j')$, hence $f_i = \sum_{j=1}^{s} a_{ji} f_j'$ for all i. Hence $r = s$ and the claim.)

2.14 (Simple Modules) In this subsection we assume that k is a field. Let G be a k-group scheme.

As usual a G-module M is called simple (and the corresponding representation is called irreducible) if $M \neq 0$ and if M has no G-submodules other than 0 and M. It is called semi-simple if it is a direct sum of simple G-submodules. For any M the sum of all its simple submodules is called the socle of M and denoted by $\operatorname{soc}_G M$ (or simple by $\operatorname{soc} M$, if it is clear which G is considered). It is the largest semi-simple G-submodule of M. For a given simple G-module E the sum of all simple G-submodules of M isomorphic to E is called the E-isotypic component of $\operatorname{soc}_G M$ (or the isotypic component of type E) and denoted by $(\operatorname{soc}_G M)_E$.

By 2.13(3) each element in a G-module is contained in a finite dimensional submodule. This implies:

(1) Each simple G-module is finite dimensional.

(2) If M is a G-module with $M \neq 0$, then $\mathrm{soc}_G M \neq 0$.

For any G-module M and any simple G-module E the map $\varphi \otimes e \mapsto \varphi(e)$ is an isomorphism

(3) $\mathrm{Hom}_G(E,M) \otimes_D E \xrightarrow{\sim} (\mathrm{soc}_G M)_E$ where $D = \mathrm{End}_G(E)$.

(Of course the algebra D over k is finite dimensional and a skew field by Schur's lemma. If k is algebraically closed, then $D = k$.)

Each one-dimensional representation is irreducible. The isotypic component of $\mathrm{soc}_G M$ of type k_λ is just M_λ. We get especially $M^G = (\mathrm{soc}_G M)_k$.

The discussion in 2.11 shows:

(4) If G is a diagonalizable k-group scheme, then each G-module is semi-simple.

The socle series or (ascending) Loewy series of M $0 \subset \mathrm{soc}_1 M = \mathrm{soc}_G M \subset \mathrm{soc}_2 M \subset \mathrm{soc}_3 M \subset \dots$ is defined iteratively through $\mathrm{soc}(M/\mathrm{soc}_{i-1}M) = \mathrm{soc}_i M/\mathrm{soc}_{i-1}M$. Again because of 2.13(3) one has

(5) $$\bigcup_{i>0} \mathrm{soc}_i M = M.$$

Any finite dimensional G-module M has a composition series (or Jordan-Hölder series). The number of factors isomorphic to a given simple G-module E is independent of the choice of the series. It is called the <u>multiplicity</u> of E as composition factor of M and usually denoted by $[M:E]$ or $[M:E]_G$.

If G is an algebraic k-group, then it is called <u>trigonalizable</u> (resp. <u>unipotent</u>), if it is isomorphic to a closed subgroup of T_n (resp. U_n) for some $n \in \mathbb{N}$ (cf. 2.2). One can show ([DG], IV, §2, 2.5 and 3.4):

(6) G <u>trigonalizable</u> $\Longleftrightarrow$ <u>Each simple</u> G-<u>module has dimension one</u>.

(7) G <u>unipotent</u> $\Longleftrightarrow$ <u>Up to isomorphism</u> k <u>is the only simple</u> G-<u>module</u>.

If we assume G be to an arbitrary k-group scheme, then we may take these results as definitions. For unipotent G we deduce $\mathrm{soc}_G M = M^G$ for each G-module. We get using (2):

(8) G <u>unipotent</u> $\Longleftrightarrow$ <u>For each</u> G-<u>module</u> $M \neq 0$ <u>we have</u> $M^G \neq 0$.

Any decomposition of M into a direct sum of two submodules leads to the corresponding decomposition of $\mathrm{soc}\, M$. If $\mathrm{soc}\, M$ is simple, then M has to be indecomposable. Therefore (8) and 2.10(10) imply

(9) <u>If</u> G <u>is unipotent, then</u> $k[G]$ <u>is indecomposable</u> <u>(for</u> ρ_ℓ <u>and</u> ρ_r).

3. Induction and Injective Modules

In the representation theory of finite groups or of Lie groups the process of inducing representations from a subgroup to the whole group is an important technique. The same holds for algebraic group schemes. So we start this section with the necessary definitions (3.3), prove elementary properties (3.4-3.6) and describe some easy special cases (3.7/8). All this is a more or less straightforward generalization of what is done in the finite group case or the Lie group case. We have however to assume that the group G and its subgroup are flat.

We then use the induction functor to show that the category of G-modules contains enough injective objects, i.e. that each G-module can be embedded into an injective one (3.9).

In the case where our ground ring k is a field we can be more precise. Then the injective G-modules are determined up to isomorphism by their socle and any semi-simple G-module M occurs as a socle of such an injective G-module, the injective hull of M. The indecomposable injective G-modules are just the injective hulls of the simple G-modules. We get especially a decomposition of $K[G]$ generalizing the decomposition of the regular representation of a finite group into principal indecomposable modules. (These results are proved in 3.10-3.17.)

Let me mention as a source [Green 1] for the last part (3.12-3.17). For the first part one may compare [Haboush 2], [Cline/Parshall/Scott 3] or [Donkin 1]. (There is not much point in attributing priorities for these generalizations.)

We assume from 3.2 on that G is a flat k-group scheme and from 3.10 on that k is a field.

3.1 (Restriction) Let G be a k-group functor and H a subgroup functor of G. Each G-module M is an H-module in a natural way: Restrict the action of $G(A)$ for each k-algebra A to $H(A)$. We get in this way a functor

$$\operatorname{res}_H^G: \{G\text{-modules}\} \longrightarrow \{H\text{-modules}\}$$

which is obviously exact. It commutes with the elementary operations on G-modules described in 2.7(1)-(4).

If G and H are group schemes, then we get the comodule map for $\operatorname{res}_H^G M$ from Δ_M as $(\mathrm{id}_M \otimes \gamma) \circ \Delta_M$ where $\gamma: k[G] \to k[H]$ is the restriction of functions.

3.2 Lemma: *Let H, H' be subgroup schemes of a k-group functor G such that H' normalizes H and is flat. Let M be a G-module. Then M^H is an H'-submodule of M.*

Proof: It is easy to check that the comodule map $\Delta_M: M \to M \otimes k[H]$ of M considered as an H-module is a homomorphism of H'-modules, if we regard $k[H]$ as an H'-module under the conjugation action. The same holds for the map $m \mapsto \Delta_M(m) - m \otimes 1$. Therefore its kernel M^H is an H'-submodule.

3.3 (Induction) Let H be a subgroup scheme of G. For each H-module M there is a natural $(G \times H)$-module structure on

$M \otimes k[G]$: Let G operate trivially on M and via the left regular representation on $k[G]$, let H operate as given on H and via the right regular representations on $k[G]$, and then take tensor products. Now $(M \otimes k[G])^H$ is a G-submodule of $M \otimes k[G]$ by lemma 3.2. We denote this G-module by $\text{ind}_H^G M$ and call it the <u>induced module</u> of M from H to G. Obviously

$$\text{ind}_H^G: \ \{H\text{-modules}\} \longrightarrow \{G\text{-modules}\}$$

is a functor.

Let us mention that we can interprete the operation of $G \times H$ on $M \otimes k[G]$ in a different way. We have $M \otimes k[G] = M_a(k[G]) \cong \text{Mor}(G, M_a)$ by 1.3 and more generally $(M \otimes k[G]) \otimes A \cong (M \otimes A) \otimes_A (k[G] \otimes A) = (M \otimes A) \otimes_A A[G_A] \cong \text{Mor}_A(G_A, (M \otimes A)_a)$ for each k-algebra A. Any $(g,h) \in G(A) \times H(A)$ acts on some $f \in \text{Mor}_A(G_A, (M \otimes A)_a)$ through

(1) $\quad ((g,h)f)(x) = h(f(g^{-1}xh))$

for all $x \in G(A')$ and all A-algebras A'. (Let me remind you that there is some abuse of notation going on: We really ought to write $((g,h)f)(A')(x) = h_{A'} f(A')(g_{A'}^{-1} x h_{A'})$ with $g_{A'} \in G(A')$ the image of g under the map $G(A) \to G(A')$ defined by the structural map $A \to A'$, similarly for $h_{A'}$.) In this interpretation we have

(2) $\quad \text{ind}_H^G M = \{f \in \text{Mor}(G, M_a) \mid f(gh) = h^{-1} f(g)$ for all

$\qquad\qquad g \in G(A), h \in H(A)$ and all k-algebras $A\}$

and the operation of G is by left translation (in a natural sense).

3.4 Proposition: Let H be a flat subgroup scheme of G.

a) The functor ind_H^G is left exact.

b) The functor ind_H^G commutes with forming direct sums, intersections of submodules, and direct limits.

Proof: a) As we assume G to be flat, the functor $M \mapsto M \otimes k[G]$ is exact. Therefore the claim follows from 2.10(4).

b) All these constructions commute with tensoring with a flat k-module and with the fixed point functor (cf. 2.10).

Remark: If the fixed point functor $?^H$ is exact, then obviously also ind_H^G is exact. So ind_H^G is certainly exact whenever H is diagonalizable (by 2.11).

3.5 For any k-module M let $\varepsilon_M\colon M \otimes k[G] \to M$ be the linear map $\varepsilon_M = \mathrm{id}_M \otimes \varepsilon_G$. If we take the identification $M \otimes k[G] \simeq \mathrm{Mor}(G, M_a)$ we have $\varepsilon_M(f) = f(1)$. We shall use the notation ε_M also for the restrictions of ε_M to various submodules of $M \otimes k[G]$.

Proposition (Frobenius Reciprocity): Let H be a flat subgroup scheme of G and M an H-module.

a) $\varepsilon_M\colon \mathrm{ind}_H^G M \to M$ is a homomorphism of H-modules.

b) For each G-module N the map $\varphi \mapsto \varepsilon_M \circ \varphi$ is an isomorphism

$$\mathrm{Hom}_G(N, \mathrm{ind}_H^G M) \xrightarrow{\sim} \mathrm{Hom}_H(\mathrm{res}_H^G N, M).$$

<u>Proof</u>: a) We have for all A, all $h \in H(A)$ and $f \in \operatorname{ind}_H^G M$:

$$(\varepsilon_M \otimes \mathrm{id}_A)(hf) = (hf)(1) = f(h^{-1}) = h(f(1)) = h(\varepsilon_M(f) \otimes 1).$$

b) In order to define an inverse consider for each $\psi \in \operatorname{Hom}_H(N,M)$ and any $x \in N$ the morphism $\tilde{\psi}(x) \in \operatorname{Mor}(G,M_a)$ with $\tilde{\psi}(x)(g) = (\psi \otimes \mathrm{id}_A)(g^{-1}(x \otimes 1))$ for all A and all $g \in G(A)$. Using the description in 3.3(2) one checks easily that $\tilde{\psi}(x) \in \operatorname{ind}_H^G M \subset \operatorname{Mor}(G,M_a)$. Another straightforward calculation shows $\tilde{\psi} \in \operatorname{Hom}_G(N, \operatorname{ind}_H^G M)$ and that the maps $\psi \mapsto \tilde{\psi}$ and $\varphi \mapsto \varepsilon_M \circ \varphi$ are inverse to each other.

<u>3.5</u> (<u>Transitivity of Induction</u>) The last result implies of course (for G,H as above):

(1) <u>The functor</u> ind_H^G <u>is right adjoint to</u> res_H^G.

This of course determines ind_H^G uniquely up to isomorphisms. (One can also say that the pair $(\operatorname{ind}_H^G M, \varepsilon_M)$ is uniquely determined up to isomorphism by 3.4.b.)

Let H' be another flat subgroup scheme of G with $H \subset H'$. We have obviously $\operatorname{res}_H^{H'} \circ \operatorname{res}_{H'}^G = \operatorname{res}_H^G$. Therefore (1) yields:

(2) <u>There is an isomorphism</u> $\operatorname{ind}_{H'}^G \circ \operatorname{ind}_H^{H'} \cong \operatorname{ind}_H^G$ <u>of functors</u>.

We can express this also in this way: <u>Induction is transitive</u>. For any H-module M we can write down isomorphisms $\operatorname{ind}_H^G M \cong \operatorname{ind}_{H'}^G \circ \operatorname{ind}_H^{H'} M$ explicitly. To any $f \in \operatorname{ind}_H^G M$ we associate $\tilde{f} \in \operatorname{Mor}(G, (\operatorname{ind}_H^{H'} M)_a)$ with $\tilde{f}(g)(h') = f(gh')$ for all $g \in G(A)$, $h' \in H(A)$ and all A.

To any $f \in \mathrm{ind}_{H'}^{G}(\mathrm{ind}_{H}^{H'} M)$ we associate $\bar{f} \in \mathrm{Mor}(G, M_a)$ with $\bar{f}(g) = f(g)(1)$ for all $g \in G(A)$ and all A. The maps $f \mapsto \tilde{f}$ and $f \mapsto \bar{f}$ turn out to be inverse isomorphisms.

Observe that 2.10.(3) implies

(3) <u>Let</u> k' <u>be a flat</u> k-<u>algebra. Then we have for each</u> H-<u>module</u> M <u>a canonical isomorphism</u>

$$(\mathrm{ind}_H^G M) \otimes k' \cong \mathrm{ind}_{H_{k'}}^{G_{k'}} (M \otimes k').$$

<u>3.6</u> <u>Proposition</u> (The Tensor Identity): <u>Let</u> H <u>be a flat subgroup scheme of</u> G. <u>For any</u> G-<u>module</u> N <u>and any</u> H-<u>module</u> N <u>there is a canonical isomorphism of</u> G-<u>modules</u>

$$\mathrm{ind}_H^G (M \otimes \mathrm{res}_H^G N) \xrightarrow{\sim} (\mathrm{ind}_H^G M) \otimes N.$$

<u>Proof</u>: Both sides may be embedded into $\mathrm{Mor}(G, (M \otimes N)_a) \cong M \otimes N \otimes k[G]$ using 3.3(2), the left hand side as

$$L = \{f\colon G \to (M \otimes N)_a \mid f(gh) = (h^{-1} \otimes h^{-1}) f(g) \quad \text{for all } g,h\},$$

the right hand side as

$$R = \{f\colon G \to (M \otimes N)_a \mid f(gh) = (h^{-1} \otimes 1) f(g) \quad \text{for all } g,h\}.$$

Here "for all g,h" means "for all $g \in G(A), h \in H(A)$ and all A". We define two endomorphisms α, β of $\mathrm{Mor}(G, (M \otimes N)_a)$ through $(\alpha f)(g) = (1 \otimes g) f(g)$ and $(\beta f)(g) = (1 \otimes g^{-1}) f(g)$ for all g. Obviously they are isomorphisms and inverse to each other. A straightforward calculation shows $\alpha(L) \subset R$ and $\beta(R) \subset L$ and that α, β are G-equivariant for the two actions of G we consider.

(On L we have $gf = f(g^{-1}?)$ and on R we have $gf = (1\otimes g)f(g^{-1}?)$.) This implies the proposition.

Remark: We ought to express the proposition (the tensor identity) as saying: The functors $(M,N)\mapsto \operatorname{ind}_H^G(M\otimes \operatorname{res}_H^G N)$ and $(M,N)\mapsto (\operatorname{ind}_H^G M)\otimes N$ from {H-modules}×{G-modules} to {G-modules} are isomorphic.

3.7 (Trivial Examples) We can apply all this especially to the subgroup schemes $H = 1$ and $H = G$. The first case yields

(1) $\operatorname{ind}_1^G M = M\otimes k[G]$ for any k-module M

(where M is considered as a trivial G-module on the right hand side), especially

(2) $\operatorname{ind}_1^G k = k[G]$.

(Here and below $k[G]$ is considered as a G-module via ρ_ℓ.) Combining (2) with 3.5.b (Frobenius reciprocity) we get for any G-module M

(3) $\operatorname{Hom}_G(M,k[G]) \simeq M^*$.

(This can also be shown directly using matrix coefficients, cf. [DG], §2, 2.3.) Taking $M = k$ in 3.6 we get for each G-module N an isomorphism

(4) $N\otimes k[G] \xrightarrow{\sim} N_{tr}\otimes k[G]$

where N_{tr} denotes the k-module N considered as a trivial G-module. Going back into the proof and the definitions one checks that this isomorphism is given by

$$x \otimes f \mapsto (1 \otimes f)\Delta_N(x).$$

If we restrict this map to the G-submodule $N \otimes k1 \cong N$ of $N \otimes k[G]$ we see:

(5) $\Delta_N \colon N \to N_{tr} \otimes k[G]$ <u>is an injective homomorphism of</u> G-<u>modules</u>.

(This can be checked directly, of course).

As $\mathrm{res}_G^G M = M$ for each G-module M we have by 3.5(1) also (canonically)

(6) $M \xrightarrow{\sim} \mathrm{ind}_G^G M$ <u>for each</u> G-<u>module</u> M.

This isomorphism $M \xrightarrow{\sim} (M \otimes k[G])^G \subset M \otimes k[G]$ is given by $(\mathrm{id}_M \otimes \sigma_G) \circ \Delta_M$. (In other words, any $m \in M$ is mapped to the morphism $G \to M_a$ with $g \mapsto g^{-1}(m \otimes 1)$ for all $g \in G(A)$ and all A.)

3.8 (<u>Induction and Semi-direct Products</u>) Let G' be a flat k-group scheme operating on G through automorphisms and let H be a flat subgroup scheme of G stable under G'. We can then form the semi-direct products $H \rtimes G'$ and $G \rtimes G'$ and we can regard $H \rtimes G'$ as a subgroup scheme of $G \rtimes G'$.

Let M be an $(H \rtimes G')$-module, i.e. a k-module which is simultaneously an H-module and a G'-module so that these two operations are compatible: $g'(hm) = (g'hg'^{-1})(g'm)$. Then G' acts naturally on $\mathrm{Mor}(G, M_a) \cong k[G] \otimes M$ via $(g'f)(g) = g'(f(g'^{-1}gg'))$, i.e. through the tensor product of the conjugation action with the given action on M. This defines a structure of an $(H \rtimes G')$-module and also of a $(G \rtimes G')$-module where H,G operate as usually in the construction of $\operatorname{ind}_H^G M$. As G' normalizes H, it operates also on $\operatorname{ind}_H^G M = \operatorname{Mor}(G,M_a)^H$, cf. 3.2. Therefore we get on $\operatorname{ind}_H^G M$ a structure as a $(G \rtimes G')$-module. We claim that we have an isomorphism of $(G \rtimes G')$-modules

$$(1) \quad \operatorname{ind}_H^G M \xrightarrow{\sim} \operatorname{ind}_{H \rtimes G'}^{G \rtimes G'} M.$$

We simply associate to $f \in \operatorname{ind}_H^G M \subset \operatorname{Mor}(G,M_a)$ the map $F \in \operatorname{ind}_{H \rtimes G'}^{G \rtimes G'} M \subset \operatorname{Mor}(G \rtimes G', M_a)$ with $F(g,g') = g'^{-1}F(g)$, and to any F the map f with $f(g) = F(g,1)$. The claim follows now from elementary calculations.

Taking $H = 1$ we get especially for any G'-module M:

$$(2) \quad \operatorname{ind}_{G'}^{G \rtimes G'} M \simeq k[G] \otimes M \simeq \operatorname{Mor}(G,M_a)$$

with G acting via ρ_ℓ on $k[G]$ and trivially on M and with G' acting via the conjugation action on $k[G]$ and as given on M.

We can also describe $\operatorname{ind}_G^{G \rtimes G'} N$ for any G-module N. There is an isomorphism

$$(3) \quad \operatorname{ind}_G^{G \rtimes G'} N \xrightarrow{\sim} \operatorname{Mor}(G',N_a) \simeq k[G'] \otimes N$$

mapping any $F \in \operatorname{ind}_G^{G \rtimes G'} N \subset \operatorname{Mor}(G \rtimes G', N_a)$ to $f\colon G' \to N_a$ with $f(g') = F(1,g')$ and any f to F with $F(g,g') = g'^{-1}gg'f(g')$. This isomorphism is compatible with the G'-action if we let G' act on $k[G']$ via ρ_ℓ and trivially on N. The action of G on some $f: G' \to N_a$ is given by $(gf)(g') = (g'gg'^{-1})f(g')$. This implies:

(4) If N is a trivial G-module, then G acts trivially on $\operatorname{ind}_G^{G \ltimes G'} N$.

3.9 We define an injective G-module to be an injective object in the category of all G-modules.

Proposition a) For each flat subgroup scheme H of G the functor ind_H^G maps injective H-modules to injective G-modules.

b) Any G-module can be embedded into an injective G-module.

c) A G-module M is injective if and only if there is an injective k-module I such that M is isomorphic to a direct summand of $I \otimes k[G]$ with I regarded as a trivial G-module.

Proof: a) This is obvious as ind_H^G is right adjoint to the exact functor res_H^G.

b) Let M be a G-module. We can embed M as a k-submodule into an injective k-module I. Then $I \otimes k[G] \simeq \operatorname{ind}_1^G I$ is injective by (a) and $\operatorname{ind}_1^G M \simeq M_{tr} \otimes k[G]$ is a submodule of $I \otimes k[G]$. Now combine this with the embedding of M into $M_{tr} \otimes k[G]$ from 3.7(4).

c) If M is injective, then the embedding $M \to I \otimes k[G]$ constructed in the proof of (b) has to split. This gives one direction in (c). The other is obvious, as $I \otimes k[G]$ is injective by (a), hence also each direct summand.

3.10 Let us assume from now on in chapter 3 that k is a field. Then we can simplify the last result:

Proposition a) *A* G-*module* M *is injective if and only if there is a vector space* V *over* k *such that* M *is isomorphic to a direct summand of* $V \otimes k[G]$ *with* V *regarded as a trivial* G-*module*.

b) *Any direct sum of injective* G-*modules is injective*.

c) *If* M,Q *are* G-*modules with* Q *injective, then* $M \otimes Q$ *is injective*.

Proof: (a) is just 3.9.c and (b) is an immediate consequence of (a). If Q is a direct summand of $V \otimes k[G]$ as in (a), then $M \otimes Q$ is a direct summand of $M \otimes V \otimes k[G]$, which is isomorphic to $M_{tr} \otimes V \otimes k[G]$ by 3.7(3). This yields (c).

3.11 Before looking at indecomposable injective G-modules in general, let us treat one important example.

Suppose $G = H \ltimes G'$ with H a diagonalizable and G' a unipotent group scheme. We set for each $\lambda \in X(H)$:

(1) $$Q_\lambda = \operatorname{ind}_H^G k_\lambda .$$

We have $k[G] \cong \operatorname{ind}_1^G k \cong \operatorname{ind}_H^G \operatorname{ind}_1^H \cong \operatorname{ind}_H^G k[H]$ by the transitivity of induction and $k[G] \cong \bigoplus_{\lambda \in X(H)} k_\lambda$ by 2.11(5) (also with respect to ρ_ℓ, of course), hence

(2) $$k[G] \cong \bigoplus_{\lambda \in X(H)} Q_\lambda .$$

We know by 3.8 that Q_λ is isomorphic to $k[G']$ when considered as a G'-module. Therefore 2.14(8) implies

(3) *Each* Q_λ *is an indecomposable and injective* G-*module*.

Each $\lambda \in X(H)$ can be extended to an element of $X(G)$ with G' in the kernel. We denote also this extension by and also the corresponding G-module by k_λ. For each G-module M the subspace $M^{G'}$ is a G-submodule by the remark to 3.2. Because of 2.11 it is a direct sum of one dimensional G-submodules of the form k_λ with $\lambda \in X(H)$. This shows especially that $M^{G'}$ is a semi-simple G-module. On the other hand, we have $M^{G'} \neq 0$ for any simple G-module because of 2.14(7). Therefore the k_λ with $\lambda \in X(H)$ are all simple G-modules (up to isomorphism) and we have

(4) $\quad \mathrm{soc}_G M = M^{G'}$

for any G-module M. The discussion in 3.8 shows that $Q_\lambda \simeq k_\lambda \otimes k[G']$ where H operates on $k[G']$ via the conjugation action. Then $(Q_\lambda)^{G'} \simeq k_\lambda \otimes (k[G']^{G'}) = k_\lambda \otimes k1 \simeq k_\lambda$, hence by (4):

(5) $\quad \mathrm{soc}_G Q_\lambda = k_\lambda$.

This shows that in this case there is for each simple G-module E an indecomposable and injective G-module with socle isomorphic to E. We want to generalize this result. At first we shall prove the uniqueness of such a module (up to isomorphism).

3.12 Proposition: *Let M, M' be injective G-modules and $\varphi \in \mathrm{Hom}_G(M,M')$. Then φ is an isomorphism, if and only if φ induces an isomorphism $\mathrm{soc}_G M \to \mathrm{soc}_G M'$.*

Proof: The "only if" part is obvious, so let us look at the "if". We know by 2.14(2) that

$$\ker \varphi \neq 0 \Longrightarrow 0 \neq \mathrm{soc}_G(\ker \varphi) = \ker(\varphi_{|\mathrm{soc}_G M}).$$

Assuming φ to induce an isomorphism of the socles we get $\ker \varphi = 0$ and the injectivity of φ. Therefore $\varphi(M) \simeq M$ is an injective G-module, hence a direct summand of M'. If M_1 is a G-stable complement, then $M' = \varphi(M) \oplus M_1$ implies $\mathrm{soc}_G(M') = \mathrm{soc}_G\varphi(M) \oplus \mathrm{soc}_G M_1$. The assumption $\mathrm{soc}_G M' = \varphi(\mathrm{soc}_G M)$ yields $\mathrm{soc}_G M_1 = 0$, hence $M_1 = 0$ by 2.14(2). Therefore φ is bijective.

<u>3.13</u> <u>Corollary</u>: <u>Two injective G-modules are isomorphic, if and only if their socles are isomorphic</u>.

<u>Proof</u>: Because of the injectivity any isomorphism of the socles can be extended to a homomorphism of the whole modules. Then apply 3.12.

<u>3.14</u> <u>Proposition</u>: <u>Let M be an injective G-module and let $\varphi_1 \in \mathrm{End}_G(\mathrm{soc}_G M)$ be idempotent. Then there is $\varphi \in \mathrm{End}_G(M)$ idempotent with</u> $\varphi_{|\mathrm{soc}_G M} = \varphi_1$.

<u>Proof</u>: Consider the socle series of M as in 2.14(5). Let us abbreviate $M_i = \mathrm{soc}_i M$. Each endomorphism of M has to preserve all M_i. Therefore the injectivity of M yields for each i an exact sequence

(1) $\quad 0 \to \underline{m}_i \to \mathrm{End}_G(M) \xrightarrow{\mathrm{res}} \mathrm{End}_G(M_i) \to 0$

where $\underline{m}_i$ is the two-sided ideal

(2) $\quad \underline{m}_i = \{\varphi \in \mathrm{End}_G(M) \mid \varphi(M_i) = 0\}.$

Any $\varphi \in \underline{m}_i$ maps M_j into M_{j-i} for all $j \geq i$. This implies

(3) $\underline{m}_i \underline{m}_j \subset \underline{m}_{i+j}$ for all $i,j \geq 1$.

We deduce from $M = \bigcup_{i \geq 1} M_i$ that

(4) $\mathrm{End}_G M = \varprojlim \mathrm{End}_G(M_i)$.

Therefore the proposition follows from a version of Hensel's lemma proved below.

<u>3.15</u> <u>Proposition</u>: <u>Let</u> R <u>be a ring and let</u> $\underline{m}_1 \supset \underline{m}_2 \supset \ldots$ <u>a chain of two-sided ideals of</u> R <u>with</u> $\underline{m}_i \underline{m}_j \subset \underline{m}_{i+j}$ <u>for all</u> $i,j \geq 1$ <u>and</u> $R \simeq \varprojlim R/\underline{m}_i$ <u>naturally. Then there is for each idempotent element</u> $e_1 \in R/\underline{m}_1$ <u>an idempotent element</u> $e \in R$ <u>with</u> $e_1 = e + \underline{m}_1$.

<u>Proof</u>: Because of $R \simeq \varprojlim R/\underline{m}_i$ it is enough to construct $e_2, e_3, \ldots \in R$ such that each $e_i + \underline{m}_i \in R/\underline{m}_i$ is idempotent and such that $e_i + \underline{m}_{i-1} = e_{i-1} + \underline{m}_{i-1}$ for each $i > 1$. We define iteratively $e_{i+1} = 2e_i(e_i - e_i^2) + e_i^2$. As $e_i + \underline{m}_i$ is assumed to be idempotent we have $e_{i+1} + \underline{m}_i = e_i^2 + \underline{m}_i = e_i + \underline{m}_i$. Furthermore we get $e_{i+1}^2 \in 4e_i^3(e_i - e_i^2) + e_i^4 + \underline{m}_i^2 = 3e_i(e_i - e_i^2) + e_i^2(e_i - e_i^2) + \underline{m}_i^2 \subset 2e_i(e_i - e_i^2) + e_i^2 + \underline{m}_{i+1}$, hence $e_{i+1} + \underline{m}_{i+1}$ is idempotent. Therefore we can go on.

<u>3.16</u> <u>Proposition</u>: a) <u>For each simple</u> G-<u>module</u> E <u>there is an injective</u> G-<u>module</u> Q_E <u>(unique up to isomorphism) with</u> $E \simeq \mathrm{soc}\, Q_E$.

b) <u>An injective</u> G-<u>module is indecomposable if and only if it is isomorphic to</u> Q_E <u>for some simple</u> G-<u>module</u> E.

c) Any injective G-module Q is a direct sum of indecomposable submodules. For each simple G-module E the number of summands isomorphic to E is equal to the multiplicity of E in $\mathrm{soc}_G Q$.

Proof: Let Q be an injective G-module. Any decomposition $\mathrm{soc}_G Q = M_1 \oplus M_2$ leads by 3.14 to a decomposition $Q = Q_1 \oplus Q_2$. As we can embed any G-module into an injective G-module by 3.9.b we get the existence of the Q_E in (a) immediately. The uniqueness follows from 3.13. The other parts of the proposition are now obvious.

3.17 The module Q_E from 3.16.a is called the injective hull of E. More generally we can find for each G-module M an injective G-module Q_M (unique up to isomorphism) with $\mathrm{soc}_G M \simeq \mathrm{soc}_G Q_M$. The embedding of $\mathrm{soc}_G M$ into Q can be extended to an embedding of M into Q_M. We call Q_M the injective hull of M. It is clear that this is compatible with the general definition e.g. in [2] , ch.X, §1, n° 9.

In the situation of 3.16.c the number of summands isomorphic to Q_E is equal to

$$\dim \mathrm{Hom}_G(E,Q)/\dim \mathrm{End}_G(E),$$

cf. 2.14(3). If we take especially $Q = k[G]$ we get from 3.7(3)

(1) $$k[G] \simeq \bigoplus_E Q_E^{d(E)}$$

where

(2) $\quad d(E) = \dim(E)/\dim(\mathrm{End}_G(E))$

and where the direct sum is taken over a system of representatives of all simple G-modules. (If k is algebraically closed, then $d(E) = \dim(E)$ of course.)

In the situation of 3.11 we have obviously $Q_\lambda = Q_{k_\lambda}$, and 3.11(2) illustrates (1) very well. In the case of an unipotent group one has $k[G] = Q_k$, cf. 2.14(9).

Let us mention one standard property of injective hulls: Let E be a simple G-module and M a finite dimensional G-module. Then

(3) $\quad [M:E]_G = \dim(\mathrm{Hom}_G(M,Q_E))/\dim(\mathrm{End}_G(E)).$

(For the notation cf. 2.14.)

4. Cohomology

Throughout this chapter let G be a flat k-group scheme.

We have shown in the last chapter that each G-module has a resolution by injective G-modules. Therefore we can define (right) derived functors of left exact functors from the category of G-modules. We can for example describe the Ext-functors as derived from the Hom-functor and we can introduce the cohomology functors $H^n(G,?)$ as derived from the fixed point functor. Furthermore there are for each flat group scheme H of G the derived functors $R^n \mathrm{ind}_H^G$ of the induction functor.

After recalling some general facts about derived functors (4.1) and making the definitions (4.2) we prove many elementary properties of the derived functors mentioned above (4.3-4.13). We prove equalities between two derived functors and mention several spectral sequences. We show that the cohomology can be computed using an explicit complex, the Hochschild complex (4.14-4.16). Besides proving a universal coefficient theorem (4.17) this complex is used for the computation of the cohomology of the additive group over a field (4.20-4.27). Because of later applications we formulate the results at once not for G_a but for direct products $G_a \times G_a \times \ldots \times G_a$.

As in the last chapter there is not much point in attributing priorities for generalities. In addition to the papers listed there one ought to mention [Andersen 12] where some results were extended to the case of an arbitrary ground ring (instead of a field.) When discussing the Hochschild complex I follow [DG] more or less. The computation of $H^\bullet(G_a,k)$ is due to [Cline/Parshall/Scott/van der Kallen].

4.1 (Derived Functors) Let $\mathbf{C}$ be an abelian category containing enough injectives, i.e. such that each object can be embedded into an injective object. Then certainly each object admits an injective resolution. We can then define the (right) *derived functors* $R^n\mathbf{F}$ of any additive (covariant) functor $\mathbf{F}$ from $\mathbf{C}$ into some other category $\mathbf{C}'$. We have $R^0\mathbf{F} = \mathbf{F}$ if and only if $\mathbf{F}$ is left exact. An object M in $\mathbf{F}$ is called *acyclic* for $\mathbf{F}$, if $R^n\mathbf{F}(M) = 0$ for all $n > 0$. Any short exact sequence in $\mathbf{C}$ gives rise to a long exact sequence in $\mathbf{C}'$.

Suppose now that $\mathbf{F}: \mathbf{C} \to \mathbf{C}'$ and $\mathbf{F}': \mathbf{C}' \to \mathbf{C}''$ are additive (covariant) functors where $\mathbf{C}, \mathbf{C}' \mathbf{C}''$ are abelian categories with $\mathbf{C}, \mathbf{C}'$ having enough injectives.

Proposition (Grothendieck's spectral sequence): *If $\mathbf{F}'$ is left exact and if $\mathbf{F}$ maps injective objects in $\mathbf{C}$ to objects acyclic for $\mathbf{F}'$, then there is a spectral sequence for each object M in $\mathbf{C}$ with*

(1) $\quad E_2^{n,m} = (R^n\mathbf{F}')(R^m\mathbf{F})M \Longrightarrow R^{n+m}(\mathbf{F}'\circ\mathbf{F})M.$

One can find a proof (and more background material) in the second edition of S. Lang's "Algebra".

Let us mention two trivial special cases:

(2) *If $\mathbf{F}'$ is exact, then $\mathbf{F}'\circ R^m\mathbf{F} \cong R^m(\mathbf{F}'\circ\mathbf{F})$ for all* $n \in \mathbb{N}$.

(This is obvious).

(3) <u>If $\underline{\underline{F}}$ is exact and maps injective objects to objects acyclic for $\underline{\underline{F}}'$, then $(R^n\underline{\underline{F}}')\circ\underline{\underline{F}} \cong R^n(\underline{\underline{F}}'\circ\underline{\underline{F}})$ for all</u> $n \in \mathbb{N}$.

(This can be proved by degree shifting, i.e. induction on n, using the long exact sequence.)

For future reference let us mention that one has for any spectral sequence $(E_r^{n,m})$ with $E_2^{n,m} = 0$ for $n < 0$ or $m < 0$ converging to some abutment (E^r) an exact sequence (cf. [2], ch. X, §2, exerc. 15c).

(4) $\quad 0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2,$

called the <u>five term exact sequence</u>.

<u>4.2</u> Throughout this chapter let G be a flat group scheme over k and H a flat subgroup scheme of G.

We know by 2.9 and 3.9.b that the G-modules form an abelian category containing enough injective objects. So we can apply the general principles from 4.1. For example the fixed point functor from {G-modules} to {k-modules} is left exact. We denote its derived functors by $M \mapsto H^n(G,M)$ and call $H^n(G,M)$ the n-th (rational) <u>cohomology</u> group of M.

For any G-module M the functor $\mathrm{Hom}_G(M,?)$ is left exact. Its derived functors are denoted (as usually) by $\mathrm{Ext}_G^n(M,?)$. They can (as always) also be defined using equivalence classes of exact sequences of G-modules.

For the trivial module k the functor $\mathrm{Hom}_G(k,?)$ is isomorphic to the fixed point functor: For each G-module M we have an isomorphism $\mathrm{Hom}_G(k,M) \xrightarrow{\sim} M^G$ with $\varphi \mapsto \varphi(1)$. We get therefore isomorphisms of derived functors

(1) $\quad \mathrm{Ext}_G^n(k,?) \cong H^n(G,?)$

The induction functor from H to G is left exact. We can therefore define also its derived functors $R^n\mathrm{ind}_H^G$.

<u>4.3</u> <u>Lemma:</u> <u>Suppose that</u> G <u>is diagonalizable. Let</u> Λ <u>be</u> <u>an abelian group with</u> $G \cong \mathrm{Diag}(\Lambda)$. <u>Then one has for all</u> G-<u>modules</u> M,N:

a) $\quad \mathrm{Ext}_G^n(M,N) \cong \prod_{\lambda\in\Lambda} \mathrm{Ext}_k^n(M_\lambda,N_\lambda)$ <u>for all</u> $n \in \mathbb{N}$.

b) $\quad H^n(G,M) = 0$ <u>for all</u> $n \in \mathbb{N}$, $n > 0$.

c) <u>If</u> k <u>is a field, then</u> $\mathrm{Ext}_G^n(M,N) = 0$ <u>for all</u> $n \in \mathbb{N}$, $n > 0$.

<u>Proof:</u> The first claim follows easily from 2.11(4). The other statements are immediate consequences.

<u>4.4</u> <u>Lemma:</u> <u>Let</u> M,N,V <u>be</u> G-<u>modules. If</u> V <u>is finitely generated and projective as a</u> k-<u>module, then we have for all</u> $n \in \mathbb{N}$ <u>a canonical isomorphism</u>

$$\mathrm{Ext}_G^n(M,V\otimes N) \xrightarrow{\sim} \mathrm{Ext}_G^n(M\otimes V^*,N).$$

<u>Proof:</u> We have a canonical isomorphism

$$\mathrm{Hom}(M,V\otimes N) \xrightarrow{\sim} \mathrm{Hom}(M\otimes V^*,N)$$

sending any φ to the map $m \otimes \alpha \mapsto (\alpha \otimes \mathrm{id}_N)(\varphi(m))$. It is easy to check that this induces an isomorphism

$$\mathrm{Hom}_G(M, V \otimes N) \xrightarrow{\sim} \mathrm{Hom}_G(M \otimes V^*, N).$$

This is functorial in N and can be interpreted as an isomorphism of functors

$$\mathrm{Hom}_G(M, ?) \circ (V \otimes ?) \xrightarrow{\sim} \mathrm{Hom}_G(M \otimes V^*, ?).$$

The functor $V \otimes ?$ is exact and maps injective G-modules to injective G-modules (cf. 3.9.c). We can therefore apply 4.1(3).

4.5 Proposition: *Let M be an H-module.*

a) *For each G-module N we have a spectral sequence with*

$$E_2^{n,m} = \mathrm{Ext}_G^n(N, R^m \mathrm{ind}_H^G M) \Longrightarrow \mathrm{Ext}_H^{n+m}(N, M)$$

b) *There is a spectral sequence with*

$$E_2^{n,m} = H^n(G, R^m \mathrm{ind}_H^G M) \Longrightarrow H^{n+m}(H, M).$$

c) *Let H' be a flat subgroup scheme of G with $H \subset H'$. Then there is a spectral sequence with*

$$E_2^{n,m} = (R^n \mathrm{ind}_{H'}^G)(R^n \mathrm{ind}_H^{H'})M \Longrightarrow (R^{n+m} \mathrm{ind}_H^G)M.$$

Proof: a) The Frobenius reciprocity in 3.4 can be interpreted as an isomorphism of functors

$$\mathrm{Hom}_G(N,?)\circ \mathrm{ind}_H^G \simeq \mathrm{Hom}_H(N,?).$$

As ind_H^G maps injective H-modules to injective G-modules by 3.9.a, we can apply 4.1(1).

b) This is the special case $N = k$ of a).

c) Take the isomorphism in 3.5(2) and argue as in the proof of a).

4.6 We call H exact in G, if ind_H^G is an exact functor. For example any diagonalizable subgroup scheme of G is exact in G. (See the remark to 3.4.) The last proposition implies obviously:

Corollary: Suppose that H is exact in G. Let M be an H-module.

a) For each G-module N and each $n \in \mathbb{N}$ there is an isomorphism

$$\mathrm{Ext}_G^n(N,\mathrm{ind}_H^G M) \simeq \mathrm{Ext}_H^n(N,M).$$

b) For each $n \in \mathbb{N}$ there is an isomorphism

$$H^n(G,\mathrm{ind}_H^G M) \simeq H^n(H,M).$$

Remark: These results are also known as "generalized Frobenius reciprocity" and "Shapiro's lemma".

4.7 When we regard $k[G]$ as a G-module and do not mention the representation explicitly, we will deal with ρ_ℓ or ρ_r. As both structures are equivalent it is most of the time not necessary to specify which of these two we consider. The same applies to H instead of G.

Lemma: Let $n \in \mathbb{N}$.

a) *We have for each* G-*module* N:

$$H^n(G, N \otimes k[G]) \cong \begin{cases} N & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

b) *We have for each* H-*module* M:

$$R^n \mathrm{ind}_H^G (M \otimes k[H]) \cong \begin{cases} M \otimes k[G] & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Proof: a) The trivial subgroup 1 of G is exact in G as it is diagonalizable (or even more trivially, as $\mathrm{ind}_1^G = k[G] \otimes ?$ is obviously exact). Therefore a) is an immediate consequence of 4.6.b (applied to $H = 1$) and of the tensor identity.

b) Apply the spectral sequence 4.5.c to $(H,1)$ instead of (H',H). As 1 is exact in H the spectral sequence together with the tensor identity yields isomorphisms

$$R^n \mathrm{ind}_H^G (M \otimes k[H]) \cong R^n \mathrm{ind}_1^G (M).$$

As 1 is exact in G the right hand side is 0 for $n > 0$ and equal to $M \otimes k[G]$ by the tensor identity. This implies b).

Remark: If k is a field, then $N \otimes k[G]$ is an injective G-module by 3.10.c. Similarly $M \otimes k[H]$ is an injective H-module. So the lemma is obvious in this case.

4.8 Proposition (The Generalized Tensor Identity).

Let N be a G-module which is flat as a k-module. Then we have for each H-module M and each $n \in \mathbb{N}$ an isomorphism

$$R^n \mathrm{ind}_H^G (M \otimes N) \cong (R^n \mathrm{ind}_H^G M) \otimes N.$$

Proof: The tensor identity may be interpreted as an isomorphism of functors

$$\mathrm{ind}_H^G \circ (\mathrm{res}_H^G N \otimes ?) \cong (N \otimes ?) \circ \mathrm{ind}_H^G.$$

Tensoring with N is exact and maps because of 3.9 and 4.7.b injective H-modules to modules acyclic for ind_H^G. So we can apply 4.1(2),(3).

4.9 (Semi-direct Products)

Let G' be a flat k-group scheme which operates on G. We can therefore form the semi-direct product $G \rtimes G'$.

We may regard the fixed point functor $?^G$ by 3.2 also as a functor from {$(G \rtimes G')$-modules} to {G'-modules}. There is an obvious isomorphism $\mathrm{res}_1^{G'} \circ ?^G \simeq ?^G \circ \mathrm{res}_G^{G \rtimes G'}$ of functors. The isomorphism of k-algebras $k[G \rtimes G'] \cong k[G] \otimes k[G']$ is compatible with the action of G via ρ_ℓ on $k[G \rtimes G']$ and $k[G]$ and with the trivial action on $k[G']$. Therefore 3.9 and 4.7.a imply that $\mathrm{res}_G^{G \rtimes G'}$ maps injective modules to modules acyclic for the fixed point functor. We therefore get isomorphisms of derived functors by 4.1(2),(3). So we have for all $n \in \mathbb{N}$ and any $(G \rtimes G')$-

module M a natural structure as a G'-module on $H^n(G,M)$.

Suppose now that G' stabilizes the subgroup scheme H of G. We can interprete 3.8(1) as an isomorphism $\operatorname{res}_G^{G\rtimes G'} \circ \operatorname{ind}_{H\rtimes G'}^{G\rtimes G'} \simeq \operatorname{ind}_H^G \circ \operatorname{res}_H^{H\rtimes G'}$ of functors. As above 3.9 and 4.7.b imply that $\operatorname{res}_H^{H\rtimes G'}$ maps injective modules to modules acyclic for ind_H^G. Therefore 4.1(2),(3) yield isomorphisms of functors (for all $n \in \mathbb{N}$):

(1) $\operatorname{res}_G^{G\rtimes G'} \circ R^n \operatorname{ind}_{H\rtimes G'}^{G\rtimes G'} \simeq R^n \operatorname{ind}_H^G \circ \operatorname{res}_H^{H\rtimes G'}$.

For $H = 1$ this shows that G' is exact in $G\rtimes G'$ which is already clear by 3.8(2). Similarly G is exact in $G\rtimes G'$ by 3.8(3).

4.10 Proposition: We have for each H-module M and each $n \in \mathbb{N}$ an isomorphism of k-modules

$$H^n(H, M \otimes k[G]) \xrightarrow{\sim} (R^n \operatorname{ind}_H^G) M.$$

Proof: The definition of ind_H^G yields an isomorphism of functors

$$\underline{\underline{F}} \circ \operatorname{ind}_H^G \simeq ?^H \circ (k[G] \otimes ?),$$

where $\underline{\underline{F}}$ is the forgetful functor from {G-modules} to {k-modules}. As $k[G] \otimes ?$ is exact and maps injective H-modules to modules acyclic for the fixed point functor (by 4.7.a), we can apply 4.1(2),(3).

4.11 Corollary: If $k[G]$ is an injective H-module, then H is exact in G.

Proof: Under our assumption $k[G]$ is a direct summand of some $M_1 \otimes k[H]$, hence $M \otimes k[G]$ of some $M_2 \otimes k[H]$ (for suitable H-modules M_1, M_2). Then 4.10 and 4.7.a imply the claim.

Remarks: 1) Suppose that k is a field. Then the corollary can be proved directly as follows. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of H-modules, then $0 \to M_1 \otimes k[G] \to M_2 \otimes k[G] \to M_3 \otimes k[G] \to 0$ is an exact sequence of injective H-modules (by 3.10), hence split as a sequence of H-modules. Therefore also the sequence of all $(M_i \otimes k[G])^H = \mathrm{ind}_H^G(M_i)$ has to be exact.

2) The example $H = 1$ shows that the converse will not hold in general. However:

4.12 Proposition: Suppose that k is a field. Then H is exact in G if and only if $k[G]$ is an injective H-module.

Proof: Because of 4.11 we have to prove one direction only. Suppose that H is exact in G. We have for each finite dimensional module V by 4.4, 4.2(1) and 4.10

$$\mathrm{Ext}_H^n(V,k[G]) \cong \mathrm{Ext}_H^n(k,V^* \otimes k[G]) \cong H^n(H,V^* \otimes k[G]) = 0$$

for all $n > 0$. Therefore the functor $\mathrm{Hom}_H(?,k[G])$ is exact when restricted to finite dimensional H-modules. This implies easily the exactness on all H-modules (i.e. the injectivity of $k[G]$) because each H-module is the direct limit of finite dimensional H-modules.

4.13 Proposition: Let k' be a flat k-algebra. Let $n \in \mathbb{N}$.

a) For each G-module N there is an isomorphism

$$H^n(G,N) \otimes k' \cong H^n(G_{k'},N \otimes k').$$

b) <u>For each</u> H-<u>module</u> M <u>there is an isomorphism</u>

$$R^n(\mathrm{ind}_H^G M) \otimes k' \cong (R^n \mathrm{ind}_{H_{k'}}^{G_{k'}})(M \otimes k').$$

<u>Proof</u>: We get from 2.10(3) and 3.5(3) isomorphisms of functors to which we want to apply 4.1(2),(3). This is possible as $? \otimes k'$ is exact and maps injective G-modules to modules acyclic for the $G_{k'}$-fixed point functor (by 3.9 and 4.7.a) and maps injective H-modules to modules acyclic for the induction from $H_{k'}$ to $G_{k'}$ (by 3.9 and 4.7.b).

<u>4.14</u> Let M be a G-module. The cohomology $H^\bullet(G,M)$ can be computed using the <u>Hochschild complex</u> $C^\bullet(G,M)$ which we are going to describe now.

We set $C^n(G,M) = M \otimes \bigotimes^n k[G]$ for all $n \in \mathbb{N}$ and define boundary maps $\partial^n: C^n(G,M) \to C^{n+1}(G,M)$ of the form $\partial^n = \sum_{i=0}^{n+1} (-1)^i \partial_i^n$ where

$$\partial_0^n(m \otimes f_1 \otimes \ldots \otimes f_n) = \Delta_M(m) \otimes f_1 \otimes \ldots \otimes f_n,$$

$$\partial_i^n(m \otimes f_1 \otimes \ldots \otimes f_n) = m \otimes f_1 \otimes \ldots \otimes f_{i-1} \otimes \Delta_G(f_i) \otimes f_{i+1} \otimes \ldots \otimes f_n$$

for $1 \le i \le n$,

$$\partial_{n+1}^n(m \otimes f_1 \otimes \ldots \otimes f_n) = m \otimes f_1 \otimes \ldots \otimes f_n \otimes 1.$$

We can interprete $C^n(G,M)$ also as $\mathrm{Mor}(G^n,M_a)$ where G^n is the direct product of n copies of G, cf. 3.3. Then the ∂_i^n look like

$$\partial_0^n f(g_1,g_2,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1}),$$

$$\partial_i^n f(g_1,g_2,\dots,g_{n+1}) = f(g_1,\dots,g_{i-1},g_i g_{i+1},g_{i+2},\dots,g_{n+1})$$

$$\text{for } 1 \le i \le n,$$

$$\partial_{n+1}^n f(g_1,g_2,\dots,g_{n+1}) = f(g_1,\dots,g_n).$$

It is easy to check that $\partial^n \partial^{n-1} = 0$ for all n. Therefore $(C^\cdot(G,M),\partial^\cdot)$ is a complex. We want to prove that its cohomology is just $H^\cdot(G,M)$.

4.15 If our last claim is true, then $C^\cdot(G,k[G])$ ought to be exact except in degree 0 by 4.7.a. Let us consider $k[G]$ as a G-module via ρ_r so that $\Delta_{k[G]} = \Delta_G$. We define for each n a linear map

$$s^n\colon C^{n+1}(G,k[G]) = \bigotimes^{n+2} k[G] \to \bigotimes^{n+1} k[G] = C^n(G,k[G])$$

through $s^n = \varepsilon_G \otimes \bigotimes^{n+1} \mathrm{id}_{k[G]}$. An elementary calculation using 2.3(2) shows $s^n \partial^n = \mathrm{id} - \partial^{n-1} s^{n-1}$ for all $n > 0$. This implies the exactness of $C^\cdot(G,k[G])$ at each point $n > 0$ whereas $\partial^0\colon C^0(G,k[G]) = k[G] \to C^1(G,k[G]) = k[G] \otimes k[G]$ maps f to $\Delta(f) - f \otimes 1$, hence has kernel $k1$. Therefore we have an exact sequence

$$(1) \qquad 0 \to k \to k[G] \to \bigotimes^2 k[G] \to \bigotimes^3 k[G] \to \dots$$

This sequence can be regarded as a sequence of homomorphisms of

G-modules when we let G operate on $\bigotimes^n k[G]$ via ρ_ℓ on the first factor and trivially on all the other factors. It is for this operation that $k[G] \to k[G] \otimes k[G]$, $f \mapsto \Delta_G(f) - f \otimes 1$ is G-equivariant. If we tensor now (1) with M we get a resolution

(2) $\quad 0 \to M \to M \otimes k[G] \to M \otimes \bigotimes^2 k[G] \to \ldots$

of M by acyclic modules. Furthermore we can by 3.7(4) make the operation of G on the factor M in any $M \otimes \bigotimes^i k[G]$ trivial, hence get a resolution

(3) $\quad 0 \to M \to M_{tr} \otimes k[G] \to M_{tr} \otimes \bigotimes^2 k[G] \to \ldots$

using the same notation as in 3.7(4). Therefore $H^\bullet(G,M)$ is the cohomology of the complex

(4) $\quad 0 \to (M_{tr} \otimes k[G])^G \to (M_{tr} \otimes \bigotimes^2 k[G])^G \to \ldots$

As G operates trivially on all but one factor and as $k[G]^G = k$ the n-th term in (4) is equal to $(M_{tr} \otimes \bigotimes^{n+1} k[G])^G \simeq M_{tr} \otimes \bigotimes^n k[G] \simeq C^n(G,M)$. Furthermore tracing back the maps one finds that ∂^n is just the map from $C^n(G,M)$ to $C^{n+1}(G,M)$ occurring in (4). (The shortest way of doing it is via the interpretation as functions $G^n \to M$.) This proves our claim.

<u>4.16</u> Let M be a G-module.

<u>Proposition:</u> <u>The cohomology of the complex</u> $C^\bullet(G,M)$ <u>is equal to</u> $H^\bullet(G,M)$.

<u>Remark:</u> In [DG], II, §3 the case of arbitrary group functors

(instead of our flat group scheme) is treated and more general coefficients are considered.

4.17 We can identify $C^\bullet(G_{k'}, M\otimes k')$ for any k-algebra k' with $C^\bullet(G,M)\otimes k'$. Suppose that M is a flat k-module. Then also all $C^n(G,M)$ are flat. If k has the property, that any submodule of a flat module is flat, then we get a universal coefficient theorem e.g. by [2], ch. X., §4, cor. 1 du th. 3 (after re-indexing). Any Dedekind ring has this property as for such a ring the notions "flat" and "torsion free" coincide (e.g. by [3], ch. VII, §4, prop. 22). We get therefore the first part of:

Proposition: Suppose that k is a Dedekind ring. Let k' be a k-algebra and let $n \in \mathbb{N}$.

a) There is for each G-module N which is flat over k an exact sequence

$$0 \to H^n(G,N)\otimes k' \to H^n(G_{k'}, N\otimes k') \to \mathrm{Tor}_1^k(H^{n+1}(G,N),k') \to 0.$$

b) There is for each H-module M which is flat over k an exact sequence of $G_{k'}$-modules

$$0 \to (R^n\mathrm{ind}_H^G M)\otimes k' \to R^n\mathrm{ind}_{H_{k'}}^{G_{k'}}(M\otimes k') \to \mathrm{Tor}_1^k(R^{n+1}\mathrm{ind}_H^G M,k') \to 0$$

Note that b) follows on the level of k'-modules from a) and 4.10. It may be left to the reader to find the $G_{k'}$-module structure on the Tor-group and to prove the equivariance of the maps.

4.18 If k' is flat over k, then we get from 4.17.a that $H^0(G,N) \otimes k' \simeq H^0(G_{k'}, N \otimes k')$ which we know already from 2.10(3) to hold for all N. If k' is not flat, however, such a statement will not be true, even for flat N (in spite of the lemma 1.17 in [Andersen 12]). Take e.g. $G = G_a$ and its representation $a \mapsto \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$ on k^2 and get a contradiction for $k = \mathbb{Z}$, $k' = \mathbb{F}_2$.

Such a formula will however hold for acyclic modules as then the last term in 4.17.a is zero. We can for example (by 4.7.a) take for N a direct summand of some $E \otimes k[G]$ where E is a flat k-module, regarded as a trivial G-module. If N' is another G-module which is finitely generated and projective over k, then $\mathrm{Hom}(N',N) \simeq (N')^* \otimes N$ is again of this type because of the tensor identity. So we have a canonical isomorphism $\mathrm{Hom}_G(N',N) \otimes k' \simeq \mathrm{Hom}_{G_{k'}}(N' \otimes k', N \otimes k)$. This generalizes to all N' which are flat over k by taking direct limits. This shows:

(1) Let N, N' be G-modules such that N' is flat over k and such that N is isomorphic to a direct summand of some G-module $E \otimes k[G]$ with E flat over k. Then we have for each k-algebra k' a natural isomorphism $\mathrm{Hom}_G(N',N) \otimes k' \simeq \mathrm{Hom}_{G_{k'}}(N' \otimes k', N \otimes k')$.

Let us mention as a special case, that we have for each k' an isomorphism

(2) $$\mathrm{End}_G(k[G]) \otimes k' \simeq \mathrm{End}_{G_{k'}}(k'[G_{k'}])$$

4.19 For any k there is on $H^\bullet(G,k) = \bigoplus_{i \geq 0} H^i(G,k)$ a structure as (associative) algebra over k. The multiplication is called the <u>cup-product</u> and satisfies the usual anti-commutativity formula: If $a \in H^i(G,k)$ and $b \in H^j(G,k)$ then $ab = (-1)^{i+j}ba$. Furthermore there is (for each G-module N) a natural structure of a $H^\bullet(G,k)$-right module on $H^\bullet(G,N) = \bigoplus_{i\geq 0} H^i(G,N)$.

Let us describe these structures using the Hochschild complexes for k and N. We can obviously identify $C^n(G,k) = \otimes^n k[G]$ and then have to write ∂_0^n in the form $\partial_0^n(x) = 1 \otimes x$. Furthermore we identify $C^n(G,N) \otimes C^m(G,k)$ and $C^{n+m}(G,N)$ for all $n,m \in \mathbb{N}$. For all $a \in C^n(G,N)$ and $b \in C^m(G,k)$ one checks easily $\partial^{n+m}(a\otimes b) = (\partial^n a)\otimes b+(-1)^n a \otimes (\partial^m b)$. Hence $a\otimes b$ is a cocycle if a and b are so. Another simple computation shows then, that the cohomology class $[a\otimes b]$ of $a\otimes b$ depends only on the classes $[a]$ of a and $[b]$ of b. Then the action of $[b] \in H^m(G,k)$ on $[a] \in H^n(G,N)$ is defined through $[a][b] = [a\otimes b]$.
In the case $N = k$ we get thus the cup-product on $H^\bullet(G,k)$.

Let G' be a flat group scheme operating on G through group automorphisms. If N is a $(G \rtimes G')$-module (e.g. $N = k$), then G' acts on each $H^n(G,N)$, cf. 4.9. This operation can be described using the Hochschild complex. The discussion above shows that G' acts on $H^\bullet(G,k)$ through algebra automorphisms and that the action of $H^\bullet(G,k)$ on an arbitrary $H^\bullet(G,N)$ is compatible with the G'-action, i.e. that $H^\bullet(G,N)\otimes H^\bullet(G,k)\to H^\bullet(G,N)$ is a homomorphism of G'-modules.

<u>4.20</u> We want to discuss $H^\bullet(G_a,k)$ or (more generally) $H^\bullet(V_a,k)$ for a free k-module V of finite rank, say $rk(V) = n$. Of course, there is a Künneth formula, reducing the second problem to the first one. But we shall prefer to formulate our results at once for V in order to keep track of the $GL(V)$-operation on the cohomology groups (as in 4.19).

Choosing a basis we identify $k[V_a]$ with the polynomial ring $k[T_1,T_2,\ldots,T_n]$. We get then an $\mathbb{N}^n$-grading and an $\mathbb{N}$-grading on the complex $C^\bullet(V_a,k)$. For each $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{N}^n$ let $C^i(V_a,k)_\alpha$ be spanned by all tensor products of monomials such that the degrees of T_i in the factors add up to α_i for each i. Set $C^i(V_a,k)_m$ equal to the sum of all $C^i(V_a,k)_\alpha$ with $m = |\alpha|$ (where $|(\alpha_1,\ldots,\alpha_n)| = \sum_{i=1}^n \alpha_i$). Obviously the $C^i(V_a,k)_m$ are $GL(V)$-stable whereas the $C^i(V_a,k)_\alpha$ are not (for $n > 1$). As the comultiplication is given by $\Delta(T_j) = 1\otimes T_j+T_j\otimes 1$ for all j, the formulas for the ∂^i in 4.14 show $\partial^i C^i(V_a,k)_\alpha \subset C^{i+1}(V_a,k)_\alpha$ for all α and $\partial^i C^i(V_a,k)_m \subset C^{i+1}(V_a,k)_m$. Therefore we get also gradings for the cohomology groups

$$(1) \qquad H^i(V_a,k) = \bigoplus_{\alpha\in\mathbb{N}^n} H^i(V_a,k)_\alpha = \bigoplus_{m\in\mathbb{N}} H^i(V_a,k)_m .$$

(Note that these gradings simply describe the representations of the diagonal subgroup of $GL(V)$ on the cohomology resp. of the subgroup of scalar diagonal matrices.)

<u>4.21</u> We can now easily compute $H^1(V_a,k)$.

<u>Lemma: Suppose that</u> k <u>is an integral domain.</u>

a) <u>If</u> $\operatorname{char}(k) = 0$, <u>then</u> $H^1(V_a,k) = \sum_{i=1}^{n} kT_i \cong V$ <u>as a</u> $GL(V)$-<u>module</u>.

b) <u>If</u> $\operatorname{char}(k) = p \neq 0$, <u>then</u> $H^1(V_a,k) = \sum_{i=1}^{n} \sum_{r=0}^{\infty} kT_i^{p^r}$.

<u>Proof</u>: We have obviously $H^0(V_a,k) = k$ and $\partial^0 = 0$, hence $H^1(V_{a,k}) = \ker(\partial^1)$. This map is given by $\partial^1(f) = 1\otimes f - \Delta(f) + f\otimes 1$. Because of 4.20(1) the monomials $\prod_{i=1}^{n} T_i^{r(i)}$ with $\partial^1(\prod_{i=1}^{n} T_i^{r(i)}) = 0$ form a basis of $\ker(\partial^1)$. If at least two $r(i)$ are positive, then each $T_i^{r(i)} \otimes \prod_{j\neq i} T_j^{r(j)}$ occurs with coefficient -1 in $\partial^1(\prod_{i=0}^{n} T_i^{r(i)})$ so that this element is different from 0. As $\partial^1(1) = 1\otimes 1$ we have to look only at

$$(1) \qquad \partial^1(T_i^r) = -\sum_{j=1}^{r-1} \binom{r}{j} T_i^j \otimes T_j^{r-j}$$

This is certainly 0, if $r = 1$. We then have to determine all $r > 1$ with all those binomial coefficients equal to 0. The result is well known and implies the lemma.

<u>4.22</u> Keep the assumption of lemma 4.21. The cup product induces a homomorphism of $GL(V)$-modules

$$H^1(V_a,k) \otimes H^1(V_a,k) \to H^2(V_a,k).$$

Because of the anti-commutativity of the cup product (i.e. because of $f \otimes f' + f' \otimes f = \partial^1(ff')$ for $f,f' \in \ker(\partial^1)$ this map has to factor through $\Lambda^2 H^1(V_a,k)$, if $\operatorname{char}(k) \neq 2$, and through $S^2 H^1(V_a,k)$, if $\operatorname{char}(k) = 2$.

Let us denote the image of this map by M. We want to show

$$(1)\qquad M \doteq \begin{cases} \Lambda^2 H^1(V_a,k) & \text{if } \operatorname{char}(k) \neq 2, \\ S^2 H^1(V_a,k) & \text{if } \operatorname{char}(k) = 2. \end{cases}$$

The image of ∂^1 in $C^2(V_a,k) = k[V_a] \otimes k[V_a]$ consists of symmetric elements, i.e. of elements stable under $f \otimes f' \mapsto f' \otimes f$. If we take two different basis elements f, f' in 4.21, then $f \otimes f'$ is not symmetric, hence the class $[f][f'] = [f \otimes f'] \in H^2(V_a,k)$ is non-zero. In order to get their linear independence we just have to observe that these tensor products are homogeneous of pairwise different degrees (except for the trivial equality $[f \otimes f'] = -[f' \otimes f\,]$).

This proves (1) for $\operatorname{char}(k) \neq 2$. For $\operatorname{char}(k) = 2$ we have still to show $f \otimes f \notin \operatorname{im}(\partial^1)$ for any basis element f in 4.21. We can do something more general. Suppose $\operatorname{char}(k) = p \neq 0$, set $\{{p \atop i}\} = \frac{1}{p}\binom{p}{i}$ for $1 \leq i \leq p-1$ and

$$(2)\qquad \beta(f) = \sum_{i=1}^{p-1} \{{p \atop i}\} f^i \otimes f^{p-i}$$

for all $f \in k[V_a]$. (So $\beta(f) = f \otimes f$ if $\operatorname{char}(k) = 2$.) This map is of course induced from the map $f \mapsto ((1 \otimes f + f \otimes 1)^p - 1 \otimes f^p - f^p \otimes 1)/p$ on $\mathbb{Z}[T_1,\ldots,T_n]$. Using this fact (or a direct calculation) we get that β maps $\ker(\partial^1) = H^1(V_a,k)$ into $\ker(\partial^2)$, hence we get a map $\bar{\beta}\colon H^1(V_a,k) \to H^2(V_a,k)$. A simple computation shows

$$\beta(f_1+f_2) = \beta(f_1)+\beta(f_2) - \partial^1 \sum_{i=1}^{p-1} \{{p \atop i}\} f_1^i f_2^{p-i}$$

for all $f_1, f_2 \in H^1(V_a,k)$. Therefore $\bar{\beta}$ is additive. Obviously $\bar{\beta}$ is $GL(V)$-equivariant and satisfies $\bar{\beta}(af) = a^p\bar{\beta}(f)$ for all $a \in k$. Take now for f a basis element from 4.21. Then $\beta(f)$ is homogeneous with degree p-times the degree of f. The only element (up to scalar multiple) in $k[V_a]$ having this degree is f^p. As $\partial^1(f^p) = 0$, we get $\beta(f) \notin \mathrm{im}(\partial^1)$. This concludes the proof of (1) and shows for $p \neq 2$ that the $\bar{\beta}(T_i^{p^r})$ with $1 \leq i \leq n$ and $r \in \mathbb{N}$ span as a basis a $GL(V)$-submodule in $H^2(V_a,k)$ intersecting M in 0.

We claim that we have found all of $H^2(V_a,k)$ in case k is a field. We refer to [DG], II, §3, 4.6 for the proof and just state the result:

Lemma: *Suppose that* k *is a field*

a) *If* $\mathrm{char}(k) = 0$, *then* $H^2(V_a,k) \cong \Lambda^2 H^1(V_a,k)$.

b) *If* $\mathrm{char}(k) = 2$, *then* $H^2(V_a,k) \cong S^2 H^1(V_a,k)$.

c) *If* $\mathrm{char}(k) \neq 2,0$, *then* $H^2(V_a,k) \cong \Lambda^2 H^1(V_a,k) \oplus k\bar{\beta}H^1(V_a,k)$.

4.23 In order to get all of $H^\bullet(V_a,k)$, we shall reduce its computation to that of the cohomology of finite cyclic groups. This is done using a filtration of the Hochschild complex.

Set $k[V_a,m]$ for all $m \in \mathbb{N}$ equal to the span of all monomials $T_1^{r(1)}T_2^{r(2)}\dots T_n^{r(n)}$ with $r(i) < m$ for all i. Then the formula $\Delta(T_i) = 1 \otimes T_i + T_i \otimes 1$ implies $\Delta(k[V_a,m]) \subset k[V_a,m] \otimes k[V_a,m]$. Set $C^j(V_a,k,m) = \bigotimes^j k[V_a,m] \subset \bigotimes^j k[V_a] = C^j(V_a,k)$. Then we see that $\partial^j C^j(V_a,k,m) \subset C^{j+1}(V_a,k,m)$. Hence $C^\bullet(V_a,k,m) =$

$= \bigoplus_{j\geq 0} C^j(V_a,k,m)$ is a subcomplex of $C^\cdot(V_a,k,m)$. Let us denote its cohomology by $H^\cdot(V_a,k,m) = \bigoplus_{i\geq 0} H^i(V_a,k,m)$.

For $m,m' \in \mathbb{N}$ with $m' \leq m$ we have an inclusion $C^\cdot(V_a,k,m') \subset C^\cdot(V_a,k,m)$, hence a homomorphism $\alpha_{m,m'}: H^\cdot(V_a,k,m') \to H^\cdot(V_a,k,m)$. We have obviously $\alpha_{m,m'} \circ \alpha_{m',m''} = \alpha_{m,m''}$ for any $m'' \leq m'$. Similarly the inclusion $C^\cdot(V_a,k,m) \to C^\cdot(V_a,k)$ induces a homomorphism $\alpha_m: H^\cdot(V_a,k,m) \to H^\cdot(V_a,k)$ with $\alpha_m \circ \alpha_{m,m'} = \alpha_{m'}$. We get thus a homomorphism $\alpha: \varinjlim H^\cdot(V_a,k,m) \to H^\cdot(V_a,k)$. Obviously $H^\cdot(V_a,k)$ is the union of all $\alpha_m(H^\cdot(V_a,k,m))$ and for each $f \in \ker(\alpha_m)$ there is $m' \geq m$ with $f \in \ker(\alpha_{m,m'})$. This implies

(1) $\quad \varinjlim H^\cdot(V_a,k,m) \xrightarrow{\sim} H^\cdot(V_a,k)$.

Note that $C^i(V_a,k,m) \otimes C^j(V_a,k,m) = C^{i+j}(V_a,k,m)$. Therefore we can define a cup-product on each $H^\cdot(V_a,k,m)$ and the α_m are homomorphisms of algebras. Hence so is the isomorphism (1).

Let me point out that this construction can be generalized to any V_a-module M which is finitely generated over k. For such an M there is some $r(M) \in \mathbb{N}$ with $\Delta_M(M) \subset M \otimes k[V_a,r(M)]$. Then all $C^\cdot(V_a,M,m)$ with $m \geq r(M)$ are subcomplexes of $C^\cdot(V_a,M)$ and we get as above

(2) $\quad \varinjlim H^\cdot(V_a,M,m) \xrightarrow{\sim} H^\cdot(V_a,k)$

<u>4.24</u> Obviously we can define a complement $C^j(V_a,k,m)^c$ to $C^j(V_a,k,m)$ in $C^j(V_a,k)$: Take the span of all tensor products

of all monomials not belonging to $C^j(V_a,k,m)$, i.e. where in at least one factor some T_i occurs with an exponent $\geq m$. In general the $C^j(V_a,k,m)^c$ do not form a subcomplex.

Suppose however that p is a prime number and that $p1 = 0$ in k. Then $\Delta(T_i^{p^r}) = 1 \otimes T_i^{p^r} + T_i^{p^r} \otimes 1$ for all i and r. This implies that all $C^j(V_a,k,p^r)^c$ are subcomplexes and that $H^j(V_a,k,p^r)$ is a direct summand of $H^j(V_a,k)$. We may write 4.23(1) in the form

$$\text{(1)}\quad H^{\cdot}(V_a,k) = \bigcup_{r>0} H^{\cdot}(V_a,k,p^r) \qquad\qquad (\text{if } pk = 0).$$

(We can generalize 4.23(2) in a similar way.)

Of course our computations in 4.21/22 are compatible with this formula. In the situation of 4.21.a we have

$$\text{(2)}\quad H^1(V_a,k,p^r) = \sum_{i=1}^{n} \sum_{j=0}^{r-1} kT_i^{p^j},$$

in 4.22.c:

$$\text{(3)}\quad H^2(V_a,k,p^r) \simeq \Lambda^2 H^1(V_a,k,p^r) \oplus k\bar{\beta}H^1(V_a,k,p^r),$$

and in 4.22.b:

$$\text{(4)}\quad H^2(V_a,k,2^r) \simeq S^2 H^1(V_a,k,2^r).$$

<u>4.25</u> The groups $H^{\cdot}(V_a,k,p^r)$ in 4.24 have a different interpretation. Let p be still a prime and suppose $p1 = 0$ in k. Identify V with k^n via the T_i and consider the (Frobenius) endomorphism F of V_a with $F(a_1,\dots,a_n) = (a_1^p,\dots,a_n^p)$ for all $(a_1,\dots,a_n) \in A^n = k^n \otimes A \simeq V_a(A)$ and all A. This is an

endomorphism of algebraic k-groups with $F^*(T_i) = T_i^p$ for all i. The kernel $V_{a,r}$ of F^r is therefore also an abgebraic k-group with $k[V_{a,r}] \cong k[T_1,\ldots,T_n]/(T_1^{p^r},\ldots,T_n^{p^r})$. (Obviously $V_{a,r}$ is independent of the choice of the identification $V \cong k^n$. Notice that $V_{a,r}$ is isomorphic to the direct product of n copies of the algebraic k-group $G_{a,r}$ introduced in 2.2.)

Obviously the restriction of functions $k[V_a] \to k[V_{a,r}]$ induces an isomorphism $k[V_a,r] \to k[V_{a,r}]$ compatible with the comultiplication, hence an isomorphism $C^\cdot(V_a,k,p^r) \to C^\cdot(V_{a,r},k)$ of complexes and an isomorphism of algebras

(1) $\quad H^\cdot(V_a,k,p^r) \tilde{\to} H^\cdot(V_{a,r},k)$.

Any $C^j(V_a,k,p^r)^c$ is just the kernel of the restriction map $C^j(V_a,k) \to C^j(V_{a,r},k)$. This gives a better reason for $\oplus\, C^j(V_a,k,p^r)$ to form a subcomplex and hence for the injectivity of the map $H^\cdot(V_a,k,p^r) \to H^\cdot(V_a,k)$.

Again we can generalize (1) to any V_a-module M, finitely generated over k, and get

(2) $\quad H^\cdot(V_a,M,p^r) \tilde{\to} H^\cdot(V_{a,r},M) \qquad$ if $p^r > r(M)$.

Notice that the gradings on $H^\cdot(V_a,k)$ considered in 4.20 induce similar gradings on $H^\cdot(V_{a,r},k)$.

<u>4.26</u> Let us assume that k is a field of characteristic $p \neq 0$. It will be convenient to suppose for the moment that k

is finite. Consider the endomorphism F of V_a as in 4.25 and define for each $r \in \mathbb{N}$, $r > 0$ a closed subgroup $V_a(p^r)$ of V_a via

$$V_a(p^r)(A) = \{v \in V_a(A) \mid F^r(v) = v\}.$$

It is defined by the ideal generated by all $T_i^{p^r} - T_i$ with $1 \leq i \leq n$. Therefore the restriction of functions induces also an isomorphism $k[V_a,p^r] \to k[V_a(p^r)]$ compatible with the comultiplication, hence an isomorphism

(1) $\quad H^\bullet(V_a,k,p^r) \xrightarrow{\sim} H^\bullet(V_a(p^r),k).$

If A is an extension field of k, then $V_a(p^r)(A)$ is simply the group of all points in A^n having all coordinates in the finite field $\mathbb{F}_{p^r}$. Let us denote this group by $V(p^r)$. It is an elementary abelian p-group of order p^{rn}. We may regard $k[V_a(p^r)]$ as the algebra of all functions from $V(p^r)$ to k. The comultiplication on $k[V_a(p^r)]$ is given by the group law in the finite group $V(p^r)$. Therefore the Hochschild complex for $V_a(p^r)$ computes the cohomology of the finite group $V(p^r)$. (Equivalently one can say that the category of $V_a(p^r)$-modules is "the same" as the category of k-$V(p^r)$-modules.)

Now the cohomology of a cyclic group is well known (cf. e.g. [8]) and the cohomology of an elementary abelian group follows using the Künneth formula. The results can be formulated as follows:

(2) *If* $p = 2$, *then* $H^{\bullet}(V_a,k,p^r) \simeq SH^1(V_a,k,p^r)$.

We denote here by $S(M)$ resp. $\Lambda(M)$ the symmetric resp. exterior algebra of a k-module M given its natural grading. If we put each element of S^iM in degree $2i$ then we write $S'(M)$.

(3) *If* $p \neq 2$, *then*

$$H^{\bullet}(V_a,k,p^r) \simeq \Lambda H^1(V_a,k,p^r) \otimes S'(V')$$

with $\quad V' \simeq H^2(V_a,k,p^r)/\Lambda^2 H^1(V_a,k,p^r)$.

These results are certainly also true, if k is finite, e.g. by 4.16(1).

<u>4.27</u> Combining 4.26(2),(3) with 4.24(1) we get a complete description of $H^{\bullet}(G_a,k)$. Before formulating the result we want to introduce some notation how to describe the operation of $GL(V)$ on the spaces $\sum_{i=1}^{n} kT_i^{p^r}$ and $\sum_{i=1}^{n} k\bar{\beta}(T_i^{p^r})$.

We can define a group endomorphism of $GL(V)$, also denoted by F, such that $F(gv) = F(g)F(v)$ for all $g \in GL(V)(A)$ and $v \in V \otimes A$ and all A. If we identify $GL(V) \simeq GL_n$ using the same basis as for $V \simeq k^n$, then $F(a_{ij}) = (a_{ij}^p)$ for all $(a_{ij}) \in GL_n(A)$ and all A. Using G we can define for each $GL(V)$-module V' and each $r \in \mathbb{N}$ a new $GL(V)$-module $V'^{(r)}$ which is equal to V' as k-module and where any $g \in GL(V)(A)$ acts on $V'^{(r)} \otimes A$ as $F^r(g)$ acts on $V' \otimes A$. (We shall discuss such "Frobenius twists" more systematically in 9.9).

Writing down the effect of any g on the $T_i^{p^j}$ in terms of gT_i one sees immediately that $\sum_{i=1}^{n} kT_i^{p^j} \cong (\sum_{i=1}^{n} kT_i)^{(j)} \cong V^{*(j)}$. Similarly one gets $\sum_{i=1}^{n} k\bar{\beta}(T_i^{p^j}) \cong V^{*(j+1)}$. We can therefore express the results as follows:

<u>Proposition</u>: <u>Suppose that</u> k <u>is a field of characteristic</u> $p \neq 0$.

a) <u>If</u> $p = 2$, <u>then</u>

$$H^{\cdot}(V_a,k) \cong S(\bigoplus_{j\geq 0} V^{*(j)})$$

<u>and</u> (<u>for all</u> $r > 0$)

$$H^{\cdot}(V_{a,r},k) \cong S(\bigoplus_{j=0}^{r-1} V^{*(j)}).$$

b) <u>If</u> $p \neq 2$, <u>then</u>

$$H^{\cdot}(V_a,k) \cong \Lambda(\bigoplus_{j\geq 0} V^{*(j)}) \otimes S'(\bigoplus_{j\geq 1} V^{*(j)})$$

<u>and</u> (<u>for all</u> $r > 0$)

$$H^{\cdot}(V_{a,r},k) \cong \Lambda(\bigoplus_{j=0}^{r-1} V^{*(j)}) \otimes S'(\bigoplus_{j=1}^{r} V^{*(j)}).$$

<u>Remarks</u>: 1) The explicit description of H^1 and H^2 gives also the gradings of the generators of the generators of $H^{\cdot}(V_a,k)$ and $H^{\cdot}(V_{a,r},k)$. All elements in $V^{*(j)}$ are homogeneous of degree p^j with respect to the $\mathbb{N}$-grading.

2) If k is a field of characteristic 0, then $H^{\cdot}(V_a,k) \cong \Lambda(V^*)$. This follows e.g. from the proposition applying the universal coefficients theorem to $\mathbb{Z}^n$.

5. Quotients and Associated Sheaves

Some properties of the derived functors of induction can be proved only by interpreting the $R^n \mathrm{ind}_H^G M$ as cohomology groups $H^n(G/H, \mathcal{L}(M))$ of certain quasi-coherent sheaves on G/H. Before we can define these "associated sheaves" (5.10/11) and prove the equality $R^n \mathrm{ind}_H^G M = H^n(G/H, \mathcal{L}(M))$ in 5.13, we have to introduce the quotients G/H.

This is a non-trivial problem. Assuming G to be a (flat) group scheme and H a (flat) subgroup scheme we want G/H to be a scheme. The choice at first sight, the functor $A \mapsto G(A)/H(A)$, will in general be no scheme. On the other hand, there is an obvious definition of a quotient scheme via a universal property (cf. 5.1) which however gives no information about existence and how the quotient looks like, if it happens to exist.

It has turned out to be useful to construct quotients not at once in the category of schemes over k but in the larger category of all k-faisceaux. These are the k-functors having a sheaf property with respect to the faithfully flat finitely presented (Grothendieck) topology, cf. 5.2/3. The quotient faisceau G/H has a not too complicated description (5.4/5). In the most important cases (e.g. over a field) the quotient faisceau is a scheme (hence the quotient scheme) and has nice properties (5.6/7). It is only in this case that we can prove the relation between sheaf cohomology and the functors of induction mentioned above.

One consequence of this relation is that ind_H^G is an exact functor, if G/H is an affine scheme. This can be proved more directly (5.8) following [Cline/Parshall/Scott 3] who prove also the inverse for linear algebraic groups over an algebraically closed field. In 5.14 we mention some more consequences, but will make use of deeper applications only in later chapters.

I follow more or less [DG] in the sections 5.1 - 5.7. Proposition 5.13 was first proved in [Haboush 2]. Let me add that closely related matter is treated in [Cline/Parshall/Scott 9].

5.1 (Quotients) For a linear algebraic group G over an algebraically closed field and a closed subgroup H of G it is well known how to make the coset space G/H into a variety. We should like to have a generalization to the case where G is a k-group scheme and H a closed subgroup scheme. Unfortunately the "obvious" choice, i.e. the functor $A \mapsto G(A)/H(A)$ turns out to be the wrong one (in general) as it will be no scheme in general.

Let us define instead a quotient via a universal property. This can be done in the more general situation of a k-group scheme G operating on a scheme X over k. A quotient scheme of X by G is a pair (Y,π) where Y is a scheme and $\pi: X \to Y$ is a morphism such that π is constant on G-orbits and such that for each morphism $f: X \to Y'$ of schemes constant on G-orbits there is exactly one morphism $f': Y \to Y'$ with $f' \circ \pi = f$. ("Constant on G-orbits" means that each $\pi(A): X(A) \to Y(A)$ is constant on the $G(A)$-orbits.) Of course, such a quotient scheme is unique up to unique isomorphism, if it exists (and that is the problem).

Let us give another formulation of this definition. We want to assume that G operates from the right. (The necessary changes for left actions will be obvious.) Consider the two morphisms $\alpha,\alpha'\colon X\times G \to X$ with $\alpha(x,g) = xg$ and $\alpha'(x,g) = x$. Then a morphism $f\colon X \to Y'$ will be constant on G-orbits if and only if $f\circ\alpha = f\circ\alpha'$. So (Y,π) is a quotient scheme if and only if $\pi\circ\alpha = \pi\circ\alpha'$ and if for all morphisms $f\colon X \to Y'$ with $f\circ\alpha = f\circ\alpha'$ there is a unique morphism $f'\colon Y \to Y'$ with $f'\circ\pi = f$. (We assume Y,Y' to be schemes.) So a quotient scheme of X by G is (in categorical language) the cokernel of the pair (α,α') in the category of schemes over k.

This way of formulating the universal property allows for generalizations. Take for example a "schematic" equivalence relation on X, i.e. a subscheme $R\subset X\times X$ such that each $R(A)$ is an equivalence relation on $X(A)$. Then a quotient scheme of X by R is the cokernel in the category of schemes of the pair of the projections from R to X. There is a generalization of these two situations (i.e. of $X\times G \rightrightarrows X$ for group actions and of $R \rightrightarrows X$ for equivalence relations) called groupoid. This is discussed e.g. in [DG], III, §2, n° 1.

5.2. (The fppf-topology) Of course, we can define quotients by group actions also in larger categories than {schemes over k} using the same type of universal property as before but allowing any Y,Y' in that larger category. If we take e.g. the category of all k-functors, then certainly $A\mapsto X(A)/G(A)$ is the quotient. If we had now a functor from {k-functors} to {schemes over k}

left adjoint to the inclusion, then it would map $A \mapsto X(A)/G(A)$ to the quotient scheme. But we do not have such a functor. It has shown to be useful in this situation to replace the category {schemes over k} by a larger one for which there is such a functor and has nice properties.

Any scheme X is by definition local (cf. 1.8), i.e. $Y \mapsto Mor(Y,X)$ is a sheaf in some sense: If $(Y_j)_j$ is an open covering of Y, then any $\alpha \in Mor(Y,X)$ is uniquely determined by its restriction to the Y_j and one can glue morphisms $\alpha_j \in Mor(Y_j,X)$ together if they coincide on intersections. The open coverings were defined using the Zariski topology.

One can now consider more general topologies, called _Grothendieck topologies_ where the property "open" is no longer attached to subsets (or rather subfunctors) but to certain morphisms. We shall consider only the _faithfully flat, finitely presented_ topology (for short "fppf" as the French is much more symmetric in this case), and the k-functors with the sheaf property for this topology will be called faisceaux (reserving the term "sheaf" to objects related to the Zariski topology).

As in 1.8 it is enough to consider open coverings of affine schemes by affine schemes. Let R be a k-algebra. An _fppf-open covering_ of R is a finite family $R_1,R_2,\ldots,R_n$ of R-algebras such that each R_i is a finitely presented R-module and such that $R_1 \times R_2 \times \ldots \times R_n$ is a faithfully flat R-module. (The last condition is equivalent to: Each R_i is a flat R-module and $Spec(R)$ is the union of the images of all $Spec(R_i)$, cf. [3], ch. II,

§2, cor. 4 de la prop. 4.)

<u>5.3</u> (<u>Faisceaux</u>) A k-functor X is called a <u>faisceau</u> if for each k-algebra R and each fppf-open covering $R_1,R_2,\ldots,R_n$ of R the sequence

$$(1) \qquad X(R) \to \prod_i X(R) \rightrightarrows \prod_{i,j} X(R_i \otimes_R R_j)$$

is exact. (The maps are the obvious ones, induced by the structural maps $R \to R_i$ and by $R_i \to R_i \otimes_R R_j$ resp. $R_i \to R_j \otimes_R R_i$ with $a \mapsto a \otimes 1$ resp. $a \mapsto 1 \otimes a$.) (A k-faisceau is defined as a k-functor which is a faisceau.)

For any k-algebras $R_1,R_2,\ldots,R_n$ we can regard each R_i as a $(\prod_{i=1}^n R_i)$-algebra via the projection . The R_i form obviously an fppf-open covering of $\prod_{i=1}^n R_i = R$. As $R_i \otimes_R R_j = 0$ for $i \neq j$ the exactness of (1) amounts in this case to:

(2) <u>The projections induce for all</u> k-<u>algebras</u> $R_1,\ldots,R_n$ <u>a bijection</u> $X(R_1 \times \ldots \times R_n) \xrightarrow{\sim} X(R_1) \times \ldots \times X(R_n)$.

A single R-algebra R' is an fppf-open covering of R if and only if it is faithfully flat and finitely presented as an R-module. Let us call this an "<u>fppf</u>-R-<u>algebra</u>". So the exactness of (1) implies:

(3) <u>If</u> R <u>is a</u> k-<u>algebra and if</u> R' <u>is an fppf</u>-R-<u>algebra, then</u> $X(R) \to X(R') \rightrightarrows X(R' \otimes_R R')$ <u>is exact</u>.

So the arguments above prove one direction of:

(4) <u>A</u> k-<u>functor</u> X <u>is a faisceau, if and only if it satisfies</u> (2) <u>and</u> (3).

For the converse one applies (3) to $R' = \prod_{i=1}^{m} R_i$ and (2) to $\prod_i R_i$ and $\prod_i R_i \otimes_R \prod_i R_j$.

Suppose that R' is a faithfully flat R-algebra. We have then an exact sequence

$$0 \to R \to R' \to R' \otimes_R R'$$

where $R \to R'$ is the structural map and where any $a \in R'$ is mapped to $a \otimes 1 - 1 \otimes a$. (This is only the beginning of a long exact sequence, see [DG], I, §1, 2.7. It is enough to show the exactness of $0 \to R \otimes_R R' \to R' \otimes_R R' \to R' \otimes_R R' \otimes_R R'$. The last map sends $a \otimes a'$ to $a \otimes 1 \otimes a' - 1 \otimes a \otimes a'$. If this is 0, then $0 = a \otimes a' - 1 \otimes aa'$, hence $a \otimes a'$ is in the image of the previous map.) We can express the exactness above also as:

(5) $R \to R' \rightrightarrows R' \otimes_R R'$ is exact

(where the two maps are $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$). Now the left exactness of $\mathrm{Hom}_{k\text{-alg}}(A,?)$ shows that each affine scheme $\mathrm{Sp}_k A$ over k is a faisceau. More generally one can show ([DG], III, §1, 1.3)

(6) <u>Any scheme over</u> k <u>is a faisceau</u>.

Let M be a k-module and k' a faithfully flat k-algebra. Then the same argument as above gives as an exact sequence

$$0 \to M \to M\otimes k' \to M\otimes k'\otimes k'$$

with maps $m \mapsto m\otimes 1$ and $m\otimes b \mapsto m\otimes b\otimes 1 - m\otimes 1\otimes b$. Applying this to all $M\otimes A$ we get:

(7) For each k-module M the functor M_a is a faisceau and a local functor.

Of course we could have mentioned the "local" part earlier. It follows from the fact that $\sum_{i=1}^{n} Af_i = A$ implies that $\prod_{i=1}^{r} (A_{f_i})$ is faithfully flat over A. (See also the description of the quasi-coherent sheaf on $\mathrm{Spec}(A)$ associated to $M\otimes A$ in [Ha], II, 5.1.)

The following property is obvious:

(8) Let X be a k-functor and k' a k-algebra. If X is a faisceau, then $X_{k'}$ is a faisceau.

5.4 (Associated Faisceaux) There is a natural construction how to associate to each k-functor X a k-faisceau $\tilde{X}$ (called the associated faisceau) together with a morphism $i: X \to \tilde{X}$ such that for all k-faisceau Y the map $f \mapsto f\circ i$ is a bijection $\mathrm{Mor}(\tilde{X},Y) \to \mathrm{Mor}(X,Y)$. We get thus a functor $X \mapsto \tilde{X}$ from {k-functors} to {k-faisceaux} left adjoint to the inclusion of {k-faisceaux} into {k-functors}. This construction should be regarded as an analogue of the construction of a sheaf associated to a presheaf. The details may be found in [DG], III, §1, 1.8 - 1.12. I shall describe $\tilde{X}$ only in a particularly simple case where X is already close to being a faisceau. To be more precise I want to assume the following:

(1) X satisfies 5.3(2) and $X(R) \to X(R')$ is injective for each k-algebra R and each fppf-R-algebra R'.

Under this assumption $\tilde{X}$ has the following form. Take a k-algebra A and consider for each fppf-A-algebra B the kernel $X(B,A)$ of $X(B) \rightrightarrows X(B \otimes_A B)$. If B' is an fppf-B-algebra, then B' is also fppf over A and the natural inclusion from $X(B)$ into $X(B')$ maps $X(B,A)$ into $X(B',A)$. More precisely $B' \otimes_A B'$ is fppf over $B \otimes_A B$, hence the standard map $X(B \otimes_A B) \to X(B' \otimes_A B')$ is injective and we can identify $X(B,A)$ with the intersection of $X(B',A)$ and $X(B)$. The $X(B,A)$ with B fppf over A form a direct system. (If B_1, B_2 are fppf over A, then $B_1 \otimes_A B_2$ is fppf over B_1 and B_2.) So we can form the direct limit of these $X(B,A)$ and this is our $\tilde{X}(A)$:

(2) $\tilde{X}(A) = \varinjlim X(B,A)$.

As all maps $X(B,A) \to X(B',A)$ are injective so are all maps $X(B,A) \to \tilde{X}(A)$, we can identify $X(B,A)$ with its image in $\tilde{X}(A)$ and regard $\tilde{X}(A)$ as the union of all $X(B,A)$. We see especially:

(3) For X as in (1) each $X(A) \to \tilde{X}(A)$ is injective.

(For arbitrary X this will not be true.)

If $A \to A'$ is a homomorphism of k-algebras, then $B \otimes_A A'$ is fppf over A' for any fppf-A-algebra B, and the natural map

$X(B) \to X(B \otimes_A A')$ maps $X(B,A)$ to $X(B \otimes_A A', A')$. Taking direct limits we get a map $\tilde{X}(A) \to \tilde{X}(A')$ which is easily checked to be functorial. In this way $\tilde{X}$ is a k-functor. It is rather obvious that $\tilde{X}$ inherits the property (1) from X. Consider any element in the kernel of $\tilde{X}(B) \rightrightarrows \tilde{X}(B \otimes_A B)$ for some fppf-A-algebra B. Then it belongs to $X(B',B)$ for some fppf-B-algebra B'. The restrictions of the two maps from $\tilde{X}(B)$ to $X(B',B)$ are induced by the maps $X(B',B) \to X(B' \otimes_B (B \otimes_A B), B \otimes_A B) \tilde{\to} X(B' \otimes_A B, B \otimes_A B) \subset X(B' \otimes_A B', B \otimes_A B) \subset \tilde{X}(B \otimes_A B)$ and $X(B',B) \to X(B' \otimes_A (B \otimes_A B), B \otimes_A B) \tilde{\to} X(B \otimes_A B', B \otimes_A B) \subset X(B' \otimes_A B', B \otimes_A B) \subset \tilde{X}(B \otimes_A B)$ where isomorphism in the second step is induced by $b' \otimes (b_1 \otimes b_2) \mapsto (b'b_1) \otimes b_2$ in the first case and by $b' \otimes (b_1 \otimes b_2) \mapsto b_1 \otimes (b'b_2)$ in the second case. (We use here that $B' \otimes_A B' \simeq B' \otimes_B (B \otimes_A B') \simeq (B' \otimes_A B) \otimes_B B'$ is fppf over $B \otimes_A B'$ and $B \otimes_A B'$.) Therefore the intersection of $\ker(\tilde{X}(B) \rightrightarrows \tilde{X}(B \otimes_A B))$ with $X(B',B)$ is equal to $\ker(X(B',B) \rightrightarrows X(B' \otimes_A B', B \otimes_A B)) = \ker(X(B') \rightrightarrows X(B' \otimes_A B')) = X(B',A)$, hence contained in $\tilde{X}(A)$. This shows that $\tilde{X}$ is a faisceau.

For any morphism $f: X \to Y$ into a k-faisceau Y any $f(B)x$ with $x \in X(B,A)$ as above has to belong to $Y(A) \subset Y(B)$ so we can define $\tilde{f}: \tilde{X} \to Y$ through $\tilde{f}(A)x = f(B)x \in Y(A)$. This is easily checked to be a morphism and to be unique with $\tilde{f}_{|X} = f$. So $\tilde{X}$ has indeed the universal property we wanted.

Notice: If each $f(A)$ is injective, so is each $\tilde{f}(A)$. So we can regard X as a subfunctor of Y. One gets easily the following:

(4) <u>Let</u> X <u>be a subfunctor of a</u> k-<u>faisceau</u> Y <u>such that</u> X <u>satisfies</u> 5.3(2). <u>Then</u> $\tilde{X}$ <u>is a subfunctor of</u> Y. <u>One has</u> $\tilde{X}(A) = \{x \in Y(A) \mid$ <u>there is a fppf</u>-A-<u>algebra</u> B <u>with</u> $x \in X(B)\}$.

It is clear in a situation as in (1), but can be proved also in the general situation, that taking the associated faisceau commutes with base change:

(5) <u>Let</u> X <u>be a</u> k-<u>functor and</u> k' <u>a</u> k-<u>algebra. Then</u> $(\tilde{X})_{k'}$ <u>is the faisceau associated to</u> $X_{k'}$.

<u>5.5</u> (<u>Images and Quotients</u>) Let $f: X \to Y$ be a morphism of k-faisceaux. The subfunctor $A \mapsto \mathrm{im}(f(A)) = f(A)X(A)$ of Y satisfies obviously 5.3(2). So 5.4(4) yields a rather precise description of the associated faisceau which is called the <u>image faisceau</u> of f. We shall usually denote this associated faisceau by $f(X)$ or $\mathrm{im}(f)$. So in general $f(A)X(A)$ is properly contained in $f(X)(A)$.

Notice: If X is a subfunctor of some k-functor Y and if both X and Y are faisceaux, then $X(A) = X(B) \cap Y(A)$ for each k-algebra A and each fppf-A-algebra B. This is obvious from the description of $X(A)$ as the kernel of $X(B) \rightrightarrows X(B \otimes_A B)$.

Now let G be a k-group faisceau and $H \subset G$ a subgroup faisceau, i.e. G is a k-group functor and H is a subgroup functor such that both are faisceaux (as functors). Then the functor $A \mapsto G(A)/H(A)$ satisfies 5.4(1). This is clear for the part about direct products. If B is an fppf-A-algebra and if $g, g' \in G(A)$ have the property $gH(B) = g'H(B)$, then

$g^{-1}g' \in G(A) \cap H(B) = H(A)$ as observed above. Therefore $G(A)/H(A) \to G(B)/H(B)$ is injective. We call the faisceau associated to $A \mapsto G(A)/H(A)$ the quotient faisceau of G by H and denote it by G/H. (So in general $(G/H)(A) \neq G(A)/H(A)$). Obviously the universal property of $X \mapsto \tilde{X}$ shows that $(G/H, \pi)$ where $\pi: G \to G/H$ is the canonical map $G(A) \to G(A)/H(A) \to (G/H)(A)$ has the universal property of a quotient within the category of {k-faisceaux}. We know by 5.4(3) that the canonical map $G(A)/H(A) \to (G/H)(A)$ is injective for each A. This can be expressed in the following form: Consider the fibre product $G \times_{G/H} G$ with respect to π (twice). Each $(G \times_{G/H} G)(A)$ consists of those $(g,g') \in G(A) \times G(A)$ with $\pi(g) = \pi(g')$, hence (by the injectivity) with $gH(A) = g'H(A)$. Therefore the maps $G(A) \times H(A) \to G(A) \times G(A)$ with $(g,h) \mapsto (g,gh)$ induce an isomorphism

(1) $G \times H \xrightarrow{\sim} G \times_{G/H} G.$

Suppose G acts from the left on a k-functor X satisfying 5.4(1). Let $x \in X(k)$. Then the subgroup functor $\mathrm{Stab}_G(x)$ of G is a faisceau where

$$\mathrm{Stab}_G(x)(A) = \{g \in G(A) \mid gx = x\}$$

for all A. We may identify the functor $A \mapsto G(A)/\mathrm{Stab}_G(x)(A)$ with a subfunctor of X. Suppose now that X is a faisceau. Then the associated faisceau $G/\mathrm{Stab}_G(x)$ can be identified with a subfunctor of X. More precisely the morphism $\pi_x: G \to X$, $g \mapsto gx$ factors through $G/\mathrm{Stab}_G(x)$ and induces an isomorphism

(2) $G/\mathrm{Stab}_G(x) \cong \mathrm{im}(\pi_x)$

onto the image faisceau of π_x which is also called the <u>orbit faisceau</u> of x.

We can define for each operation of a k-group faisceau G on a k-faisceau X (say from the right) a quotient faisceau X/G as the associated faisceau of the functor $A \mapsto X(A)/G(A)$. In general this functor will not satisfy 5.4(1) so in general the description of X/G is more complicated than what is done in 5.4. If however each $G(A)$ acts <u>fixed point free</u> on $X(A)$, then 5.4(1) and 5.4(3) hold and one has similar to (1) an isomorphism $X\times G \tilde{\rightarrow} X\times_{X/G}X$. One has always $(X/G)(k) = X(k)/G(k)$ if k is an algebraically closed field, e.g. by [DG], III, §1, 1.15. Take such k and assume $X(k)$ and $G(k)$ to be varieties. In general there will be orbits of $G(k)$ on $X(k)$ which are not closed. Then $X(k)/G(k)$ cannot be a variety such that the canonical map $X(k) \to X(k)/G(k)$ is a morphism. Therefore in this situation X/G is not a scheme. It is only for very nice operations (like a subgroup on a whole group) where the quotient faisceau leads (most of the time) to the quotient scheme.

Let us mention one special case. Take G,H as above and let H operate on some k-faisceau X from the left. Then H operates on $G\times X$ from the right via $(g,x)h = (gh,h^{-1}x)$. This operation is fixed point free (as the operation of H on G is so). Let us denote the quotient $(G\times X)/H$ by $G\times^H X$ and call it the <u>associated bundle</u> over G/H corresponding to X. Notice that the morphism $G\times H \to G/H$, $(g,x)\mapsto \pi(g)$ with π as above is constant on the $H(A)$-orbits and takes values in a faisceau, hence factors through $G\times^H X$ so that we have a canonical map π_X: $G\times^H X \to G/H$. It is easily checked that the map $(g,x)\mapsto (g,(g,x)H)$ is an isomorphism from $G\times X$ to the functor $A\mapsto G(A)\times_{(G/H)(A)}$ $(G(A)\times X(A)/H(A)$. So the right hand side is a faisceau. On the other hand its associated faisceau is $G\times_{(G/H)}(G\times^H X)$. So we get an isomorphism

(3) $\quad G\times X \cong G\times_{G/H}(G\times^H X)$.

5.6 (Quotient Faisceaux as Schemes) Let G be an affine group scheme and H an affine subgroup scheme. If the quotient faisceau G/H happens to be a scheme, then it is because of 5.3(6) also the quotient scheme.

In general G/H is not a scheme, see the counter-examples in [DG], III, §3, 3.3 and in [10] , p. 157. There are however some important cases where it is a scheme which we want to mention now.

(1) If k is a field and if G and H are algebraic k-groups, then G/H is a scheme.

This is proved in [DG], III, §3, 5.4. (Remember that "algebraic" means that $k[G]$ and $k[H]$ are finitely generated as k-algebras.) It is a special case of the following :

(2) If k is a Dedekind ring, if G is an algebraic k-group and if H is a closed and flat subgroup scheme of H, then G/H is a scheme.

This is proved in [1], Thm. 4.C.

Let us call an affine group scheme G over k <u>finite</u> if $k[G]$ is a finitely generated projective k-module. Now one has

(3) <u>If</u> H <u>is finite, then</u> G/H <u>is an affine scheme</u>.

This is really a special case of the following, more general result:

(4) <u>Let</u> X <u>be an affine scheme on which</u> G <u>operates fixed point free</u>. <u>If</u> G <u>is finite, then</u> X/G <u>is an affine scheme. It is isomorphic to</u> $Sp_k(k[X]^G)$.

Though not stated in this way this follows easily by combining [DG], III, §2, n° 4 and §1, 2.10. The results at the first place imply that $k[X]$ is finitely generated and projective as a module over $k[X]^G$ and that X/G is a subfunctor of $Sp_k(k[X]^G)$. The second result quoted implies that the inclusion $k[X]^G \subset k[X]$ induces an epimorphism $X \to Sp_k(k[X]^G)$ in the category of k-faisceaux while on the other hand the image faisceau is equal to X/G.

There is in [DG], III, §2 also a discussion of the case where X is not affine or where G does not act fixed point free.

<u>5.7</u> (<u>Flatness of Quotients</u>) Let G and H be a group scheme such that H is a subgroup scheme of G. Let us quote from [DG], III, §3, 2.5 and 2.6 the following result:

(1) <u>If</u> H <u>is flat and if</u> G/H <u>is a scheme, then the canonical map</u> $\pi: G \to G/H$ <u>is faithfully flat and affine</u>.

If $U \subset G/H$ is an open and affine subscheme and if we are in the situation of (1), then $\pi^{-1}(U) \subset G$ is an affine subscheme of G and $k[\pi^{-1}(U)]$ is faithfully flat over $k[U]$. If G is flat, then $k[\pi^{-1}(U)]$ is flat over k, hence also $k[U]$ is flat over k. So we get:

(2) _If_ G _and_ H _are flat anf if_ G/H _is a scheme, then_ G/H _is flat_.

5.8 Proposition: Let G _be a flat group scheme over_ k _and_ H _a flat subgroup scheme in_ k. _If_ G/H _is an affine scheme, then_ H _is exact in_ G.

Proof: Set $R = k[G/H] = k[G]^H$. The isomorphism $G\times H \xrightarrow{\sim} G\times_{G/H}G$ in 5.5(1) is compatible with the action of H by right multiplication on the second factors, hence the isomorphism $k[G]\otimes_R k[G] \xrightarrow{\sim} k[G]\otimes k[H]$ is compatible with the representation of H via ρ_r on the second factors (and the trivial representation on the first factors).

Let M be an H-module. Then we can tensor the last isomorphism with M (over k) to get an isomorphism of H-modules

(1) $M\otimes k[G]\otimes k[H] \simeq k[G]\otimes_R(M\otimes k[G])$.

As H operates trivially on the first factor $k[G]$, we get for the Hochschild complex

(2) $C^{\bullet}(H,k[G]\otimes_R(M\otimes k[G])) \simeq k[G]\otimes_R C^{\bullet}(H,M\otimes k[G])$.

We know by 5.7(1) that $k[G]$ is faithfully flat over R, hence

(2) yields

(3) $H^{\bullet}(H, k[G] \otimes_R (M \otimes k[G])) \cong k[G] \otimes_R H^{\bullet}(H, M \otimes k[G])$.

Now (1), (3) and 4.7.a imply

(4) $k[G] \otimes_R H^n(H, M \otimes k[G]) = 0$ for all $n > 0$.

Again the faithful flatness of $k[G]$ over R together with 4.10 yields $R^n \mathrm{ind}_H^G M = 0$ for all $n > 0$, hence our claim.

Remarks: 1) This proposition follows also from the interpretation of the $R^n \mathrm{ind}_H^G M$ as sheaf cohomology groups, cf. 5.13. 2) For linear algebraic groups over an algebraically closed field the converse of the proposition is proved in [Cline/Parshall/Scott 3], 4.3.

5.9 Corollary: Let G be a flat group scheme over k and H a finite subgroup scheme. Then H is exact in G.

This is clear from 5.6(3).

5.10 (Associated Sheaves) Let us assume from now on that G is a flat group scheme over k and that H is a flat subgroup scheme of G such that G/H is a scheme. Let us denote the canonical map $G \to G/H$ by π.

It was mentioned in 1.11 that there corresponds to each scheme X a topological space $|X|$ with a sheaf of rings. Furthermore the open subsets of $|X|$ correspond bijectively to the open subfunctors of X. We can therefore describe a sheaf on $|X|$ as a contravariant functor from {open subfunctors of X}

(with inclusions as morphisms) to some other category having the usual sheaf property for open coverings of open subfunctors (defined in 1.7(4)). For example, the structural sheaf $\mathcal{O}_X$ associates to each open subfunctor Y the ring $\mathcal{O}_X(Y) := \mathrm{Mor}(Y, \mathbb{A}^1) = k[Y]$.

We want to apply this to $X = G/H$ and to associate to each H-module M a sheaf $\mathcal{L}(M) = \mathcal{L}_{G/H}(M)$ on G/H. We set for each open subfunctor $U \subset G/H$:

$$(1)\quad \mathcal{L}(M)(U) = \{f \in \mathrm{Mor}(\pi^{-1}U, M_a) \mid f(gh) = h^{-1}f(g) \quad \text{for all} \quad h \in H(A), g \in (\pi^{-1}U)(A) \text{ and all } A\}.$$

If $\pi^{-1}U$ is affine, then we have a representation of H on $k[\pi^{-1}U]$ by right translation. Tensoring this with the given action we get obviously

$$(2)\quad \mathcal{L}(M)(U) = (k[\pi^{-1}U] \otimes M)^H \qquad \text{for } \pi^{-1}U \text{ affine,}$$

especially

$$(3)\quad \mathcal{L}(M)(G/H) = \mathrm{ind}_H^G M.$$

If U, U' are open subfunctors of G/H with $U \subset U'$ then we have an obvious restriction map $\mathcal{L}(M)(U') \to \mathcal{L}(M)(U)$. So $\mathcal{L}(M)$ is at least a presheaf.

We can express the definition of $\mathcal{L}(M)$ as follows. Consider the morphisms $\alpha: \pi^{-1}(U) \times H \to \pi^{-1}(U)$, $(g,h) \mapsto gh$ and α': $M_a \times H \to M_a$, $(m,h) \mapsto h^{-1}m$. Then $f \in \mathrm{Mor}(\pi^{-1}(U), M_a)$ is in $\mathcal{L}(M)(U)$ if and only if $f \circ \alpha = \alpha' \circ (f \times id_H)$. So we have an exact sequence

(4) $\mathcal{L}(M)(U) \to \mathrm{Mor}(\pi^{-1}(U), M_a) \rightrightarrows \mathrm{Mor}(\pi^{-1}(U) \times H, M_a)$.

Because of 5.3(7) the functors $U \mapsto \mathrm{Mor}(\pi^{-1}(U), M_a)$ and $U \mapsto \mathrm{Mor}(\pi^{-1}(U), M_a)$ are sheaves, hence so is $\mathcal{L}(M)$. It is called the <u>associated sheaf</u> to M on G/H. It is obviously a sheaf of $\mathcal{O}_{G/H}$-modules. If $\varphi: M \to M'$ is a homomorphism of H-modules, then

(5) $\mathcal{L}(\varphi): \mathcal{L}(M) \to \mathcal{L}(M'), \quad f \mapsto \varphi_a \circ f$

is obviously a homomorphism of $\mathcal{O}_{G/H}$-modules. So $\mathcal{L}$ is a functor from {H-modules} to {$\mathcal{O}_{G/H}$-modules}.

<u>5.11</u> <u>Proposition</u>: a) <u>The functor $\mathcal{L}$ is exact.</u>
b) <u>For each</u> H-<u>module</u> M <u>the</u> $\mathcal{O}_{G/H}$-<u>module</u> $\mathcal{L}(M)$ <u>is quasi-coherent</u>.
c) <u>If</u> M <u>is an</u> H-<u>module which is finitely generated over</u> k, <u>then</u> $\mathcal{L}(M)$ <u>is a coherent</u> $\mathcal{O}_{G/H}$-<u>module</u>.

<u>Proof</u>: a) It is enough to show that $M \mapsto \mathcal{L}(M)(U)$ is exact for any open and affine $U \subset G/H$. For such U also $U' = \pi^{-1}U$ is affine and $k[U']$ is faithfully flat over $k[U]$ by 5.7(1). It is therefore enough to show that

$$M \mapsto k[U'] \otimes_{k[U]} \mathcal{L}(M)(U) = k[U'] \otimes_{k[U]} (k[U'] \otimes M)^H$$

is exact, cf. 5.10(2). The isomorphism in 5.5(1) induces an isomorphism $U' \times H \to U' \times_U U'$ compatible with the right action of H on the second factor, hence so is the corresponding isomorphism $k[U'] \otimes_{k[U]} k[U'] \simeq k[U'] \otimes k[H]$. As $(k[H] \otimes M)^H \simeq M$ (cf. 3.7(6)), the functor above can be identified with $M \mapsto M \otimes k[U']$.

This is exact, as we assume G to be flat.

b) For each scheme X and each k-module M the sheaf $U \mapsto \mathrm{Mor}(U, M_a)$ is quasi-coherent. (If X is affine, then it is the quasi-coherent sheaf associated to the $k[X]$-module $k[X] \otimes M$, cf. Yoneda's lemma, 1.3). The sheaves $U \mapsto \mathrm{Mor}(\pi^{-1}(U), M_a)$ and $U \mapsto \mathrm{Mor}(\pi^{-1}U) \times H, M_a)$ in 5.10(4) are direct images of such sheaves, hence quasi-coherent (cf. [Ha], II, 5.8), hence so is the kernel $\mathcal{L}(M)$, cf. [Ha], II, 5.7.

c) We have to show that $\mathcal{L}(M)(U)$ is finitely generated over $k[U]$ for each $U \subset G/H$ open and affine. As $k[\pi^{-1}U]$ is faithfully flat over $k[U]$ by 5.7(1) it is enough to show that $k[\pi^{-1}U] \otimes_{k[U]} \mathcal{L}(M)(U)$ finitely generated over $k[\pi^{-1}U]$, e.g. by [3], ch. I, §3, prop. 11. This module is isomorphic to $k[\pi^{-1}U] \otimes M$ as seen in the proof of a), hence finitely generated by assumption.

5.12 (Examples) Let us mention a free trivial cases. The trivial H-module k yields $\mathcal{L}(k)(U) = \mathrm{Mor}(\pi^{-1}(U), \mathbb{A}^1)^H \cong \mathrm{Mor}(\pi^{-1}(U)/H, \mathbb{A}^1) \cong \mathrm{Mor}(U, \mathbb{A}^1)$, hence

(1) $\mathcal{L}(k) \cong \mathcal{O}_{G/H}$.

Consider on the other hand the H-module $k[H]$ under ρ_ℓ or, more generally, any $M \otimes k[H]$ for any k-module M regarded as a trivial H-module. For any $U \subset G/H$ open and affine we can identify $\mathrm{Mor}(\pi^{-1}U, (M \otimes k[H])_a) \cong \mathrm{Mor}((\pi^{-1}U) \times H, M_a)$ and the H-invariance condition translates into $f(g,h') = f(gh, h^{-1}h')$ for all g,h,h'. The map $(g,h) \mapsto (gh,h)$ is an automorphism of

$(\pi^{-1}U)\times H$ and transfers the condition into $f(g,hh') = f(g,h')$. In this way $\mathcal{L}(M)(U)$ is identified with $\mathrm{Mor}(\pi^{-1}U,M_a) = \mathcal{L}_{G/1}(M)(\pi^{-1}U) = (\pi_*\mathcal{L}_{G/1}(M))(U)$. This implies

(2) $\mathcal{L}(M\otimes k[H]) \simeq \pi_*\mathcal{L}_{G/1}(M)$.

This is a special case of the following result. Let $H'\subset H$ be a flat subgroup scheme such that also G/H' is a scheme. There is a canonical morphism $\pi': G/H' \to G/H$. Then we get an isomorphism of functors

(3) $\pi'_*\circ\mathcal{L}_{G/H'} \simeq \mathcal{L}_{G/H}\circ\mathrm{ind}_{H'}^{H}$.

On the level of global functions this is just the transitivity of induction 3.5(2). The proof may be left to the reader.

5.13 Proposition: We have for all H-modules M and all $n\in\mathbb{N}$ isomorphisms of k-modules

$$H^n(G/H,\mathcal{L}(M)) \simeq (R^n\mathrm{ind}_H^G)(M).$$

Proof: We can interprete 5.10(3) as an isomorphism of functors

$$\underline{\underline{F}}\circ\mathrm{ind}_H^G \simeq H^0(G/H,?)\circ\mathcal{L}$$

where $\underline{\underline{F}}$ is the forgetful functor from {G-modules} to {k-modules}. In order to apply 4.1(2),(3) we have to know that $\mathcal{L}$ maps injective H-modules to acyclic sheaves. By 3.9.c it is enough to consider H-modules of the form $M\otimes k[H]$ for a trivial H-module M. Because of 5.12(2) we have to look at all $H^n(G/H,\pi_*\mathcal{L}_{G/1}(M))$.

But as G and π are affine we get e.g. from [Ha], III, exerc. 4.1 and thm. 3.5 that

$$H^n(G/H, \pi_* \mathcal{L}_{G/1}(M)) \simeq H^n(G, \mathcal{L}_{G/1}(M)) = 0$$

for all $n > 0$.

5.14 Of course 5.13 gives another approach to proposition 5.8.

Let us mention two corollaries which follow from well known results on sheaf cohomology (cf. [Ha], III, 2.7 and 5.2(a)):

(1) Suppose that G/H is noetherian. Then $R^n \mathrm{ind}_H^G = 0$ for all $n > \dim G/H$.

(2) Suppose that k is noetherian and that M is finitely generated over k. If G/H is a projective scheme, then each $H^n(G/H, \mathcal{L}(M)) \simeq R^n \mathrm{ind}_H^G M$ is a finitely generated k-module.

One can use 5.13 also to get new approaches to earlier results. For example 5.12(3) yields at first isomorphisms of derived functors $(R^i \pi'_*) \circ \mathcal{L}_{G/H'} \simeq \mathcal{L}_{G/H} \circ (R^i \mathrm{ind}_{H'}^H)$, cf. [Andersen 2], 1.2, and then we get 4.5.c from the Leray spectral sequence $H^j(G/H, R^i \pi'_* ?) \Rightarrow H^{i+j}(G/H', ?)$.

5.15 (Associated Sheaves and Bundles) The associated sheaves $\mathcal{L}(M)$ can also be described using the associated bundle $G \times^H M_a$ as in 5.5. Set $\Gamma(U, G \times^H M_a)$ for each open $U \subset G/H$ equal to the set of all morphisms $s: U \to G \times^H M_a$ with $\pi_M \circ s = \mathrm{id}_U$, i.e. of all sections of π_M over U. (Here π_M is the π_{M_a} from 5.5.) We claim that we have for each such U a canonical bijection

(1) $\mathcal{L}(M)(U) \tilde{\to} \Gamma(U, G\times^H M_a)$.

Any $f \in \mathcal{L}(M)(U)$ is a map $f: \pi^{-1}U \to M_a$, hence defines a map $f_1: \pi^{-1}U \to G\times^H M_a$ with $\pi_M \circ f_1 = \pi$ by mapping any $g \in \pi^{-1}U$ at first to $(g,f(g)) \in G\times M_a$ and then to its canonical image in $G\times^H M_a$. This map f_1 is obvious constant on H-cosets, hence factors through $\pi^{-1}U/H$ which we can identify with U via π. (Note that $\pi^{-1}U$ is an open subfunctor of G, hence a scheme, hence a faisceau so that $\pi^{-1}U/H$ makes sense and is equal to U.) The factorisation $\bar{f}: U \to G\times^H M_a$ satisfies $\pi_M \circ \bar{f} \circ \pi = \pi$, hence $\pi_M \circ \bar{f} = id_U$, i.e. $\bar{f} \in \Gamma(U, G\times^H M_a)$.

Consider on the other hand $s \in \Gamma(U, G\times^H M_a)$. Take the isomorphism $\alpha: G\times_{G/H}(G\times^H M_a) \tilde{\to} G\times M_a$ from 5.5(3). Combining the map $g \mapsto (g,s(g))$ from $\pi^{-1}U$ to $G\times_{G/H}(G\times^H M_a)$ with α and then the second projection $G\times M_a \to M_a$ we get a morphism $s_1: \pi^{-1}U \to M_a$. It can now be checked that $s_1 \in \mathcal{L}(M)(U)$ and that the maps $s \mapsto s_1$ and $f \mapsto \bar{f}$ are inverse to each other. See [Cline/Parshall/Scott 9], 1.3 for more details. (In down to earth terms $s_1(g)$ is for any $g \in \pi^{-1}U(A)$ the unique element in $M \otimes A$ such that $s(\pi(g))$ is the class of $(g, s_1(g))$.

<u>5.16</u> (<u>Pull-backs</u>) Let $\varphi: G' \to G$ be a homomorphism of k-group schemes and let H' be a subgroup scheme of G' with $\varphi(H') \subset H$. Suppose that G'/H' is a scheme.

The universal property of G'/H' yields a morphism $\bar{\varphi}: G'/H' \to G/H$ with $\pi \circ \varphi = \bar{\varphi} \circ \pi'$ where $\pi: G \to G/H$ and $\pi': G' \to G'/H'$ are the canonical maps. We can now form for each

H-module M the inverse image sheaf $\bar{\varphi}^* \mathcal{L}_{G/H}(M)$. On the other hand we can consider M via $\varphi_{|H'}$ as an H'-module and form $\mathcal{L}_{G'/H'}(M)$. We claim that these sheaves of $\mathcal{O}_{G'/H'}$-modules are isomorphic :

$$(1) \qquad \bar{\varphi}^* \mathcal{L}_{G/H}(M) \simeq \mathcal{L}_{G'/H'}(M).$$

One can show that the inverse image of the sheaf $U \mapsto \Gamma(U, G \times^H M_a)$ is the sheaf $U' \mapsto \Gamma(U', G'/H' \times_{G/H} (G \times^H M_a))$. One can check that there is an isomorphism $G' \times^{H'} M_a \tilde{\to} G'/H' \times_{G/H} (G \times^H M_a)$ of the form $(g', m)H' \mapsto (g'H', (\varphi(g'), m)H)$. (Details may be left to the reader.) From this we get (1) using 5.15.

6. Factor Groups

If G is a k-group faisceau and N a normal subgroup faisceau of G, then G/N is again a k-group faisceau and has the universal property of a factor group. This and related things are described in 6.1/2 following [DG].

In this chapter we discuss the relation between the representation theories of G,N and G/N under the assumption that they all are flat group schemes. The results are usually generalizations of known theorems in the case of abstract group theory like e.g. the Lyndon-Hochschild-Serre spectral sequence in 6.6 or the Clifford theory in 6.14/15.

More or less all necessary references have been given before. Let me add that 6.11 generalizes 3.1 in [Andersen/Jantzen].

6.1 (Factor Groups) Let G be a k-group faisceau and N a normal subgroup faisceau of G. Obviously $A \mapsto G(A)/N(A)$ is a k-group functor. Then so is the associated faisceau G/N. This follows (on one hand) from the universal property (cf. [DG], III, §3, 1.2) and is (on the other hand) clear from the construction in 5.4/5: For any $g,g' \in (G/N)(A)$ there is an fppf-A-algebra B with g,g' both in the kernel of $G(B)/N(B) \rightrightarrows G(B\otimes_A B)/N(B\otimes_A B)$. As these maps are group homomorphisms also gg' and g^{-1} belong to the kernel. This yields easily the group structure on each $(G/N)(A)$. Furthermore it is simple to see that all maps $(G/N)(A) \to (G/N)(A')$ and $G(A) \to (G/N)(A)$ are group homomorphisms. Hence G/N is a k-group faisceau and the canonical map $\pi: G \to G/N$

is a group homomorphism. We call G/N the <u>factor group</u> of G by N.

Note that G/N has the universal property of a factor group: If $\varphi: G \to G'$ is a homomorphism of k-group faisceaux with $N \subset \ker(\varphi)$, then there is a unique group homomorphism $\bar{\varphi}: G/N \to G'$ with $\bar{\varphi} \circ \pi = \varphi$. (As φ is constant on the N-cosets, the universal property of G/N as a quotient faisceau gives the existence of $\bar{\varphi}$ as a morphism. It is immediate from the construction that $\bar{\varphi}$ is a group homomorphism. This follows also from the uniqueness of $\bar{\varphi}$.)

For any homomorphism $\varphi: G \to G'$ of k-group faisceaux the kernel $\ker(\varphi)$ is a normal subgroup faisceau of G. We can identify $G/\ker(\varphi)$ with the image faisceau $\mathrm{im}(\varphi)$ which is a subgroup faisceau of G'. This is really a special case of an orbit faisceau as we can make any $g \in G(A)$ operate on $G'(A)$ as multiplication with $\varphi(g)$.

<u>6.2</u> (<u>Product Subgroups</u>) Let G be a k-group faisceau and let H,N be subgroup faisceaux of G such that H normalizes N. We can then form the semi-direct product $H \ltimes N$ and have a natural homomorphism $H \ltimes N \to G$, $(h,n) \mapsto hn$ with kernel isomorphic to the intersection $H \cap N$ (cf. 2.6). Both $H \ltimes N$ and $H \cap N$ are k-group faisceaux. We denote the image faisceaux of the homomorphism $H \ltimes N \to G$ by HN and call it the product of H and N. It is a subgroup faisceau of G with

(1) $(H \ltimes N)/(H \cap N) \cong HN.$

The definitions imply for any k-algebra A

(2) $(HN)(A) = \{g \in G(A) \mid$ there are an fppf-A-algebra B and $h \in H(B),\ n \in N(B)$ with $g = hn$ in $G(B)\}$.

Obviously N is a normal subgroup faisceau of HN. The canonical homomorphism $HN \to (HN)/N$ has kernel N, hence its restriction to H has kernel $H \cap N$. We get thus an embedding $H/(H \cap N) \to (HN)/N$ which has to be an isomorphism: For all g,h,n as in (2) the element $h(H(B) \cap N(B))$ defines an element in $(H/H \cap N))(A)$ which is mapped to $gN(A)$. Therefore all $(HN)(A)/N(A)$ are in the image, hence all $((HN)/N)(A)$ in the image faisceau. So we get the isomorphism theorem

(3) $H/(H \cap N) \to (HN)/N$.

Suppose now that N is normal in G and let $\pi\colon G \to G/N$ be the canonical map. Let us denote by $\pi(H)$ the image faisceau of $\pi_{|H}$. Then

(4) $HN = \pi^{-1}(\pi(H))$

Indeed, if $g \in \pi^{-1}(\pi(H))(A)$ then there is B (fppf over A) with $\pi(g) \in \pi(H(B))$, hence $h \in H(B)$ with $gh^{-1} \in \mathrm{Ker}(\pi)(B) = N(B)$ and $g \in (HN)(A)$ by (2). The other inclusion is even more obvious.

If $H \supset N$, then obviously $HN = H$ and $H = \pi^{-1}(\pi(H))$. So we have for normal N the usual bijection between {subgroup faisceauxof G containing N} and {subgroup faisceaux of G/N}.

Furthermore one can then show (for $H \supset N$) that H is normal in G if and only if H/N is normal in G/N and that one has a canonical isomorphism $(G/N)/(H/N) \xrightarrow{\sim} G/H$ of faisceau which is a group isomorphism, if H is normal, cf. [DG], III, §3, 3.7.

6.3 (G/N-modules) Let us assume from now on until the end of this chapter that G is a flat group scheme over k and that N is a normal and flat subgroup scheme of G.

Via the canonical map $\pi\colon G \to G/N$ any G/N-module M is in a natural way also a G-module. We denote this G-module by π^*M in case a special notation is useful, otherwise we simply write M. Obviously π^* is a functor from {G/N-modules} to {G-modules} which is exact and faithful, i.e. we have for all G/N-modules M, M':

(1) $\mathrm{Hom}_{G/N}(M,M') = \mathrm{Hom}_G(\pi^*M,\pi^*M')$.

(Any $\bar{g} \in (G/N)(A)$ has a representative $g \in G(B)$ with B fppf over A. If $\varphi \in \mathrm{Hom}_G(\pi^*M,\pi^*M')$, then $\varphi \otimes \mathrm{id}_B$ commutes with g, hence $\varphi \otimes \mathrm{id}_A$ with $\bar{g}$ as $M \otimes A$ is mapped injectively into $M \otimes B$.)

The image of π^* consists of all G-modules V on which N operates trivially. For such V the k-group functor $A \mapsto G(A)/N(A)$ operates naturally on V_a and this operation can be extended uniquely to the associated faisceau G/N as V_a is itself a faisceau. This follows from the universal property of G/N and also from its explicit description in 5.4/5.

The full subcategory of all G-modules on which N operates trivially is obviously an abelian category. So we see that the category of all G/N-modules is an abelian category even without knowing whether G/N is a flat group scheme (what we needed in 2.9) or not.

6.4 For any G-module V the subspace V^N is a G-submodule of V be 3.2 on which N operates trivially. We therefore can regard V^N as a G/N-module and $V \mapsto V^N$ as a left exact functor from {G-modules} to {G/N-modules}.

Lemma: The functor $V \mapsto V^N$ from {G-modules} to {G/N-modules} is right adjoint to π^*. It maps injective G-modules to injective G/N-modules. The category of G/N-modules contains enough injective objects.

Proof: We have for any G/N-module M and any G-module V

$$\mathrm{Hom}_G(\pi^*M,V) \simeq \mathrm{Hom}_G(\pi^*M,V^N) = \mathrm{Hom}_{G/N}(M,V^N)$$

by 6.3(1) where the first isomorphism is induced by the inclusion $V^N \subset V$. This shows that $V \mapsto V^N$ is right adjoint to the exact functor π^*, hence also that injective objects are mapped to injective objects. Any embedding of π^*M into an injective G-module Q induces an embedding of M into the injective G/N-module Q^N. Therefore {G/N-modules} contains enough injective objects.

Remark: We can generalize the above as follows. Let E be a G-module which is finitely generated and projective over k. Then $M \mapsto \pi^*(M) \otimes E$ is an exact functor from {G/N-modules} to {G-modules}. The functor $V \mapsto \mathrm{Hom}_N(E,V) \simeq (E^* \otimes V)^N$, cf. 2.10(6), is right adjoint to it. It is therefore left exact and maps injective G-modules to injective G/N-modules. Indeed, one has for any G-module V and any G/N-module M

$$\mathrm{Hom}_{G/N}(M,(E^*\otimes V)^N) = \mathrm{Hom}_G(M,(E^*\otimes V)^N) \cong \mathrm{Hom}_G(M,E^*\otimes V)$$
$$\cong \mathrm{Hom}_G(M\otimes E,V)$$

using 4.4 for the last step. Notice that we can regard this also as an isomorphism of functors

(1) $\quad \mathrm{Hom}_{G/N}(M,?)\circ\mathrm{Hom}_N(E,?) \cong \mathrm{Hom}_G(M\otimes E,?)$.

6.5 (Factor Groups as Affine Schemes) Let us quote from [DG], III, §3, 5.6 the following result:

(1) *If k is a field and if G,N are algebraic k-groups, then G/N is an algebraic k-group.*

Notice that in our convention an algebraic k-group is assumed to be affine.

Another case where we know G/N to be affine is when N is a finite group scheme (by 5.6(3)).

Let us recall from 5.7(2) and 5.8:

(2) *If G/N is an affine scheme, then it is flat and N is exact in G.*

Of course in this case we do not need 6.3/4 to see that

{G/N-modules} is an abelian category and has enough injective objects. The functor $V \mapsto V^N$ maps $M \otimes k[G]$ for any k-module M to $M \otimes k[G]^N \cong M \otimes k[G/N]$ if k is a field. Therefore we can use also 3.9.c to show that it maps injective G-modules to injective G/N-modules (in that case).

<u>6.6</u> <u>Proposition</u>: <u>Suppose</u> N <u>is exact in</u> G. <u>Let</u> E <u>be a</u> G-<u>module which is finitely generated and projective over</u> k. <u>Then the derived functors of</u> $V \mapsto \mathrm{Hom}_N(E,V)$ <u>from</u> {G-<u>modules</u>} <u>to</u> {G/N-<u>modules</u>} <u>can be identified with</u> $V \mapsto \mathrm{Ext}_N^n(E,V)$. <u>There are for each</u> G/N-<u>module</u> M <u>and each</u> G-<u>module</u> V <u>spectral sequences</u>

(1) $$E_2^{n,m} = \mathrm{Ext}^n_{G/N}(M, \mathrm{Ext}^m_N(E,V)) \Longrightarrow \mathrm{Ext}_G^{n+m}(M \otimes E, V)$$

<u>and</u>

(2) $$E_2^{n,m} = \mathrm{Ext}^n_{G/N}(M, H^m(N,V)) \Longrightarrow \mathrm{Ext}_G^{n+m}(M,V)$$

<u>and</u>

(3) $$E_2^{n,m} = H^n(G/N, H^m(N,V)) \Longrightarrow H^{n+m}(G,V).$$

<u>Proof</u>: As N is exact in G the functor res_N^G maps injective G-modules to modules acyclic for the fixed point functor $?^N$. (Use 3.9.c and 4.10.) The composition of $?^N$ from {N-modules} to{k-modules}with res_N^G is isomorphic to the composition of $\mathrm{res}_1^{G/N}$ with $?^N$ from {G-modules} to {G/N-modules}. Therefore 4.1(2),(3) implies that all $V \mapsto H^n(N,V)$ can be regarded as the derived functors of $V \mapsto V^N$ from {G-modules} to {G/N-modules}. The same is true for $V \mapsto H^n(N, E \otimes V) \cong \mathrm{Ext}^n_N(E,V)$, cf. 4.4,

and $V \mapsto (E^* \otimes V)^N \cong \mathrm{Hom}_N(E,V)$.

As $\mathrm{Hom}_N(E,?)$ maps injective G-modules to injective G/N-modules we can apply 4.1(1) to 6.4(1) and get the spectral sequence in (1). Taking $E = k$ we get (2), and setting $M = k$ yields (3).

Remark: The spectral sequence in (3) is known as the Lyndon-Hochschild-Serre spectral sequence.

6.7 In the special case $E = k$ the proposition 6.6 implies that each $H^n(N,V)$ for any G-module V has a natural structure as a G/N-module. This can be constructed using the Hochschild complex. We make G act on each $C^n(N,V) \cong V \otimes \bigotimes^n k[N]$ via the given representation on V and via the conjugation action on each factor $k[N]$. Then each ∂^n is a homomorphism of G-modules as Δ_V and Δ_N are so. This makes each $H^n(N,V)$ into a G-module.

One can now check that all connecting maps $H^n(N,V'') \to H^{n+1}(N,V')$ for any exact sequence $0 \to V' \to V \to V'' \to 0$ of G-modules are homomorphisms of G-modules. (See [Sullivan 3], 4.1 for the case of a field.) The universal property of derived functors (via δ-functors) shows then that the G-modules $H^n(N,V)$ constructed in this way yield the derived functors of $V \mapsto V^N$ from {G-modules} to {G-modules}. This functor can be written as the composition of $V \mapsto V^N$ from {G-modules} to {G/N-modules} with the natural inclusion of {G/N-modules} into {G-modules}. The last functor being exact implies that G/N-structure on $H^n(N,V)$ given by the proposition must lead to the same G-structure as the construction using the Hochschild complex.

Notice that this implies in the case $G = N$ that the action of G on the $H^n(G,V)$ constructed with the conjugation action on the Hochschild complex is trivial.

<u>6.8</u> <u>Corollary</u>: <u>Suppose that</u> N <u>is diagonalizable. Then we have for all</u> G-<u>modules</u> V <u>and</u> E <u>with</u> E <u>finitely generated and projective over</u> k, <u>for all</u> G/N-<u>modules</u> M <u>and all</u> $n \in \mathbb{N}$ <u>isomorphisms</u>

(1) $\mathrm{Ext}^n_{G/N}(M,\mathrm{Hom}_N(E,V) \simeq \mathrm{Ext}^n_G(M\otimes E,V)$

<u>and</u>

(2) $\mathrm{Ext}^n_{G/N}(M,V^N) \simeq \mathrm{Ext}^n_G(M,V)$

<u>and</u>

(3) $H^n(G/N,V^N) \simeq H^n(G,V)$.

<u>Proof</u>: All this follows immediately from 6.6 and 4.3 as each E_λ is a projective k-module and as N is exact in G (cf. 4.6).

<u>Remark</u>: If we apply (3) to the G-module π^*M, then we get

(4) $H^n(G/N,M) \simeq H^n(G,M)$.

<u>6.9</u> <u>Corollary</u>: <u>Suppose that</u> G/N <u>is a diagonalizable group scheme. Then we have for all</u> G-<u>modules</u> V <u>and</u> E <u>and for all</u> G/N-<u>modules</u> M <u>with</u> E,M <u>projective over</u> k <u>and</u> $\mathrm{rk}_k(E) < \infty$ <u>isomorphisms for all</u> $n \in \mathbb{N}$:

(1) $\mathrm{Hom}_{G/N}(M,\mathrm{Ext}_N^n(E,V)) \simeq \mathrm{Ext}_G^n(M\otimes E,V)$

<u>and</u>

(2) $\mathrm{Hom}_{G/N}(M,H^n(N,V)) \simeq \mathrm{Ext}_G^n(M,V)$

<u>and</u>

(3) $H^n(N,V)^{G/N} \simeq H^n(G,V)$.

<u>Proof</u>: As G/N is affine, hence N exact in G, we can apply 6.6. The formulas follow now immediately from 4.3.

<u>Remark</u>: Suppose $G/N \simeq \mathrm{Diag}(\Lambda)$ for some abelian group Λ. We have by 2.11(3) decompositions

(4) $\mathrm{Ext}_N^n(E,V) \simeq \bigoplus_{\lambda\in\Lambda} \mathrm{Ext}_N^n(E,V)_\lambda$

(for all $n\in \mathbb{N}$). The map $\varphi \mapsto \varphi(1)$ is for any G/N-module M' an isomorphism $\mathrm{Hom}_{G/N}(k_\lambda,M') \xrightarrow{\sim} M'_\lambda$, cf. 2.11(4). We can therefore identify the direct summands in (4),(1) and 4.4 as follows:

(5) $\mathrm{Ext}_N^n(E,V)_\lambda \simeq \mathrm{Ext}_G^n(E\otimes\lambda,V) \simeq \mathrm{Ext}_G^n(E,V\otimes(-\lambda))$.

(We use the convention $E\otimes\lambda = E\otimes k_\lambda$ etc.) In the special case $E = k$ we get (for all $\lambda\in\Lambda$ and $n\in\mathbb{N}$)

(6) $H^n(N,V)_\lambda \simeq H^n(N,V\otimes(-\lambda))$.

<u>6.10</u> <u>Proposition</u>: <u>Let</u> H <u>be a flat subgroup scheme of</u> G <u>with</u> $N\subset H$. <u>Suppose that both</u> G/N <u>and</u> H/N <u>are affine. Then one has for each</u> H/N-<u>module</u> M <u>and each</u> $n\in\mathbb{N}$ <u>an isomorphism of</u> G-<u>modules</u>

(1) $(R^n \mathrm{ind}_H^G)M \simeq (R^n \mathrm{ind}_{H/N}^{G/N})M.$

Proof: Let $\pi\colon G \to G/N$ and $\pi'\colon H \to H/N$ be the canonical maps. Our claim ought to be formulated as an isomorphism of functors:

(1') $(R^n \mathrm{ind}_H^G)\circ\pi'^* \simeq \pi^*\circ R^n \mathrm{ind}_{H/N}^{G/N}.$

Let us consider at first the case $n = 1$, i.e. get an isomorphism

(2) $\mathrm{ind}_H^G(M) \simeq \mathrm{ind}_{H/N}^{G/N}(M).$

The right hand side is a subset of $\mathrm{Mor}(G/N,M_a)$ which we may identify with $\mathrm{Mor}(G,M_a)^N$ because of the universal property of G/N. (Remember that M_a is a faisceau.) Any $f \in \mathrm{Mor}(G,M_a)$ will belong to $\mathrm{ind}_{H/N}^{G/N}(M)$ if and only if $f(gn) = f(g)$ and $f(gh) = h^{-1}f(g)$ for all $g \in G(A)$, $h \in H(A)$, $n \in N(A)$ and all A. As $N(A) \subset H(A)$ operates trivially on $M\otimes A$ we can drop the first part of the condition. The second one alone describes just $\mathrm{ind}_H^G(M)$ so that we get (2). As above we ought to have formulated this as an isomorphism of functors

(2') $\mathrm{ind}_H^G\circ\pi' \simeq \pi\circ\mathrm{ind}_{H/N}^{G/N}.$

This formula implies (1) using 4.1(2), (3) as soon as we can show that π'^* maps injective H/N-modules to H-modules acyclic for ind_H^G. By 3.9.c it is enough to look at H/N-modules of the form $Q\otimes k[H/N] \simeq \mathrm{ind}_1^{H/N}(Q)$ for injective k-modules Q. Applying (2) to (H,N) instead of (G,H) we can identify $\pi'^*(\mathrm{ind}_1^{H/N}Q)$

with $\mathrm{ind}_N^H(Q)$ where we regard Q as a trivial N-module.

By our assumption N is exact in H and G. The spectral sequence 4.5.c yields therefore

(3) $(R^n\mathrm{ind}_H^G)\circ(\mathrm{ind}_N^H) = 0$ for all $n > 0$.

This certainly implies the required acyclicity of $\mathrm{ind}_N^H(Q)$ above, hence (1).

Remark: We use often only the following part of the proposition: Let M be an H-module. If N operates trivially on M, then it operates trivially also on $\mathrm{ind}_H^G M$ and even on all $(R^n\mathrm{ind}_H^G)M$.

6.11 The isomorphism in 6.10(2) can be regarded as a special case of a more general statement which we are going to prove now.

Proposition: Let H be a flat subgroup scheme of G. Suppose both G/N and $H/(H \cap N)$ are affine schemes.

a) The functors $\underline{\underline{F}}_1$, $\underline{\underline{F}}_2$ from {H-modules} to {G/N-modules} with {G/N-modules} with

$$\underline{\underline{F}}_1(M) = (\mathrm{ind}_H^G M)^N$$

and

$$\underline{\underline{F}}_2(M) = \mathrm{ind}_{H/(H\cap N)}^{G/N}(M^{H\cap N})$$

are isomorphic.

b) For each H-module M there are spectral sequences

(1) $E_2^{n,m} = H^n(N, R^j \mathrm{ind}_H^G M) \Longrightarrow (R^{n+m}\underline{\underline{F}}_1)M$

and

(2) $E_2^{n,m} = (R^n \mathrm{ind}_{H/(H\cap N)}^{G/N}) H^m(H\cap N, M) \Longrightarrow (R^{n+m}\underline{\underline{F}}_2)M.$

Proof: a) Let $\pi: G \to G/N$ and $\pi': H \to H/(H\cap N)$ be the canonical maps. Obviously $\mathrm{res}_H^G \circ \pi^* = \pi'^* \circ \mathrm{res}_{H/(H\cap N)}^{G/N}$. This yields an isomorphism of the adjoint functors, i.e. of $\underline{\underline{F}}_1$ and $\underline{\underline{F}}_2$.
b) Both $\underline{\underline{F}}_1$ and $\underline{\underline{F}}_2$ are compositions of two left-exact functors. It is enough to show that the first one maps injective objects to acyclic objects with respect to the second one. Then we can apply 4.4(1).

The functor ind_H^G maps injective H-modules to injective G-modules (3.9). This gives the claim for $\underline{\underline{F}}_1$. Notice that we have to apply 6.6 in order to regard the $H^n(N,?)$ as derived functors on the category of G-modules.

In the second case we have to apply 6.4 to $(H, H\cap N)$ instead of (G,N).

Remark: Notice that a) implies $R^n\underline{\underline{F}}_1 \simeq R^n\underline{\underline{F}}_2$ for all n, so the two spectral sequences (1) and (2) have the same abutment.

6.12 Proposition: Let H be a flat subgroup scheme of G such that HN is an affine scheme. Then there are isomorphisms

<u>of functors</u>

(1) $\quad \mathrm{res}_N^{HN} \circ \mathrm{ind}_H^{HN} \simeq \mathrm{ind}_{H\cap N}^N \circ \mathrm{res}_{H\cap N}^H$

<u>and</u>

(2) $\quad \mathrm{res}_H^{HN} \circ \mathrm{ind}_N^{HN} \simeq \mathrm{ind}_{H\cap N}^H \circ \mathrm{res}_{H\cap N}^N$.

<u>Proof</u>: Let H' be the kernel of the obvious homomorphism $H \ltimes N \to G$ which can be identified with $H \cap N$ via $h \mapsto (h,h^{-1})$, cf. 2.6. We have an isomorphism $(H \ltimes N)/H' \simeq HN$, so HN is by our assumption and by 6.5(2) flat.

Let M be an H-module and M' an N-module. Because of $H' \cap N = 1 = H' \cap H$ (in $H \ltimes N$) we get from 6.11.a isomorphisms

$$\mathrm{ind}_H^{HN}(M) \simeq (\mathrm{ind}_H^{H\ltimes N} M)^{H'} \quad \text{and} \quad \mathrm{ind}_N^{HN}(M') \simeq (\mathrm{ind}_N^{H\ltimes N} M')^{H'}.$$

Now 3.8(2),(3) yield

$$\mathrm{ind}_H^{HN} M \simeq (k[N] \otimes M)^{H'} \quad \text{and} \quad \mathrm{ind}_N^{HN}(M') \simeq (k[H] \otimes M')^{H'}.$$

Here any $h \in H(A)$ and $n \in N(A)$ operate via $\rho_c(h) \otimes h$ resp. $\rho_\ell(n) \otimes 1$ on $k[N] \otimes M$ where ρ_c is the conjugation action. If $h \in H(A) \cap N(A)$, then $(h,h^{-1}) \in H'(A)$ acts therefore as $\rho_r(h) \otimes h$. So N and $H \cap N$ act on $k[N] \otimes M$ as in the definition of $\mathrm{ind}_{H\cap N}^N$. This yields (1).

Similarly, any $h \in H(A)$ will operate on $k[H] \otimes M'$ as in the definition of $\mathrm{ind}_{H\cap N}^H(M')$. Some $(n,n^{-1}) \in H'(A)$ with

$n \in N(A) \cap H(A)$ will not operate in that way, but the set of fixed points will be the same. (Regarding $f \in k[H] \otimes M'$ as morphism $H \to M_a$, then $(n,n^{-1})f(h) = (h^{-1}n^{-1}h)f(h \cdot h^{-1}n^{-1}h)$.) From this we get (2).

Remark: Suppose that also $H/(H \cap N)$ is affine. Then $\mathrm{res}^H_{H\cap N}$ maps injective H-modules to modules acyclic for $\mathrm{ind}^N_{H\cap N}$ as

$$(R^n \mathrm{ind}^N_{H\cap N})(Q \otimes k[H]) \simeq H^n(H\cap N, Q \otimes k[H] \otimes k[N])$$
$$\simeq (R^n \mathrm{ind}^H_{H\cap N})(Q \otimes k[N]) = 0$$

for all $n > 0$ and all k-modules Q. We therefore get from (1) and 4.1(2),(3) isomorphisms of derived functors (for each $n \in \mathbb{N}$)

(3) $\mathrm{res}^{HN}_N \circ R^n \mathrm{ind}^{HN}_H \simeq R^n \mathrm{ind}^N_{H\cap N} \circ \mathrm{res}^H_{H\cap N}$.

In (2) the higher derived functors are 0 (for $H/(H\cap N) \simeq (HN)/N$ affine).

6.13 Keep the notations of 6.12. The inclusion of H into HN induces by 6.2(3) an isomorphism $H/(H\cap N) \simeq (HN)/N$. Similarly one can show that the inclusion of N into HN induces an isomorphism of faisceaux $N/(H\cap N) \simeq (HN)/H$. (One can regard $(HN)/H$ as an orbit faisceau of N, cf. 5.5(2).)

Suppose now that these quotient faisceau are schemes. Then any N-module M' resp. any H-module M defines a sheaf $\mathcal{L}_{(HN)/N}(M')$ resp. $\mathcal{L}_{(HN)/H}(M)$ as in chapter 5. The isomorphisms above identify it with $\mathcal{L}_{H/(H\cap N)}(\mathrm{res}^N_{H\cap N}M')$ resp. $\mathcal{L}_{N/(H\cap N)}$ $(\mathrm{res}^H_{H\cap N}M)$. This is a consequence of 5.16(1). Using 5.13 one gets another approach to 6.12(3) and the symmetric statement with H and N interchanged.

This can be generalized as follows: Let H,H' be flat subgroup schemes such that the multiplication map $m: H\times H' \to G$ has image faisceau equal to G. Then one gets an isomorphism of faisceaux $H/(H\cap H') \xrightarrow{\sim} G/H'$. If these quotient faisceaux are sheaves, then one gets as above

(1) $\mathrm{res}^G_H \circ R^n\mathrm{ind}^G_{H'} \cong R^n\mathrm{ind}^H_{H\cap H'} \circ \mathrm{res}^{H'}_{H\cap H'}$.

A (slightly) more general result is proved in [Cline/ Parshall/Scott 9], 4.1.

<u>6.14</u> Any $g \in G(k)$ operates through conjugation on each $N(A)$. We can define for each N-module V another N-module ${}^gV$, the module <u>twisted</u> by g, by taking the same k-module but by making any $n \in N(A)$ act as $g^{-1}ng$ acts on V. Then obviously ${}^g({}^{g'}V) \cong {}^{(gg')}V$ for any $g,g' \in G(k)$. Furthermore ${}^nV \cong V$ for all $n \in N(k)$: The action of n on V gives the isomorphism. More generally, if V is an N-submodule of a G-module M, then gV is another N-submodule of M which is isomorphic to ${}^gV$.

Suppose from now on that k is a field. Any N-module V is simple (resp. semi-simple) if and only if ${}^gV$ is so. This implies:

(1) <u>If</u> M <u>is a</u> G-<u>module, then</u> $\mathrm{soc}_N M$ <u>is</u> $G(k)$-<u>stable</u>.

Let L,M be G-modules with $\dim(L) < \infty$. Then $\mathrm{Hom}(L,M) \cong L^* \otimes M$ is also a G-module and $\mathrm{Hom}_N(L,M) \cong (L^* \otimes M)^N$ is a G-submodule, cf. 6.3/4. The map $\varphi \otimes x \mapsto \varphi(x)$ from $\mathrm{Hom}(L,M) \otimes L$ to M is easily seen to be a homomorphism of G-modules. Therefore 2.14(3) implies

(2) <u>If</u> L <u>is simple as an</u> N-<u>module with</u> $\mathrm{End}_N(L) \cong k$, <u>then we have an isomorphism</u> $\mathrm{Hom}_N(L,M) \otimes L \cong (\mathrm{soc}_N M)_L$ <u>of</u> G-<u>modules</u>.

<u>6.15</u> We call $G(k)$ dense in G if there is no closed subfunctor $X \subset G$ with $X(k) \supset G(k)$ and $X \neq G$, cf. the definition of closures in 1.4. If k is an algebraically closed field and G is a reduced algebraic k-group, then $G(k)$ is dense in G (by Hilbert's Nullstellensatz). The same is true for G reduced connected and algebraic over any infinite perfect field ([Bo], 18.3). For reductive G one may even drop the assumption "perfect".

<u>Proposition</u>: <u>Suppose that</u> k <u>is a field and that</u> $G(k)$ <u>is dense in</u> G. <u>Let</u> M <u>be a</u> G-<u>module</u>.

a) <u>The</u> N-<u>socle</u> $\mathrm{soc}_N M$ <u>is a</u> G-<u>submodule of</u> M.

b) <u>If</u> M <u>is a semi-simple</u> G-<u>module, then</u> M <u>is also semi-simple for</u> N.

<u>Proof</u>: As $G(k)$ is dense any subspace of M is by 2.12(5) a G-submodule if and only if it is $G(k)$-stable. Hence (a) follows from 6.14(1). If $M \neq 0$, then $\mathrm{soc}_N M \neq 0$ by 2.14(2). Therefore (b) follows from (a).

<u>6.16</u> Let $\varphi: G \to G'$ be a homomorphism of flat group schemes. Each G'-module M is in a natural way also a G-module. This yields an exact functor φ^* from {G'-modules} to {G-modules}. In two special cases we have constructed a left adjoint functor φ_*: When φ is an inclusion, then $\varphi_* = \operatorname{ind}_G^{G'}$, and when φ induces an isomorphism $G/\ker\varphi \xrightarrow{\sim} G'$, then $\varphi_* = ?^{\ker(\varphi)}$. In general φ is a composition of maps of this type ([DG], III, §3, 3.2) so we get such a left adjoint in general. See [Donkin 1], section 3 or [Cline/Parshall/Scott 6], 1.2 for a unified treatment.

7. Algebras of Distributions

Over a field of characteristic 0 the representation theory of a connected algebraic group G is very well reflected by the representation theory of its Lie algebra $\underline{g}$. Any representation of G gives rise to a representation of $\underline{g}$. Then the notions of "submodule", "fixed point" or "module representation" give the same result whether applied to G-modules or to $\underline{g}$-modules.

This is no longer true in characteristic $p \neq 0$. Still any G-module yields a $\underline{g}$-module in a natural way, but now there may be $\underline{g}$-submodules which are no G-submodules, or $\underline{g}$-homomorphisms which are no G-homomorphisms, etc.

It is however still possible to save some of the advantages of the linearization process (of going from G to $\underline{g}$) by looking not only at $\underline{g}$ but at the algebra Dist(G) of all distributions on G with support at the origin. (See 7.1 and 7.7 for the definition.)

In characteristic 0 it will not contain more information, as then Dist(G) is isomorphic to the universal enveloping algebra of $\underline{g}$. This is no longer true in characteristic $p \neq 0$ and there Dist(G) will do everything that $\underline{g}$ does not do (7.14 - 7.17).

In this chapter we give at first the definitions of distributions with support in a rational point on an affine scheme, prove elementary properties and then go over to distributions on group schemes with support in the origin.

The definitions and results are more or less contained in [DG], [T] and [Y]. In [T] and [Y] there are many more results on distributions on schemes over a field than I could include here. In some cases it was necessary to extend their results from fields to rings. There [Haboush 3] was very useful.

7.1 (Distributions with support in a point) Let X be an affine scheme over k and $x \in X(k)$. Set $I_x = \{f \in k[X] \mid f(x) = 0\}$. Then $k[X] = k1 \oplus I_x \simeq k \oplus I_x$.

A distribution on X with support in x of order $\leq n$ is a linear map $\mu\colon k[X] \to k$ with $\mu(I_x^{n+1}) = 0$. These distributions form a k-module which we denote by $\mathrm{Dist}_n(X,x)$. We have

(1) $(k[X]/I_x^{n+1})^* \simeq \mathrm{Dist}_n(X,x) \subset k[X]^*$.

Obviously $\mathrm{Dist}_0(X,x) \simeq k^* \simeq k$ and for any n

(2) $\mathrm{Dist}_n(X,x) \simeq k \oplus \mathrm{Dist}_n^+(X,x)$

where

(3) $\mathrm{Dist}_n^+(X,x) = \{\mu \in \mathrm{Dist}_n(X,x) \mid \mu(1) = 0\} \simeq (I_x/I_x^{n+1})^*$.

For each $\mu \in \mathrm{Dist}_n(X,x)$ we call $\mu(1)$ its constant term and elements in $\mathrm{Dist}_n^+(X,x)$ are called distributions without constant term. The k-module $\mathrm{Dist}_1^+(X,x) \simeq (I_x/I_x^2)^*$ is called the tangent space at X in x and denoted by T_xX. (Cf. [DG], II, §4, 3.3 for another description.)

The union of all $\mathrm{Dist}_n(X,x)$ in $k[X]^*$ is denoted by $\mathrm{Dist}(X,x)$ and its elements are called distributions on X with support in x:

(4) $\mathrm{Dist}(X,x) = \{\mu \in k[X]^* \mid \exists n \in \mathbb{N} : \mu(I_x^{n+1}) = 0\} = \bigcup_{n \geq 0} \mathrm{Dist}_n(X,x).$

This is obviously a k-module. Similarly $\mathrm{Dist}^+(X,x) = \bigcup_{n \geq 0} \mathrm{Dist}_n^+(X,x)$ is a k-module.

For each $f \in k[X]$ and $\mu \in k[X]^*$ we define $f\mu \in k[X]^*$ through $(f\mu)(f_1) = \mu(ff_1)$ for all $f_1 \in k[X]$. In this way $k[X]^*$ is a $k[X]$-module. As each I_x^{n+1} is an ideal in $k[X]$; obviously each $\mathrm{Dist}_n(X,x)$ and hence also $\mathrm{Dist}(X,x)$ is a $k[X]$-submodule of $k[X]^*$.

We have restricted ourselves above to the case of affine schemes. There is however a definition available for all schemes. One defines distributions in general as special deviations([DG], II, §4, 5.2), shows that all these deviations form a k-module([DG], II, §4, 5.4), and uses [DG], II, §4, 5.7 in order to prove that one gets in the affine case the same definition as above.

In the case of a ground field, however, we can easily give another description which works for all schemes. Suppose that k is a field. Then we can associate to $x \in X(k)$ the local ring $\mathcal{O}_{X,x}$ and its maximal ideal $\underline{m}_x$. In the affine case these are localizations $\mathcal{O}_{X,x} = k[X]_x$ and $\underline{m}_x = (I_x)_x$. Furthermore the natural map $k[X] \to \mathcal{O}_{X,x}$ induces then isomorphisms $k[X]/I_x^{n+1} \simeq \mathcal{O}_{X,x}/(\underline{m}_x)^{n+1}$ for all n. So we can in general define $\mathrm{Dist}_n(X,x)$

as $(\mathcal{O}_{X,x}/(\underline{m}_x)^{n+1})^*$. Similarly we get $\mathrm{Dist}(X,x)$, $\mathrm{Dist}^+(X,x)$, $\mathrm{Dist}_n^+(X,x)$.

<u>7.2</u> (<u>Elementary Properties</u>) Let $\varphi\colon X \to Y$ be a morphism of affine schemes over k and let $\varphi^*\colon k[Y] \to k[X]$ be its comorphism. Then $(\varphi^*)^{-1} I_x = I_{\varphi(x)}$ for all $x \in X(k)$, hence $\varphi^*(I_{\varphi(x)}^{n+1})$ I_x^{n+1} and φ^* induces a linear map $k[Y]/I_{\varphi(x)}^{n+1} \to k[X]/I_x^{n+1}$. The transposed maps for all n yield a linear map

(1) $(d\varphi)_x\colon \mathrm{Dist}(X,x) \to \mathrm{Dist}(Y,\varphi(x))$

with $(d\varphi)_x(\mathrm{Dist}_n(X,x)) \subset \mathrm{Dist}_n(Y,\varphi(x))$ and $(d\varphi)_x(\mathrm{Dist}_n^+(X,x)) \subset \mathrm{Dist}_n^+(Y,\varphi(x))$ for all n. We get on $T_xX = \mathrm{Dist}_1^+(X,x)$ the usual tangent map and call $(d\varphi)_x$ in general the <u>tangent map</u> of φ in x. One checks easily $d(\psi\circ\varphi)_x = (d\psi)_{\varphi(x)} \circ d\varphi_x$ for any morphism $\psi\colon Y \to Z$ into another affine scheme.

Let X be an affine scheme over k and $x \in X(k)$. Suppose I is an ideal in $k[X]$ with $x \in V(I)(k)$, i.e. with $I \subset I_x$, cf. 1.4 for the notation. We can then apply the construction above to the inclusion of $V(I)$ into X. We have $k[V(I)] = k[X]/I$, the ideal of x is I_x/I, its n-th power is $(I_x^n+I)/I$. This implies that the inclusion yields isomorphisms

(2) $\mathrm{Dist}_n(V(I),x) \simeq \{\mu \in \mathrm{Dist}_n(X,x) \mid \mu(I) = 0\}$

and

(3) $\mathrm{Dist}(V(I),x) \simeq \{\mu \in \mathrm{Dist}(X,x) \mid \mu(I) = 0\}$,

similarly for Dist_n^+ and Dist^+. We shall usually identify both sides in (2) and (3). If I' is another ideal with $x \in V(I')(k)$, then 1.4(5) implies

(4) $\mathrm{Dist}(V(I)\cap V(I'),x) = \mathrm{Dist}(V(I),x)\cap \mathrm{Dist}(V(I'),x)$

similarly for Dist_n, Dist^+, Dist_n^+. If $x \in D(f)(k)$ for some $f \in k[X]$, then the canonical map $k[X] \to k[X]_f$ induces an isomorphism of each $k[X]/I_x^{n+1}$ onto the corresponding object for $D(f)$. Therefore the inclusion of $D(f)$ into X induces an isomorphism

(5) $\mathrm{Dist}(D(f),x) \cong \mathrm{Dist}(X,x)$,

similarly for Dist_n, etc.

The constructions and results above have generalizations to the case where the schemes are not affine. This is particularly obvious when k is a field and when we can work with $\mathcal{O}_{X,x}$. One can also generalize (5) to $\mathrm{Dist}(Y,x) \cong \mathrm{Dist}(X,x)$ for any open subscheme Y of X with $x \in Y(k)$.

<u>7.3</u> (<u>Distributions on $\mathbb{A}^n$</u>) Let us consider as an example at first $X = \mathbb{A}^1 = \mathrm{Sp}_k k[T]$ and $x = 0$, hence $I_x = (T)$. The k-module $k[X]/I_x^{n+1}$ is free and has the residue classes of $1 = T^0$, $T = T^1, T^2, \dots, T^n$ as a basis. Define $\gamma_m \in k[T]^* = k[\mathbb{A}^1]^*$ through $\gamma_m(T^n) = 0$ for $n \neq m$ und $\gamma_m(T^m) = 1$. Then obviously $\mathrm{Dist}(\mathbb{A}^1,0)$ is a free k-module with basis $(\gamma_m)_{m\in\mathbb{N}}$ and each $\mathrm{Dist}_n(\mathbb{A}^1,0)$ is a free k-module with basis $(\gamma_m)_{0\le m\le n}$. If k is a field of characteristic 0, then obviously

$$\gamma_m(f) = \frac{1}{m!}\ ((\frac{\partial}{\partial T})^m f)(0).$$

This can be easily generalized to $\mathbb{A}^m = \mathrm{Sp}_k k[T_1,\ldots,T_m]$ for all m. For each multi-index $a = (a(1),a(2),\ldots,a(m)) \in \mathbb{N}^m$ set $T^a = T_1^{a(1)}T_2^{a(2)}\ldots T_m^{a(m)}$ and denote by γ_a the linear map with $\gamma_a(T^b) = 0$ for all $b \in \mathbb{N}^m$, $b \neq a$ and $\gamma_a(T^a) = 1$. One checks easily that $\mathrm{Dist}(\mathbb{A}^m,0)$ is free over k with all γ_a as a basis and that $\mathrm{Dist}_n(\mathbb{A}^m,0)$ is free over k with all γ_a with $|a| \leq n$ as a basis. (For a as above set $|a| = \sum_{i=1}^m a(i)$.) If k is a field of characteristic 0, then

$$\gamma_a(f) = \frac{1}{\Pi a(i)!}\ ((\frac{\partial}{\partial T_1})^{a(1)} (\frac{\partial}{\partial T_2})^{a(2)} \ldots (\frac{\partial}{\partial T_m})^{a(m)} f)(0).$$

If k is a field, then any $\mathrm{Dist}(X,x)$ will only depend on the $\underline{m}_x$-adic completion of $\mathcal{O}_{X,x}$. So for a simple point x all $\mathrm{Dist}_n(X,x)$ and $\mathrm{Dist}(X,x)$ will look like $\mathrm{Dist}_n(\mathbb{A}^m,0)$ and $\mathrm{Dist}(\mathbb{A}^m,0)$ where $m = \dim_x X$, cf. [DG], I, §4, 4.2.

<u>7.4</u> (<u>Infinitesimal Flatness</u>) Let X be an affine scheme over k and $x \in X(k)$. We call X <u>infinitesimally flat</u> in x if each $k[X]/I_x^{n+1}$ with $n \in \mathbb{N}$ is a finitely presented and flat (or, equivalently, projective) k-module. (In [Haboush 3] this property is called "infinitesimally smooth". As obviously over a field any algebraic scheme (cf. 1.6) has this property, I think that name to be not appropriate.)

If X is infinitesimally flat in x, then also each I_x^n/I_x^m with $n \leq m$ is finitely generated and projective and each I_x^n is a direct k-summand of $k[X]$.

Let k' be a k-algebra. Any $x \in X(k)$ defines a point in $X(k') = X_{k'}(k')$ with ideal $I_x \otimes k' \subset k[X] \otimes k' \simeq k'[X_{k'}]$. Then $k'[X_{k'}]/(I_x \otimes k')^{n+1} \simeq (k[X]/I_x^{n+1}) \otimes k'$. Now ring extension commutes with taking the dual module as long as the module is finitely generated and projective. So we get:

(1) <u>If</u> X <u>is infinitesimally flat in</u> x, <u>then</u> $X_{k'}$ <u>is infinitesimally flat in</u> x <u>for each</u> k-<u>algebra</u> k'. <u>There are natural isomorphisms</u> $\mathrm{Dist}_n(X,x) \otimes k' \simeq \mathrm{Dist}_n(X_{k'},x)$ <u>and</u> $\mathrm{Dist}(X,x) \otimes k' \simeq \mathrm{Dist}(X_{k'},x)$.

Of course, we use here the letter x also for the image of x in $X_{k'}(k') = X(k')$.

Consider two affine schemes X,X' and points $x \in X(k)$ and $x' \in X'(k)$. Then the ideal of (x,x') in $k[X \times X'] \simeq k[X] \otimes k[X']$ is $I_{(x,x')} = I_x \otimes k[X'] + k[X] \otimes I_{x'}$. If X and X' are infinitesimally flat in x resp. x', then $I_{(x,x')}^{n+1}$ can be identified with $\sum_{j=0}^{n+1} I_x^j \otimes I_{x'}^{n+1-j}$ and then with $\bigcap_{j=0}^{n} (k[X] \otimes I_{x'}^{n+1-j} + I_x^{j+1} \otimes k[X'])$. Now some elementary considerations yield:

(2) <u>If</u> X <u>and</u> X' <u>are infinitesimally flat in</u> x <u>resp.</u> x', <u>then</u> $X \times X'$ <u>is infinitesimally flat in</u> (x,x'). <u>There is an isomorphism</u> $\mathrm{Dist}(X,x) \otimes \mathrm{Dist}(X',x') \simeq \mathrm{Dist}(X \times X',(x,x'))$ <u>mapping</u> $\sum_{m=0}^{n} \mathrm{Dist}_m(X,x) \otimes \mathrm{Dist}_{n-m}(X',x')$ <u>onto</u> $\mathrm{Dist}_n(X \times X'(x,x'))$ <u>for</u>

<u>each</u> $n \in \mathbb{N}$.

We can apply (2) to $X' = X$. Consider the diagonal morphism $\delta_X \colon X \to X\times X$, $x \mapsto (x,x)$. Let us regard the tangent map $(d\delta_X)_x$ as a map $\Delta'_{X,x} \colon \mathrm{Dist}(X,x) \to \mathrm{Dist}(X,x) \otimes \mathrm{Dist}(X,x)$. It makes $\mathrm{Dist}(X,x)$ into a coalgebra, i.e. satisfies 2.3(1) as $(\mathrm{id}\times\delta_X)\circ\delta_X = (\delta_X\times\mathrm{id})\circ\delta_X$. This coalgebra is cocommutative, i.e. $s\circ\Delta'_{X,x} = \Delta'_{X,x}$ where $s(f_1\otimes f_2) = f_2\otimes f_1$. The map $\varepsilon_x \colon f \mapsto f(x)$ is a counit, i.e. satisfies 2.3(2). If $\varphi\colon X \to Y$ is a morphism, then $(d\varphi)_x$ is a homomorphism of coalgebras, as $(\varphi\times\varphi)\circ\delta_X = \delta_Y\circ\varphi$. So we have seen:

(3) <u>If</u> X <u>is infinitesimally flat in</u> x, <u>then</u> $\mathrm{Dist}(X,x)$ <u>has a natural structure as a cocommutative coalgebra with a counit</u>. <u>Tangent maps are homomorphisms for these structures</u>.

<u>7.5</u> An affine scheme X is called <u>noetherian</u>, if $k[X]$ is a noetherian ring, and it is called <u>integral</u>, if $k[X]$ is an integral domain.

<u>Proposition</u>: <u>Let</u> X <u>be an affine scheme over</u> k <u>and</u> $x \in X(k)$. <u>Let</u> I, I' <u>ideals in</u> $k[X]$ <u>with</u> $x \in V(I)(x) \cap V(I')(x)$. <u>If</u> $V(I)$ <u>is integral, noetherian and infinitesimally flat in</u> x, <u>then</u>:

$$V(I) \subset V(I') \iff \mathrm{Dist}(V(I),x) \subset \mathrm{Dist}(V(I'),x).$$

<u>Proof</u>: If $V(I) \subset V(I')$, then $I' \subset I$ by 1.4(3), hence $\mathrm{Dist}(V(I),x) \subset \mathrm{Dist}(V(I'),x)$ by 7.2(3).

Suppose now $\mathrm{Dist}(V(I),x) \subset \mathrm{Dist}(V(I'),x)$. We want to show

(1) $I' \subset I + I_x^{n+1}$

for all $n \in \mathbb{N}$. If not, then $(I'+I_x^{n+1}+I)/(I_x^{n+1}+I) \neq 0$ for some n. Now I_x/I is the ideal of x in $k[V(I)] = k[X]/I$ and its $(n+1)$-st power is $(I_x^{n+1}+I)/I$. So $k[X]/(I_x^{n+1}+I)$ is a finitely generated and projective module. For any $a \in (I'+I_x^{n+1}+I)/(I_x^{n+1}+I)$, $a \neq 0$ there is some $\mu \in (k[X]/(I_x^{n+1}+I))^*$ $= \mathrm{Dist}_n(V(I),x)$ with $\mu(a) \neq 0$, hence $\mu(I') \neq 0$ and $\mu \notin$ $\mathrm{Dist}(V(I'),x)$. So we get a contradiction and have established (1).

We can now apply Krull's intersection theorem to $k[V(I)] =$ $k[X]/I$ and get $I = \bigcap_{n \geq 0} (I+I_x^{n+1}) \supset I'$, hence $V(I) \subset V(I')$.

__Remark:__ This generalizes obviously to the case where I is no longer integral, but where I_1 contains all associated prime ideals of I.

__7.6__ __Proposition:__ __Suppose that__ k __is a field. Let__ $\varphi: X \to Y$ __be a morphism of algebraic schemes over__ k __and let__ $x \in X(k)$. __If__ φ __is flat in__ x, __then__ $(d\varphi)_x: \mathrm{Dist}(X,x) \to \mathrm{Dist}(Y,\varphi(x))$ __is surjective.__

__Proof:__ Set $A = \mathcal{O}_{Y,\varphi(x)}$ and $B = \mathcal{O}_{X,x}$. The flatness of φ in x amounts to the following: Using the comorphism (we may assume X,Y to be affine) we may regard A as a subalgebra of B such that B is a faithfully flat A-module. This faithful flatness implies $\underline{m}_{\varphi(x)}^{n+1} = A \cap B\underline{m}_{\varphi(x)}^{n+1}$ for all $n \in \mathbb{N}$, cf. [3], ch. I, §3, prop. 9. As we assume our schemes to be algebraic the rings A,B are noetherian and each $A/\underline{m}_{\varphi(x)}^{n+1}$ is finite dimensional. So Krull's intersection theorem yields

$$B\underline{m}_{\varphi(x)}^{n+1} = \bigcap_{r\geq 0} (\underline{m}_x^{r+1} + B\underline{m}_{\varphi(x)}^{n+1}),$$

hence

$$\underline{m}_{\varphi(x)}^{n+1} = \bigcap_{r>0} ((\underline{m}_x^{r+1} + B\underline{m}_{\varphi(x)}^{n+1}) \cap A),$$

and $\dim(A/\underline{m}_{\varphi(x)}^{n+1}) < \infty$ implies that there is some r with $\underline{m}_{\varphi(x)}^{n+1} = A \cap (\underline{m}_x^{r+1} \cap B\underline{m}_{\varphi(x)}^{n+1})$. We can therefore embed $A/\underline{m}_{\varphi(x)}^{n+1} \simeq (A+B\underline{m}_{\varphi(x)}^{n+1}+\underline{m}_x^{r+1})/(B\underline{m}_{\varphi(x)}^{n+1}+\underline{m}_x^{r+1})$ into $B/(B\underline{m}_{\varphi(x)}^{n+1}+\underline{m}_x^{r+1})$. As k is a field, any $\mu \in \mathrm{Dist}_n(Y,\varphi(x)) = (A/\underline{m}_{\varphi(x)}^{n+1})^*$ has an extension to $B/(B\underline{m}_{\varphi(x)}^{n+1}+\underline{m}_x^{r+1})$ which gives some $\mu' \in (B/\underline{m}_x^{r+1})^* \simeq \mathrm{Dist}_r(X,x)$. Then obviously $(d\varphi)_x\mu' = \mu$. Therefore $(d\varphi)_x$ is surjective.

<u>Remark</u>: Note that we do not claim that each $\mathrm{Dist}_n(X,x)$ is mapped onto $\mathrm{Dist}_n(Y,\varphi(x))$. Indeed, it is well known that e.g. the "classical" tangent map $T_xX = \mathrm{Dist}_1^+(X,x) \to \mathrm{Dist}_1^+(Y,\varphi(x)) = T_{\varphi(x)}Y$ will not be surjective in general.

<u>7.7</u> (<u>Distributions on a Group Scheme</u>) Let G be a group scheme over k. In this case we set

$$\mathrm{Dist}(G) = \mathrm{Dist}(G,1).$$

We can make $\mathrm{Dist}(G)$ into an associative algebra over k. For any $\mu,\nu \in k[G]^*$ we can define a product $\mu\nu$ as

(1) $\mu\nu\colon k[G] \xrightarrow{\Delta} k[G]\otimes k[G] \xrightarrow{\mu\otimes\nu} k\otimes k \tilde{\to} k.$

We have obviously $\mu\nu \in k[G]^*$ and the bilinearity of $(\mu,\nu)\mapsto \mu\nu$. Furthermore 2.3(1) implies that this multiplication is associative and 2.3(2) that ε_G is a neutral element. So $k[G]^*$ has a structure as an associative algebra over k with one. It will in general be not commutative.

Now $\mathrm{Dist}(G)$ is a subalgebra of $k[G]^*$ with

(2) $\mathrm{Dist}_n(G)\,\mathrm{Dist}_m(G) \subset \mathrm{Dist}_{n+m}(G)$.

This follows easily from the formula $\Delta(I_1^n) \subset \sum_{r=0}^{n} I_1^r \otimes I_1^{n-r}$, cf. 2.4(1). (We have written here $I_1^r \otimes I_1^{n-r}$ instead of its image in $k[G]\otimes k[G]$.) More precisely, 2.4(1) implies

$$\Delta(f^n) \in (1\otimes f+f\otimes 1)^n + \sum_{r=1}^{n-1} I_1^r \otimes I_1^{n-r}$$

for all $f \in I_1$ and $n \in \mathbb{N}$. We get therefore

(3) <u>If</u> $\mu \in \mathrm{Dist}_n(G,)$ $\nu \in \mathrm{Dist}_m(G)$, <u>then</u> $[\mu,\nu] = \mu\nu-\nu\mu \in \mathrm{Dist}_{n+m-1}(G)$.

So $\mathrm{Dist}(G)$ has a structure as filtered associative algebra over k such that the associated graded algebra is commutative. We call Dist(G) the <u>algebra of distributions on</u> G, dropping the addendum "with support in the origin". (Some people call Dist(G) the <u>hyperalgebra</u> of G.)

Because of $\Delta(1) = 1 \otimes 1$ the subspace $\mathrm{Dist}^+(G)$ is a two-sided ideal in Dist(G). Therefore (3) implies $[\mathrm{Dist}^+_n(G), \mathrm{Dist}^+_m(G)] \subset \mathrm{Dist}^+_{n+m-1}(G)$. This shows especially that $\mathrm{Dist}^+_1(G)$ is a Lie algebra which we denote by Lie(G) and call <u>the</u> Lie algebra of G. Note that $\mathrm{Lie}(G) = T_1G$ as a k-module, cf. 7.1. It can be shown that we have constructed the usual structure as a Lie algebra on T_1G.

<u>7.8</u> (<u>Examples</u>) Let us look at first at the additive group $G = G_a$. As a scheme we may identify $G_a = \mathrm{Sp}_k k[T]$ with $\mathbb{A}^1$. Therefore we have described $\mathrm{Dist}(G_a)$ as a k-module already in 7.3. Let as before γ_n be the element with $\gamma_n(T^n) = 1$ and $\gamma_n(T^m) = 0$ for $m \neq n$. We have $\Delta(T) = 1 \otimes T + T \otimes 1$, hence $\Delta(T^n) = \sum_{i=0}^{n} \binom{n}{i} T^i \otimes T^{n-i}$. This implies easily

(1) $\gamma_n \gamma_m = \binom{n+m}{n} \gamma_{n+m}$,

hence

(2) $\gamma_1^n = n! \gamma_n$.

So $\mathrm{Dist}(G_{a,\mathbb{C}})$ can be identified with the polynomial ring $\mathbb{C}[\gamma_1]$, and $\mathrm{Dist}(G_{a,\mathbb{Z}})$ with the $\mathbb{Z}$-lattice spanned by all $\frac{\gamma_1^n}{n!}$. In general $\mathrm{Dist}(G_a) = \mathrm{Dist}(G_{a,\mathbb{Z}}) \otimes_{\mathbb{Z}} k$.

Let us consider now the multiplicative group $G_m = \mathrm{Sp}_k k[T,T^{-1}]$. Then I_1 is generated by $T-1$. The residue classes of $1, (T-1), (T-1)^2, \ldots, (T-1)^n$ form a basis of $k[G_m]/I_1^{n+1}$. There is a unique $\delta_n \in \mathrm{Dist}(G_m)$ with $\delta_n(I_1^{n+1}) = 0 = \delta_n((T-1)^i)$

for $0 \leq i < n$ and $\delta_n((T-1)^n) = 1$. From this and the binomial development of $T^n = ((T-1)+1)^n$ one gets $\delta_r(T^n) = \binom{n}{r}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{N}$. If k is a $\mathbb{Q}$-algebra, then obviously

$$\delta_r f = \frac{1}{r!}\left(\left(\frac{\partial}{\partial T}\right)^r f\right)(1)$$

All δ_r with $r \in \mathbb{N}$ form a basis of $\mathrm{Dist}(G_m)$, all δ_r with $r \leq n$ one of $\mathrm{Dist}_n(G_m)$. We get $\Delta(T-1) = (T-1)\otimes(T-1) + (T-1)\otimes 1 + 1\otimes(T-1)$ from $\Delta(T) = T\otimes T$, hence

$$(3) \quad \delta_r\delta_s = \sum_{i=0}^{\min(r,s)} \frac{(r+s-i)!}{(r-i)!(s-i)!i!}\,\delta_{r+s-i}$$

We get as a special case $\delta_1\delta_r = (r+1)\delta_{r+1}+r\delta_r$, hence $(\delta_1-r)\delta_r = (r+1)\delta_{r+1}$ and inductively

$$(4) \quad r!\delta_r = \delta_1(\delta_1-1)\dots(\delta_1-r+1).$$

If k is a $\mathbb{Q}$-algebra, then $\delta_r = \binom{\delta_1}{r}$. Therefore $\mathrm{Dist}(G_{m,\mathbb{C}}) = \mathbb{C}[\delta_1]$ and $\mathrm{Dist}(G_{m,\mathbb{Z}})$ is the $\mathbb{Z}$-lattice in $\mathrm{Dist}(G_{m,\mathbb{C}})$ generated by all $\binom{\delta_1}{r}$. In general $\mathrm{Dist}(G_m) = \mathrm{Dist}(G_{m,\mathbb{Z}})\otimes_{\mathbb{Z}} k$.

7.9 (Elementary Properties) If $\alpha\colon G \to G'$ is a homomorphism of group schemes over k, then

$$(1) \quad d\alpha = (d\alpha)_1\colon \mathrm{Dist}(G) \to \mathrm{Dist}(G')$$

is a homomorphism of algebras. This follows easily from the definition of the multiplication. On $\mathrm{Lie}(G) = \mathrm{Dist}_1^+(G)$ we get the usual tangent map $\mathrm{Lie}(G) \to \mathrm{Lie}(G')$ which is a homomorphism of Lie algebras.

If H, H' are closed subgroup schemes of a group scheme G, then the inclusions of $\mathrm{Dist}(H)$ and $\mathrm{Dist}(H')$ into $\mathrm{Dist}(G)$, cf. 7.2(3), are homomorphisms of algebras, and 7.2(4) implies

(2) $\mathrm{Dist}(H \cap H') = \mathrm{Dist}(H) \cap \mathrm{Dist}(H')$,

similarly $\mathrm{Lie}(H \cap H') = \mathrm{Lie}(H) \cap \mathrm{Lie}(H')$. (The same statement for linear algebraic groups is known to be false in general. There the intersection as varieties is considered, not as schemes as we do here.)

We call G <u>infinitesimally flat</u> if it is so at 1. Now 7.4(2) implies easily

(3) <u>If</u> G_1, G_2 <u>are infinitesimally flat group schemes, then</u> $G_1 \times G_2$ <u>is infinitesimally flat and there is an isomorphism of algebras over</u> k

$$\mathrm{Dist}(G_1) \otimes \mathrm{Dist}(G_2) \xrightarrow{\sim} \mathrm{Dist}(G_1 \times G_2).$$

In the case of a semi-direct product there is still an isomorphism of k-modules.

If we take $G_1 = G_2 = G$ and consider the multiplication map $m_G\colon G \times G \to G$, then we see easily:

(4) <u>If</u> G <u>is an infinitesimally flat group scheme over</u> k, <u>then</u> $d(m_G)\colon \mathrm{Dist}(G) \otimes \mathrm{Dist}(G) \to \mathrm{Dist}(G)$ <u>is given by</u> $d(m_G)(\mu \otimes \nu) = \mu\nu$ <u>for all</u> $\mu, \nu \in \mathrm{Dist}(G)$.

For G as in (4) and any k-algebra k' the isomorphism

$\mathrm{Lie}(G)\otimes k' \xrightarrow{\sim} \mathrm{Lie}(G_{k'})$ resp. $\mathrm{Dist}(G)\otimes k' \cong \mathrm{Dist}(G_{k'})$, cf. 7.4(1), are isomorphisms of Lie algebras resp. of associative algebras. Furthermore the comultiplication $\Delta'_G = \Delta'_{G,1}$: $\mathrm{Dist}(G) \to \mathrm{Dist}(G)\otimes\mathrm{Dist}(G)$ can be checked to be a homomorphism of algebras over k.

The map $i_G\colon G \to G$ with $g \mapsto g^{-1}$ has as a tangent map (cf. 2.3)

(5) $\sigma'_G = d(i_G)\colon \mu \mapsto \mu\circ\sigma_G$.

One checks easily that σ'_G is an anti-automorphism of $\mathrm{Dist}(G)$, i.e. satisfies $\sigma'_G(\mu\nu) = \sigma'_G(\nu)\sigma'_G(\mu)$ for all μ,ν. If G is infinitesimally flat, then σ'_G is a coinverse for the coalgebra structure, i.e. 2.3(3) is satisfied by $(\Delta'_G,\sigma'_G,\varepsilon_G)$ instead of $(\Delta,\sigma,\varepsilon)$.

<u>7.10</u> (<u>Distributions and the Enveloping Algebra</u>) To each Lie algebra $\underline{g}$ over k one can associate its universal enveloping algebra $U(\underline{g})$. One may consult [4], ch. I, §2, or [6], ch. 2 for the definition and the elementary properties of this object. It has a natural filtration $U_0(\underline{g}) = k1 \subset U_1(\underline{g}) = k1 \oplus \underline{g} \subset U_2(\underline{g}) \subset \dots$ where $U_n(\underline{g})$ is spanned over k by all products $x_1x_2\dots x_r$ with $r \leq n$ and all $x_i \in \underline{g}$.

Let G be a group scheme over k. As $\mathrm{Lie}(G) = \mathrm{Dist}_1^+(G)$ is a Lie subalgebra of $\mathrm{Dist}(G)$ the universal property of $U(\mathrm{Lie}(G))$ yields a homomorphism $\gamma\colon U(\mathrm{Lie}(G)) \to \mathrm{Dist}(G)$ of algebras which induces the identity on $\mathrm{Lie}(G)$. It maps

$U_n(\mathrm{Lie}(G))$ to $\mathrm{Dist}_n(G)$ because of 7.7(2).

It is not very difficult to prove (cf. [DG], II, §6, n° 1):

(1) If k is a field of characteristic 0 and G an algebraic k-group, then γ is an isomorphism $U(\mathrm{Lie}(G)) \tilde{\to} \mathrm{Dist}(G)$ and maps each $U_n(\mathrm{Lie}\ G)$ bijectively to $\mathrm{Dist}_n(G)$.

Using this one can then show that algebraic k-groups are smooth and reduced over fields of characteristic 0, cf. [DG], II, §6, 1.1.

If k is a field of characteristic $p \neq 0$, then the situation is completely different. In this case for each $\mu \in \mathrm{Lie}(G) = \mathrm{Dist}_1^+(G)$ also its p-th power in Dist(G) belongs to Lie(G). This is more easily seen by identifying Dist(G) with the algebra of left or right invariant derivations of $k[G]$ as in 7.18 below. Let us denote this p-th power in $\mathrm{Lie}(G) \subset \mathrm{Dist}(G)$ by $x^{[p]}$ in order to distinguish it from the p-th power x^p in $U(\mathrm{Lie}\ G)$). The pair $(\mathrm{Lie}(G), x \mapsto x^{[p]})$ is an example of what is called a p-Lie algebra. (One can find the general definition in [DG], II, §7, n° 3.) For any p-Lie algebra $(\mathfrak{g}, x \mapsto x^{[p]})$ set $U^{[p]}(\mathfrak{g})$ equal to the quotient of $U(\mathfrak{g})$ by the two-sided ideal generated by all $x^p - x^{[p]}$ with $x \in \mathfrak{g}$. This algebra is called the restricted enveloping algebra of $\mathfrak{g}$. We can still regard $\mathfrak{g}$ as a subspace of $U^{[p]}(\mathfrak{g})$. If $x_1, \dots, x_m$ is a basis of $\mathfrak{g}$, then all $x_1^{a(1)} x_2^{a(2)} \dots x_m^{a(m)}$ with $0 \le a(i) < p$ for all i form a basis of $U^{[p]}(\mathfrak{g})$, cf. [DG], II, §7, 3.6. So $\dim U^{[p]}(\mathfrak{g}) = p^{\dim(\mathfrak{g})}$.

By the definition of $x^{[p]}$ for $x \in \mathrm{Lie}(G)$ it is clear that γ has to factor through $U^{[p]}(\mathrm{Lie}(G))$. One can show:

(2) <u>If</u> k <u>is a field with</u> $\mathrm{char}(k) = p \neq 0$ <u>and</u> G <u>an algebraic</u> k-<u>group, then</u> γ <u>induces an injective homomorphism</u> $U^{[p]}(\mathrm{Lie}(G)) \to \mathrm{Dist}(G)$.

For this and for more details one may consult [DG], II, §7, n° 2-4.

<u>7.11</u> (G-<u>modules and</u> Dist(G)-<u>modules</u>) Let G be a group scheme over k. Then any G-module M carries a natural structure as a Dist(G)-module: One sets for each $\mu \in \mathrm{Dist}(G)$ and $m \in M$:

(1) $\mu m = (\mathrm{id}_M \bar{\otimes} \mu)\Delta_M(m)$,

i.e. the operation of μ on M is given by

(2) $M \xrightarrow{\Delta_M} M \otimes k[G] \xrightarrow{\mathrm{id}_M \otimes \mu} M \otimes k \xrightarrow{\sim} M.$

It is obvious that $(\mu,m) \mapsto \mu m$ is bilinear and it is easy to see that $\mu(\nu m) = (\mu\nu)m$ and $\varepsilon_G m = m$ for all $m \in M$ and $\mu,\nu \in \mathrm{Dist}(G)$ using 2.8(2),(3) and 7.7(1).

Obviously 2.8(4) implies for all G-modules M,M':

(3) $\mathrm{Hom}_G(M,M') \subset \mathrm{Hom}_{\mathrm{Dist}(G)}(M,M')$.

Applying this to inclusions we get

(4) <u>Any</u> G-<u>submodule of a</u> G-<u>module</u> M <u>is also a</u> Dist(G)-<u>submodule of</u> M.

Of course on a factor module the structure as a Dist(G)-module coming from the G-structure is equal to the structure as a factor module for Dist(G).

The Dist(G)-structure on a direct sum of G-modules is the one as a direct sum of Dist(G)-modules.

We get from 2.10(2):

(5) <u>If</u> $m \in M^G$, <u>then</u> $\mu m = \mu(1)m$ <u>for all</u> $\mu \in \mathrm{Dist}(G)$.

More generally, 2.10(2') implies for each $\lambda \in X(G) \subset k[G]$

(6) <u>If</u> $m \in M_\lambda$, <u>then</u> $\mu m = \mu(\lambda)m$ <u>for all</u> $\mu \in \mathrm{Dist}(G)$.

For any G-module M and any $\lambda \in X(G)$ we can construct the G-module $M \otimes k_\lambda$ which we usually denote by $M \otimes \lambda$. We can identify $M \otimes \lambda$ with M as a k-module. If $\Delta_M(m) = \sum_i m_i \otimes f_i$, then $\Delta_{M\otimes\lambda}(m) = \sum_i m_i \otimes \lambda f_i$. This implies (cf. 7.1 for the k[G]-module structure on Dist(G)):

(7) <u>Any</u> $\mu \in \mathrm{Dist}(G)$ <u>operates on</u> $M \otimes \lambda$ <u>as</u> $\lambda\mu$ <u>operates on</u> M.

If G is infinitesimally flat, then any $\mu \in \mathrm{Dist}(G)$ operates on a tensor product of two G-modules through $\Delta'_G(\mu) \in \mathrm{Dist}(G) \otimes \mathrm{Dist}(G)$.

Let M be a G-module which is finitely generated and projective over k. Then M^* is a G-module in a canonical way, cf. 2.7(4). The operation of Dist(G) on M^* is then given by

(8) $(\mu\varphi)(m) = \varphi(\sigma'_G(\mu)m)$

for all $\mu \in \mathrm{Dist}(G), \varphi \in M^*$ and $m \in M$.

If G is flat, then 2.13(2) implies that each $m \in M$ is contained in a Dist(G)-submodule of M finitely generated over k. In this sense M is a <u>locally finite</u> Dist(G)-<u>module</u>.

<u>7.12</u> (<u>The Case</u> $G = G_a$) Let us use the basis $(\gamma_n)_{n\in\mathbb{N}}$ of $\mathrm{Dist}(G_a)$ as in 7.3 and 7.8. As $k[G_a] = k[T]$ is free with basis $(T^i)_{i\geq 0}$ we can write uniquely $\Delta_M(m) = \sum_{i\geq 0} m_i \otimes T^i$ for and G_a-module M and $m \in M$ with almost $m_i = 0$. Then obviously $\gamma_n m = m_n$ for all n, i.e. $\Delta_M(m) = \sum_{n\geq 0} (\gamma_n m)\otimes T^n$. So the structure as a $\mathrm{Dist}(G_a)$-module determines the comodule map uniquely, hence also the structure as a G_a-module. This implies for $G = G_a$ that there is equality in 7.11(3) and that the converse holds in 7.11(4),(5).

In general not all locally finite $\mathrm{Dist}(G_a)$-modules arise from G_a-modules. If e.g. k is a field of characteristic 0, then one can define for each $b \in k$ a structure as G_a-module on k where each γ_r operates as multiplication with $b^r/(r!)$. For $b \neq 0$ this module does not come from a G_a-module. If k is a field of characteristic $p \neq 0$, then we can make k^2 into a $\mathrm{Dist}(G_a)$-module letting each γ_i operate as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ if i is of the form p^r with $r \in \mathbb{N}$, $r > 0$ as 1 if $i = 0$, and as 0 otherwise. This structure does not come from G_a.

<u>7.13</u> (<u>The Case</u> $G = G_m$) Let us use the basis $(\delta_r)_{r\geq 0}$ of $\mathrm{Dist}(G_m)$ as in 7.8. If M is a G-module and $m \in M$, then $\Delta_M(m) = \sum_{i\in\mathbb{Z}} m_i \otimes T^i$ with uniquely determined $m_i \in M$, almost all

zero. Then

(1) $\delta_n m = \sum\limits_{i\in\mathbb{Z}} \binom{i}{n} m_i$ for all $n \in \mathbb{N}$.

Remember that $M = \bigoplus\limits_{i\in} M_i$ where $M_i = \{m' \in M \mid \Delta_M(m') = m' \otimes T^i\}$ and that $m_i \in M_i$ in the situation above, cf. 2.11.

For $a_1, a_2, \ldots, a_r \in \mathbb{Z}$ pairwise different there is $f \in \mathbb{Q}[T]$ with $f(a_1) = 1$, $f(a_2) = \ldots = f(a_r) = 0$ and $f(\mathbb{Z}) \subset \mathbb{Z}$. There are integers $b_j \in \mathbb{Z}$ with $f = \sum\limits_{j\geq 0} b_j \binom{T}{j}$, cf. [St 1], p. 16. Denote then by $\tilde{f}$ the element $\sum\limits_{j\geq 0} b_j \delta_j \in \mathrm{Dist}(G_m)$. If we apply this construction to $\{a_1, \ldots, a_r\} = \{i \in \mathbb{Z} \mid m_i \neq 0\}$, then we get $\tilde{f}_i \in \mathrm{Dist}(G_m)$ with $\tilde{f}_i m = m_i$.

This shows for any Dist(G)-submodule N of M that $N = \oplus(N \cap M_i)$, hence that N is also a G-submodule, i.e. the converse of 7.11(4). Also the converse of 7.11(5), (6) is true, i.e. for all $j \in \mathbb{Z}$:

(2) $M_j = \{m \in M \mid \delta_n m = \binom{j}{n} m$ for all $n \in \mathbb{N}\}$.

Indeed, consider any m as on the right hand side. Take the m_i as above. Then $\binom{j}{n} m_i = \binom{i}{n} m_i$ for all $n \in \mathbb{N}$. For $i \neq j$ we take f as above with $f(i) = 1$ and $f(j) = 0$ and get $m_i = fm_i = 0$. Hence $m \in M_j$.

Note that (2) implies that the $\mathrm{Dist}(G_m)$-structure determines the G_m-structure, especially that we have equality in 7.11(3) for $G = G_m$.

In general not every locally finite $\mathrm{Dist}(G_m)$-module arises from a G_m-module. If k is a field of characteristic 0 and if $a \in k$, then we make k into a $\mathrm{Dist}(G_m)$-module letting any δ_i operate as $\binom{a}{i}$. For $a \notin \mathbb{Z}$ this structure does not come from G_m. If k is a field of characteristic $p \neq 0$ one can make a similar construction with any p-adic integer a.

<u>7.14</u> Lemma: <u>Let</u> G <u>be an infinitesimally flat, noetherian and integral group scheme over</u> k. <u>If</u> M <u>is a</u> G-<u>module which is projective over</u> k, <u>then for all</u> $\lambda \in X(G)$:

$$M_\lambda = \{m \in M \mid \mu m = \mu(\lambda)m \quad \underline{\text{for all}} \quad \mu \in \mathrm{Dist}(G)\}$$

<u>Proof</u>: Observe at first that there is for each $x \in M \otimes k[G]$ with $x \notin M \otimes I_1^{n+1}$ some $\mu \in \mathrm{Dist}_n(G)$ with $(\mathrm{id}_M \otimes \mu)x \neq 0$. (Use embeddings of M and $k[G]/I_1^{n+1}$ into free modules.)

Now if $\mu m = \mu(\lambda)m$ for all $\mu \in \mathrm{Dist}(G)$, then $(\mathrm{id}_M \otimes \mu)(\Delta_M(m) - m \otimes \lambda) = 0$ for all μ, hence $\Delta_M(m) - m \otimes \lambda \in M \otimes I_1^{n+1}$ for all n by the argument above, hence $\Delta_M(m) - m \otimes \lambda \in \bigcap_{n>0} (M \otimes I_1^{n+1}) = M \otimes (\bigcap_{n>0} I_1^{n+1})$. (Use a split embedding of M into a free module for the last equality.) Now Krull's intersection theorem shows that the last term is 0, hence $\Delta_M(m) = m \otimes \lambda$ and $m \in M_\lambda$.

<u>7.15</u> Lemma: <u>Let</u> G <u>be an infinitesimally flat, noetherian and integral group scheme over</u> k. <u>Let</u> M <u>be a</u> G-<u>module and</u> M' <u>a</u> k-<u>submodule of</u> M <u>such that</u> M/M' <u>is projective over</u> k. <u>Then</u> M' <u>is a</u> G-<u>submodule of</u> M <u>if and only if it is a</u> Dist(G)-<u>submodule</u>.

Proof: As M/M' is projective, the k-submodule M' is a direct summand of M and we can identify $M'\otimes k[G]$ with the kernel of $M\otimes k[G] \to (M/M')\otimes k[G]$. We have to show: If M' is a Dist(G)-submodule, then $\Delta_M(M')\subset M'\otimes k[G]$, i.e. the image N of $\Delta_M(M')$ in $(M/M')\otimes k[G]$ is zero. Now $\mathrm{Dist}(G)M'\subset M'$ is equivalent to $(\mathrm{id}_M\bar{\otimes}\mu)\Delta_M(M')\subset M'$ for all $\mu\in \mathrm{Dist}(G)$, hence implies $(\mathrm{id}_{M/M'}\bar{\otimes}\mu)N = 0$. As in the last proof this yields

$$N\subset \bigcap_{n>0} (M/M')\otimes I_1^{n+1} = (M/M')\otimes \bigcap_{n>0} I_1^{n+1} = 0,$$

hence the lemma.

7.16 Lemma: Let G be an infinitesimally flat, noetherian and integral group scheme over k. Then one has for all G-modules M,M' which are projective over k, if M is finitely generated over k

$$\mathrm{Hom}_G(M,M') = \mathrm{Hom}_{\mathrm{Dist}(G)}(M,M').$$

Proof: Under our assumption we can identify $\mathrm{Hom}(M,M') \simeq M^*\otimes M'$, this is a G-module and projective as a k-module. As $\mathrm{Hom}_G(M,M') \simeq (M^*\otimes M')^G$ we can apply 7.14 and have to show only that any $\mu\in \mathrm{Dist}(G)$ operates on any $\psi\in \mathrm{Hom}_{\mathrm{Dist}(G)}(M,M')$ as multiplication with $\mu(1)$. But if $\Delta_G'(\mu) = \sum_i \mu_i\otimes \mu_i'$, then

$\mu\psi = \sum_i \mu_i'\circ\psi\circ\sigma_G'(\mu_i)$ for all $\psi\in \mathrm{Hom}(M,M')$, hence $\mu\psi = \sum_i \mu_i'\sigma_G'(\mu_i)\circ\psi$

for $\psi\in \mathrm{Hom}_{\mathrm{Dist}(G)}(M)$. As $\sum_i \mu_i'\sigma_G'(\mu_i) = \mu(1)\varepsilon_G$ we get the claim.

<u>Remark</u>: If M is a direct limit of G-modules to which we can apply 7.16, then 7.16 holds also for M. Hence the local finiteness of M implies, that we can take any M in 7.16 as long as k is a field. Similarly we can take for M any torsion free k-module, if k is a Dedekind ring and G flat, as in that case finitely generated torsion free modules are projective.

<u>7.17</u> (<u>The Case of a Ground Field</u>) An affine scheme X over k is called <u>irreducible</u>, if $\sqrt{0}$ is a prime ideal in $k[X]$. This is equivalent to the irreducibility of $\mathrm{Spec}(k[X])$ with respect to the Zariski topology, cf. [Ha], II, 3.0.1. It is integral if and only if it is irreducible and reduced, cf. [Ha], II, 3.1.

If k is a field of characteristic 0, then any algebraic k-group is smooth, hence reduced. So in this case the notions "irreducible" and "integral" coincide.

Suppose now that k is a perfect field of characteristic p. If G is an irreducible algebraic k-group, then there is by [DG], III, §3, 6.4 an isomorphism $G \cong X \times Y$ of affine schemes with Y integral and where $k[X]$ is a finite dimensional local k-algebra. The only maximal ideal of $k[X]$ is nilpotent. This shows that we have $\bigcap_{n>0} I_1^{n+1} = 0$ in $k[G]$. It was for this property that we needed G to be integral in the last proofs. So we see:

(1) <u>Suppose that</u> k <u>is a perfect field. Then the results of</u> 7.14 - 7.16 <u>hold for any irreducible algebraic</u> k-<u>group</u>.

We can use the same argument with respect to 7.5.

(2) <u>Suppose that</u> k <u>is a perfect field</u>. <u>Let</u> G <u>be an algebraic</u> k-<u>group and</u> H, H' <u>closed subgroups of</u> G. <u>If</u> H <u>is irreducible</u>, <u>then</u>

$$H \subset H' \Longleftrightarrow \mathrm{Dist}(H) \subset \mathrm{Dist}(H').$$

<u>7.18</u> (<u>Distributions as Differential Operators</u>) Let G be a group scheme over k. Any operation of G on an affine scheme X leads (cf. 2.7) to a representation of G on $k[X]$, hence makes $k[X]$ into a $\mathrm{Dist}(G)$-module. When dealing with a right operation $\alpha\colon X \times G \to X$ (resp. a left operation $\beta\colon G \times X \to X$), then the operation of $\mu \in \mathrm{Dist}(G)$ on $k[X]$ is given by $(\mathrm{id}_{k[X]} \overline{\otimes} \mu) \circ \alpha^*$ (resp. $\sigma_G'(\mu) \overline{\otimes}\, \mathrm{id}_{k[X]}) \circ \beta^*$).

There is a general notion of <u>differential operators</u> on a scheme, cf. [DG], II, §4, 5.3. In the case of an affine scheme X they can be described as follows ([DG], II, §4, 5.7): Each $f \in k[X]$ defines $\mathrm{ad}(f)\colon \mathrm{End}(k[X]) \to \mathrm{End}(k[X])$ through $(\mathrm{ad}(f)\psi)(f_1) = f\psi(f_1) - \psi(ff_1)$, i.e. $\mathrm{ad}(f)\psi$ is the commutator of the left multiplication by f and of ψ. Then a differential operator on X of order $\leq n$ is some $D \in \mathrm{End}(k[X])$ with $\mathrm{ad}(f_0)\mathrm{ad}(f_1)\ldots\mathrm{ad}(f_n)D = 0$ for all $f_0, \ldots, f_n \in k[X]$. A differential operator on X is then defined as a differential operator of order $\leq n$ for some $n \in \mathbb{N}$. The differential operators form a subalgebra of $\mathrm{End}(k[X])$.

For G operating on X as above, any $\mu \in \mathrm{Dist}_n(G)$ operates on $k[X]$ as a differential operator of order $\leq n$ as an elementary argument shows, cf. [DG], II, §4, 6.3.

When dealing with the operation of G on itself by left resp. right translation, then we get an operation of any $\mu \in \mathrm{Dist}(G)$ as a differential operator on G which commutes with the operation of G by multiplication from the other side. This construction turns out to yield an isomorphism of $\mathrm{Dist}(G)$ onto the algebra of all differential operators on G which are right resp. left invariant (i.e. which commute with the action of G by right resp. left translation), cf. [DG], II, §4, 6.5.

The conjugation action of G on itself yields a representation of G on $k[G]$ which stabilizes I_1, hence also all I_1^{n+1}. We get thus G-structures on all $k[G]/I_1^{n+1}$, hence also on all $\mathrm{Dist}_n(G) = (k[G]/I_1^{n+1})^*$, provided G is infinitesimally flat. If so, then we get also a representation of G on the direct limit $\mathrm{Dist}(G)$. The representation of G on $\mathrm{Lie}(G) = \mathrm{Dist}_1^+(G)$ constructed thus is the <u>adjoint</u> representation of G. We use the notation Ad for the representation of G on $\mathrm{Dist}(G)$ and all $\mathrm{Dist}_n(G)$, $\mathrm{Dist}_n^+(G)$ and the notation ad for the corresponding operations of $\mathrm{Dist}(G)$ on itself or its submodules.

Suppose that G is infinitesimally flat. An elementary calculation shows that the adjoint representation on $\mathrm{Dist}(G)$ and the action of $\mathrm{Dist}(G)$ on any G-module M are related by the formula

(1) $(\mathrm{Ad}(g)\varphi)m = g(\varphi(g^{-1}m))$

for any $\varphi \in \mathrm{Dist}(G)\otimes A \simeq \mathrm{Dist}(G_A)$, $m \in M\otimes A$, $g \in G(A)$ and any A.

Let us write down explicitly how any $\mu \in \mathrm{Dist}(G)$ operates on $k[G]$ and $\mathrm{Dist}(G)$ under the conjugation resp. adjoint action (for G infinitesimally flat). Suppose $\Delta'_G(\mu) = \sum_i \mu_i \otimes \mu'_i$. Then the conjugation action of μ is because of 2.8(7) given by

(2) $$\sum_i (\sigma'_G(\mu_i) \,\overline{\otimes}\, \mathrm{id}_{k[G]} \,\overline{\otimes}\, \mu'_i) \circ (\mathrm{id}_{k[G]} \otimes \Delta_G) \circ \Delta_G .$$

As $\Delta'_G \circ \sigma'_G = (\sigma'_G \otimes \sigma'_G) \circ \Delta'_G$ the adjoint action is given by (using 7.11(8) and 7.7(1))

(3) $$\begin{aligned} \mathrm{ad}(\mu)\mu' &= \sum_i (\mu_i \,\overline{\otimes}\, \mu' \,\overline{\otimes}\, \sigma_G(\mu'_i)) \circ (\mathrm{id}_{k[G]} \otimes \Delta_G) \circ \Delta_G \\ &= \sum_i \mu_i \mu' \sigma_G(\mu'_i) . \end{aligned}$$

7.19 For any family $(X_j)_{j \in J}$ of subfunctors of a group scheme G there is a smallest closed subgroup scheme H of G containing all X_j. (Take the intersection of all closed subgroup schemes containing all X_j.) We call H the closed subgroup of G generated by all X_j.

Proposition: *Suppose that* k *is an algebraically closed field. Let* G *be an algebraic* k-*group and let* $(H_j)_{j \in J}$ *be a family of integral closed subgroups of* G. *Let* H *be the closed subgroup of* G *generated by all* $(H_j)_{j \in J}$. *Then* H *is integral and* $\mathrm{Dist}(H)$ *is the subalgebra of* $\mathrm{Dist}(G)$ *generated by all* $\mathrm{Dist}(H_j)$.

Proof: The reduced subgroup of G defined by $H(k)$ contains all H_j, hence H is reduced. We can assume (by [DG], II, §5, 4.6 or [Bo], 2.2) that $(H_j)_{j \in J} = \{H_1, H_2, \ldots, H_r\}$ and that the multiplication map $\alpha: H_1 \times H_2 \times \ldots \times H_r \to H$ is surjective on points over k. This implies that H is irreducible, hence integral. Furthermore, the theorem of generic flatness ([DG], I, §3, 3.7) provides us with a point over k where α is flat, hence $d\alpha$ by 7.6 surjective on the distributions with support in that point. As $d\alpha$ in $(1,1,\ldots,1)$ is multiplication, the same argument as in [Bo], 7.5 yields

(1) $\mathrm{Dist}(H) = (\mathrm{Ad}(h_1)\mathrm{Dist}(H_1))(\mathrm{Ad}(h_2)\mathrm{Dist}\, H_2))\ldots(\mathrm{Ad}(h_r)\mathrm{Dist}(H_r))$

for suitable $h_1,\ldots,h_r \in H(k)$.

Let R be the subalgebra of Dist(G) generated by all $\mathrm{Dist}(H_i)$. As $H_i \subset H$ for all i, also $R \subset \mathrm{Dist}(H)$. Because of (1) we have to show that R is stable under Ad(h) for all $h \in H(k)$, or by the surjectivitiy of $\alpha(k)$ that k is an H_i-submodule of Dist(G) for each i. By 7.15 it is enough to show stability under each $\mathrm{Dist}(H_i)$ for the adjoint action. This is now clear from 7.18(2) as $\Delta_G'(\mathrm{Dist}(H_i)) \subset \mathrm{Dist}(H_i) \otimes \mathrm{Dist}(H_i)$ and $\sigma_G'(\mathrm{Dist}(H_i)) = \mathrm{Dist}(H_i)$ for all i. Indeed Δ_G' resp. σ_G' restrict to Δ_{H_i}' and σ_{H_i}' on $\mathrm{Dist}(H_i)$.

<u>Remarks</u>: 1) There is another proof in [Y], 10.10. The proof above follows the one in [Bo], 7.6 that Lie(H) is generated as a Lie algebra by all $\mathrm{Lie}(H_i)$ provided char(k) = 0.
2) Drop the assumption that k is algebraically closed. Let K be an algebraic closure of k. If each $(H_j)_K$ is still integral, then the claim of the proposition is still satisfied: We get from [Bo], 2.2 that H_K is the closed subgroup generated by all $(H_j)_K$.

With R as in the proof we get $R \otimes K = \mathrm{Dist}(H_K) = \mathrm{Dist}(H) \otimes K$, hence $R = \mathrm{Dist}(H)$ using 7.4(1).

Now $(H_j)_K$ is integral if and only if it is reduced, cf. [DG], II, §5, 1.1. This will certainly be satisfied, if k is perfect, cf. [Bo], AG 2.2.

7.20 The algebras of distributions have recently been used
[12] by M. Takeuchi (in [ ]) to give a proof of the uniqueness theorem (and the isogeny theorem) for reductive groups without using rank-2-computations. Let us sketch a minor modification of his argument (using standard notions about reductive groups).

Suppose that k is an algebraically closed field. Let us work in the category or linear algebraic groups over k. Let G_1, G_2 be reductive algebraic groups over k with maximal tori T_1, T_2 and suppose there is an isomorphism $\psi: T_1 \to T_2$ inducing an isomorphism of the root data in the sense of [Sp], 9.1.6. Suppose both root systems are identified and let S be a set of simple roots. For each $\alpha \in S$ let $G_{1,\alpha} = Z_{G_1}((\ker \alpha)^0)$ and let $U_{1,\alpha}, U_{1,-\alpha}$ be the root subgroups corresponding to α. Define similarly $G_{2,\alpha}, U_{2,\alpha}, U_{2,-\alpha}$.

The complete description of the semi-simple rank-1-case gives isomorphisms $\psi_\alpha: G_{1,\alpha} \to G_{2,\alpha}$ for all $\alpha \in S$ with $\psi_{\alpha|T_1} = \psi$ and $\psi_\alpha(U_{1,\alpha}) = U_{2,\alpha}$, $\psi_\alpha(U_{1,-\alpha}) = U_{2,-\alpha}$, cf. [Hu], 32.3. Set $T = \{(t_1, \psi(t_1)) | t_1 \in T_1\}$ and $G_\alpha = \{(g_1, \psi_\alpha(g_1)) | g_1 \in G_{1,\alpha}\}$ for all $\alpha \in S$ and define similarly $U_\alpha, U_{-\alpha}$.

Let G be the group generated by all $G_\alpha \cong G_{1,\alpha}$ in $G_1 \times G_2$. As all G_α are closed and irreducible subgroups also G is closed and irreducible. We want to show that the projections $p_i: G_1 \times G_2 \to G_i$ for $i = 1,2$ induce isomorphisms $G \xrightarrow{\sim} G_i$. (That is obviously enough.) Let us restrict to $i = 1$.

Now p_1 is surjective as G_1 is generated by all $G_{1,\alpha}$ with $\alpha \in S$. As $\mathrm{Lie}(G_1) = \sum_{g \in G_1} \sum_{\alpha \in S} \mathrm{Ad}(g)\, \mathrm{Lie}(G_{1,\alpha})$ also $dp_1: \mathrm{Lie}(G) \to \mathrm{Lie}(G_1)$ is surjective. Therefore it suffices to show $\ker(p_1) \cap G = 1$. Obviously $\ker(p_1) \cap T = 1$. If we can show that T is a maximal torus in G, then any normal subgroup $\neq 1$ of G intersects T non-trivially, hence $\ker(p_1) \cap G = 1$. Now $T \subset G$ is certainly contained in some maximal torus T' of G, which then must be contained in $Z_{G_1 \times G_2}(T) = T_1 \times T_2$. (Note that no root of $G_1 \times G_2$ vanishes on T.) It is therefore enough to show $\mathrm{Dist}(G) \cap \mathrm{Dist}(T_1 \times T_2) = \mathrm{Dist}(T)$ as then $\mathrm{Dist}(T) = \mathrm{Dist}(T')$, hence $T = T'$ by 7.5.

The multiplication induces an isomorphism of $U_{1,\alpha} \times T_1 \times U_{1,\alpha}$ onto an open neighbourhood of 1 in $G_{1,\alpha}$. Therefore $\mathrm{Dist}(G_{1,\alpha}) = \mathrm{Dist}(U_{1,-\alpha})\mathrm{Dist}(T_1)\mathrm{Dist}(U_{1,\alpha})$. Similar results hold in G_2 and G. Therefore $\mathrm{Dist}(G)$ is the subalgebra generated by $\mathrm{Dist}(T)$ and all $\mathrm{Dist}(U_\alpha)$ and $\mathrm{Dist}(U_{-\alpha})$ with $\alpha \in S$. If $\alpha, \beta \in S$, $\alpha \neq \beta$, then $U_{1,\alpha}$ and $U_{1,-\beta}$ commute, hence $\mathrm{Dist}(U_{1,\alpha})$ and $\mathrm{Dist}(U_{1,-\beta})$ do so. The same holds for $U_{2,\alpha}$ and $U_{2,-\beta}$, for U_α and $U_{-\beta}$. Using this and the formula for $\mathrm{Dist}(G_\alpha)$ one gets

(1) $\mathrm{Dist}(G) = \mathrm{Dist}(G)^- \mathrm{Dist}(T) \mathrm{Dist}(G)^+$

where $Dist(G)^+$ resp. $Dist(G)^-$ is the subalgebra generated by all $Dist(U_\alpha)$ resp. $Dist(U_{-\alpha})$ with $\alpha \in S$.

Using the big cell in $G_1 \times G_2$ one gets similarly an isomorphism induced by the multiplication

(2) $Dist(G_1 \times G_2) \xleftarrow{\sim} Dist(G_1 \times G_2)^- \otimes Dist(T_1 \times T_2) \otimes Dist(G_1 \times G_2)^+$

where $Dist(G)^- \subset Dist(G_1 \times G_2)^-$ and $Dist(G)^+ \subset Dist(G_1 \times G_2)^+$ and of course $Dist(T) \subset Dist(T_1 \times T_2)$. Comparing (1) and (2) one gets $Dist(T) = Dist(G) \cap Dist(T_1 \times T_2)$ as claimed.

The isogeny theorem can be proved similarly. One simply starts with ψ and ψ_α which are isogenies.

We shall look at $Dist(G)$ for G reductive in more detail in part II.

8. Representations of Finite Algebraic Groups

Let us suppose throughout this chapter that k is a field.

A k-group scheme G is called a finite algebraic group if $\dim k[G] < \infty$. We have met already some examples $(\mu_{(n)}, G_{a,r})$. One can associate to each finite abstract group a finite algebraic group in a natural way (8.5.a). The examples which are most important for us will be introduced in chapter 9 (the Frobenius kernels).

We look in this chapter at some special features of the representation theory of such finite G. Let me mention right away that one can find in [Voigt] many more results which we do not look at here.

One of these special features is that injective G-modules are also projective as in the representation theory of abstract finite groups. Whereas in that case (abstract finite groups) the injective hull of a simple module is also its projective cover this is no longer true in our situation (in general). Here the simple head and the simple socle of an injective indecomposable module differ by a character of G which we call the modular function of G (8.13).

Another special feature is seen when dealing with a closed subgroup H of G. We do not only have the right adjoint ind_H^G to the restriction functor res_H^G but also a left adjoint coind_H^G (the coinduction). Both functors are exact and they are related by dualizing (8.14-8.16). In fact one can get one from the other by at first tensoring with a character depending on the modular functions of H and G (8.17).

One main ingredient in the proofs of these results is the fact that $k[G]$ and $k[G]^*$ are isomorphic as G-modules (8.7 and 8.12). This is a special case of a more general theorem of Larson and Sweedler (cf. [11]). As a source for the other non-trivial results let me mention [9] and [13].

When working not over a field but over an arbitrary commutative ring (say R) then one should define a finite algebraic group over R as an R-group scheme G such that $R[G]$ is finitely generated and projective as an R-module. It is elementary how to generalize 8.1 - 8.6 to this more general situation. For an extension of 8.12 and 8.17 to this situation one may consult [13], cf. also [9].

8.1 (Finite Algebraic Groups) A k-group scheme G is called finite (hence: a finite algebraic k-group), if $\dim k[G] < \infty$. It is called infinitesimal, if it is finite and if the ideal $I_1 = \{f \in k[G] | f(1) = 0\}$ is nilpotent.

If k' is an extension field of k, then obviously G is finite (resp. infinitesimal), if and only if $G_{k'}$ is so.

The closed subgroups $G_{a,r}$ of the additive group (introduced at the end of 2.2) are infinitesimal groups. They are examples of Frobenius kernels, the (for us) most important class of infinitesimal groups, which will be introduced in chapter 9.

The groups $\mu_{(n)}$ for each $n \in \mathbb{N}$ are finite (cf. 2.2). If

$char(k) = p \neq 0$ and if n is a power of p, then $\mu_{(n)}$ is infinitesimal.

<u>8.2</u> <u>Lemma</u>: <u>Let</u> G <u>be an algebraic</u> k-<u>group</u>.
a) G <u>is finite, if and only if</u> $G(K)$ <u>is finite for each extension</u> K <u>of</u> k.
b) G <u>is infinitesimal, if and only if</u> $G(K) = 1$ <u>for each extension</u> K <u>of</u> k.

<u>Proof</u>: a) If $\dim k[G] < \infty$, then each element in $k[G]$ is algebraic over k, hence has only a finite number of possible images in any K (under an element of $G(K) = \mathrm{Hom}_{k\text{-alg}}(k[G],K)$). As any $g \in G(K)$ is given by its values on the basis of $k[G]$ there are only finitely many possibilities for g.

Consider on the other hand an algebraic closure K of k and suppose that $G(K)$ is finite. We can replace G by G_K, hence suppose $k = K$. We can write $k[G]$ in the form $k[T_1,\ldots,T_n]/I$ for some ideal I. Any prime ideal containing I has to be a maximal ideal. The same is true for any associated prime ideal of I. This implies easily $\dim k[G] = \dim k[T_1,\ldots,T_n]/I < \infty$.
b) If I_1 is nilpotent, then it has to be annihilated by any homomorphism of k-algebras $k[G] \to K$ into a field extension. As $k[G] = k1 \oplus I_1$ there is only one such homomorphism, hence $G(K) = 1$.

Suppose on the other hand $G(K) = 1$ for an algebraic closure K of k. We may assume $k = K$ and can identify $k[G]/\sqrt{0}$ with all functions from $G(K)$ to K. This implies $I_1 = \sqrt{0}$, hence that I_1 is nilpotent.

<u>8.3</u> (<u>Duality of Finite Dimensional Hopf Algebras</u>) For any finite dimensional vector space V (over k) the canonical map $V \to (V^*)^*$ is an isomorphism. Mapping any linear map $\varphi: V_1 \to V_2$ between two finite dimensional vector spaces to its transposed map $\varphi^*: V_2^* \to V_1^*$ is therefore a bijection $\mathrm{Hom}(V_1,V_2) \tilde{\to} \mathrm{Hom}(V_2^*,V_1^*)$.

Let R be a finite dimensional vector space over k. We get from above isomorphisms $\mathrm{Hom}(k,R) \tilde{\to} \mathrm{Hom}(R^*,k)$ and $\mathrm{End}(R) \tilde{\to} \mathrm{End}(R^*)$ and $\mathrm{Hom}(R\otimes R,R) \tilde{\to} \mathrm{Hom}(R^*,R^*\otimes R^*)$ using the isomorphism $R^*\otimes R^* \tilde{\to} (R\otimes R)^*$. So multiplication on R (i.e. bilinear maps $R\times R \to R$ or, equivalently, linear maps $m: R\otimes R \to R$) correspond bijectively to comultiplications on R^* (i.e. linear maps $m^*: R^* \to R^*\otimes R^*$). Similarly comultiplications $\Delta: R \to R\otimes R$ on R correspond bijectively to multiplications $\Delta^*: R^*\otimes R^* \to R^*$ on R^*. Furthermore m is associative (resp. Δ is coassociative, i.e. satisfies 2.3(1)), if and only if m^* is coassociative (resp. Δ^* is associative). An element $a \in R$ is a 1 for the multiplication m, if and only if the map $\varepsilon_a: R^* \to k$, $\varphi \mapsto \varphi(a)$ is a counit for m^* (i.e. satisfies 2.3(2) with the appropriate modifications in the notation). Similarly $\varepsilon \in R^*$ is a counit for Δ, if and only if it is a 1 for Δ^*.

If we have on R both a multiplication m and a comultiplicatio Δ, then Δ is a homomorphism of algebras (with respect to m), if and only if m^* is a homomorphism of algebras (with respect to Δ^*). If so, then some $\sigma \in \mathrm{End}(R)$ is an antipode for Δ and m (i.e. satisfies 2.3(3) and $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a,b \in R$), if and only if σ^* is an antipode for m^* and Δ^*. This shows:

If R is a Hopf algebra, then so is R^* in a natural way. For two such finite dimensional Hopf algebras R_1, R_2 a linear map $\psi\colon R_1 \to R_2$ is a homomorphism of Hopf algebras if and only if $\psi^*\colon R_2^* \to R_1^*$ is a homomorphism of Hopf algebras. Thus we get

(1) *The functor* $R \mapsto R^*$, $\psi \mapsto \psi^*$ *is a self-duality on the category of all finite dimensional Hopf algebras*.

This anti-equivalence has obviously the property that R is commutative, if and only if R^* is cocommutative (cf. 2.2).

8.4 (*Finite Algebraic Groups and Hopf Algebras*) We have by 2.3 an anti-equivalence of categories {group schemes over k} $\to$ {commutative Hopf algebras over k}. Combining this with 8.3(1) we get an equivalence of categories:

(1) {*finite algebraic* k-*groups*} $\to$ {*finite dimensional cocommutative Hopf algebras over* k}.

Each finite algebraic k-group G is mapped to $k[G]^*$. We denote this Hopf algebra by $M(G)$ and call it the algebra of all *measures* on G. We usually denote its comultiplication by Δ_G', its antipode by $\sigma_G' = \sigma_G^*$ and its counit by $\varepsilon_G'\colon \mu \to \mu(1)$.

We have an obvious embedding $G(k) = \mathrm{Hom}_{k\text{-alg}}(k[G],k)$ $\mathrm{Hom}(k[G],k) = M(G)$: To each $g \in G(k)$ there corresponds the (Dirac) measure of $\delta_g\colon f \mapsto f(g)$. An element $\mu \in M(G) = k[G]^*$ is a homomorphism of algebras if and only if $\Delta_G'(\mu) = \mu \otimes \mu$. The multiplication on $G(k)$ is just the multiplication in $M(G)$. More generally, one can identify

$$G(A) = \mathrm{Hom}_{k\text{-alg}}(k[G],A) \subset \mathrm{Hom}(k[G],A) \simeq$$

$$\simeq k[G]^* \otimes A = M(G) \otimes A$$

for any k-algebra A with

$$\{\mu \in M(G) \otimes A \mid (\Delta_G^! \otimes \mathrm{id}_A)(\mu) = \mu \otimes \mu, \quad \varepsilon_G^!(\mu) = 1\}$$

In chapter 7 we have associated to each group scheme G the algebra $\mathrm{Dist}(G)$, cf. 7.1 and 7.7. If G is finite, then obviously $\mathrm{Dist}(G)$ is a subalgebra of $M(G)$ and G is infinitesimal if and only if $M(G) = \mathrm{Dist}(G)$. One checks easily that

(2) $\mathrm{Lie}(G) = \mathrm{Dist}_1^+(G) = \{\mu \in M(G) \mid \Delta_G^!(\mu) = \mu \otimes 1 + 1 \otimes \mu\}$.

8.5 (Examples) a) If Γ is an abstract finite group, then its group algebra $k\Gamma$ is a cocommutative Hopf algebra in a natural way. Considered as a vector space $k\Gamma$ has a basis which we can identify with Γ. These basis elements multiply as in Γ and we define the comultiplication via $\gamma \mapsto \gamma \otimes \gamma$, the counit via $\gamma \mapsto 1$ and the antipode via $\gamma \mapsto \gamma^{-1}$ for each $\gamma \in \Gamma$. Hence there is a finite algebraic k-group G with $M(G) \simeq k\Gamma$. For any k-algebra A the group $G(A)$ can be identified with the set of all $\sum_{\gamma\in\Gamma} a_\gamma \gamma \in A\Gamma \simeq k\Gamma \otimes A$ with $\sum_{\gamma\in\Gamma} a_\gamma (\gamma \otimes \gamma) = \sum_{\gamma,\gamma'\in\Gamma} a_\gamma a_{\gamma'} (\gamma \otimes \gamma')$ and $\sum_{\gamma\in\Gamma} a_\gamma = 1$. If A is an integral domain (or, more generally, has no idempotents $\neq 0,1$), then $G(A) \simeq \Gamma$. This construction can obviously be carried out over any ring, not only over a field.

b) Suppose that $\operatorname{char} k = p \neq 0$ and let $\underline{g}$ be a finite dimensional p-Lie algebra, cf. 7.10. Then its restricted enveloping algebra $U^{[p]}(\underline{g})$ is a cocommutative Hopf algebra. Any $x \in \underline{g}$ is mapped to $x \otimes 1 + 1 \otimes x$ under the comultiplication, to 0 under the counit, and to $-x$ under the antipode. So there is a finite algebraic group G with $M(G) \simeq U^{[p]}(\underline{g})$. One gets obviously $\underline{g} \subset \mathrm{Lie}(G)$ from 8.4(2). The embedding of $U^{[p]}(\mathrm{Lie}\ G)$ into $\mathrm{Dist}(G) \subset M(G) \cong U^{[p]}(\underline{g})$ has therefore to be an isomorphism. We get $\mathrm{Lie}(G) = \underline{g}$ and $M(G) = \mathrm{Dist}(G)$ so that G is infinitesimal. See [DG], II, §7, 3.9 - 3.12 for more details.

8.6 (Modules for G and $M(G)$). Let R be a finite dimensional Hopf algebra. If M is an R-module, then M is an R^*-comodule in a natural way: Define the comodule map $M \to M \otimes R^* \simeq \mathrm{Hom}(R,M)$ by mapping m to $a \mapsto am$. If M is an R-comodule, then M is an R^*-module in a natural way: Define the action of any $\mu \in R^*$ as $(\mathrm{id}_M \bar{\otimes} \mu) \circ \Delta_M$, if Δ_M is the comodule map $M \to M \otimes R$. For two such comodules M_1, M_2 a linear map $\psi\colon M_1 \to M_2$ is a homomorphism of R-comodules if and only if it is a homomorphism of R^*-modules. In this way we get an equivalence of categories

(1) $\{R\text{-}\underline{\text{comodules}}\} \xrightarrow{\sim} \{R^*\text{-}\underline{\text{modules}}\}$.

Let G be a finite algebraic k-group. Then the categories of G-modules and k[G]-comodules are equivalent by 2.8. Combining this with (1) we get an equivalence of categories

(2) $\{G\text{-}\underline{\text{modules}}\} \xrightarrow{\sim} \{M(G)\text{-}\underline{\text{modules}}\}$.

Here to any G-module M there corresponds the M(G)-module M with

$\mu \in M(G)$ operating as $(\mathrm{id}_M \bar{\otimes} \mu) \circ \Delta_M$. We recover the action of $G(k)$ via the embedding $G(k) \subset M(G)^\times$ and, more generally, the action of any $G(A)$ via the embedding $G(A) \subset (M(G) \otimes A)^\times$ and the operation of $M(G) \otimes A$ on $M \otimes A$.

It is clear that we get on $\mathrm{Dist}(G) \subset M(G)$ the same operation as in 7.11. Furthermore all the statements in 7.11 generalize to $M(G)$. The claims in 7.14 - 7.17 hold obviously for any finite algebraic group G with $\mathrm{Dist}(G)$ replaced by $M(G)$.

The representations of G on $k[G]$ via ρ_ℓ and ρ_r lead to two (contragredient) representations of G on $M(G)$, hence to two structures of an $M(G)$-module on $M(G)$. Using the generalization of 7.11(8) one checks that any $\mu \in M(G)$ operates on $M(G)$ as left multiplication by μ when we deal with ρ_ℓ, and as right multiplication with $\sigma_G'(\mu)$ when we deal with ρ_r.

For G corresponding to a finite abstract group Γ as in 8.5.a the theory of G-modules is the same as that of $k\Gamma$-modules, hence equal to the representation theory of Γ over k.

For G corresponding to a p-Lie algebra as in 8.5.b the theory of G-modules is the same as that of $U^{[p]}(\underline{g})$-modules, hence equal to the representation theory of $\underline{g}$ considered as a p-Lie algebra.

8.7 Let from now on until the end of this chapter G be a finite algebraic k-group.

When we regard $k[G]$ resp. $M(G)$ as a G-module it will be with respect to ρ_ℓ or ρ_r resp. the contragredient representation which we call also the left or right regular representation of G. In case we want to distinguish in our notations between ρ_ℓ or ρ_r we add an index "ℓ" or "r" to the modules, i.e. write e.g. $k[G]_\ell$ and $M(G)_\ell$.

Lemma: The G-modules $M(G)$ *and* $k[G]$ *are isomorphic. We have* $\dim M(G)^G = 1$.

Proof: By the tensor identity we have $M(G) \otimes k[G] \simeq k[G]^r$ where $r = \dim k[G]$. On the other hand $M(G) \otimes k[G] = k[G]^* \otimes k[G]$ is self-dual as a G-module, hence also isomorphic to $(k[G]^*)^r$. The Krull-Schmidt theorem about unique decomposition into (finite dimensional) indecomposable modules implies that $k[G] \simeq k[G]^*$ has to hold as $k[G]^r \simeq (k[G]^*)^r$ for some $r > 0$. The last equality follows now from 2.10(5).

8.8 (*Invariant Measures*) We call an element in $M(G)^G_\ell$ (resp. $M(G)^G_r$) a *left* (resp. *right*) *invariant measure on* G. (In [11] such elements are called "integrals", in [Haboush 3] "norm forms".)

The description of the left and right regular representations of $M(G)$ on itself in 8.6 implies

$$(1)\quad M(G)^G_\ell = \{\mu_0 \in M(G) \mid \mu\mu_0 = \mu(1)\mu_0 \quad \text{for all} \quad \mu \in M(G)\}$$

and

$$(2)\quad M(G)^G_r = \{\mu_0 \in M(G) \mid \mu_0\mu = \mu(1)\;{}_0 \quad \text{for all} \quad \mu \in M(G)\}$$

as $\sigma'_G(\mu)(1) = \mu(1)$ for all $\mu \in M(G)$. Furthermore we have

(3) $\sigma'_G(M(G)^G_r) = M(G)^G_\ell$

as σ'_G intertwines the left and the right regular representations (or, using (1), (2), as it is an antiautomorphism of $M(G)$ considered as an algebra).

Obviously $M(G)^G_\ell$ is stable under right multiplication by elements of $M(G)$, hence a $M(G)$- and G-submodule of $M(G)$ with respect to the right regular representation. (This can also be seen directly.) As $\dim M(G)^G_\ell = 1$ the representation of G on $M(G)^G_\ell$ is given by some $\delta_G \in X(G) \subset k[G]$. So for all $g \in G(A)$ and any A

(4) $\rho_r(g)(\mu_0 \otimes 1) = \mu_0 \otimes \delta_G(g)$ for all $\mu_0 \in M(G)^G_\ell$,

and, equivalently, for all $\mu \in M(G)$

(5) $\mu_0\mu = \sigma'_G(\mu)(\delta_G)\mu_0 = \mu(\delta_G^{-1})\mu_0$ for all $\mu_0 \in M(G)^G_\ell$.

(Observe that $\sigma_G(\chi) = \chi^{-1}$ for all $\chi \in X(G)$.) This character δ_G is called the <u>modular function</u> of G. We call G <u>unimodular</u> if $\delta_G = 1$. (In the examples in 8.9 each G will be unimodular. We shall meet a case where $\delta_G \neq 1$ later on in part II.)

We could have defined δ_G also via $M(G)^G_r$ as (3) implies for all $\mu \in M(G)$

(6) $\mu\mu_0 = \mu(\delta_G^{-1})\mu_0$ for all $\mu_0 \in M(G)^G_r$

or, equivalently, for all $g \in G(A)$ and all A

(7) $\rho_\ell(g)(\mu_0 \otimes 1) = \mu_0 \otimes \delta_G^{-1}(g)$ for all $\mu_0 \in M(G)_r^G$.

8.9 (Examples) If G corresponds to an abstract finite group Γ as in 8.5.a, then

(1) $M(G)_\ell^G = M(G)_r^G = k \sum_{\gamma \in \Gamma} \gamma$

Consider as another example $G = G_{a,r}$ with $r \in \mathbb{N}$, $r > 0$ assuming $\mathrm{char}(k) = p \neq 0$. Set $q = p^r$. As $G_{a,r}$ is a subgroup of $G_a = \mathrm{Sp}_k k[T]$, we can identify $M(G_{a,r}) = \mathrm{Dist}(G_{a,r})$ with the subalgebra of $\mathrm{Dist}(G_a)$ spanned by all μ with $\mu(T^{a+i}) = 0$ for all $i \geq 0$. Using the basis $(\gamma_n)_{n \in \mathbb{N}}$ of $\mathrm{Dist}(G_a)$ as in 7.8 we get

$$M(G_{a,r}) = \sum_{n=0}^{a-i} k\gamma_n .$$

As $\gamma_0(1) = 1$ and $\gamma_n(1) = 0$ for $n > 0$, as $\gamma_0\gamma_{q-1} = \gamma_{q-1}$ and $\gamma_n\gamma_{q-1} = \binom{q+n-1}{n-1}\gamma_{n+q-1} = 0$ for $0 < n \leq q-1$ we see that γ_{q-1} is an invariant measure on $G_{a,r}$. Using $\dim M(G)^G = 1$ or some additional computations we get

(2) $M(G)_\ell^G = M(G)_r^G = k\gamma_{q-1}$ for $G = G_{a,r}$.

Assume again $\mathrm{char}\, k = p \neq 0$, let $r \in \mathbb{N}$, $r > 0$ and set $q = p^r$. Let us consider $G = \mu_{(q)}$ and determine $M(G)^G$. As $\mu_{(q)}$ is an infinitesimal and closed subgroup of G_m we can identify $M(\mu_{(q)}) = \mathrm{Dist}(\mu_{(q)})$ with a subalgebra of $\mathrm{Dist}(G_m)$. Let us use the notations of 7.8. Then $M(\mu_{(q)})$ consists of all $\nu \in \mathrm{Dist}(G_m)$

with $\nu(T^i(T^q-1)) = 0$ for all $i \in \mathbb{Z}$. Obviously

$$\delta_n(T^i(T^q-1)) = \binom{q+i}{n} - \binom{i}{n}$$

for all $i \in \mathbb{Z}$. The standard formula for binomial coefficients mod p (cf. e.g. [Haboush 3], 5.1) shows $\delta_n(T^i(T^q-1)) = 0$ for all $i \in \mathbb{Z}$ if $0 \leq n < q$. As $\dim M(\mu_{(q)}) = \dim k[\mu_{(q)}] = q$, we get

$$M(\mu_{(q)}) = \sum_{n=0}^{q-1} k\delta_n .$$

We claim

$$(3) \qquad M(G)_r^G = M(G)_\ell^G = k \sum_{i=0}^{q-1} (-1)^i \delta_i \qquad \text{for} \quad G = \mu_{(q)} .$$

Set $\mu_0 = \sum_{i=0}^{q-1} (-1)^i \delta_i$. As δ_0 is the 1 in $M(G)$ and $\delta_0(1) = 1$ and $\delta_n(1) = 0$ for $n > 0$, we have to show $\delta_n \mu_0 = 0$ for all n with $0 < n < q$. We have by 7.8(3):

$$\delta_n \mu_0 = \sum_{i=0}^{q-1} \sum_{j=0}^{\min(i,n)} (-1)^i \binom{n+i-j}{i-j} \binom{n}{j} \delta_{n+i-j}$$

If $n+i-j > q-1$, then $\binom{n+i-j}{i-j} = 0$ and we can delete the corresponding summand. Substituting $s = i-j$ we get

$$\delta_n \mu_0 = \sum_{i=0}^{a-1} (-1)^i \sum_{s=\max(0,i-n)}^{\min(i,q-1-n)} \binom{n+s}{s} \binom{n}{i-s} \delta_{n+s}$$

$$= \sum_{s=0}^{q-1-n} \Big(\sum_{i=s}^{n+s} (-1)^i \binom{n}{i-s} \Big) \binom{n+s}{s} \delta_{n+s} = 0.$$

8.10 (Projective and Injective Modules) We call a projective object in the category of all G-modules simply a projective G-module. They correspond under the equivalence of categories to the projective M(G)-modules. This shows that each G-module is a homomorphic image of a projective G-module, hence that projective resolutions exist in the category of G-modules. (This is not true for arbitrary group schemes.)

The representation theory of finite dimensional algebras shows that the indecomposable projective G-modules are (up to isomorphism) the indecomposable direct summands of $M(G)$. For each simple G-module E there is a unique (up to isomorphism) projective G-module Q with $Q/\mathrm{rad}(Q) \simeq E$. It is called the projective cover of E. One gets in this way a bijection between the isomorphism classes of simple G-modules and of indecomposable projective G-modules.

Now the isomorphism $M(G) \simeq k[G]$ from 8.7 together with 3.10 shows that a finite dimensional G-module is projective if and only if it is injective. The indecomposable injective indecomposable G-modules are exactly the indecomposable projective G-modules. There is a bijection $E \mapsto E'$ on the set of isomorphism classes of simple G-modules such that the injective hull Q_E of E (cf. 3.16) is the projective cover of E', i.e.

(1) $Q_E/\mathrm{rad}(Q_E) \simeq E'$.

We intend to describe this bijection and have to be more precise about the isomorphism $M(G) \simeq k[G]$ at first.

8.11 ($M(G)$ as a Module over $k[G]$). There is a natural structure as a $k[G]$-module on $M(G)$: For any $f \in k[G]$ and $\mu \in M(G)$ we define $f\mu$ through

(1) $(f\mu)(f_1) = \mu(ff_1)$ for all $f_1 \in k[G]$.

The following properties follow from straightforward computations which may be left to the reader.

(2) $f\varepsilon_G = f(1)\varepsilon_G$ for all $f \in k[G]$,

(3) $\sigma_G'(f\mu) = \sigma_G(f)\sigma_G'(\mu)$ for all $f \in k[G]$, $\mu \in M(G)$,

(4) If $\mu_1, \mu_2 \in M(G)$ and $f \in k[G]$ with $\Delta_G(f) = \sum_{i=1}^{r} f_i \otimes f_i'$, then $f(\mu_1\mu_2) = \sum_i (f_i\mu_1)(f_i'\mu_2)$.

We have $\Delta_G(\chi) = \chi \otimes \chi$ and $\chi(1) = 1$ for all $\chi \in X(G) \subset k[G]$. Therefore (2) and (4) imply:

(5) If $\chi \in X(G)$, then $\mu \mapsto \chi\mu$ is an algebra endomorphism of $M(G)$. Its inverse is $\mu \mapsto \chi^{-1}\mu$.

We claim furthermore for any $f \in k[G]$, $\mu \in M(G)$ and $g \in G(A)$ (for all A):

(6) $\rho_\ell(g)(f\mu) = (\rho_\ell(g)f)(\rho_\ell(g)\mu)$

and

(7) $\rho_r(g)(f\mu) = (\rho_r(g)f)(\rho_r(g)\mu)$.

(We really ought to write $\rho_\ell(g)(f\mu \otimes 1)$ etc.) Indeed we have

$$\rho_\ell(g)(f\mu) = (f\mu)\circ\rho_\ell(g^{-1}) = \mu\circ(f?)\circ\rho_\ell(g^{-1})$$

$$= \mu\circ\rho_\ell(g^{-1})\circ(\rho_\ell(g)f?) = (\rho_\ell(g)\mu)\circ(\rho_\ell(g)f?)$$

$$= (\rho_\ell(g)f)(\rho_\ell(g)\mu).$$

The proof of (7) is similar.

8.12 If M is a G-module, then we denote by M^ℓ the $(G\times G)$-module which is equal to M as a vector space and where the first factor G operates as on M and the second factor operates trivially. Similarly M^r is defined. For $\lambda \in X(G)$ we shall usually write λ^ℓ and λ^r instead of $(k_\lambda)^\ell$ and $(k_\lambda)^r$. We regard $k[G]$ and $M(G)$ as a $(G\times G)$-modules with the first factor operating via ρ_ℓ and the second one via ρ_r.

Proposition: Let $\mu_0 \in M(G)_\ell^G, \mu_0 \neq 0$. Then $f\mapsto f\mu_0$ is an isomorphism of $k[G]$-modules and of $(G\times G)$-modules:

$$k[G]\otimes(\delta_G)^r \xrightarrow{\sim} M(G).$$

Proof: It is obvious from the definitions and from 8.11(6), (7) that the map considered is a homomorphism of $k[G]$- and of $(G\times G)$-modules. We have to prove only its bijectivity. As both sides have the same dimension it is enough to prove its injectivity.

Consider the endomorphism γ of $M(G)\otimes k[G]$ which is the composite of the map $id_{M(G)}\otimes\Delta_G$: $M(G)\otimes k[G] \to M(G)\otimes k[G]\otimes k[G]$ with the map $M(G)\otimes k[G]\otimes k[G] \to M(G)\otimes k[G]$, $\mu\otimes f_1\otimes f_2 \mapsto f_1\mu\otimes f_2$. We can identify $M(G)\otimes k[G]$ with $Mor(G,M(G)_a)$ associating to each $\mu\otimes f$ the map $g\mapsto f(g)\mu$. Then $\gamma(\mu\otimes f)$ is easily checked to be the map $g \mapsto (\rho_r(g)f)\mu$.

Let us fix now $f \in k[G]$ and consider $F \in \mathrm{Mor}(G, M(G)_a)$ with $F(g) = (\rho_\ell(g)f)\mu_0 = \rho_\ell(g)(f\mu_0)$ for all $g \in G(A)$ and all g. If $\Delta_G(f) = \sum_{i=1}^{r} f_i \otimes f_i'$, then $\rho_\ell(g)f = \sum_{i=1}^{r} f_i(g^{-1})f_i'$, hence F corresponds to $\sum_{i=1}^{r} (f_i'\mu_0) \otimes \sigma_G(f_i) \in M(G) \otimes k[G]$. Its image under γ is therefore the morphism $g \mapsto (\sum_{i=1}^{r} (\rho_r(g)\sigma_G(f_i))f_i')\mu_0$. Now $\sum_{i=1}^{r} (\rho_r(g)\sigma_G(f_i))f_i'$ maps any g' to $\sum_{i=1}^{r} f_i((g'g)^{-1})f_i'(g') = f(g^{-1}g'^{-1}g') = f(g^{-1})$. This implies $\gamma(F) = \mu_0 \otimes \sigma_G(f)$. If $f\mu_0 = 0$, then $F = 0$, hence $\mu_0 \otimes \sigma_G(f) = 0$. As $\mu_0 \neq 0$, this implies $f = 0$. So the map considered is injective.

Remarks: 1) If we take $\mu_0 \in M(G)_r^G, \mu_0 \neq 0$, then $f \mapsto f\mu_0$ is an isomorphism of $k[G]$- and $(G \times G)$-modules

$$k[G] \otimes (\delta_G^{-1})^\ell \tilde{\rightarrow} M(G).$$

2) The affine and finite scheme G is also a projective scheme of dimension 0. It has therefore a dualizing sheaf, cf. [Ha], p. 241. This is easily seen to be the coherent sheaf with global sections equal to $M(G) = k[G]^*$: We have for each finite dimensional $k[G]$-module M a non-degenerate pairing $\mathrm{Hom}_{k[G]}(M, k[G]^*) \times M \to k[G]^* \to k$ mapping at first (φ, m) to $\varphi(m)$ and then μ to $\mu(1)$. (Use $\mathrm{Hom}_{k[G]}(M, k[G]^*) \simeq \mathrm{Hom}_{k[G]}(k[G], M^*) \simeq M^*$ with the obvious structure as a $k[G]$-module on M^*.)

In [Kempf 5], 5.1 the proposition is proved using the interpretation of $M(G)$ as the dualizing sheaf.

<u>8.13</u> <u>Proposition</u>: <u>Let</u> E <u>be a simple</u> G-<u>module and</u> Q <u>a projective cover of</u> E. <u>Then</u>

$$\operatorname{soc} Q \simeq E \otimes \delta_G.$$

<u>Proof</u>: Choose a complete set $e_1, \ldots, e_r$ of primitive, orthogonal idempotents in $M(G)$, hence a decomposition

$$M(G) = \bigoplus_{i=1}^{r} M(G)e_i$$

into indecomposable (projective and injective) modules. There are simple G-modules E_i and $E_i^!$ $(1 \leq i \leq r)$ with $M(G)e_i/\operatorname{rad} M(G)e_i \simeq E_i$ and $\operatorname{soc} M(G)e_i \simeq E_i^!$ for all i. We have to show $E_i^! = E_i \otimes \delta_G$ for all i.

For any G-module M the map $\varphi \mapsto \varphi(e_i)$ is an isomorphism $\operatorname{Hom}_G(M(G)e_i, M) \xrightarrow{\sim} e_iM$. If M is simple, then $M \simeq E_i$ if and only if $e_iM \neq 0$. Any $\mu \in M(G)$ operates on M^* through $\mu\varphi = \varphi \circ \sigma_G^!(\mu)$ and on $M \otimes \chi$ for $\chi \in X(G)$ as $\chi\mu$ operates on M. Therefore (for M simple)

$$(1) \quad M \simeq E_i \Longleftrightarrow e_iM \neq 0 \Longleftrightarrow \sigma_G^!(e_i)M^* \neq 0$$

$$\Longleftrightarrow (\chi\sigma_G^!(e_i))(M^* \otimes \chi^{-1}) \neq 0.$$

Because of 8.11(5) also $\chi\sigma_G^!(e_1), \ldots, \chi\sigma_G^!(e_r)$ is a complete orthogonal set of primitive idempotents in $M(G)$. We get from (1)

(2) $E_i^* \otimes \chi^{-1} \simeq M(G)\chi\sigma_G'(e_i)/\mathrm{rad}\, M(G)\chi\sigma_G'(e_i)$

for all i.

Choose μ_o as in 8.12 and let $\psi: M(G) \to k[G]$ be inverse to the map $f \mapsto f_{\mu_o}$ from 8.12. The $(G \times G)$-homomorphism property of 8.12 implies for all $\mu, \mu' \in M(G)$

$$\psi(\mu\mu') = \psi(\rho_\ell(\mu)\mu') = \rho_\ell(\mu)\psi(\mu')$$

$$= \psi(\rho_r(\sigma_G'(\mu'))\mu) = \rho_r(\delta_G\sigma_G'(\mu'))\psi(\mu).$$

Therefore each $\psi(M(G)e_i) = \rho_r(\delta_G\sigma_G'(e_i))\psi(M(G))$ is orthogonal to each $M(G)\delta_G\sigma_G'(e_j)$ with $j \neq i$. As ψ is an isomorphism for ρ_ℓ we get for all i

$$M(G)e_i \simeq \psi(M(G)e_i) \simeq (M(G)\delta_G\sigma_G'(e_i))^*,$$

hence

(3) $\mathrm{soc}\, M(G)e_i \simeq (M(G)\delta_G\sigma_G'(e_i)/\mathrm{rad}\, M(G)\delta_G\sigma_G'(e_i))^*$.

Now (2) and (3) imply

$$E_i' \simeq (E_i^* \otimes \delta_G^{-1})^* \simeq E_i \otimes \delta_G.$$

<u>Remark</u>: If $\delta_G = 1$ (i.e. if G is unimodular), then the projective cover and the injective hull of every simple G-module coincide. If we apply the proposition to the trivial G-module k, then we get that also the converse holds.

One can show for unimodular G that $M(G)$ is a symmetric algebra in the sense of [5], ch. IX, cf. [Humphreys 9]. In general $M(G)$ is only a Frobenius algebra.

8.14 (Coinduced Modules) Any closed subgroup H of G is itself a finite algebraic k-group. We can identify $M(H)$ with the subalgebra $\{\mu \in M(G) \mid \mu(I(H)) = 0\}$ where $I(H) \subset k[G]$ is the ideal of H, cf. the corresponding result for $\mathrm{Dist}(H)$ in 7.2(3).

The equivalence of categories 8.6(2) enables us to define a functor coind_H^G from {H-modules} to {G-modules} through

(1) $\mathrm{coind}_H^G M = M(G) \otimes_{M(H)} M$

for any H-module M. We call this functor the coinduction from H to G. (When comparing this to what is done for Lie algebras e.g. in [6], ch. 5 one has to observe that there the terms induction and coinduction have just the opposite meanings. Also in [Voigt] our $\mathrm{coind}_H^G M$ is called an induced module.)

We have obviously:

(2) The functor coind_H^G is right exact.

For any H-module the map $i_M\colon M \to \mathrm{coind}_H^G M$ with $i_M(m) = 1 \otimes m$ is a homomorphism of H-modules. The universal property of the tensor product implies for each G-module V that we get an isomorphism

(3) $\mathrm{Hom}_G(\mathrm{coind}_H^G M, V) \xrightarrow{\sim} \mathrm{Hom}_H(M, \mathrm{res}_H^G V)$, $\varphi \mapsto \varphi \circ i_M$.

Hence:

(4) The functor coind_H^G is left adjoint to res_H^G.

Furthermore:

(5) The functor coind_H^G maps projective H-modules to projective G-modules.

8.15 Lemma: Let H be a closed subgroup of G and M a finite dimensional H-module. Then there is an isomorphism of G-modules

$$\mathrm{coind}_H^G M \simeq (\mathrm{ind}_H^G(M^*))^*.$$

Proof: For all finite dimensional G-modules V_1, V_2 the bijection $\mathrm{Hom}(V_1,V_2) \simeq \mathrm{Hom}(V_2^*,V_1^*)$ mapping each φ to its transposed φ^* induces a bijection $\mathrm{Hom}_G(V_1,V_2) \simeq \mathrm{Hom}_G(V_2^*,V_1^*)$.

Using this and 8.14(3) we get for each finite dimensional G-module V canonical isomorphisms

$$\mathrm{Hom}_G(V,(\mathrm{coind}_H^G M)^*) \tilde{\to} \mathrm{Hom}_G(\mathrm{coind}_H^G M, V^*)$$

$$\tilde{\to} \mathrm{Hom}_H(M,V^*) \tilde{\to} \mathrm{Hom}_H(V,M^*)$$

mapping any ψ to $(i_M)^* \circ \psi$. This generalizes to all V by taking direct limits. Therefore $(\mathrm{coind}_H^G M)^*$ has the universal property of $\mathrm{ind}_B^G(M^*)$ as in 3.5, hence is isomorphic to $\mathrm{ind}_H^G(M\ )$.

8.16 (Exactness of Induction) Let H be a closed subgroup of G. As H is a finite algebraic k-group the quotient G/H is affine by 5.6(3), hence 5.8 implies:

(1) ind_H^G is exact.

We get now from 4.12:

(2) $k[G]$ <u>is an injective</u> H-<u>module</u>.

Hence:

(3) $M(G)$ <u>is a projective left and right</u> $M(H)$-<u>module</u>,

and:

(4) coind_H^G <u>is exact</u>.

Of course (4) follows also directly from (1) and 8.15. One can improve (3) and show that $M(G)$ is a free module over $M(H)$, cf. [9], 2.4. We do not have to go into this.

If M' is a projective and finite dimensional right $M(H)$-module, then we have for each H-module M an isomorphism

(5) $M' \otimes_{M(H)} M \xrightarrow{\sim} \mathrm{Hom}_{M(H)}(\mathrm{Hom}_{M(H)}(M', M(H)), M)$

mapping each $m' \otimes m$ with $m' \in M'$ and $m \in M$ to the map $\varphi \mapsto \varphi(m')m$. Here we form $\mathrm{Hom}_{M(H)}(M', M(H))$ via the operation of $M(H)$ on itself by right multiplication and we consider it as an $M(H)$-module via the left multiplication on $M(H)$. In order to prove bijectivity in (5) one restricts to the case $M' = M(H)^n$ for some n where both sides are isomorphic to M^n.

Because of (3) we can apply this to $M(G)$ considered as an $M(H)$-module under right multiplication. The map in (5) is now easily checked to be an isomorphism of G-modules

(6) $\mathrm{coind}_H^G M \xrightarrow{\sim} \mathrm{Hom}_H(\mathrm{Hom}_H(M(G), M(H)), M)$

where the operation of G on the right hand side is derived from the left regular representation on $M(G)$.

<u>8.17</u> <u>Proposition</u>: <u>Let</u> H <u>be a closed subgroup of</u> G. <u>Then we have for each</u> H-<u>module</u> M <u>an isomorphism</u>:

$$\operatorname{coind}_H^G M \xrightarrow{\sim} \operatorname{ind}_H^G(M \otimes (\delta_G)_{|H}\delta_H^{-1}).$$

<u>Proof</u>: We have isomorphisms of $(G\times H)$-modules

$$\begin{aligned}\operatorname{Hom}_H(M(G),M(H)) &\simeq (M(G)^* \otimes M(H))^H \\ &= (k[G]\otimes M(H))^H \simeq (k[G]\otimes k[H]\otimes \delta_H)^H \\ &= \operatorname{ind}_H^H(k[G]\otimes\delta_H) \simeq k[G]\otimes \delta_H.\end{aligned}$$

This is regarded as an H-module via the right regular representation on $k[G]$ and via δ_H and as a G-module via the left regular representation on $k[G]$.

We get now from 8.16(6) isomorphisms of G-modules

$$\begin{aligned}\operatorname{coind}_H^G M &\simeq \operatorname{Hom}_H(k[G]\otimes \delta_H, M) \\ &\simeq (M(G)\otimes \delta_H^{-1}\otimes M)^H \\ &\simeq (k[G]\otimes(\delta_G)_{|H}\delta_H^{-1}\otimes M)^H \\ &= \operatorname{ind}_H^G(M\otimes(\delta_G)_{|H}\delta_H^{-1}).\end{aligned}$$

<u>8.18</u> <u>Corollary</u>: <u>Let</u> H <u>be a closed subgroup of</u> G <u>and</u> M <u>a finite dimensional</u> H-<u>module</u>. <u>Then</u>:

$$(\mathrm{ind}_H^G M)^* \simeq \mathrm{ind}_H^G(M^* \otimes (\delta_G)_{|H}\delta_H^{-1}).$$

Proof: This follows from 8.17 and 8.15.

Remark: One can interprete this as Serre duality for the sheaf cohomology of $\mathcal{L}_{G/H}(M)$, cf. 5.10.

8.19 Proposition: Let G' be a k-group scheme operating on G through group automorphisms. Then G' operates naturally on $k[G]$ and $M(G)$. The space $M(G)_\ell^G$ is a G'-submodule of $M(G)$ and the operation of G' on $M(G)_\ell^G$ is given by some $\chi \in X(G')$. If $\mu_0 \in M(G)_\ell^G$, $\mu_0 \neq 0$, then the map $f \mapsto f\mu_0$ is an isomorphism $k[G]\otimes\chi \xrightarrow{\sim} M(G)$ of G'-modules. If G is a closed normal subgroup of G' and if we take the action of G' by conjugation on G, then $\chi_{|G} = \delta_G$.

Proof: We can form the semi-direct product $G \rtimes G'$ and make it operate on G such that G acts through left multiplication and G' as given. This yields representations of $G \rtimes G'$ on $k[G]$ and $M(G) = k[G]^*$ which yield the operations considered in proposition when restricted to G' and which yield the left regular representations when restricted to G. Hence $M(G)_\ell^G$ are the fixed points of the normal subgroup G of $G \rtimes G'$, hence a G'-submodule by 3.2.

It is now obvious that G' operates through some $\chi \in X(G')$ on $M(G)_\ell^G$ and that $f \mapsto f\mu_0$ is an isomorphism $k[G]\otimes\chi \xrightarrow{\sim} M(G)$ of G'-modules. Suppose finally that G is a normal subgroup of G' and that we consider the conjugation action of G' on G. Then each $g \in G(A) \subset G'(A)$ acts through the composition of $\rho_\ell(g)$ and $\rho_r(g)$ on $M(G) \otimes A$, hence through $\rho_r(g)$ on $\mu_0 \otimes 1$. Therefore the definitions show $\chi(g) = \delta_G(g)$.

<u>8.20</u> <u>Proposition:</u> <u>Let</u> G' <u>be a</u> k-<u>group scheme containing</u> G <u>as a closed normal subgroup. Let</u> H' <u>be a closed subgroup of</u> G <u>and set</u> $H = H' \cap G$. <u>Let</u> M <u>be an</u> H'-<u>module. Then there is a natural structure as an</u> $H'G$-<u>module on</u> $\text{coind}_H^G M$ <u>extending the structure as a</u> G-<u>module. For each</u> $H'G$-<u>module</u> V <u>there is a canonical isomorphism</u>

(1) $$\text{Hom}_{H'G}(\text{coind}_H^G M, V) \xrightarrow{\sim} \text{Hom}_{H'}(M, V).$$

<u>If</u> $\chi \in X(G')$ <u>resp.</u> $\chi' \in X(H')$ <u>is the character through which</u> G' <u>resp.</u> H' <u>operates on</u> $M(G)_\ell^G$ <u>resp.</u> $M(H)_\ell^H$, <u>then we have an isomorphism of</u> $H'G$-<u>modules</u>

(2) $$\text{coind}_H^G M \xrightarrow{\sim} \text{ind}_{H'}^{H'G}(M \otimes (\chi_{|H'})\chi'^{-1}).$$

<u>If</u> $\dim M < \infty$, <u>then we have an isomorphism of</u> $H'G$-<u>modules</u>

(3) $$(\text{ind}_{H'}^{H'G} M)^* \xrightarrow{\sim} \text{ind}_{H'}^{H'G}(M'^* \otimes (\chi_{|H'})\chi'^{-1}).$$

<u>Proof:</u> Let us work with the description of $\text{coind}_H^G M$ as in 8.16(6). We make H' operate on $M(G)$ and $M(H)$ via the conjugation action on G and H. We get thus a representation of H' on $\text{Hom}(M(G),M(H))$ which extends to $H' \ltimes H$ if we let H operate through the two right regular representations. By 3.2 the subspace $\text{Hom}_H(M(G),M(H))$ is an H'-module. Together with the given action of H' on M this makes $\text{Hom}(\text{Hom}_H(M(G),M(H),M)$ into an H'-module. This operation of H' can be extended to

$H' \ltimes H$ with H operating via ρ_ℓ on $M(H)$ and through the restriction of the H'-action on M. Again $\mathrm{Hom}_H(\mathrm{Hom}_H(M(G),M(H)),M)$ is an H'-submodule. We can extend the operation of H' to $H' \ltimes G$ letting G act through ρ_ℓ on $M(G)$, i.e. inducing the action of G on $\mathrm{coind}_H^G M$.

For any $h \in H(A)$ for some A the element $(h,h^{-1}) \in (H' \ltimes G)(A)$ acts trivially. (This follows easily from the definitions.) Therefore we get a representation of $(H' \ltimes G)/H \simeq H'G$, cf. 6.2(1), on $\mathrm{coind}_H^G M$ extending the given one of G.

Using this structure, the isomorphism in 8.14(3) is easily checked to be an isomorphism of H'-modules (provided V is an H'-module). It therefore has to induce an isomorphism of the H'-fixed points. This implies (1).

We get (2) by examining the proof of 8.17. After replacing δ_G by χ and δ_H by χ' all isomorphisms there are also compatible with the H'-action, hence with the structure as $H'G$-module. Similarly 8.15 generalizes from G to $H'G$ and together with (2) yields (3) as in 8.18.

<u>Remark</u>: We denote $\mathrm{coind}_H^G M$ when considered as an $H'G$-module by $\mathrm{coind}_{H'}^{H'G} M$. Obviously $\mathrm{coind}_{H'}^{H'G}$ is a functor from {H'-modules} to {$H'G$-modules} and 8.14(2)-(4), 8.16(4), generalize to this. Note that we have by construction an isomorphism of functors

$$\mathrm{res}_G^{H'G} \circ \mathrm{coind}_{H'}^{H'G} \simeq \mathrm{coind}_H^G \circ \mathrm{res}_H^{H'}$$

which is dual to 6.12.

9. Representations of Frobenius Kernels

Throughout this chapter let p be a prime number. We shall always assume that k is a perfect field with $\mathrm{char}(k) = p$.

Let G be an algebraic k-group. If $k = \mathbb{F}_p$, then the map $f \mapsto f^p$ on $k[G]$ is an endomorphism of k-algebras and defines a morphism $F_G: G \to G$ which is a group endomorphism. The kernels $G_r = \ker(F_G^r)$ are called the Frobenius kernels of G. They are infinitesimal algebraic k-groups. One can generalize this to all k by replacing F_G^r as above by some group homomorphism $G \to G^{(r)}$ into a suitable k-group $G^{(r)}$.

We give in this chapter at first the definitions and elementary properties (9.1 - 9.7). We then compute their modular functions in the case of reduced groups (9.8 combined with 8.19).

The representation theory of the first Frobenius kernel G_1 of G is equivalent to that of $\mathrm{Lie}(G)$ as a p-Lie algebra. Therefore each cohomology group $H^i(G_1,M)$ is equal to the corresponding "restricted Lie algebra cohomology group" in the sense of [Hochschild 3]. In that paper these groups are compared to the ordinary Lie-algebra cohomology groups (cf. 9.16), especially in low degrees.

One of his main results can now be interpreted as a "six term exact sequence" arising from a spectral sequence (9.18/19). This spectral sequence was found for $p \neq 2,3$ and G reductive in [Friedlander/Parshall 1]. (But compare also the remark at the end of section 3 in [Hochschild 3].) Their results were somewhat generalized in [Andersen/Jantzen]. The present version of proposition 9.18 is the same as in my lectures in Shanghai and was proved also by Friedlander and Parshall.

9.1 (The Frobenius Morphism on an Affine Variety) Before defining Frobenius morphism in general we want to motivate the definitions by an example. Let us assume in this section k to be algebraically closed.

Let X be an affine variety over k (as in 1.1). We can embed X as a Zariski closed subset into some k^n. The map $F: k^n \to k^n$, $(a_1, a_2, \ldots, a_n) \mapsto (a_1^p, a_2^p, \ldots, a_n^p)$ is a bijective morphism of varieties. It is also a closed map. (Using that $f^p \in \operatorname{im}(F^*)$ for all $f \in k[T_1, \ldots, T_n]$ one shows $\sqrt{k[T_1, \ldots, T_n] F^*(F^*)^{-1} I} = \sqrt{I}$ for each ideal I $k[T_1, \ldots, T_n]$.) Therefore each $F^r(X)$ with $r \in \mathbb{N}$ is a closed subset of k^n and F^r induces a bijective morphism $X \to F^r(X)$. We want to show that the pair $(F^r(X), F^r: X \to F^r(X))$ has an intrinsic meaning, i.e. is independent (up to isomorphism) of the embedding of X into k^n.

Define for each $f \in k[X]$ a map $\varphi_r(f): F^r(X) \to k$ through $\varphi_r(f)(x') = f(F^{-r}(x'))^p$ for all $x' \in F^r(X)$. Obviously φ_r is an injective ring homomorphism from $k[X]$ to the algebra of all functions from $F^r(X)$ to k and satisfies $\varphi_r(af) = a^{p^r} \varphi_r(f)$ for all $a \in k$ and $f \in k[X]$. If f is the i-th coordinate function on k^n restricted to X, then $\varphi_r(f)$ is the i-th coordinate function restricted to $F^r(X)$. Therefore φ_r induces a bijection from $k[X]$ to $k[F^r(X)]$.

Denote by $k[X]^{(-r)}$ the k-algebra which coincides as a ring with $k[X]$ but where each $a \in k$ operates as $a^{(p^{-r})}$ does on $k[X]$. Then we can regard φ_r as an isomorphism of k-algebras $k[X]^{(-r)} \tilde{\to} k[F^r(X)]$. This shows that $F^r(X)$ as a variety has an intrinsic meaning. If we identify $k[F^r(X)]$ with $k[X]^{(-r)}$ via φ_r, then the comorphism of F^r is the map $k[X]^{(-r)} \to k[X]$, $f \mapsto f^{p^r}$ for all f, hence also F^r has a description independent of the embedding of X into k^n.

9.2 (The Frobenius Morphism on a Scheme) From now on let k be again an arbitrary perfect field of characteristic p.

For each k-algebra A and each $m \in \mathbb{Z}$ we define $A^{(m)}$ as the k-algebra which coincides with A as a ring but where each $b \in k$ operates as b^{p^m} does on A. Trivially $A^{(0)} = A$. One has obviously isomorphisms

(1) $(A^{(m)})^{(n)} = A^{(m+n)}$ for all $m,n \in \mathbb{Z}$

and (for all k-algebras A,A')

(2) $\mathrm{Hom}_{k\text{-}alg}(A^{(-m)},A') \cong \mathrm{Hom}_{k\text{-}alg}(A,A'^{(m)})$ for all $m \in \mathbb{Z}$.

(This is the identity map.) For each k-algebra A, each $m \in \mathbb{Z}$ and $r \in \mathbb{N}$ the map

(3) $\gamma_r\colon A^{(m)} \to A^{(m+r)}$, $a \mapsto a^{p^r}$

is a homomorphism of k-algebras.

We define now for any k-functor X and any $r \in \mathbb{N}$ a new k-functor $X^{(r)}$ through

(4) $X^{(r)}(A) = X(A^{(r)})$ for all k-algebras A.

Furthermore we define a morphism $F_X^r\colon X \to X^{(r)}$ through

(5) $F_X^r(A) = X(\gamma_r)\colon X(A) \to X(A^{(r)}) = X^{(r)}(A)$

for all A. We call F_X^r the r-th Frobenius morphism on X. Obviously $X \to X^{(r)}$ is a faithful functor from {k-functors} to itself.

One gets from (1) for all $r,s \in \mathbb{N}$ and all X

(6) $(X^{(r)})^{(s)} = X^{(r+s)}$ and $F^s_{X^{(r)}} \circ F_X^r = F_X^{r+s}$.

If we consider an affine scheme $X = Sp_k R$ for some k-algebra R, then (2) implies for all $r \in \mathbb{N}$

(7) $(Sp_k R)^{(r)} \simeq Sp_k(R^{(-r)})$

Furthermore F_X^r has as comorphism $R^{(-r)} \to R$, $f \mapsto f^{p^r}$. So the construction of $X^{(r)}$ and F_X^r generalizes the situation considered in 9.1.

We can interprete the definition (4) as saying that $X^{(r)}$ arises from X through base change from k to $k^{(r)}$ which then is identified with k as a ring. We can therefore apply the general remarks about base change in 1.10. So the functor $X \to X^{(r)}$ maps subfunctors to subfunctors, commutes with taking intersections and inverse images of subfunctors and with taking direct and fibre products. It maps local functors to local functors, schemes to schemes, and faisceaux to faisceaux (cf. 5.3(8)). If X is an affine scheme and I an ideal in $k[X]$, then $V(I)^{(r)} = V(I^{(-r)})$ and $D(I)^{(r)} = D(I^{(-r)})$ where $I^{(-r)} \subset k[X]^{(-r)}$ is just I with the new operation of k.

If $k = \mathbb{F}_p$, then obviously $X^{(r)} = X$ for all r and any k-functor X. If X is affine and if F_X is the endomorphism of X with $F_X^*(f) = f^p$ for all $f \in k[X]$, then obviously $F_X^r = (F_X)^r$. More generally, if k is again arbitrary, but if X has an $\mathbb{F}_p$-structure (i.e. there is some $\mathbb{F}_p$-functor X' with $X = (X')_k$), then we can identify $X^{(r)}$ with X. In the affine case one has $k[X] = \mathbb{F}_p[X'] \otimes_{\mathbb{F}_p} k$ and the map $f \otimes a \mapsto f \otimes a^{p^r}$ (for all $f \in \mathbb{F}_p[X']$ and $a \in k$) induces an isomorphism $k[X^{(r)}] = k[X]^{(-r)} \tilde{\rightarrow} k[X]$. (This map is called for $r = 1$ the <u>arithmetic</u> Frobenius endomorphism of $k[X]$.) Taking this identification F_X^r is the endomorphism of X with comorphism $f \otimes a \mapsto f^{p^r} \otimes a$ (for all f, a as above.) This map is called for $r = 1$ the <u>geometric</u> Frobenius endomorphism of $k[X]$.

<u>Remark</u>: It is clear that (4) makes sense not only for our perfect field k but also for any $\mathbb{F}_p$-algebra as only $r \in \mathbb{N}$ appears in that formula. We can also take the interpretation via base change in that situation. It may be left to the reader to find out later on how much generalizes to this case.

<u>9.3</u> (<u>Fibres of the Frobenius Morphism</u>) Let X be an affine

scheme over k. Consider a point $x \in X(k)$ and let us denote its ideal by $I_x = \{f \in k[X] | f(x) = 0\}$. Then the ideal of $F_X^r(x) \in X^{(r)}(k)$ in $k[X^{(r)}] = k[X]^{(-r)}$ is $I_x^{(-r)}$ (i.e. I_x with the new scalar operation) as $f(F_x^r(x)) = f(x)^p$ for all f. This implies (for all $r \in \mathbb{N}$)

$$(1) \quad (F_X^r)^{-1}(F_X^r(x)) = V(\sum_{f \in I_x} k[X]f^{p^r}).$$

So the $(F_X^r)^{-1}(F_X^r(x))$ form an ascending chain of closed subschemes of X.

Suppose now that X is algebraic. Then I_x is a finitely generated ideal, say $I_x = \sum_{i=1}^{m} k[X]f_i$. Then

$$(F_X^r)^{-1}(F_X^r(x)) = V(\sum_{i=1}^{m} k[X]f_i^{p^r})$$

for all r. The ideal defining $(F_X^r)^{-1}(F_X^r(x))$ is contained in $I_x^{p^r}$ and contains $I_x^{mp^r}$. This implies (cf. 7.1, 7.2(2))

$$(2) \quad \mathrm{Dist}(X,x) = \bigcup_{r>0} \mathrm{Dist}((F_X^r)^{-1}(F_X^r(x)),x).$$

We can choose the f_i such that the $f_i+I_x^2$ $(1 \leq i \leq m)$ form a basis of I_x/I_x^2. If x is a simple point of X then $m = \dim_x X$ and the $f_i (1 \leq i \leq m)$ are algebraically independent. Therefore the residue classes of all $f_1^{n(1)} f_2^{n(2)} \dots f_m^{n(m)}$ with all $n(i) < p^r$ form a basis of $k[(F_X^r)^{-1}(F_X^r(x))]$. This shows

(3) <u>If</u> x <u>is a simple point of</u> X, <u>then</u> $\dim k[(F_X^r)^{-1}(F_X^r(x)] = p^{rm}$ <u>where</u> $m = \dim_x X$.

Let us add that (1) generalizes to

(4) $(F_X^r)^{-1}V(I^{(-r)}) = V(\sum_{f\in I} k[X]f^{p^r})$

for all ideals I in $k[X]$ (and any affine X) whereas

(5) $(F_X^r)^{-1}D(I^{(-r)}) = D(I)$.

(Use that $\sqrt{I} = \sqrt{\sum_{f\in I} k[X]f^{p^r}}$ and 1.5(5),(10).)

9.4 (Frobenius Kernels) Let G be a k-group functor. Then obviously each $G^{(r)}$ is also a k-group functor and $F_G^r\colon G \to G^{(r)}$ is a homomorphism of k-group functors. Its kernel $G_r = \ker(F_G^r)$ is a normal subgroup functor of G which we call the r-th Frobenius kernel of G. The factorization in 9.2(6) implies that we get an ascending chain

(1) $G_1 \subset G_2 \subset G_3 \subset \dots$

of normal subgroup functors of G.

If H is a subgroup functor of G, then $H^{(r)}$ is a subgroup functor of $G^{(r)}$ and F_H^r is the restriction of F_G^r to H. This implies

(2) $H_r = H \cap G_r$,

especially for all $r,r' \in \mathbb{N}$

(3) $(G_r)_{r'} = \begin{cases} G_{r'} & \text{for } r' \le r, \\ G_r & \text{for } r' \ge r. \end{cases}$

If $k = \mathbb{F}_p$ or if G is defined over $\mathbb{F}_p$, then we can identify each $G^{(r)}$ with G and interprete F_G^r as the r-th power of some Frobenius endomorphism $F_G: G \to G$ (which is F_G^1 after the identification $G \simeq G^{(1)}$). This is true e.g. for $G = G_a$ and $G = G_m$. In these cases $(F_G)^*(T) = T^p$ in the notations of 2.2. Therefore $G_{m,r} = \mu_{(p^r)}$ for all r and the $G_{a,r}$ from 2.2 are the Frobenius kernels of G_a. (So our new notation is compatible with the old one.)

9.5 Let G be a k-group scheme. The image faisceau (cf. 5.5) of F_G^r in $G^{(r)}$ is isomorphic to G/G_r (by 6.1) as $G_r = \ker(F_G^r)$. For each subgroup scheme H of G we can identify

(1) $F_G^r(H) \simeq F_H^r(H) \simeq H/H_r$

by 9.4(2) and $(F_G^r)^{-1}F_G^r(H)$ with G_rH, cf. 6.2(4).

The factorization $F_G^{r'} = F_{G^{(r)}}^{r'-r}\, F_G^r$ yields

(2) $G_{r'} = (F_G^r)^{-1}((G^{(r)})_{r'-r})$

for all $r' \geq r$.

Proposition: If G is a reduced algebraic k-group, then each F_G^r induces isomorphisms $G/G_r \simeq G^{(r)}$ and $G_{r'}/G_r \simeq (G^{(r)})_{r'-r}$ for all $r' \geq r$.

Proof: By [DG], II, §5, 5.1.b the embedding of $F_G^r(G) \simeq G/G_r$ into $G^{(r)}$ is a closed immersion. Therefore G/G_r is identified with the closed subgroup of $G^{(r)}$ defined by the kernel of the comorphism $(F_G^r)^*: k[G]^{(-r)} \to k[G]$ which maps each f to f^{p^r}

i.e. we get

(3) $F_G^r(G) = V(\{f \in k[G] \mid f^{p^r} = 0\}^{(-r)})$.

If G is reduced, i.e. if $k[G]$ does not contain nilpotent elements $\neq 0$, then obviously $F_r(G) = G^{(r)}$.

As we have shown F_G^r to be an epimorphism of faisceaux, each subfaisceau Y of $G^{(r)}$ is equal to the image faisceau $F_G^r((F_G^r)^{-1}Y)$. Therefore the last claim follows from (1) and (2).

<u>Remark</u>: If G is defined over $\mathbb{F}_p$, we can express the results as $G/G_r \cong G$ and $G_{r'}/G_r \cong G_{r'-r}$.

<u>9.6</u> (Dist(G) <u>and the</u> $\mathrm{Dist}(G_r)$) Let G be an algebraic k-group scheme and I_1 the ideal in $k[G]$ defining 1. Keep this assumption and convention until the end of this chapter. Obviously G_r is the closed subscheme of G defined by $\sum_{f \in I_1} k[G]f^{p^r}$. Therefore $k[G_r]$ is finite dimensional and the ideal of 1 in $k[G_r]$ is nilpotent. Hence (cf. 8.1):

(1) <u>Each G_r is an infinitesimal k-group</u>.

The $\mathrm{Dist}(G_r)$ form because of 9.4(1) an ascending chain of subalgebras of G and one has by 9.3(2):

(2) $\mathrm{Dist}(G) = \bigcup_{r>0} \mathrm{Dist}(G_r)$.

Therefore 7.14 - 7.17 imply, if G is <u>irreducible</u>:

(3) <u>If</u> M <u>is a</u> G-<u>module, then</u> $M^G = \bigcap_{r>0} M^{G_r}$

(4) <u>If</u> M, M' <u>are</u> G-<u>modules</u>, <u>then</u> $\mathrm{Hom}_G(M,M') = \bigcap_{r>0} \mathrm{Hom}_{G_r}(M,M')$.

(5) Let M be a G-module and N a subspace of M. Then N is a G-submodule if and only if it is a G_r-submodule for all $r \in \mathbb{N}$.

In (3) and (4) we have descending chains $M^{G_1} \supset M^{G_2} \supset M^{G_3} \supset \ldots$ and $\mathrm{Hom}_{G_1}(M,M') \supset \mathrm{Hom}_{G_2}(M,M') \supset \mathrm{Hom}_{G_3}(M,M') \supset \ldots$ If $\dim M < \infty$ resp. if $\dim M \otimes M' < \infty$, then these chains have to stabilize. So we get (still for G irreducible):

(6) If M is a G-module with $\dim M < \infty$, then there is an $n \in \mathbb{N}$ with $M^G = M^{G_r}$ for all $r > n$.

(7) If M, M' are G-modules with $\dim(M \otimes M') < \infty$, then there is an $n \in \mathbb{N}$ with $\mathrm{Hom}_G(M,M') = \mathrm{Hom}_{G_r}(M,M')$ for all $r > n$.

9.7 (Lie(G) and G_1) Choose $f_1, \ldots, f_m \in I_1$ such that the $f_i + I_1^2$ form a basis of I_1/I_1^2. Then $m = \dim \mathrm{Lie}(G)$ and the f_i generate I_1 as an ideal. One has obviously $\dim k[G_r] \leq p^{rm}$ for all r, and equality holds, if 1 is a simple point of G (cf. 9.3(3)). So we get (e.g. by [DG], II, §5, 2.1/3)

(1) If G is reduced, then $\dim k[G_r] = p^{r \dim(G)}$ for all $r \in \mathbb{N}$.

We have obviously for all $r \in \mathbb{N}$ (and any G)

(2) $\mathrm{Lie}(G_r) = \mathrm{Lie}\ G$.

The subalgebra $U^{[p]}(\mathrm{Lie}(G)) = U^{[p]}(\mathrm{Lie}(G_1))$ of $\mathrm{Dist}(G_1) \subset \mathrm{Dist}(G)$, cf. 7.10(2), has dimension p^m, whereas $\dim \mathrm{Dist}(G_1) = \dim k[G_1] \leq p^m$. This implies

(3) $U^{[p]}(\mathrm{Lie}(G)) \cong \mathrm{Dist}(G_1)$.

This shows that G_1 is the infinitesimal k-group corresponding to the p-Lie algebra $\mathrm{Lie}(G)$ as in 8.5.b and that the representation theory of G_1 is equivalent to that of $\mathrm{Lie}(G)$ as a p-Lie algebra (cf. 8.6).

9.8 Proposition: Let G be a reduced algebraic k-group and $r \in \mathbb{N}$. Then G operates on $\mathrm{Dist}(G_r)_\ell^{G_r}$ through the character

$$g \mapsto \det(\mathrm{Ad}(g))^{p^r-1}$$

where Ad denotes the adjoint representation of G on $\mathrm{Lie}(G)$.

Proof: Recall from 8.19 that the conjugation action of G on G_r leads to representations of G on $k[G_r]$ and $M(G_r) = \mathrm{Dist}(G_r)$ and that $M(G_r)_\ell^{G_r}$ is a one dimensional submodule on which G has to operate through some character $\chi \in X(G)$.

Set $q = p^r_2$ and choose $f_1, \ldots, f_m \in I_1$ such that the $f_i + I_1^2$ form a basis of I_1/I_1^2. Let $\bar{f}_i$ be the image of f_i in $k[G_r]$. As G is reduced, hence 1 a simple point, the monomials $\bar{f}_1^{a(1)} \bar{f}_2^{a(2)} \ldots \bar{f}_m^{a(m)}$ with $0 \leq a(i) < q$ for all i form a basis of $k[G_r]$.

We can identify $k[G_r]$ with the factor ring $k[T_1, \ldots, T_m]/(T_1^q, \ldots, T_m^q)$ of the polynomial ring $k[T_1, \ldots, T_m]$. It is therefore a graded ring in a natural way. Any endomorphism φ of the vector space $\sum_{i=1}^{m} k\bar{f}_i$ induces an endomorphism of the graded algebra $k[G_r]$. As $F = \prod_{i=1}^{m} \bar{f}_i^{q-1}$ is the only basis element of degree $m(q-1)$ it has to be mapped under φ into a multiple $c(\varphi)F$ of itself. Obviously $\varphi \mapsto c(\varphi)$ has to be multiplicative. This implies $c(\varphi) = \det(\varphi)^{q-1}$ for all φ as this is obviously true for φ in upper or lower triangular form (with respect to the $\bar{f}_i$), hence for all φ by multiplicativity. This extends easily to any k-algebra A and any endomorphism of $\sum_{i=1}^{m} k\bar{f}_i \otimes A$ as $c(\varphi)$ is obviously a polynomial in the matrix coefficients of φ.

This can be applied especially to the operation of any $g \in G(A)$ for any k-algebra A on $k[G_r] \otimes A$ derived from the conjugation action on G_r. Then the action of g on $\sum_{i=1}^{m} k\bar{f}_i \otimes A \simeq (I_1/I_1^2) \otimes A \simeq \mathrm{Lie}(G)^* \otimes A$ is dual to the adjoint action on $\mathrm{Lie}(G) \otimes A$, hence has determinant equal to $\det(\mathrm{Ad}(g))^{-1}$. So this implies

$$gF = \det(\mathrm{Ad}(g))^{-(q-1)}F.$$

Consider now $\mu_0 \in \mathrm{Dist}(G_r)_\ell^{G_r}$, $\mu_0 \neq 0$. If $\mu_0(F) = 0$, then $\mu_0(k[G_r]F) = 0$ as $k[G_r]F = kF$, hence $(k[G_r]\mu_0)(F) = 0$ by the definition of the $k[G_r]$-module structure on $\mathrm{Dist}(G_r)$ in 8.11, hence $\mathrm{Dist}(G_r)(F) = 0$ by 8.12. This is a contradiction, so we must have $\mu_0(F) \neq 0$. Then

$$\chi(g)\mu_0(F) = (g\mu_0)(F) = \mu_0(g^{-1}F)$$
$$= \det(\mathrm{Ad}(g))^{q-1}\mu_0(F)$$

implies $\chi(g) = \det(\mathrm{Ad}(g))^{q-1}$ as $\mu_0(F)$ is a unit in A.

<u>Remark</u>: The same proof works for any algebraic k-group G and for $r = 1$ because of 9.7(3). So we can take any p-Lie algebra $\underline{g}$ over k and consider the infinitesimal k-group G corresponding to G as in 8.5.b. Then $G = G_1$ and $\mathrm{Dist}(G) = U^{[p]}(\underline{g})$. Then the proposition implies that the modular function δ_G is given by $\delta_G(g) = \det(\mathrm{Ad}(g))^{p-1}$. The representation of $\underline{g}$ on $\mathrm{Dist}(G)_\ell^G$ is then given by the differential, i.e. by $(p-1)\mathrm{tr}(\mathrm{ad}(?)) = -\mathrm{tr}(\mathrm{ad}(?))$. As the operation of $\underline{g}$ determines that of G in this case, we see that G is unimodular if and only if $\mathrm{tr}(\mathrm{ad}(x)) = 0$ for all $x \in \underline{g}$. This is a theorem of Larson and Sweedler, cf. the discussion in [Humphreys 9].

<u>9.9</u> (<u>Frobenius Twists of Representations</u>)

Let M be a G-module. We can define for each $r \in \mathbb{N}$ a new G-module which we denote by $M^{(r)}$ and call the r-th <u>Frobenius twist</u> of M. We set $M^{(r)}$ as a group equal to M and make it into a vector space over k by letting each $a \in k$ operate on $M^{(r)}$ as $a^{p^{-r}}$ does on M. (This convention is certainly awkward as 9.2 suggests that we should call it $M^{(-r)}$. Still in the context of representations the present notation is more useful, and we shall always be careful whether we deal with k-algebras or G-modules.)

For each k-algebra A there is a semi-linear map $\gamma_A\colon M \otimes A \to M^{(r)} \otimes A$ with $\gamma_A(m \otimes a) = m \otimes a^{p^r}$ for all $m \in M$, $a \in A$. If $(m_i)_{i \in I}$ is a basis of M, then the basis $(m_i \otimes 1)_{i \in I}$ of the A-module M is mapped to a basis of the A-module $M^{(r)} \otimes A$.

Therefore for each $\varphi \in \mathrm{End}_A(M \otimes A)$ there is a unique $\varphi' \in \mathrm{End}_A(M^{(r)} \otimes A)$ with $\varphi' \circ \gamma_A = \gamma_A \circ \varphi$. The map $\mathrm{End}_A(M \otimes A) \to \mathrm{End}_A(M^{(r)} \otimes A)$, $\varphi \mapsto \varphi'$ is semi-linear, compatible with the composition of maps, and functorial in A. The given representation of G von M yields for each A a homomorphism

$$G(A) \to \mathrm{End}_A(M \otimes A)^\times \to \mathrm{End}_A(M^{(r)} \otimes A)^\times$$

which is functorial in A, hence a G-module structure. This is the twisted module we wanted to define. If $(m_i)_{i \in I}$ is a basis of M as above and if $g \in G(A)$ has the matrix $(a_{ij})_{i,j \in I}$ with respect to $(m_i \otimes 1)_{i \in I}$, then g has the matrix $(a_{ij}^{p^r})_{i,j \in I}$ with respect to the corresponding basis of $M^{(r)} \otimes A$. (This is one reason for the notation $M^{(r)}$ instead of $M^{(-r)}$.) If $\Delta_M(m) = \Sigma m_i \otimes f_i$ for some $m \in M$, then $\Delta_{M^{(r)}}(m) = \Sigma m_i \otimes f_i^{p^r}$.

Suppose now that M has a fixed $\mathbb{F}_p$-structure, i.e. an $\mathbb{F}_p$-subspace $M' \subset M$ with $M' \otimes_{\mathbb{F}_p} k = M$. We get then a Frobenius endomorphism F_M on M and on each $M \otimes A = M' \otimes_{\mathbb{F}_p} A$ through $F_M(m \otimes a) = m \otimes a^p$. Then each F_M^r is an isomorphism of A-modules $M \otimes A \to M^{(r)} \otimes A$. Suppose that G is defined over $\mathbb{F}_p$ and denote the corresponding Frobenius endomorphism by $F_G\colon G \to G$. If the representation of G on M is defined over $\mathbb{F}_p$ (i.e. if $F_G(g)F_M(m) = F_M(gm)$ for all $m \in M$, $g \in G(A)$), then we can define a new representation of G on M by composing the given $G \to GL(M)$ with $F_G^r\colon G \to G$. Then $F_M^{(r)}\colon M \to M^{(r)}$ is an isomorphism of G-modules if we take the new structure on M just defined and on $M^{(r)}$ as above. (This follows from an elementary computation.)

Note that our definitions are compatible with 4.27.

9.10 (The Associated Graded Group) The powers of I_1 define a filtration of $k[G]$ and we can form the associated graded algebra $\operatorname{gr} k[G] = \bigoplus_{n\geq 0} I_1^n/I_1^{n+1}$. There is obviously a surjection from the symmetric algebra $S(I_1/I_1^2)$ onto $\operatorname{gr} k[G]$ compatible with the grading.

The formulas 2.4(1),(2) show that Δ_G and σ_G induce also a comultiplication and an antipode on $\operatorname{gr} k[G]$ making (together with the obvious augmentation) $\operatorname{gr} k[G]$ into a (commutative and cocommutative) Hopf algebra. So there is a k-group scheme $\operatorname{gr}(G)$ with $\operatorname{gr} k[G] \simeq k[\operatorname{gr}(G)]$ (the associated graded group).

We can interprete $S(I_1/I_1^2)$ as $k[((I_1/I_1^2)^*)_a] = k[(\operatorname{Lie} G)_a]$. Then the surjection $S(I_1/I_1^2) \to \operatorname{gr} k[G] = k[\operatorname{gr}(G)]$ is compatible with the Hopf algebra structure (again because of 2.4(1),(2)). Thus:

(1) $\operatorname{gr}(G)$ is canonically isomorphic to a closed subgroup scheme of $\operatorname{Lie}(G)_a$.

The same arguments as in 9.7(1) imply

(2) If G is reduced, then $\operatorname{gr}(G) \simeq \operatorname{Lie}(G)_a$

and

(3) <u>If</u> G <u>is reduced, then</u> $gr(G_r) \simeq (Lie(G)_a)_r$ <u>for all</u> $r \in \mathbb{N}$.

<u>9.11</u> (<u>A Filtration of the Hochschild Complex</u>) The filtration of $k[G]$ as in 9.10 leads also to a filtration of the Hochschild complex $C^\bullet(G,M)$ for each G-module M. We set for all $i,n \in \mathbb{N}$

(1) $C^i(G,M)_{(n)} = \Sigma M \otimes I_1^{a(1)} \otimes I_1^{a(2)} \otimes \ldots \otimes I_1^{a(i)}$

where we sum over all i-tuples $(a(1),\ldots,a(i)) \in \mathbb{N}^i$ with $\sum_j a(j) \geqslant n$. Because of 2.4(1),(2) and as $\Delta_M(m)-m \otimes 1 \in M \otimes I_1$ for all $m \in M$ the definition of the coboundary operators in 4.14 shows

(2) $\partial^i C^i(G,M)_{(n)} \subset C^{i+1}(G,M)_{(n)}$

for all i and n.

Each quotient $C^i(G,M)_{(n)}/C^{i+1}(G,M)_{(n+1)}$ can be identified with the direct sum of all

$$M \otimes (I_1^{a(1)}/I_1^{a(1)+1}) \otimes \ldots \otimes (I_1^{a(i)}/I_1^{a(i)+1})$$

over all i-tuples $(a(1),\ldots,a(i))$ with $\sum_{i=1}^n a(i) = n$. We can on the other hand regard M as a trivial $gr(G)$-module and form $C^\bullet(gr(G),M)$. The grading on $k[gr(G)]$ leads in a natural way (cf. 4.20) to a grading on each $C^i(gr(G),M)$. We denote the homogeneous part of degree n by $C^i(gr(G),M)_n$. Then

(3) $C^i(G,M)_{(n)}/C^{i+1}(G,M)_{(n+1)} \simeq C^i(gr(G),M)_n$

for all i,n. These identifications are easily checked to be compatible with the boundary operators so that the associated graded complex of $C^\bullet(G,M)$ is isomorphic to the graded complex $C^\bullet(\mathrm{gr}(G),M) = C^\bullet(\mathrm{gr}(G),k)\otimes M$.

The general theory about filtered complexes (consult e.g. [7], I.4) shows that there is a spectral sequence with E_1-terms $E_1^{i,j} = H^{i+j}(\mathrm{gr}\ G,k)_j \otimes M$. If G is irreducible, then $\bigcap_{n>0} I_1^{n+1} = 0$, hence $\bigcap_{n>0} C^i(G,M)_{(n)} = 0$ for all i. Therefore in this case the spectral sequence converges to the cohomology of the original complex.

<u>9.12</u> <u>Proposition</u>: <u>Suppose</u> G <u>is irreducible. Then there is for each</u> G-<u>module</u> M <u>a spectral sequence with</u>

(1) $E_1^{i,j} = H^{i+j}(\mathrm{gr}(G),k)_i \otimes M \Longrightarrow H^{i+j}(G,M)$.

This is what we proved in the last section. Let us add that the spectral sequence is compatible with the cup-product in case $M = k$ resp. with the $H^\bullet(G,k)$-module structure on $H^\bullet(G,M)$ in the general case.

If some other group H operates on G through group automorphisms, then it operates on $C^\bullet(G,k)$ preserving the filtration. Then we get a natural action of H on each term of the spectral sequence such that all differentials are homomorphisms of H-modules. Also the filtration on the abutment is compatible with the action of H. This generalizes to an arbitrary G-module M if we have also an operation of H on M compatible with the operation of G (i.e. defining a representation of $G\rtimes H$).

9.13 Proposition: Let G be reduced and irreducible. Set $\mathfrak{g} = \mathrm{Lie}(G)$.

a) There is for each G-module M a spectral converging to $H^\bullet(G,M)$ with the following E_1-terms:

If $p \neq 2$, then

$$E_1^{i,j} = \oplus\, M \otimes (S^{a(1)}\mathfrak{g}^*)^{(1)} \otimes (S^{a(2)}\mathfrak{g}^*)^{(2)} \otimes \cdots$$
$$\otimes \Lambda^{b(1)}\mathfrak{g}^* \otimes (\Lambda^{b(2)}\mathfrak{g}^*)^{(1)} \quad \cdots$$

where we sum over all finite sequences $(a(n))_{n\geq 1}$ and $(b(n))_{n\geq 1}$ in $\mathbb{N}$ with

$$i+j = \sum_{n\geq 1} (2a(n)+b(n)) \quad \text{and} \quad i = \sum_{n\geq 1} (a(n)p^n + b(n)p^{n-1}).$$

If $p = 2$, then

$$E_1^{i,j} = \oplus\, M \otimes S^{a(1)}\mathfrak{g}^* \otimes (S^{a(2)}\mathfrak{g}^*)^{(1)} \otimes \cdots$$

where we sum over all finite sequences $(a(n))_{n>1}$ in $\mathbb{N}$ with

$$i+j = \sum_{n\geq 1} a(n) \quad \text{and} \quad i = \sum_{n\geq 1} a(n)p^{n-1}.$$

b) Let $r \in \mathbb{N}$ and M be a G_r-module. If we take above only r-tuples $(a(n))_{1\leq n\leq r}$ and (for $p \neq 2$) $(b(n))_{1\leq n\leq r}$ then we get the $E_1^{i,j}$-terms in a spectral sequence converging to $H^\bullet(G_r,M)$.

Proof: This follows from 9.12 and 4.27 using 9.10(2),(3).

Remark: Again these spectral sequences are compatible with the operation of some group H on G or G_r through automorphisms if H operates also on M in a compatible way (e.g. always for the trivial module $M = k$). This follows from the fact that H then operates on $gr(G) = \underline{g}_a$ or $gr(G_r) = (\underline{g}_a)_r$ through a representation on $\underline{g}$ so that the isomorphisms in 4.27 are compatible with the action of H. (This applies especially to the operation of G on G_r through conjugation.) The (r) denotes a twist of the operation of H as in 9.9.

9.14 The spectral sequence in 9.13.b is especially easy for $r = 1$.

Lemma: If $p = 2$, then we can compute $H^{\cdot}(G_1,M)$ for any G_1-module M as the cohomology of a complex

$$0 \to M \to M \otimes \underline{g}^* \to M \otimes S^2\underline{g}^* \to M \otimes S^3\underline{g}^* \to \dots$$

where $\underline{g} = \mathrm{Lie}(G)$.

Proof: We have by 9.13 that $M \otimes S^i\underline{g} = E_1^{i,0}$ whereas $E_1^{i,j} = 0$ for $j \neq 0$ or $i < 0$. So the only non-zero differentials in the spectral sequence are $d_1^{i,0}$: $E_1^{i,0} \to E_1^{i+1,0}$. They provide the maps in the complex and its cohomology groups $E_2^{i,0}$ are equal to its abutment.

Remark: Note that we do not have to assume G to be reduced and irreducible when dealing with G_1 (here and below.) The assumption of irreducibility is needed to make the spectral sequence in 9.12 converge to the G-homology. As each G_r is irreducible we do not need the irreducibility of G in 9.13.b. The assumption of reducedness was needed to get 9.10(3). But we have $gr(G_1) = (\mathrm{Lie}(G)_a)_1$ for and G by 9.7(3).

9.15 Lemma: Let M be a G_1-module and set $\underline{g} = \mathrm{Lie}(G)$. If $p \neq 2$, then there is a spectral sequence with

$$E_0^{i,j} = M \otimes (S^i \underline{g}^*)^{(1)} \otimes \Lambda^{j-i} \underline{g}^* \Longrightarrow H^{i+j}(G_1, M)$$

Proof: We have in 9.13.b as E_1-terms $E_1^{a(p-1)+b,-(p-2)a} = M \otimes (S^a \underline{g}^*)^{(1)} \otimes \Lambda^{b-a} \underline{g}^*$ for all $b \geq a \geq 0$ and all other $E_1^{i,j}$ are 0. So $E_1^{i,j} = 0$ for $(p-2) \nmid j$, hence $d_r^{i,j} = 0$ for $r \not\equiv 1 \mod (p-2)$ as d_r has bidegre $(r, 1-r)$. We can therefore re-index the spectral sequence by calling now $E_r^{i,j}$ the old $E_{(p-2)r+1}^{(p-1)i+j,-(p-2)i}$. This gives then $E_0^{i,j}$ as above.

9.16 (Lie Algebra Cohomology) In order to compute the E_1-terms of the spectral sequence from 9.15 it will be necessary to deal with (ordinary) Lie algebra cohomology (cf. e.g. [4], ch. I, §3, exerc. 12).

If $\underline{g}$ is a finite dimensional Lie algebra over any field and if M is a $\underline{g}$-module, then the Lie algebra cohomology $H^\bullet(g, M)$ of M can be computed using complex $M \otimes \Lambda \underline{g}^*$ where we take the standard grading of $\Lambda \underline{g}^*$. The map $d_0 \colon M \to M \otimes \underline{g}^*$ maps any $m \in M$ to the unique element $\sum_{j=1}^{r} m_j \otimes \varphi_j \in M \otimes \underline{g}^*$ with $xm = \sum_{j=1}^{r} \varphi_j(x) m_j$ for all $x \in g$. (It is something like a comodule map.) In general one has for any $m \in M$ and $\psi \in \Lambda^i \underline{g}^*$

(1) $d_i(m \otimes \psi) = \sum_j m_j \otimes (\varphi_j \wedge \psi) + m \otimes d_i'(\psi)$

with m_j, φ_j as above and where $d_i' : \Lambda^i \underline{g}^* \to \Lambda^{i+1}\underline{g}^*$ is the boundary operator in the case of the trivial module. This in turn is uniquely determined by $d_1' : \underline{g}^* \to \Lambda^2\underline{g}^* \simeq (\Lambda^2\underline{g})^*$ which is the transposed of $\Lambda^2\underline{g} \to \underline{g}$, $x\wedge y \mapsto -[x,y]$ and by the derivation property

(2) $d_{i+j}'(\varphi\wedge\psi) = d_i'(\varphi)\wedge\psi + (-1)^i \varphi\wedge d_j'(\psi)$

for all $\varphi \in \Lambda^i\underline{g}^*$ and $\psi \in \Lambda^j\underline{g}^*$.

<u>9.17</u> <u>Lemma</u>: <u>Let</u> M <u>be a</u> G_1<u>-module and set</u> $\underline{g} = \mathrm{Lie}(G)$. <u>Suppose</u> $p \neq 2$. <u>Then one has in</u> 9.15

$$E_1^{0,j} = H^j(\underline{g},M) \qquad \underline{\text{for all}} \;\; j \in \mathbb{N}.$$

<u>Proof</u>: We have $E_0^{0,j} = M \otimes \Lambda^j\underline{g}^*$ and $d_0^{0,j}$ maps $M \otimes \Lambda^j\underline{g}^*$ to $M \otimes \Lambda^{j+1}\underline{g}^*$ for all $j \in \mathbb{N}$. So we have to show that the complex $(E_0^{0,\bullet}, d_0^{0,\bullet})$ is the same as the one computing the Lie algebra cohomology.

The compatibility of the spectral sequence with the cup-product in the case $k = M$ and with corresponding module structures in general implies that the $d_0^{0,i}$ have derivation properties analogous to 9.16(1),(2). It is therefore enough to prove that $d_0^{0,0} : M \to M \otimes \underline{g}^*$ and $d_0^{0,1} : \underline{g}^* \to \Lambda^2\underline{g}^*$ in the case $M = k$ are the same maps as in 9.16.

In the original notation of 9.13 our present $E_0^{0,i}$ was called $E_1^{i,0}$ and arose as a subquotient of $C^i(G_1,M)_{(i)}/C^i(G_1,M)_{(i+1)}$. Any $e \in E_0^{0,i}$ has a representative $\tilde{e} \in C^i(G_1,M)_{(i)}$ with $\partial^i\tilde{e} \in C^{i+1}(G_1,M)_{(i+1)}$ and $d_0^{0,i}(e)$ is the class of $\partial^i\tilde{e}$ in the subquotient $E_0^{0,i+1}$ of $C^{i+1}(G_1,M)_{(i+1)}/C^{i+1}(G_1,M)_{(i+2)}$.

In the case $i = 0$ we have $\partial^0\colon M \to M\otimes k[G_1]$, $m \mapsto \Delta_M(m) - m\otimes 1$. We can write $\Delta_M(m) = m\otimes 1 + \sum_{i=1}^{s} m_i \otimes f_i$ where $f_i \in I_1 = \{f \in k[G_1] \mid f(1) = 0\}$. We have $C^0(G_1,M)_{(n)} = M$ and $C^1(G_1,M)_{(n)} = M\otimes I_1^n$ for all $n \in \mathbb{N}$, hence $E_0^{0,1} = M\otimes I_1/I_1^2 = C^1(G_1,M)_{(1)}/C^1(G_1,M)_{(2)}$. Therefore $d_0^{0,0}(m) = \sum_{i=1}^{s} m_i \otimes \bar{f}_i$ where $\bar{f}_i = f_i+I_1^2$. The operation of any $x \in \underline{g} = (I_1/I_1^2)^*$ is given by $xm = \sum_{i=1}^{s} \bar{f}_i(x)m_i$. This shows that $d_0^{0,0}$ is the same map as in 9.16.

Take now $M = k$ and consider $d_0^{0,1}$. It maps $\underline{g}^* = I_1/I_1^2 = C^1(G_1,k)_{(1)}/C^1(G_1,k)_{(2)}$ into a subquotient of $C^2(G_1,k)_{(2)}/C^2(G_1,k)_{(3)}$. For any $f \in I_1$ we can write $\Delta_G(f) = 1\otimes f + f\otimes 1 + \sum_{i=1}^{s} f_i \otimes f_i'$ with $f_i, f_i' \in I_1$, cf. 2.4(1). Then $\partial^1 f = \sum_{i=1}^{s} f_i \otimes f_i'$. So $\bar{f} = f+I_1^2 \in I_1/I_1^2 = E_0^{0,1}$ is mapped to the class of $-\sum_{i=1}^{s} f_i \otimes f_i'$ in the subquotient $H^2(\mathrm{gr}\, G_1,k)_2$ of $C^2(G_1,k)_{(2)}/C^2(G_1,k)_{(3)} = C^2(\mathrm{gr}\, G_1,k)_2$. By the definition of the cup product this is the sum of the products of $\bar{f}_i = f_i+I_1^2$ and $\bar{f}_i' = f_i'+I_1^2$ in $H^\bullet(\mathrm{gr}\, G_1,k)$. It belongs to the subalgebra generated by $H^1(\mathrm{gr}\ G_1,k) \cong \mathfrak{g}^*$ which is identified with $\Lambda\mathfrak{g}^*$. So $d_O^{O,1}\bar{f} = -\sum_{i=1}^{s} \bar{f}_i \wedge \bar{f}_i'$. As the Lie algebra structure on $(I_1/I_1^2)^* = \mathrm{Dist}_1^+(G)$ is defined through $[x,y] = (x \otimes y - y \otimes x) \circ \Delta_G$ we see that $d_O^{O,1}$ is transposed to $x \wedge y \mapsto [x,y]$ as claimed.

Remark: Notice that this computation gives also the boundary maps in the complex of lemma 9.14.

9.18 (Ordinary and Restricted Cohomology) If M, M' are G_1-modules, then we can interprete each $\mathrm{Ext}^i_{G_1}(M',M)$ resp. $\mathrm{Ext}^i_{\mathfrak{g}}(M',M)$ as set of equivalence classes of exact sequences

$$O \to M \to M_1 \to M_2 \to \dots \to M_i \to M' \to O$$

of homomorphisms of G_1-modules (resp. $\mathfrak{g}$-modules). So we have a natural map $\mathrm{Ext}^i_{G_1}(M',M) \to \mathrm{Ext}^i_{\mathfrak{g}}(M',M)$. Taking $M' = k$ we get a natural map $H^i(G_1,M) \to H^i(\mathfrak{g},M)$. Let us describe this explicitly for $i = 1$.

Each 1-cocycle $\varphi\colon \mathfrak{g} \to M$ defines an extension of $\mathfrak{g}$-modules $O \to M \to M(\varphi) \to k \to O$ where $M(\varphi) = M \oplus k$ as vector space with $x \in \mathfrak{g}$ operating through $x(m,a) = (xm + a\varphi(x), O)$ for all $a \in k$ and $m \in M$. One checks easily that this is an extension of G_1-modules if and only if $\varphi(x^{[p]}) = x^{p-1}\varphi(x)$ for all $x \in \mathfrak{g}$. This equation is certainly satisfied, if φ is a coboundary, i.e. of the form $x \mapsto xm$ for some $m \in M$. So we get an embedding $H^1(G_1,M) \hookrightarrow H^1(\mathfrak{g},M)$. More precisely the image is exactly the kernel of the map associating to the class of φ as above (in

$H^1(\underline{g},M))$ the map $x \mapsto \varphi(x^{[p]}) - x^{p-1}\varphi(x)$ from $\underline{g}$ to M. This map is semilinear, i.e. it is additive and satisfies $\varphi(ax) = a^p\varphi(x)$ for all $a \in k$ and $x \in \underline{g}$. Let $\mathrm{Hom}^s(\underline{g},M)$ be the space of all such maps. We have so far constructed an exact sequence

$$0 \to H^1(G_1,M) \to H^1(\underline{g},M) \to \mathrm{Hom}^s(\underline{g},M).$$

We can be more precise. An elementary computation using the cocycle property $\varphi([x,y]) = x\varphi(y)-y\varphi(x)$ for all $x,y \in \underline{g}$ shows $\varphi(x^{[p]})-x^{p-1}\varphi(x) \in M^{\underline{g}}$ for all $x \in \underline{g}$. So we can replace $\mathrm{Hom}^s(\underline{g},M)$ by $\mathrm{Hom}^s(\underline{g},M^{\underline{g}})$. We can now go on and associate to any $\psi \in \mathrm{Hom}^s(\underline{g},M^{g})$ a p-Lie algebra $\underline{g}(\psi)$ which is an extension

$$0 \to M \to \underline{g}(\psi) \to \underline{g} \to 0$$

of p-Lie algebras, where we regard M as a commutative p-Lie algebra with $m^{[p]} = 0$ for all $m \in M$. We take $\underline{g}(\psi) = M \oplus \underline{g}$ with Lie bracket $[(m,x),(m',x')] = (xm'-x'm,[x,x'])$ and p-th power $(m,x)^{[p]} = (x^{p-1}m+\psi(x^{[p]}),x^{[p]})$ for all $m,m' \in M$ and $x,x' \in \underline{g}$. (It may be left to the reader to check that this is indeed a p-th power map on the semi-direct product.)

Now $\underline{g}(\psi)$ and $\underline{g}(0)$ are equivalent extensions if and only if there is an isomorphism $\underline{g}(0) \to \underline{g}(\psi)$ of p-Lie algebras of the form $(m,x) \longmapsto (m+\varphi(x),x)$ for some $\varphi \in \mathrm{Hom}(\underline{g},M)$. Such a map is a homomorphism of Lie algebras, if and only if φ is a 1-cocycle, and it is compatible with the p-th power map, if and only if $\psi(x) = \varphi(x^{[p]})-x^{p-1}\varphi(x)$ for all $x \in \underline{g}$. So $\underline{g}(\psi),\underline{g}(0)$ are equivalent if and only if ψ is in the image of $H^1(\underline{g},M) \to \mathrm{Hom}^s(\underline{g},M)$.

The set of all equivalence classes of all central extensions of p-Lie algebras (resp. of Lie algebras) $0 \to M \to \underline{h} \to \underline{g} \to 0$ such that the adjoint operation of $\underline{g} \simeq \underline{h}/M$ on M is the given operation, is a vector space in a natural way with $\underline{g}(0)$ as zero. One can identify this group with $H^2(G_1,M)$ resp $H^2(\underline{g},M)$ and one can show that the map $\psi \mapsto \underline{g}(\psi)$ induces a linear map $\mathrm{Hom}^s(\underline{g},M^{\underline{g}}) \to H^2(G_1,M)$. One can furthermore show that the image is exactly the kernel of the forgetful map $H^2(G_1,M) \to H^2(\underline{g},M)$. In this way we get an exact sequence

$$
\begin{aligned}
(1)\quad & 0 \to H^1(G_1,M) \to H^1(\underline{g},M) \to \mathrm{Hom}^s(\underline{g},M^{\underline{g}}) \\
& \to H^2(G_2,M) \to H^2(\underline{g},M) \to \mathrm{Hom}^s(\underline{g},H^1(\underline{g},M))
\end{aligned}
$$

where I want to refer to the original proof in [Hochschild 3] (cf. p. 575) for the last map and the exactness at the last two places to be looked at. We shall construct an exact sequence in 9.19(1) which will contain the same terms as (1) and ought to be isomorphic to (1). In order to prove that all terms are the same in both sequences, we need (1) in a special case:

(2) <u>If</u> M <u>is an injective</u> G_1<u>-module, then the canonical map</u> $H^1(\underline{g},M) \to \mathrm{Hom}^s(\underline{g},M^{\underline{g}})$ <u>is an isomorphism</u>.

<u>9.19</u> <u>Proposition</u>: <u>The spectral sequence in</u> 9.15 <u>has the following</u> E_1<u>-terms</u>:

$$E_1^{i,j} \simeq H^{j-i}(\underline{g},M) \otimes (S^i\underline{g}^*)^{(1)}.$$

<u>Proof</u>: The derivation property of the differential

$$d_0^{i,j}: M \otimes \Lambda^{j-i}\underline{g}^* \otimes (S^i\underline{g}^*)^{(1)} \to M \otimes \Lambda^{j-i+1}\underline{g}^* \otimes (S^i\underline{g}^*)^{(1)}$$

implies $d_0^{i,j}(m \otimes \varphi \otimes \psi) = d_0^{0,j-i}(m \otimes \varphi) \otimes \psi + (m \otimes \varphi \otimes 1)$ $(1 \otimes d_{0,k}^{i,i}(\psi))$ for all $m \in M$, $\varphi \in \Lambda^{j-i}\underline{g}^*$ and $\psi \in (S^i\underline{g})^{(1)}$ where $d_{0,k}^{i,i}$ is the differential in the case $M = k$. Therefore it is by 9.17 enough to show $d_{0,k}^{i,i} = 0$ for all i. Again the derivation property shows that it is enough to show $d_{0,k}^{1,1} = 0$.

We know from 9.15 that $E_r^{i,j} \neq 0$ implies $j \geq i \geq 0$. As d_r has bidegree $(r,1-r)$ this shows $E_2^{i,j} = E_\infty^{i,j}$ for all $(i,j) \in \{(0,1),(0,2),(1,1)\}$ and $E_2^{1,1} = E_1^{1,1}/\mathrm{im}(d_1^{0,1})$, $E_2^{0,1} = \ker(d_1^{0,1}) \subset E_1^{0,1}$ and $E_2^{0,2} = \ker(d_1^{0,2}) \subset E_1^{0,2}$. We see also that $E_\infty^{0,1} = H^1(G_1,M)$ and that there is an exact sequence $0 \to E_\infty^{1,1} \to H^2(G_1,M) \to E_\infty^{0,2} \to 0$. Combining this with 9.17 we get a six-term-exact-sequence

$$(1) \quad 0 \to H^1(G,M) \to H^1(\underline{g},M) \to E_1^{1,1}$$
$$\to H^2(G_1,M) \to H^2(\underline{g},M) \to E_1^{1,2}.$$

Here $E_1^{1,1} = \ker(d_0^{1,1}) \subset E_0^{1,1} = M \otimes \underline{g}^{*(1)}$.

Take now an injective G_1-module M_1 with $M_1^{\underline{g}} = k$, e.g. the injective hull of k or $k[G_1]$ with the left or right regular representation. Now 9.18(2) implies $H^1(\underline{g},M_1) \simeq \mathrm{Hom}^s(\underline{g},M_1^{\underline{g}}) \simeq \underline{g}^{*(1)}$ and that (because of the naturality of the maps) the inclusion of k into M_1 induces a surjection $H^1(\underline{g},k) \to H^1(\underline{g},M_1)$, hence (by the naturality of (1)) a surjection of $E_1^{1,1}$ for $M = k$

to $E_1^{1,1}$ for $M = M_1$, hence to $H^1(\underline{g},M_1) \cong \underline{g}^{*(1)}$. But $E_1^{1,1}$ for $M = k$ is equal to $\ker(d_{0,k}^{1,1}) \subset \underline{g}^{*(1)}$, so dimension considerations show $d_{0,k}^{1,1} = 0$.

10. Reduction mod p

Let G be a group scheme over $\mathbb{Z}$. If V is a finite dimensional $G_{\mathbb{Q}}$-module, then we can find a G-module $V_{\mathbb{Z}}$ with $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong V$, cf. 10.3. We can then form the G_k-module $V_k = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ for any ring k. If p is a prime number and $k = \mathbb{F}_p$ (or k = an algebraic closure of $\mathbb{F}_p$), then we say that we get V_k from V through reduction mod p.

In general there will be more than one module (even up to isomorphism) which we can get from V through reduction mod p, as we can choose different $V_{\mathbb{Z}}$. One can still show that they have the same composition factors. (One can express this in the form that the class of V_k in the Grothendieck group of G_k is uniquely determined by V.) This independence was proved in [Serre] generalizing the corresponding statement for abstract finite groups due to Brauer. One can even generalize Brauer's lifting of idempotents. So every injective indecomposable G_k-module lifts to the p-adic completion of $\mathbb{Z}$. Furthermore then Brauer's reciprocity law holds in this situation. These results were proved in [Green 1], and we follow Green's approach here.

We can replace $\mathbb{Z}$ above by any Dedekind ring R and $\mathbb{F}_p$ by any residue field of R. Then all the results will still hold and we do everything in this generality. (Therefore the term "mod p" occurs only in the title and the introduction of this chapter.)

10.1 (Restriction of Scalars) Let k' be a k-algebra and G a k-group functor. We observed in 2.7(6) that any G-module M leads in a natural way to a $G_{k'}$-module $M \otimes k'$: For each k'-algebra A' the group $G_{k'}(A') = G(A')$ operates as given on $M \otimes A' = (M \otimes k') \otimes_{k'} A'$.

There is a functor in the opposite direction: We can regard each $G_{k'}$-module V in a natural way as a G-module. For any k-algebra A the map $a \mapsto 1 \otimes a$ is a homomorphism of k-algebras $A \to k' \otimes A$, hence induces a group homomorphism $G(A) \to G(k' \otimes A) = G_{k'}(k' \otimes A)$ and thus an operation of $G(A)$ on $V \otimes_{k'} (k' \otimes A) \simeq V \otimes A$. These operations are compatible with homomorphisms of k-algebras and lead therefore to a representation of G on V regarded as a k-module.

In the case of a group scheme we get the comodule map of V as a G-module (i.e. $V \to V \otimes k[G]$) from that as a $G_{k'}$-module (i.e. $V \to V \otimes_{k'} k'[G_{k'}]$) using the identification $V \otimes_{k'} k'[G_{k'}] = V \otimes_{k'} (k' \otimes k[G]) \simeq V \otimes k[G]$.

If M is a G-module, then the map $i_M: M \to M \otimes k'$, $m \mapsto m \otimes 1$ is a homomorphism of G-modules, if we regard the $G_{k'}$-module $M \otimes k'$ as a G-module as above. Indeed, the operation of any $G(A)$ on $M \otimes k' \otimes A$ comes from the operation of $G(k' \otimes A)$ on this module and the homomorphism $j_A: a \mapsto 1 \otimes a$ from A to $k' \otimes A$. We can regard $i_M \otimes id_A: M \otimes A \to M \otimes k' \otimes A$ also as $id_M \otimes j_A$ and it is therefore compatible with the action.

The universal property of the tensor product implies that

$\varphi \mapsto \varphi \circ i_M$ is a bijection $\mathrm{Hom}_{k'}(M \otimes k', V) \to \mathrm{Hom}_k(M, V)$ for any k-module M and any k'-module V. We claim that it yields a bijection

(1) $\mathrm{Hom}_{G_{k'}}(M \otimes k', V) \tilde{\to} \mathrm{Hom}_G(M, V)$

when M is a G-module and V a $G_{k'}$-module. As i_M is a homomorphism of G-modules we have already proved one direction. Suppose now that $\varphi \circ i_M$ is a homomorphism of G-modules and let us show that φ is a homomorphism of $G_{k'}$-modules. Consider any k'-algebra A' and the map $\varphi \otimes \mathrm{id}_{A'}: M \otimes k' \otimes_{k'} A' \to M' \otimes_{k'} A'$. We have to show that it commutes with the action of $G_{k'}(A') = G(A')$. We can identify $M \otimes k' \otimes_{k'} A' \cong M \otimes A'$ and then factorize the map into at first $(\varphi \circ i_M) \otimes \mathrm{id}_{A'}: M \otimes A' \to M' \otimes A'$ and then the canonical map $M' \otimes A' \to M' \otimes_{k'} A'$. By assumption the first map is $G(A')$-equivariant where we get the action of $G(A')$ on $M' \otimes A'$ from that of $G_{k'}(k' \otimes A')$ on $M' \otimes_{k'} (k' \otimes A') \cong M' \otimes A'$ via $A' \to k' \otimes A'$, $a \mapsto 1 \otimes a$. As $k' \otimes A' \to A'$, $b \otimes a \mapsto ba$ is a homomorphism of k'-algebras, the corresponding map $M' \otimes A' \cong M' \otimes_{k'} (k' \otimes A') \to M' \otimes_{k'} A'$ is compatible with $G_{k'}(k' \otimes A') \to G_{k'}(A')$, hence with the action of $G_{k'}(A')$. This is what we had to prove.

10.2 (Lattices) Let R be a Dedekind ring and K its field of fractions. Let me remind you that a lattice in a finite dimensional vector space V over K is a finitely generated R-submodule M of V such that the canonical map $M \otimes_R K \to V$ is an isomorphism.

This map is always injective, so we can weaken the condition to "V is generated by M over K", cf. [3], ch. VII, §4, n° 1, rem. 1. As R is a Dedekind ring, any such lattice is a projective R-module and its rank is equal to $\dim_K V$. If M is a lattice in V and V' is a subspace of V, then $M \cap V'$ is a lattice in V' and $(M+V')/V'$ is one in V/V'. If $M_1 \subset V_1$ and $M_2 \subset V_2$ are lattices, then $M_1 \otimes_R M_2$ is one in $V_1 \otimes_K V_2$. (For more details, consult [3], ch. VII, §4, n° 1.)

10.3 Lemma: Let R be a Dedekind ring and G a flat group scheme over R. Let K be the field of fractions of R and V a finite dimensional G_K-module. Then there is a G-stable lattice in V.

Proof: Let $v_1, v_2, \ldots, v_n$ be a basis of V. By 2.13(3) there is a G-submodule M of V containing all v_i which is finitely generated over R. As M generates V over K, it is a lattice.

10.4 Proposition: Let R be a complete discrete valuation ring with k as its residue field. Let G be a flat R-group scheme. Then there is for each idempotent $e \in \mathrm{End}_{G_k}(k[G_k])$ an $\tilde{e} \in \mathrm{End}_G(k[G])$ inducing e.

Proof: Denote the maximal ideal in R by $\underline{m}$. Let me remind you that by 4.18(2) the canonical map from $\mathrm{End}_G(R[G]) \otimes_R k \simeq \mathrm{End}_G R[G]/\underline{m}\, \mathrm{End}_G R[G]$ to $\mathrm{End}_{G_k}(k[G_k])$ is an isomorphism.

We want to apply proposition 3.15 to the ring $\mathrm{End}_G R[G]$ and its chain of ideals $\underline{m}_i = \underline{m}^i \mathrm{End}_G R[G]$. If that is possible, we get the lemma as an obvious consequence. So we have to prove that naturally

(1) $\mathrm{End}_G R[G] \cong \varprojlim \mathrm{End}_G R[G]/\underline{m}^i \mathrm{End}_G R[G]$.

If M is a G-submodule of $R[G]$ which is finitely generated over R, then it is a free R-module and we have an isomorphism $\mathrm{Hom}_G(M,R[G]) \cong M^*$ by Frobenius reciprocity. As R is complete, we get

$$\mathrm{Hom}_G(M,R[G]) \cong \varprojlim \mathrm{Hom}_G(M,R[G])/\underline{m}^i\mathrm{Hom}_G(M,R[G]).$$

This implies (1), as $R[G]$ is the direct limit of such M.

<u>10.5</u> <u>Corollary</u>: <u>Let</u> R <u>be as in 10.4. For each indecomposable and injective</u> G_k<u>-module</u> Q <u>there is a direct summand</u> $\tilde{Q}$ <u>of</u> $R[G]$ <u>with</u> $Q \cong \tilde{Q} \otimes_R k$.

<u>Proof</u>: We may assume that Q is a direct summand of $k[G_k]$. (Continue 3.16 and 3.10!) Therefore we can find $\varphi \in \mathrm{End}_{G_k}(k[G_k])$ idempotent with $Q \cong \varphi(k[G_k])$. Let $\psi \in \mathrm{End}_G(R[G])$ be idempotent inducing φ. Then $\psi(R[G])$ is a direct summand of $R[G]$ and

$$Q \cong \varphi(k[G_k]) = (\psi \otimes \mathrm{id}_k)k[G_k] \cong \psi(R[G]) \otimes k.$$

<u>10.6</u> (<u>Reciprocity</u>) Let us assume from now on in this chapter that R is a Dedekind ring which is not a field. We denote its field of fractions by K. Let $\underline{m}$ be a maximal ideal of R and suppose that $k = R/\underline{m}$.

Let G be a flat R-group scheme. If V is a finite dimensional G_K-module, then we can find by 10.3 a G-stable lattice V_R in V and then form the G_k-module $V_k = V_R \otimes_R k$. We have obviously

$\dim V_k = \mathrm{rk}_R(V_R) = \dim_K V$.

The choice of a G-stable lattice is not unique and different choices will lead in general to non-isomorphic G_k-modules. We claim however that the composition factors of V_k are uniquely determined by V.

Let E be a simple G_k-module and let Q_E be an injective hull of E, cf. 3.16/17. The multiplicity $[V_k:E]_{G_k}$ of E as a composition factor of V_k is then equal to (cf. 3.17(3))

$$(1) \quad [V_k:E]_{G_k} = \dim \mathrm{Hom}_{G_k}(V_k, Q_E)/\dim \mathrm{End}_{G_k}(E).$$

Let $\hat{R}$ be the $\underline{m}$-adic completion of R and denote by $\hat{K}$ its field of fractions. We can identify k with the residue field of $\hat{R}$. By 10.5 there is a direct summand $\tilde{Q}_E$ of the G_k-module $\hat{R}[G_{\hat{R}}] = R[G]\otimes_R \hat{R}$ with $\tilde{Q}_E \otimes_R k \simeq Q_E$. Now 4.18(1) implies (as $V_k \simeq (V_R \otimes_R \hat{R}) \otimes_{\hat{R}} k$)

$$(2) \quad \mathrm{Hom}_{G_R}(V_R \otimes_R \hat{R}, \tilde{Q}_E) \otimes_{\hat{R}} k \simeq \mathrm{Hom}_{G_k}(V_k, Q_E).$$

On the other hand $(V_R \otimes_R \hat{R}) \otimes_{\hat{R}} \hat{K} \simeq V \otimes_K \hat{K}$ and K is flat over R, so already 2.10(7) implies

$$(3) \quad \mathrm{Hom}_{G_{\hat{R}}}(V_R \otimes_R \hat{R}, \tilde{Q}_E) \otimes_{\hat{R}} \hat{K} \simeq \mathrm{Hom}_{G_{\hat{R}}}(V \otimes_K \hat{K}, \tilde{Q}_E \otimes_{\hat{R}} \hat{K}).$$

A comparison of ranks and dimensions implies the "Brauer reciprocity formula"

(4) $[V_k:E]_{G_k} = \dim_{\hat{K}} \mathrm{Hom}_{G_{\hat{K}}}(V\otimes_K \hat{K}, \tilde{Q}_E \otimes_{\hat{R}} \hat{K})/\dim \mathrm{End}_{G_k}(E)$.

<u>10.7</u> Assume in addition that each simple G_k-module is absolutely simple (hence satisfies $\mathrm{End}_{G_k} = k$), that also each simple G_K-module is absolutely simple, and that each G_K-module is semi-simple. (This is e.g. satisfied for a split connected reductive group, if char (K) = 0, see part II.) Let us assume in order to simplify that $R = \hat{R}$.

Consider a simple G_K-module V and a simple G_k-module E. Let us construct $\tilde{Q}_E$ and V_k as in 10.6. The semi-simplicity of $\tilde{Q}_E \otimes_R K$ and the absolute simplicity of V implies that $\dim_K \mathrm{Hom}_{G_K}(V, \tilde{Q}_E \otimes_R K)$ is equal to the multiplicity of V as a composition factor of $\tilde{Q}_E \otimes_R K$. So 10.6(4) yields

(1) $[V_k:E]_{G_k} = [\tilde{Q}_E \otimes_R K:V]_{G_K}$.

If we take an abstract finite group Γ and carry out the construction of 8.5.a over R, then we get Brauer's original theorem.

<u>10.8</u> Return to the more general situation of 10.6! We can interprete the result as a statement about Grothendieck groups.

Recall that one can associate to each abelian category a Grothendieck group. One starts with the free group generated by the objects of the category (let [M] denote the generator corresponding to an object M) and divides out the subgroup generated by all $[M]-[M']-[M'']$ for all short exact sequences $0 \to M' \to M \to M'' \to 0$ in the category.

Let us denote by $\underline{\underline{R}}(G)$ the Grothendieck group of all those G-modules which are finitely generated over R. Define $\underline{\underline{R}}(G_K)$ and $\underline{\underline{R}}(G_k)$ by analogy. Then $\underline{\underline{R}}(G_K)$ and $\underline{\underline{R}}(G_k)$ are free abelian groups with the classes $[E']$ resp. $[E]$ of all simple G_K-modules E' resp. simple G_k-modules E as a basis. For any finite dimensional G_k-module M one has

$$[M] = \sum_E [M:E]_{G_k} [E]$$

where E runs through a system of representatives of isomorphism classes of simple G_k-modules. (Similarly for G_K.) In these cases (over a field) the Grothendieck groups have a natural ring structure induced by the tensor product, i.e. with $[M \otimes M'] = [M][M']$.

We can now deduce from 10.6(4) that the class $[V_k]$ of V_k is uniquely determined by V and does not depend on the choice of V_R. One gets in this way easily a homomorphism of rings $\underline{\underline{R}}(G_K) \to \underline{\underline{R}}(G_k)$ with $[V] \mapsto [V_k]$.

<u>10.9</u> Let me mention some results about $\underline{\underline{R}}(G)$ proved in [Serre]. The map $M \mapsto M \otimes K$ induces a homomorphism $\underline{\underline{R}}(G) \to \underline{\underline{R}}(G_K)$. Its kernel is equal to the subgroup of $\underline{\underline{R}}(G)$ generated by all [M] such that M is a (finitely generated) torsion module (and a G-module). Lemma 10.3 implies that the map is surjective, i.e. that we get an exact sequence of the form

(1) $0 \to \underline{\underline{R}}_{tor}(G) \to \underline{\underline{R}}(G) \to \underline{\underline{R}}(G_K) \to 0.$

Consider the category of all G-modules which are finitely generated and projective over R and let $\underline{\underline{R}}_{pr}(G)$ be its Grothendieck group. The inclusion of categories induces a homomorphism from $\underline{\underline{R}}_{pr}(G)$ to $\underline{\underline{R}}(G)$ which turns out to be an isomorphism

(2) $\underline{\underline{R}}_{pr}(G) \tilde{\to} \underline{\underline{R}}(G).$

The reduction mod $\underline{m}$ (i.e. $M \mapsto M \otimes k = M/\underline{m}M$) defines a homomorphism $\underline{\underline{R}}_{pr}(G) \to \underline{\underline{R}}(G_k)$, by (2) also $\underline{\underline{R}}(G) \to \underline{\underline{R}}(G_k)$. One checks that $\underline{\underline{R}}_{tor}(G)$ is mapped to 0 and gets $\underline{\underline{R}}(G_K) \to \underline{\underline{R}}(G_k)$ by (1). This is the same map constructed using 10.6(4).

If R is a principal ideal domain, then $\underline{\underline{R}}_{tor}(G) = 0$. (If M is a G-stable lattice in a finite dimensional G_K-module V, then $[M/\underline{m}M] = 0$ in $\underline{\underline{R}}(G)$ as $M \simeq \underline{m}M$. One can show that $\underline{\underline{R}}_{tor}(G)$ is generated by such $[M/\underline{m}M]$ for all possible $\underline{m}$.)

References

A: General Text Books

[Bo] A. Borel: Linear Algebraic Groups, New York 1969 (Benjamin)

[Ca] R.W. Carter: Simple Groups of Lie Type, London 1972 (Wylie)

[Ch] C. Chevalley: Théorie des groupes de Lie, tome II: Groupes algébriques, Paris 1951 (Hermann

[DG] M. Demazure, P. Gabriel: Groupes Algébriques I, Paris/Amsterdam 1970 (Masson/North-Holland)

[DG'] M. Demazure, P. Gabriel: Introduction to Algebraic Geometry and Algebraic Groups, Amsterdam 1980 (North-Holland)

[SGA 3] M. Demazure, A. Grothendieck (dirig.): Schémas en Groupes, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Lecture Notes in Mathematics 151-153, Berlin/Heidelberg/New York 1970 (Springer)

[F] J. Fogarty: Invariant Theory, New York 1969 (Benjamin)

[Ha] R. Hartshorne: Algebraic Geometry, New York/Heidelberg/Berlin 1977 (Springer)

[Ho 1] G. Hochschild: Introduction to Affine Algebraic Groups, San Francisco 1971 (Holden-Day)

[Ho 2] G. Hochschild: Basic Theory of Algebraic Groups and Lie Algebras, New York/Heidelberg/Berlin 1981 (Springer)

[Hu] J.E. Humphreys: Linear Algebraic Groups, Berlin/Heidelberg/New York 1975 (Springer)

[N] D.G. Northcott: Affine Sets and Affine Groups, London Math. Soc. Lect. Notes Ser. 39, Cambridge 1980 (Camb. Univ. Press)

[Sa] I. Satake: Classification Theory of semi-simple Algebraic Groups, Chicago 1967 (Univ. of Illinois, Chicago Circle)

[Se] G. Seligman: Algebraic Groups, New Haven, Conn. 1964 (Yale Univ.)

[SC] Séminaire Chevalley 1956-58: Classification des Groupes de Lie algébriques, Paris 1958 (Secr. math.)

[SHS] Séminaire Heidelberg-Strasbourg 1965/66, Groupes algébriques linéaires, Strasbourg 1967 (Inst. Rech. Math. Avanc.)

[Sp] T.A. Springer: Linear Algebraic Groups, Boston/Basel/Stuttgart 1981 (Birkhäuser)

[St 1] R. Steinberg: Lectures on Chevalley Groups, New Haven, Conn. 1968 (Yale Univ.)

[St 2] R. Steinberg: Conjugacy Classes in Algebraic Groups, Lecture Notes in Mathematics 366, Berlin/Heidelberg/ New York 1974 (Springer)

[T] M. Takeuchi: Tangent coalgebras and hyperalgebras I, Japanese J. Math. 42 (1974), 1-143

[Ti] J. Tits: Lectures on Algebraic Groups, 1st Part (Notes by P. André and D. Winter), New Haven, Conn. 1968 (Yale Univ.)

[W] W.C. Waterhouse: Introduction to Affine Group Schemes, New York/Heidelberg/Berlin 1979 (Springer)

[Y] H. Yanagihara: Theory of Hopf algebras Attached to Group Schemes, Lecture Notes in Mathematics 614, Berlin/ Heidelberg/New York 1977 (Springer)

B: Special Articles

J. Alperin
Projective modules für $SL(2,2^n)$, J. Pure Appl. Algebra 15 (1979), 219-234

J. Alperin, L.G. Kovacs
Periodicity of Weyl modules for $SL(2,q)$, J. Algebra, 74 (1982), 52-54

H.H. Andersen
1) Cohomology of line bundles on G/B, Ann. Scient. Éc. Norm. Sup. (4) 12 (1979), 85-100
2) The first cohomology group of a line bundle on G/B, Invent. math. 51 (1979), 287-296
3) Vanishing theorems and induced representations, J. Algebra 62 (1980), 86-100
4) The strong linkage principle, J. reine angew. Math. 315 (1980), 53-59
5) The Frobenius morphism on the cohomology of homogeneous vector bundles on G/B, Ann. of Math. 112 (1980), 113-121
6) On the structure of Weyl modules, Math. Z. 170 (1980), 1-14
7) Representations of algebraic groups via cohomology of line bundles, pp. 171-175 in: E. Balslev (ed.) 18th Scandinavian Congress of Mathematicians (Proc. Aarhus 1980), Boston/Basel/Stuttgart 1981 (Birkhäuser)
8) Line bundles on flag manifolds, pp. 21-42 in: [S6]
9) On the structure of the cohomology of line bundles on G/B, J. Algebra 71 (1981), 245-258
10) Extensions of modules for algebraic groups, Amer. J. Math. 106 (1984), 489-504
11) An inversion formula for the Kazhdan-Lusztig polynomials for affine Weyl groups, Adv. in Math.
12) Filtrations of cohomology modules for Chevalley groups, Ann. Scient. Ec. Norm. Sup.(4) 16 (1983), 495-528
13) Schubert varieties and Demazure's character formula
14) Torsion in the cohomology of line bundles on homogeneous spaces for Chevalley groups

H.H. Andersen, J.C. Jantzen
Cohomology of induced representations for algebraic groups Math. Ann. 269 (1984), 487-525

H.H. Andersen, J. Jørgensen, P. Landrock
The projective indecomposable modules of $SL(2,p^n)$ Proc. London Math. Soc. (3) 46 (1983), 38-52

J. Archer
Principal indecomposable modules for some three-dimensional special linear groups. Bull. Austral. Math. Soc. 22 (1980), 439-455

G. Avrunin

1) 2-cohomology of some unitary groups, Ill. J. Math. 24 (1980), 317-332

2) Generic cohomology of twisted groups, Trans. Amer. Math. Soc. 268 (1981), 247-253

K. Baclawski, J. Towber

The shape algebra and standard bases for G_2

J.W. Ballard

1) Some generalized characters of finite Chevalley groups, Math. Z. 147 (1976), 163-174

2) Projective modules for finite Chevalley groups, Trans. Amer. Math. Soc. 245 (1978), 221-249

3) Injective modules for restricted enveloping algebras, Math. Z. 163 (1978), 57-63

4) Clifford's theorem for algebraic groups and Lie algebras, Pacific J. Math. 106 (1983), 1-15

G. Bell

1) On the cohomology of the finite special linear groups I, II, J. Algebra 54 (1978), 216-238 and 239-259

2) Cohomology of degree 1 and 2 of the Suzuki groups, Pacific J. Math. 75 (1978), 319-329

A. Borel

1) Properties and linear representations of Chevalley groups, pp. 1-55 in [S1]

2) Linear representations of semisimple algebraic groups, pp. 421-440 in: R. Hartshorne (ed.), Algebraic Geometry, Arcata 1974, Proc. Symp. Pure Math. 29, Providence, R.I. 1978 (Amer. Math. Soc.)

B. Braden

Restricted representations of classical Lie algebras of type A_2 and B_2, Bull. Amer. Math. Soc. 73 (1967), 482-486

N. Burgoyne

Modular representations of some finite groups, pp. 13-17 in: I. Reiner (ed.), Representation Theory of Finite Groups and Related Topics (Madison, Wisc. 1970), Proc. Symp. Pure Math. 21, Providence, R.I. 1971 (Amer. Math. Soc.)

R. Burkhardt

1) Die Zerlegungsmatrizen der Gruppen $PSL(2,p^f)$, J. Algebra 40 (1976), 75-96

2) Über ein kombinatorisches Problem aus der modularen Darstellungstheorie, J. Combinat. Theory (A) 21 (1976), 68-79

3) Über die Zerlegungszahlen der Suzukigruppen, J. Algebra <u>59</u> (1979), 421-433

4) Über die Zerlegungszahlen der unitären Gruppen $PSU(3,2^{2f})$, J. Algebra <u>61</u> (1979), 548-581

J.F. Carlson
The cohomology of irreducible modules over $SL(2,p^n)$, Proc. London Math. Soc. (3) <u>47</u> (1983), 480-492

R.W. Carter
The relation between characteristic 0 representations and characteristic p representations of finite groups of Lie type, pp. 301-311 in [S3]

R.W. Carter, E. Cline
The submodule structure of Weyl modules for groups of type A_1, pp. 303-311 in [S5]

R.W. Carter, G. Lusztig
1) On the modular representations of the general linear and symmetric groups, Math. Z. <u>136</u> (1974), 193-242

2) Modular representations of finite groups of Lie type, Proc. London Math. Soc. (3) <u>32</u> (1976), 347-384

R.W. Carter, M.T.J. Payne
On homomorphisms between Weyl modules and Specht modules, Math. Proc. Camb. Phil. Soc. <u>87</u> (1980), 419-425

P. Cartier
Représentations linéaires des groupes algébriques semisimples en caractéristique non nulle, exp. 255 in: Séminaire Bourbaki (Proc. Paris 1962/63), (2e éd.:) Paris 1964 (Secr. math.)

L. Chastkofsky
1) Projective characters for finite Chevalley groups, pp. 359-363 in [S3]

2) Projective characters for finite Chevalley groups, J. Algebra <u>69</u> (1981), 347-357

3) Variations on Hulsurkar's matrix with applications to representations of algebraic Chevalley groups, J. Algebra <u>82</u> (1983), 255-274

4) Rationality of certain zeta functions associated with modular representation theory

5) Generic Cartan invariants for Chevalley groups

L. Chastkofsky, W. Feit
1) On the projective characters in characteristic 2 of the groups $Suz(2^m)$ and $Sp_4(2^n)$, Publ. math. I.H.E.S. <u>51</u> (1980), 9-35

2) On the projective characters in characteristic 2 of the groups $SL_3(2^m)$ and $SU_3(2^m)$, J. Algebra 63 (1980), 124-142

E. Cline

1) Ext^1 for SL_2, Commun. Algebra 7 (1979), 107-111

2) A second look at the Weyl modules for SL_2, preprint

E. Cline, B. Parshall, L. Scott

1) Cohomology of finite groups of Lie type I, Publ. math. I.H.E.S. 45 (1975), 169-191

2) Cohomology of finite groups of Lie type II, J. Algebra 45 (1977), 182-198

3) Induced modules and affine quotients, Math. Ann. 230 (1977), 1-14

4) Induced modules and extensions of representations, Invent. math. 47 (1978), 41-51

5) Induced modules and extensions of representations II, J. London Math. Soc. (2) 20 (1979), 403-414

6) Cohomology, hyperalgebras and representations, J. Algebra 63 (1980), 98-123

7) On the tensor product theorem for algebraic groups, J. Algebra 63 (1980), 264-267

8) Detecting rational cohomology of algebraic groups, J. London Math. Soc. (2) 28 (1983), 293-300

9) A Mackey imprimitivity theory for algebraic groups, Math. Z. 182 (1983), 447-471

10) On injective modules for infinitesimal algebraic groups I, Proc. London Math. Soc.

E. Cline, B. Parshall, L. Scott, W. van der Kallen

Rational and generic cohomology, Invent. math. 39 (1977), 143-163

C.W. Curtis

1) Representations of Lie algebras of classical type with application to linear groups, J. Math. Mech. 9 (1960), 307-326

2) On the dimensions of the irreducible modules of Lie algebras of classical type, Trans. Amer. Math. Soc. 96 (1960), 135-142

3) On projective representations of certain finite groups, Proc. Amer. Math. Soc. 11 (1960), 852-860

4) Irreducible representations of finite groups of Lie type, J. reine angew. Math. 219 (1965), 180-199

5) Modular representations of finite groups with split (B,N)-pairs, pp. 57-95 in [S1]

S.W. Dagger
On the blocks of the Chevalley groups, J. London Math. Soc. (2) 3 (1971), 21-29

C. De Concini
1) Symplectic standard tableaux, Adv. in Math. 34 (1979), 1-27

2) Characteristic free "decomposition" of the coordinate ring of the symplectic group, pp. 121-129 in: A. de Luca (ed.), Non-Commutative Structures in Algebra and Geometric Combinatorics, Quaderni de la Ricerca Scientifica 1979, Roma 1981 (Cons. Naz. delle Ricerc.)

C. De Concini, D. Eisenbud, C. Procesi
1) Young diagrams and determinantal varieties, Invent. math. 56 (1980), 129-165

2) Hodge algebras, astérisque 91, Paris 1982 (soc. math. de france)

C. De Concini, D. Kazhdan
Special bases for S_n and GL(n), Israel J. Math. 40 (1981), 275-290

C. De Concini, V. Lakshmibai
Arithmetic Cohen-Macaulayness and arithmetic normality for Schubert varieties, Amer. J. Math. 103 (1981), 835-850

C. De Concini, C. Procesi
1) A characteristic free approach to invariant theory, Adv. in Math. 21 (1976), 330-354

2) Symmetric functions, conjugacy classes and the flag variety, Invent. math. 64 (1981), 203-219

3) Hodge algebras, a survey, pp. 79-83 in [S6]

4) Complete symmetric varieties, pp. 1-44 in: F. Gherardelli (ed.), Invariant Theory (Proc. Montecatini 1982), Lecture Notes in Mathematics 996, Berlin/Heidelberg/New York/Tokyo 1983 (Springer)

C. De Concini, E. Strickland
1) Traceless tensors and the symmetric group, J. Algebra 61 (1979), 112-128

2) On the variety of complexes, Adv. in Math. 41 (1981), 57-77

M. Demazure
1) Sur la formule des caractères de H. Weyl, Inventiones math. 9 (1970), 249-252
2) Invariants symetriques entiers des groupes de Weyl et torsion, Invent. math. 21 (1973), 287-301
3) Désingularisation des variétiés de Schubert généralisées, Ann. Scient. Éc. Norm. Sup (4) 7 (1974), 330-354
4) Une nouvelle formule des caractères, Bull. Sci. Math. (2) 98 (1974), 163-172
5) A very simple proof of Bott's theorem, Invent. math. 33 (1976), 271-272
6) A Moebius-like formula in the Schubert calculus, Invent. math. 35 (1976), 317-319
7) Démonstration de la conjecture de Mumford (d'aprés W. Haboush), pp. 138-144 (exp. 462) in: Séminaire Bourbaki 1974/75, Lecture Notes in Mathematics 514, Berlin/Heidelberg/New York 1976 (Springer)

D.I. Deriziotis
1) The Brauer complex of a Chevalley group, J. Algebra 70 (1981), 261-269
2) The submodule structure of Weyl modules for groups of type A_1, Commun. Algebra 9 (1981), 247-265
3) A proof of a Carter-Cline theorem on A_1-Weyl modules, Bull. Acad. Polon. Sci. (Ser. Sci. Math.) 30 (1982), 485-491

R. Dipper
1) Vertices of irreducible representations of finite Chevalley groups in the describing characteristic, Math. Z. 175 (1980), 143-159
2) On irreducible modules over finite Chevalley groups and parabolic subgroups, Commun. Algebra 10 (1982), 1073-1088

S. Donkin
1) Hopf complements and injective comodules, Proc. London Math. Soc. (3) 40 (1980), 298-319
2) Rationally injective modules for semisimple algebraic groups as direct limits, Bull. London Math. Soc. 12 (1980), 99-102
3) On a question of Verma, J. London Math. Soc. (2) 21 (1980), 445-455
4) A filtration for rational modules, Math. Z. 177 (1981), 1-8
5) The blocks of a semi-simple algebraic groups, J. Algebra 67 (1980), 36-53

6) A note on the characters of the projective modules for the infinitesimal subgroups of a semisimple algebraic group, Math. Scand. 51 (1982), 142-150

7) On Ext^1 for semisimple groups and infinitesimal subgroups, Math. Proc. Camb. Phil. Soc. 92 (1982), 231-238

8) A note on decomposition numbers of reductive algebraic groups, J. Algebra 80 (1983), 226-234

9) Tensor products and filtrations for rational representations of algebraic groups

10) A note on decomposition numbers for general linear groups and symmetric groups

11) Finite resolutions of modules for reductive algebraic groups

S. Doty
The submodule structure of certain Weyl modules for groups of type A_n

S. Doty, J.B. Sullivan
The submodule structure of Weyl modules for SL_3

G. Fischer
Darstellungstheorie des ersten Frobeniuskerns der SL_2, Diss. Bielefeld 1982

E. Formanek, C. Procesi
Mumford's conjecture for the general linear group, Adv. in Math. 19 (1976), 292-305

J.W. Franklin
Homomorphisms between Verma modules and Weyl modules in characteristic p, Ph.D. Thesis, Warwick 1981

E. Friedlander, B. Parshall

1) Cohomology of algebraic and related finite groups, Invent. math. 74 (1983), 85-117

2) Cohomology of Lie algebras and algebraic groups, Amer. J. Math.

3) Limits of infinitesimal group cohomology

D.J. Glover
A study of certain modular representations, J. Algebra 51 (1978), 425-475

J.A. Green

1) Locally finite representations, J. Algebra 41 (1976), 131-171

2) Polynomial Representations of GL_n, Lecture Notes in Mathematics 830, Berlin/Heidelberg/New York 1980 (Springer)

W.L. Griffith

1) Cohomology of flag varieties in characteristic p, Ill. J. Math. 24 (1980), 452-461

2) The indecomposability of $H^*(G/B,L)$, Colloquium Math. 47 (1982), 59-63

W. Haboush

1) Reductive groups are geometrically reductive, Ann. of Math. 102 (1975), 67-84

2) Homogeneous vector bundles and reductive subgroups of reductive algebraic groups, Amer. J. Math. 100 (1978), 1123-1137

3) Central differential operators on split semi-simple groups over fields of positive characteristic, pp. 35-85 in: Séminaire d'Algèbre P. Dubreil et M.-P. Malliavin (Proc. 1979), Lecture Notes in Mathematics 795, Berlin/Heidelberg/ New York 1980 (Springer)

4) A short proof of the Kempf vanishing theorem, Invent. math. 56 (1980), 109-112

G. Hiß

Die adjungierten Darstellungen der Chevalley-Gruppen, Arch. Math. 42 (1984), 408-416

G. Hochschild

1) Cohomology of restricted Lie algebras, Amer. J. Math. 76 (1954), 555-580

2) Cohomology of algebraic linear groups, Ill. J. Math. 5 (1961), 492-519

3) Rationally injective modules for algebraic linear groups, Proc. Amer. Math. Soc. 14 (1963), 880-883

G.M.D. Hogeweij

Almost classical Lie algebras I, Indag. math. 44 (1982), 441-452

S.G. Hulsurkar

Proof of Verma's conjecture on Weyl's dimension polynomial, Invent. math. 27 (1974), 45-52

J.E. Humphreys

1) Modular representations of classical Lie algebras and semisimple groups, J. Algebra 19 (1971), 51-79

2) Defect groups for finite groups of Lie type, Math. Z. 119 (1971), 149-152

3) Projective modules for SL(2,q), J. Algebra 25 (1973), 513-518

4) Some computations of Cartan invariants for finite groups of Lie type, Comm. Pure Appl. Math. 26 (1973), 745-755

5) Weyl groups, deformations of linkage classes and character degrees for Chevalley groups, Commun. Algebra 1 (1974), 475-490

6) Representations of SL(2,p), Amer. Math. Monthly 82 (1975), 21-39

7) Ordinary and Modular Representations of Chevalley Groups, Lecture Notes in Mathematics 528, Berlin/Heidelberg/ New York 1976 (Springer)

8) On the hyperalgebra of a semisimple algebraic group, pp. 203-210 in: H. Bass, Ph. Cassidy, J. Kovacic (eds.), Contributions to Algebra, A Collection of Papers Dedicated to Ellis Kolchin, New York/London 1977 (Academic Press)

9) Symmetry for finite dimensional Hopf algebras, Proc. Amer. Math. Soc. 68 (1978), 143-146

10) Weyl modules and Bott's theorem in characteristic p, pp. 474-483 in: W. Rossmann (ed.), Lie Theories and their Applications (Proc. Kingston, Ont. 1977), Queen's Papers in Pure and Applied Mathematics 48, Kingston, Ont. 1978 (Queen's University)

11) Modular representations of finite groups of Lie type, pp. 259-290 in [S2]

12) Cartan invariants and decomposition numbers of Chevalley groups, pp. 347-351 in: [S3]

13) Deligne-Lusztig characters and principal indecomposable modules, J. Algebra 62 (1980), 299-303

14) Ordinary and modular characters of SL(3,p), J. Algebra 72 (1981), 8-16

15) Restricted Lie algebras (and beyond), pp. 91-98 in: S. Amitsur, D. Saltman, G. Seligman (eds.), Algebraists Homage (Proc. Conf. in honor of N. Jacobson, New Haven, Conn. 1981), Contemp. Math. 13, Providence, R.I. 1982 (Amer. Math. Soc.)

16) Cohomology of G/B in characteristic p, Adv. in Math.

17) On the structure of Weyl modules, Comm. in Algebra 12 (1984), 2665-2677

J.E. Humphreys, J.C. Jantzen
Blocks and indecomposable modules for semisimple algebraic groups, J. Algebra 54 (1978), 494-503

J.E. Humphreys, D.-n. Verma
Projective modules for finite Chevalley groups, Bull. Amer. Math. Soc. 79 (1973), 467-468

R.S. Irving
The structure of certain highest weight modules fór SL_3

G.D. James
The decomposition of tensors over fields of prime characteristic, Math. Z. 172 (1980), 161-178

J.C. Jantzen
1) Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen, Bonner math. Sch. 67 (1973)

2) Zur Charakterformel gewisser Darstellungen halbeinfacher Gruppen und Lie-Algebren, Math. Z. 140 (1974), 127-149

3) Darstellungen halbeinfacher Gruppen und kontravariante Formen, J. reine angew. Math. 290 (1977), 157-199

4) Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren, J. Algebra 49 (1977), 441-469

5) Weyl modules for groups of Lie type, pp. 291-300 in [S2]

6) Über Darstellungen höherer Frobenius-Kerne halbeinfacher algebraischer Gruppen, Math. Z. 164 (1979), 271-292

7) Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne, J. reine angew. Math. 317 (1980), 157-199

8) Zur Reduktion modulo p der Charaktere von Deligne und Lusztig, J. Algebra 70 (1981), 452-474

9) Zur Reduktion modulo p unipotenter Charaktere endlicher Chevalley-Gruppen, Math. Z. 181 (1982), 97-128

10) Filtrierungen der Darstellungen in der Hauptserie endlicher Chevalley-Gruppen, Proc. London Math. Soc. (3) 49 (1984), 445-482

A.V. Jeyakumar

1) Principal indecomposable representations for the group SL(2,q), J. Algebra 30 (1974), 444-458

2) Periodic modules for the group SL(2,q), Commun. Algebra 8 (1980), 1721-1735

W. Jones

An algorithm for generating dominant weights, Commun. Algebra 5 (1977), 759-771

W. Jones, B. Parshall

On the 1-cohomology of finite groups of Lie type, pp. 313-328 in [S5]

V.G. Kac

Irreducible representations of Lie algebras of classical type (russ.) Usp. Mat. Nauk 27 (1972), No. 5 (167), 237-238

V. Kac, B. Weisfeiler

Coadjoint action of a semisimple algebraic group and the center of the enveloping algebra in characteristic p, Indag. math. 38 (1976), 136-151

W. van der Kallen

Infinitesimally central extensions of Chevalley groups, Lecture Notes in Mathematics 356, Berlin/Heidelberg/ New York 1973 (Springer)

S. Kato

1) Spherical functions and a q-analogue of Kostant's weight multiplicity formula, Invent. math. 66 (1982), 461-468

2) On the Kazhdan-Lusztig polynomials for affine Weyl groups

D. Kazhdan, G. Lusztig

1) Representations of Coxeter groups and Hecke algebras, Invent. math. 53 (1979), 164-184

2) Schubert varieties and Poincaré duality, pp. 185-203 in: R. Osserman, A. Weinstein (eds.), Geometry of the Laplace Operator (Honolulu, Hawaii 1979), Proc. Symp. Pure Math. 36, Providence, R.I. 1980 (Amer. Math. Soc.)

G. Kempf

1) Linear systems on homogeneous spaces, Ann. of Math. 103 (1976), 557-591

2) The geometry of homogeneous spaces versus induced representations, pp. 1-5 in: J.I. Igusa(ed.), Algebraic Geometry, The Johns Hopkins Centenary Lectures (Proc. 1976), Baltimore, Md./London 1977 (John Hopkins Univ. Press)

3) The Grothendieck-Cousin complex of an induced representation, Adv. in Math. 29 (1978), 310-396

4) Algebraic representations of reductive groups, pp. 575-577 in [S4] .

5) Representations of algebraic groups in prime characteristic, Ann. Scient. Éc. Norm. Sup. (4) 14 (1981), 61-76

M. Koppinen

1) On the composition factors of Weyl modules, Math. Scand. 51 (1982), 212-216

2) On the translation functors for a semisimple algebraic group, Math. Scand. 51 (1982), 217-226

3) Good bimodule filtrations for coordinate rings

4) Decomposing and lifting hyperalgebras, Ann. Univ. Turku (A) 184 (1983)

5) Computations of simple characters of a Chevalley group

6) Homomorphisms between neighbouring Weyl modules

M. Koppinen, T. Neuvonen

An imprimitivity theorem for Hopf algebras, Math. Scand. 41 (1977), 193-198

B. Krcmar

Irreduzible rationale Darstellungen endlicher Chevalley-Gruppen, Diss. Tübingen 1979

N. Kuhn (with an appendix by P. Landrock)

The modular Hecke algebra and Steinberg representation of finite Chevalley groups, J. Algebra 91 (1984), 125-141

K. Kühne-Hausmann

Zur Untermodulstruktur der Weylmoduln für $S\ell_3$

V. Lakshmibai

1) Kempf varieties, J. Indian Math. Soc. 40 (1976), 299-349

2) Standard monomial bases for G_2

V. Lakshmibai, C. Musili, C.S. Seshadri

1) Cohomology of line bundles on G/B, Ann. Scient. Ec. Norm. Sup. (4) 7 (1974), 89-138 and 8 (1974), 421

2) Geometry of G/P-III Proc. Indian Acad. Sci. (A) 88 (1979), 93-177

3) Geometry of G/P-IV Proc. Indian Acad. Sci. (A) 88 (1979), 279-362

4) Geometry of G/P, Bull. Amer. Math. Soc. (New Ser.) 1 (1979), 432-435

V. Lakshmibai, C.S. Seshadri

1) Geometry of G/P-II Proc. Indian Acad. Sci. (A) 87 (1978), 1-54

2) Singular locus of a Schubert variety

3) Geometry of G/P-V

G. Lusztig

1) Divisibility of projective modules of finite Chevalley groups by the Steinberg module, Bull. London Math. Soc. 8 (1976), 130-134

2) Hecke algebras and Jantzen's generic decomposition patterns, Adv. in Math. 37 (1980), 121-164

3) Some problems in the representation theory of finite Chevalley groups, pp. 313-317 in [S3]

4) Singularities, character formulas and a q-analogue of weight multiplicities, pp. 208-229 in: analyse et topologie sur les espaces singuliers II-III, astérisque 101-102, Paris 1983 (soc. math. de france)

R.P. Martineau

On 2-modular representations of the Suzuki groups, Amer. J. Math. 94 (1972), 55-72

V.B. Mehta, A. Ramanathan

Frobenius splitting and cohomology vanishing for Schubert varieties

A.A. Milner

Irreducible representations of modular Lie algebras, Math. USSR - Izv. 9 (1975), 1167-1187 (orig. russ.: Izv. Mat. Akad. Nauk SSSR (Ser. Mat.) 39 (1975), 1240-1259)

C. Musili

1) Postulation formula for Schubert varieties, J. Indian Math. Soc. 36 (1972), 143-171

2) Some properties of Schubert varieties, J. Indian Math. Soc. 38 (1974), 131-145

C. Musili, C.S. Seshadri

1) Standard monomial theory, pp. 441-476 in: Séminaire d'Algèbre P. Dubreil et M.-P. Malliavin (Proc. 1980), Lecture Notes in Mathematics 867, Berlin/Heidelberg/ New York 1981 (Springer)

2) Schubert varieties and the variety of complexes, pp. 327-359 in: M. Artin, J. Tate (eds.), Arithmetic and Geometry, Papers Dedicated to I.R. Shafarevitch, Vol. II: Geometry, Boston/Basel/Stuttgart 1983 (Birkhäuser)

T. Neuvonen

1) On the structure of produced and induced indecomposable Lie modules, Ann. Acad. Sci. Fenn.(AI), 1 (1975), 199-206

2) A criterion for complete reducibility of restricted Lie modules, Arch. Math. 28 (1977), 149-156

3) A complete reducibility criterion with an application to representations of semisimple groups, J. Algebra 60 (1979), 282-288

P. Norton

O-Hecke algebras, J. Austral. Math. Soc. (A) 27 (1979), 337-357

J. O'Halloran

1) Weyl modules and the cohomology of Chevalley groups, Amer. J. Math. 103 (1981), 399-410

2) Representation theory of an extension of a Chevalley group by a vector group, J. Algebra 72 (1981), 335-341

3) A vanishing theorem for the cohomology of Borel subgroups, Commun. Algebra 11 (1983), 1603-1606

4) One-cohomology of infinitesimal subgroups of a Chevalley group

5) Cohomology of a Borel subgroup of a Chevalley group

B. Parshall, L. Scott

An imprimitivity theorem for algebraic groups, Indag. Math. 42 (1980), 39-47

W. Pfautsch

1) Die Köcher der Frobeniuskerne in der SL_2, Diss. Bielefeld 1983

2) Ein Satz über Blöcke von Frobeniuskernen einer halbeinfachen algebraischen Gruppe

R. Pfetzing
Der Darstellungstyp der Frobeniuskerne in der SL_3, Diss. Bielefeld 1983

H. Pollatsek
First cohomology groups of some linear groups defined over fields of characteristic 2, Ill. J. Math. 15 (1971), 393-417

A.A. Premet, I.D. Suprunenko
1) The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights, Commun. Algebra 11 (1983), 1309-1342
2) Quadratic modules for Chevalley groups over fields of odd characteristic, Math. Nachr. 110 (1983), 65-96

C. Procesi
1) Les bases de Hodge dans la theorie des invariants, pp. 128-144 in: Séminaire d'Algèbre P. Dubreil (Proc. 1976/77), Lecture Notes in Mathematics 641, Berlin/Heidelberg/New York 1978 (Springer)
2) Young diagrams, standard monomials and invariant theory, pp. 537-542 in [S4]

S. Ramanan, A. Ramanathan
Projection normality of flag varieties and Schubert varieties

F. Richen
1) Modular representations of split (B,N) pairs, Trans. Amer. Math. Soc. 140 (1969), 435-460
2) Blocks of defect zero of split (B,N) paris, J. Algebra 21 (1972), 275-279

M. Ronan, S. Smith
1) Sheaves on buildings and modular representations of Chevalley groups, J. Algebra
2) On computation of sheaf homology on finite geometries

A.N. Rudakov
1) On representations of classical Lie algebras of characteristic p, Math. USSR - Izv. 4 (1970), 741-749 (russ. orig.: Izv. Akad. Nauk SSSR (Ser. Mat.), 34:4 (1970), 735-743)
2) Dimensions of some irreducible representations of Lie algebras of classical type over fields of finite characteristic (russ.), Tr. Semin. Im. I.G. Petrovskogo 3 (1978), 147-160
3) A condition for the complete reducibility of a representation of a Lie algebra of finite characteristic (russ.) pp. 77-82 in: A.I. Kostrikin (ed.), Algebra, Collection, Moskva 1980 (Izdat. Mosk. Univ.)

4) Reducible p-representations of simple three dimensional Lie p-algebras (russ.), Vestn. Mosk. Univ. (I) 6 (1982), 45-49

A.N. Rudakov, I.R. Shafarevich
Irreducible representations of a simple three dimensional Lie algebra over a field of finite characteristic, Math. Notes Acad. Sci. USSR 2 (1967), 760-767 (russ. orig.: Mat. Zametki 2:5 (1967), 439-454)

H. Sawada
1) A characterization of the modular representations of finite groups with split (B,N) pairs, Math. Z. 155 (1977), 29-41
2) Some infinite dimensional indecomposable modules of Chevalley groups, Arch. Math. 36 (1981), 414-422

L. Scott
Representations in characteristic p, pp. 319-331 in [S3]

J.-P. Serre
Groupes de Grothendieck des schémas en groupes réductifs déployées, Publ. math. I.H.E.S. 34 (1968), 37-52

C.S. Seshadri
1) Geometric reductivity over arbitrary base, Adv. in Math. 26 (1977), 225-274
2) Geometry of G/P-I
pp. 207-239 in: K.G. Ramanathan (ed.), K.P. Ramanujam, A Tribute, Berlin/Heidelberg/New York 1978 (Springer)
3) Standard monomial theory and the work of Demazure, pp. 355-384 in: S. Iitaka (ed.), Algebraic Varieties and Analytic Varieties (Proc. Tokyo 1981), Amsterdam 1983 (North-Holland)
4) Line bundles on Schubert varieties

S.D. Smith
Irreducible modules and parabolic subgroups, J. Algebra 75 (1982), 286-289

T.A. Springer
Weyl's character formula for algebraic groups, Invent. math. 5 (1968), 85-105

T.A. Springer, W.A.M. van Ham, M. van der Wel
On the Cartan invariants of $SL_2(\mathbb{F}_q)$, Commun. Algebra 10 (1982), 1565-1588

B. Srinivasan
On the modular characters of the special linear group $SL(2,p^n)$, Proc. London Math. Soc. (3) 14 (1964), 101-114

R. Steinberg
1) Prime power representations of finite linear groups II, Canad. J. Math. 9 (1957), 347-351
2) Representations of algebraic groups, Nagoya Math. J. 22 (1963), 33-56

J.B. Sullivan
1) Some representation theory for the modular general linear groups, J. Algebra 45 (1977), 516-535
2) Representations of the hyperalgebra of an algebraic group, Amer. J. Math. 100 (1978), 643-652
3) Relations between the cohomology of an algebraic group and its infinitesimal subgroups, Amer. J. Math. 100 (1978), 995-1014
4) Simply connected groups, the hyperalgebra and Verma's conjecture, Amer. J. Math. 100 (1978), 1015-1019
5) Lie algebra cohomology at irreducible modules, Ill. J. Math. 23 (1979), 363-373
6) The second Lie algebra cohomology group and Weyl modules, Pacific J. Math. 86 (1980), 321-326
7) Frobenius operations on Hochschild cohomology, Amer. J. Math. 102 (1980), 765-780

I.D. Suprunenko
1) The invariance of the set of weights of irreducible representations of algebraic groups and Lie algebras of type A_ℓ with restricted highest weight under reduction modulo p (russ.), Vesci Akad. Navuk BSSR, ser. fis.-mat. 2 (1983), 18-22
2) On the restriction of the fundamental representations of the groups $A_n(K)$ to connected algebraic subgroups (russ.), preprint, July 1983
3) On the restriction of irreducible representations of SL(m,K) to classical groups (russ.), preprint, July 1984

N.B. Tinberg
1) Some indecomposable modules of groups with split (B,N)-pairs, J. Algebra 61 (1979), 508-526
2) Some indecomposable modules of groups with split (B,N)-pairs, pp. 363-366 in [S3]
3) Modular representations of finite groups with unsaturated split (B,N)-pairs, Canad. J. Math. 32 (1980), 714-733

J. Towber
1) Two new functors from modules to algebras, J. Algebra 47 (1977), 80-104
2) Young symmetry, the flag manifold, and representations of GL(n), J. Algebra 61 (1979), 414-462

B.S. Upadhyaya
1) Composition factors of the principal indecomposable modules for the special linear group SL(2,q), J. London Math. Soc. (2) 17 (1978), 437-445
2) Filtrations with Weyl module quotients of the principal indecomposable modules of the group SL(2,q), Commun. Algebra 7 (1979), 1469-1488

B.Ju. Vejsfejler, V.G. Kac
Irreducible representations of Lie p-algebras, Funct. Anal. Appl. 5 (1971), 111-117 (orig. russ.: Funkc. Anal. Prilož. 5:2 (1971), 28-36)

F. Veldkamp
1) Representations of algebraic groups of type F_4 in characteristic 2, J. Algebra 16 (1970), 326-339
2) The center of the universal enveloping algebra of a Lie algebra in characteristic p, Ann. Scient. Éc. Norm. Sup. (4) 5 (1972), 217-240

D.-n. Verma
The rôle of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, pp. 653-705 in: I.M. Gel'fand (ed.), Lie Groups and their Representations. (Proc. Budapest 1971) London 1975 (A. Hilger)

D. Voigt

Induzierte Darstellungen in der Theorie der endlichen algebraischen Gruppen, Lecture Notes in Mathematics 592, Berlin/Heidelberg/New York 1977 (Springer)

Wang Jian-pan

1) Sheaf cohomology of G/B and tensor products of Weyl modules, J. Algebra 77 (1982), 162-185

2) Coinduced representations and injective modules for hyperalgebra b_r, Chin. Ann. of Math. 4B (1983), 357-364

P.W. Winter

On the modular representations of the two dimensional special linear group over an algebraically closed field, J. London Math. Soc. (2) 16 (1977), 237-252

W.J. Wong

1) Representations of Chevalley groups in characteristic p, Nagoya Math. J. 45 (1971), 39-78

2) Irreducible modular representations of finite Chevalley groups, J. Algebra 20 (1972), 355-367

[S 1] A. Borel et al.
Seminar on Algebraic Groups and Related Finite Groups (Proc. Princeton, N.J. 1968/9), Lecture Notes in Mathematics 131, Berlin/Heidelberg/New York 1970 (Springer)

[S 2] M. Collins (ed.)
Finite Simple Groups II (Proc. Durham 1978), London/New York 1980 (Academic Press)

[S 3] B. Cooperstein, G. Mason (eds.)
The Santa Cruz Conference on Finite Groups (1979), Proc. Symp. Pure Math. 37, Providence, R.I. 1980 (Amer. Math. Soc.)

[S 4] O. Lehto (ed.)
Proceedings of the International Congress of Mathematicians (Helsinki 1978), Helsinki 1980

[S 5] W.R. Scott, F. Gross (eds.)
Proceedings of the Conference on Finite Groups (Park City, Utah 1975), New York/London 1976 (Academic Press)

[S 6] tableaux de Young et foncteurs de Schur en algèbre et géometrie (Proc. Toruń 1980), astérisque 87-88, Paris 1981 (soc. math. de france)

C: Additional References

[1] S. Anantharaman: Schémas en groupes, espaces homogènes et espaces algèbriques sur une base de dimension 1, Bull. Soc. math. France, Mém. 33 (1973), 5-79

[2] N. Bourbaki: Algèbre, Paris 1958 (ch. I), 1962 (ch. II), 1971 (ch. III, 2nd ed.), 1959 (ch. IV/V), 1964 (ch.VI/VII, 2nd ed.), 1958 (ch. VIII), 1959 (ch. IX), 1980 (ch. X) (Hermann: ch. I - IX, Masson: ch. X)

[3] N. Bourbaki: Algèbre Commutative, Paris 1961 (ch. I/II), 1962 (ch. III/IV), 1964 (ch. V/VI), 1965 (ch. VII) (Hermann)

[4] N. Bourbaki: Groupes et algèbres de Lie, Paris 1971 (ch. I), 1972 (ch. II/III), 1968 (ch. IV-VI), 1975 (ch. VII/VIII) (Hermann)

[5] C.W. Curtis, I. Reiner: Representation theory of finite groups and associative algebras, New York 1962 (Interscience)

[6] J. Dixmier: Algèbres enveloppantes, Paris-Bruxelles-Montréal 1974 (Gauthier-Villars)

[7] R. Godement: Théorie des faisceaux, Paris 1964 (Hermann)

[8] P.J. Hilton, U. Stammbach: A course in homological algebra, Berlin-Heidelberg-New York 1971 (Springer)

[9] U. Oberst, H.-J. Schneider: Über Untergruppen endlicher algebraischer Gruppen, manuscripta math. 8 (1973), 217-241

[10] M. Raynaud: Faisceaux amples sur les schémas en groupes et les espaces homogènes, Lecture Notes in Mathematics 119, Berlin-Heidelberg-New York 1970 (Springer)

[11] M. Sweedler: Hopf algebras, New York 1969 (Benjamin)

[12] M. Takeuchi: A hyperalgebraic proof of the isomorphism and isogeny theorems for reductive groups, J. Algebra 85 (1983), 179-196

[13] D. Voigt: Endliche Hopfalgebren, Math. Z. 134 (1973), 189-203