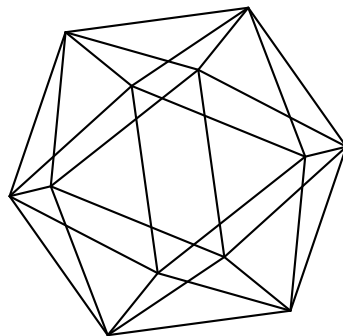# Max-Planck-Institut für Mathematik Bonn

A note on the Schottky problem

by

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A NOTE ON THE SCHOTTKY PROBLEM

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ABSTRACT. In this article, we discuss and survey the recent progress towards the Schottky problem, and make some comments on the relations between the André-Oort conjecture, Okounkov convex bodies, Coleman's conjecture, stable modular forms, Siegel-Jacobi spaces, stable Jacobi forms and the Schottky problem.

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1. Introduction

For a positive integer g, we let

$$\mathbb{H}_g = \left\{ \tau \in \mathbb{C}^{(g,g)} \mid \tau = {}^t\tau, \ \mathrm{Im}\,\tau > 0 \right\}$$

be the Siegel upper half plane of degree g and let

$$Sp(2g, \mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^t M J_g M = J_g \}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^t M$ denotes the transposed matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Then $Sp(2g, \mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

(1.1) $M \cdot \tau = (A\tau + B)(C\tau + D)^{-1},$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$. Let

$$\Gamma_g = Sp(2g, \mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}) \mid A, B, C, D \text{ integral} \right\}$$

be the Siegel modular group of degree g. This group acts on $\mathbb{H}_g$ properly discontinuously.

Let $\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g$ be the Siegel modular variety of degree g, that is, the moduli space of g-dimensional principally polarized abelian varieties, and let $\mathcal{M}_g$ be the the moduli space of projective curves of genus g. Then according to Torelli's theorem, the Jacobi mapping

(1.2) $T_g : \mathcal{M}_g \longrightarrow \mathcal{A}_g$

defined by

$$C \longmapsto J(C) := \text{the Jacobian of } C$$

is injective. The Jacobian locus $J_g := T_g(\mathcal{M}_g)$ is a $(3g-3)$-dimensional subvariety of $\mathcal{A}_g$ if $g \geq 2$.

The Schottky problem is to characterize the Jacobian locus or its closure $\bar{J}_g$ in $\mathcal{A}_g$. At first this problem had been investigated from the analytical point of view : to find explicit equations of J_g (or $\bar{J}_g$) in $\mathcal{A}_g$ defined by Siegel modular forms on $\mathbb{H}_g$, for example, polynomials in the theta constant $\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, 0)$ (see Definition (2.4)) and their derivatives. The first result in this direction was due to Friedrich Schottky [132] who gave the simple and beautiful equation satisfied by the theta constants of Jacobians of dimension 4. Much later the fact that this equation characterizes the Jacobian locus J_4 was proved by J. Igusa [79] (see also E. Freitag [52] and Harris-Hulek [74]). Past decades there has been some progress on the characterization of Jacobians by some mathematicians. Arbarello and De Concini [9] gave a set of such equations defining $\bar{J}_g$. The Novikov conjecture which states that a theta function satisfying the Kadomtsev-Petviasvili (briefly, K-P) differential equation is the theta function of a Jacobian was proved by T. Shiota [136]. Later the proof of the above Novikov conjecture was simplified by Arbarello and De Concini [10]. Bert van Geeman [58] showed that $\bar{J}_g$ is an irreducible component of the subvariety of $\mathcal{A}_g^{\text{Sat}}$ defined by certain equations. Here $\mathcal{A}_g^{\text{Sat}}$ is the Satake compactification of $\mathcal{A}_g$. I. Krichever [85] proved that the existence of one trisecant line of the associated Kummer variety characterizes Jacobian varieties among principally polarized abelian varieties.

S.-T. Yau and Y. Zhang [187] obtained the interesting results about asymptotic behaviors of logarithmical canonical line bundles on toroidal compactifications of the Siegel modular varieties. Working on log-concavity of multiplicities in representation theory, A. Okounkov [113, 114] showed that one could associate a convex body to a linear system on a projective variety, and use convex geometry to study such linear systems. Thereafter R. Lazarsfeld and M. Mustață [86] developed the theory of Okounkov convex bodies associated to linear series systematically. E. Freitag [50] introduced the concept of stable modular forms to investigate the geometry of the Siegel modular varieties. In 2014, using stable modular forms, G. Codogni and N. I. Shepherd-Barron [29] showed there is no stable Schottky-Siegel forms. We recall that Schottky-Siegel forms are scalar-valued Siegel modular forms vanishing on the Jacobian locus. Recently G. Codogni [28] found the ideal of stable equations of the hyperelliptic locus. About twenty years ago J.-H.Yang [158, 164] introduced the notion of stable Jacobi forms to try to study the geometry of the universal abelian varieties. In this paper, we discuss the relations among logarithmical line bundles on toroidal compactifications, the André-Oort conjecture, Okounkov convex bodies, Coleman's conjecture, Siegel-Jacobi spaces, stable Schottky-Siegel forms, stable Schottky-Jacobi forms and the Schottky problem.

This article is organized as follows. In Section 2, we briefly survey some known approaches to the Schottky problem and some results so far obtained concerning the characterization of Jacobians. In Section 3, we briefly describe the results of Yau and Zhang concerning the behaviors of logarithmical canonical line bundles on toroidal compactifications of the Siegel modular varieties. In Section 4, we review some recent progress on the André-Oort conjecture. In Section 5, we review the theory of Okounkov convex bodies associated to linear series (cf. [86] and [22]). In Section 6, we discuss the relations among logarithmical line bundles on toroidal compactifications, the André-Oort conjecture, Okounkov convex bodies, Coleman's conjecture and the Schottky problem. In the final section we give some remarks and propose some open problems about the relations among the Schottky problem, the André-Oort conjecture, Okounkov convex bodies, stable Schottky-Siegel forms, stable Schottky-Jacobi forms and the geometry of the Siegel-Jacobi space. We define the notion of stable Schottky-Jacobi forms and the concept of stable Jacobi equations for the universal hyperelliptic locus. In Appendix A, we survey some known results about subvarieties of the Siegel modular variety. In Appendix B, we review recent results concerning an extension of the Torelli map to a toroidal compactification of the Siegel modular variety. In Appendix C, we describe why the study of singular modular forms is closely related to that of the geometry of the Siegel modular variety. In Appendix D, we briefly talk about singular Jacobi forms. Finally in Appendix E, we review the concept of stable Jacobi forms introduced by the author and relate the study of stable Jacobi forms to that of the geometry of the universal abelian variety.

Finally the author would like to mention that he tried to write this article in another new perspective concerning the Schottky problem different from that of other mathematicians. The list of references in this article is by no means complete though we have strived to give as many references as possible. Any inadvertent omissions of references related to the contents in this paper will be the author's fault.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $\mathbb{R}^+$ denotes the set of all positive real numbers. $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of all nonnegative integers and the set of all nonnegative real numbers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, tr(A) denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^t M$ denotes the transpose of a matrix M. I_n denotes the identity matrix of degree n. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^t ABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a number field F, we denote by $\mathbb{A}_{F,f}$ the ring of finite adéles of F.

2. **Some Approaches to the Schottky Problem**

Before we survey some approaches to the Schottky problem, we provide some notations and definitions. Most of the materials in this section can be found in [65]. We refer to [15, 36, 46, 65, 108, 124] for more details and discussions on the Schottky problem.

In this section, we let g be a fixed positive integer. For a positive integer ℓ, we define the principal level ℓ subgroup

$$\Gamma_g(\ell) := \{\gamma \in Sp(2g, \mathbb{Z}) \,|\, \gamma \equiv I_{2g} \pmod{\ell}\}.$$

and define the theta level ℓ subgroup

$$\Gamma_g(\ell, 2\ell) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(\ell) \,\middle|\, \operatorname{diag}({}^t AB) \equiv \operatorname{diag}({}^t CD) \equiv 0 \pmod{\ell} \right\}.$$

We let

$$\mathcal{A}_g(\ell) := \Gamma_g(\ell)\backslash\mathbb{H}_g \quad \text{and} \quad \mathcal{A}_g(\ell, 2\ell) := \Gamma_g(\ell, 2\ell)\backslash\mathbb{H}_g.$$

Definition 2.1. *([78, pp, 49-50], [106, p. 123], [162, p. 862] or [172, p. 127]) Let ℓ a positive integer. For any ϵ and δ in $\frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g$, we define the* theta function with characteristics ϵ *and* δ *by*

$$(2.1) \qquad \theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, z) := \sum_{N \in \mathbb{Z}^g} e^{\pi i\{(N+\epsilon)\tau\,{}^t(N+\epsilon) + 2\,(N+\epsilon)\,{}^t(z+\delta)\}}, \quad (\tau, z) \in \mathbb{H}_g \times \mathbb{C}^g.$$

The Riemann theta function $\theta(\tau, z)$ is defined to be

$$(2.2) \qquad \theta(\tau, z) := \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, z), \quad (\tau, z) \in \mathbb{H}_g \times \mathbb{C}^g.$$

For each $\tau \in \mathbb{H}_g$, we have the transformation behavior

$$(2.3) \qquad \theta(\tau, z + a\tau + b) := e^{-\pi i(a\tau\,{}^t a + 2\,a\,{}^t z)}\theta(\tau, z) \quad \text{for all } a, b \in \mathbb{Z}^g.$$

The function

$$(2.4) \qquad \theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau) := \theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, 0), \quad \tau \in \mathbb{H}_g$$

is called the theta constant of *order* ℓ. It is known that the theta constants $\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau)$ of order ℓ are Siegel modular forms of weight $\frac{1}{2}$ for $\Gamma_g(\ell, 2\ell)$ [106, p. 200].

For a fixed $\tau \in \mathbb{H}_g$, we let $\Lambda_\tau := \mathbb{Z}^g \tau + \mathbb{Z}^g$ be the lattice in $\mathbb{C}^g$. According to the formula (2.3), the zero locus $\{ z \in \mathbb{C}^g \mid \theta(\tau, z) = 0 \}$ is invariant under the action of the lattice Λ_τ on $\mathbb{C}^g$, and thus descends to a well-defined subvariety $\Theta_\tau \subset A_\tau := \mathbb{C}^g / \Lambda_\tau$. In fact A_τ is a principally polarized abelian variety with ample divisor Θ_τ.

Definition 2.2. *For $\epsilon \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g$ the theta function of the second order with characteristic ϵ is defined to be*

$$(2.5) \qquad \Theta[\epsilon](\tau, z) := \theta \begin{bmatrix} 2\epsilon \\ 0 \end{bmatrix} (2\tau, 2z), \quad (\tau, z) \in \mathbb{H}_g \times \mathbb{C}^g.$$

We define the theta constant of the second order to be

$$(2.6) \qquad \Theta[\epsilon](\tau) := \theta \begin{bmatrix} 2\epsilon \\ 0 \end{bmatrix} (2\tau, 0) = \Theta[\epsilon](\tau, 0), \quad \tau \in \mathbb{H}_g.$$

Then we see that $\Theta[\epsilon](\tau)$ is a Siegel modular form of weight $\frac{1}{2}$ for $\Gamma_g(2, 4)$.

We have the following results.

Theorem 2.1. *(Riemann's bilinear addition formula)* [78, p. 139]

$$(2.7) \qquad \left(\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, z) \right)^2 = \sum_{\sigma \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g} (-1)^{4\, {}^t\sigma\delta} \Theta[\sigma + \epsilon](\tau, 0) \cdot \Theta[\sigma](\tau, z).$$

Theorem 2.2. *For $\ell \geq 2$, the map*

$$(2.8) \qquad \Phi_\ell : \mathcal{A}_g(2\ell, 4\ell) \longrightarrow \mathbb{P}^N(\mathbb{C}), \quad N := \ell^{2g} - 1$$

defined by

$$\Phi_\ell(\tau) := \left\{ \theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau) \,\Big|\, \epsilon, \delta \in \frac{1}{\ell} \mathbb{Z}^g / \mathbb{Z}^g \right\}$$

is an embedding.

Proof. See Igusa [78] for $\ell = 4n^2$, and Salvati Manni [127] for $\ell \geq 2$. $\qquad\square$

Remark 2.1. *We consider the theta map*

$$(2.9) \qquad Th : \mathcal{A}_g(2, 4) \longrightarrow \mathbb{P}^{2^g - 1}(\mathbb{C})$$

defined by

$$Th(\tau) := \left\{ \Theta[\epsilon](\tau) \,\Big|\, \epsilon \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g \right\}.$$

We observe that according to Theorem 2.1, $\Phi_2(\tau)$ can be recovered uniquely up to signs from $\Theta(\tau)$. Since Φ_2 is injective on $\mathcal{A}_g(4, 8)$, the theta map Th is finite-to-one on $\mathcal{A}_g(2, 4)$. In

fact, it is known that the theta map Th is generically injective, and it is conjectured that Th is an embedding.

Now we briefly survey some approaches to the Schottky problem. As mentioned before, most of the following materials in this section comes from a good survey paper [65].

(A) Classical Approach

For $\tau \in \mathbb{H}_g$ and a positive integer $\ell \in \mathbb{Z}^g$,

$$A_\tau[\ell] := \left\{ \tau\epsilon + \delta \in A_\tau \mid \epsilon, \delta \in \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g \right\}$$

denotes the subgroup of A_τ consisting of torsion points of order ℓ. For $m = \tau\epsilon + \delta \in A_\tau[\ell]$, we briefly write

$$(2.10) \qquad\qquad \theta_m(\tau, z) := \theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, z).$$

We define the Igusa modular form to be

$$(2.11) \qquad F_g(\tau) := 2^g \sum_{m \in A_\tau[2]} \theta_m^{16}(\tau) - \left(\sum_{m \in A_\tau[2]} \theta_m^8(\tau) \right)^2, \quad \tau \in \mathbb{H}_g.$$

It was proved that $F_g(\tau)$ is a Siegel modular form of weight 8 for the Siegel modular group Γ_g such that when rewritten in terms of theta constants of the second order using Theorem 2.1,

$(F_g$–1) $F_g \equiv 0$ for $g = 1, 2$;
$(F_g$–2) F_3 is the defining equation for $\overline{Th(J_3(2,4))} = \overline{Th(\mathcal{A}_3(2,4))} \subset \mathbb{P}^7(\mathbb{C})$;
$(F_g$–3) F_4 is the defining equation for $\overline{Th(J_4(2,4))} \subset \overline{Th(\mathcal{A}_4(2,4))} \subset \mathbb{P}^{15}(\mathbb{C})$.

For more detail, we refer to [132, 79, 52, 97] for the case $g = 4$ and refer to [124] for the case $g = 5$. For $g \geq 5$, no similar solution is known or has been proposed.

Theorem 2.3. *If $g \geq 5$, then F_g does not vanish identically on J_g. In fact, the zero locus of F_5 on J_5 is the locus of trigonal curves.*

The above theorem was proved by Grushevsky and Salvati Manni [70].

(B) The Schottky-Jung Approach

Definition 2.3. *For an étale connected double cover $\tilde{C} \longrightarrow C$ of a curve $C \in \mathcal{M}_g$ (such a curve is given by a two-torsion point $\eta(\neq 0) \in J(C)[2]$) we define the Prym variety to*

$$Prym(\tilde{C} \longrightarrow C) := Prym(C, \eta) := \mathrm{Ker}_0(J(\tilde{C}) \longrightarrow J(C)) \in \mathcal{A}_{g-1},$$

where Ker_0 denotes the connected component of 0 in the kernel and the map $J(\tilde{C}) \longrightarrow J(C)$ is the norm map corresponding to the cover $\tilde{C} \longrightarrow C$. We denote by $\mathcal{P}_{g-1} \subset \mathcal{A}_{g-1}$ the locus of Pryms of all étale double covers of curves in $\mathcal{M}_g$. The problem of describing $\mathcal{P}_{g-1}$ is called the Prym-Schottky problem.

Remark 2.2. *The restriction of the principal polarization $\Theta_{J(\tilde{C})}$ to the Prym gives twice the principal polarization. However this polarization admits a canonical square root , which thus gives a natural principal polarization on the Prym.*

Theorem 2.4. (*Schottky-Jung proportionality*). *Let τ be the period matrix of a curve C of genus g and let τ_* be the period matrix of the Prym for $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$. Then for any $\epsilon, \delta \in \frac{1}{2}\mathbb{Z}^{g-1}/\mathbb{Z}^{g-1}$ the theta constants of $J(C)$ and of the Prym are related by*

$$(2.12) \qquad \left(\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau_*) \right)^2 = C\,\theta \begin{bmatrix} 0 & \epsilon \\ 0 & \delta \end{bmatrix} (\tau) \cdot \theta \begin{bmatrix} 0 & \epsilon \\ 1 & \delta \end{bmatrix} (\tau).$$

Here the constant C is independent of ϵ, δ.

Proof. See Schottky-Jung [133] and also Farkas [46] for a rigorous proof. $\qquad\square$

Definition 2.4. (*The Schottky-Jung locus* [65]). *Let I_{g-1} be the defining ideal for the image $\overline{Th(\mathcal{A}_{g-1}(2,4))} \subset \mathbb{P}^{2^{g-1}-1}$ (see Remark 2.1). For any equation $F \in I_{g-1}$, we let F_η be the polynomial equation on $\mathbb{P}^{2^g-1}$ obtained by using the Schottky-Jung proportionality to substitute an appropriate polynomial of degree 2 in terms of theta constants of τ for the square of any theta constant of τ_*. Let S_g^η be the ideal obtained from I_{g-1} in this way. We define the* big Schottky-Jung locus $\mathcal{S}_g^\eta(2,4) \subset \mathcal{A}_g(2,4)$ *to be the zero locus of S_g^η. It is not known that $I_g \subset S_g^\eta$ and thus we define $\mathcal{S}_g^\eta(2,4)$ within $\mathcal{A}_g(2,4)$, and not as a subvariety of the projective space $\mathbb{P}^{2^g-1}$. We now define the* small Schottky-Jung locus *to be*

$$(2.13) \qquad \mathcal{S}_g(2,4) := \bigcap_\eta \mathcal{S}_g^\eta(2,4),$$

where η runs over the set $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}\setminus\{0\}$. We note that the action of Γ_g permutes the different η and the ideals S_g^η. Therefore the ideal defining $\mathcal{S}_g(2,4)$ is Γ_g-invariant, and the locus $\mathcal{S}_g(2,4)$ is a preimage of some $\mathcal{S}_g \subset \mathcal{A}_g$ under the level cover.

Theorem 2.5. *(a) The Jacobian locus J_g is an irreducible component of the small Schottky-Jung locus $\mathcal{S}_g$.*
(b) $J_g(2,4)$ is an irreducible component of the big Schottky-Jung locus $\mathcal{S}_g^\eta(2,4)$ for any η.

Proof. The statement (a) was proved by van Geeman [58] and the statement (b) was proved by Donagi [36]. $\qquad\square$

Donagi [37] conjectured the following.

Conjecture 2.1. *The small Schottky-Jung locus is equal to the Jacobian locus, that is, $\mathcal{S}_g = J_g$.*

(C) The Andreotti-Mayer Approach

We let $\mathrm{Sing}\,\Theta$ be the singularity set of the theta divisor Θ for a principally polarized abelian variety (A, Θ).

Theorem 2.6. *For a non-hyperelliptic curve C of genus g, we have $\dim(\mathrm{Sing}\,\Theta_{J(C)}) = g-4$, and for a hyperelliptic curve C, $\dim(\mathrm{Sing}\,\Theta_{J(C)}) = g-3$. For a generic principally polarized abelian variety, the theta divisor is smooth.*

Proof. The proof was given by Andreotti and Mayer [8]. $\qquad\square$

Definition 2.5. *We define the k-th* Andreotti-Mayler locus *to be*

$$N_{k,g} := \{ (A, \Theta) \in \mathcal{A}_g \mid \dim \operatorname{Sing} \Theta \geq k \ \}.$$

Theorem 2.7. $N_{g-2,g} = \mathcal{A}_g^{\mathrm{dec}}$. *Here*

$$\mathcal{A}_g^{\mathrm{dec}} := \left(\bigcup_{k=1}^{g-1} \mathcal{A}_k \times \mathcal{A}_{g-k} \right) \subset \mathcal{A}_g$$

denotes the locus of decomposable ppavs (product of lower-dimensional ppavs) of dimension g.

Proof. The proof was given by Ein and Lazasfeld [43]. □

Theorem 2.8. J_g *is an irreducible component of* $N_{g-4,g}$, *and the locus of hyperelliptic Jacobians* Hyp_g *is an irreducible component of* $N_{g-3,g}$.

Proof. The proof was given by Andreotti and Mayer [8]. □

Theorem 2.9. *The Prym locus* $\mathcal{P}_g$ *is an irreducible component of* $N_{g-6,g}$.

Proof. The proof was given by Debarre [31]. □

Theorem 2.10. *The locus of Jacobians of curves of genus 4 with a vanishing theta-null is equal to the locus of 4-dimensional principally polarized abelian varieties for which the double point singularity of the theta divisor is not ordinary (i.e., the tangent cone does not have maximal rank).*

Proof. See Grushevsky-Salvati Manni [69] and Smith-Varley [139]. □

Problem. Can it happen that $N_{k,g} = N_{k+1,g}$ for some k, g ?

(D) The Approach via the K-P Equation

In his study of solutins of nonlinear equations of Korteveg de Vrie type, I. Krichever [84] proved the following fact :

Theorem 2.11. *Let τ be the period matrix of a curve C of genus g and let $\theta(z)$ (cf. (2.2)) be the Riemann theta function of the Jacobian $J(C)$. Then there exist three vectors W_1, W_2, W_3 in $\mathbb{C}^g$ with $W_1 \neq 0$ such that, for every $z \in \mathbb{C}^g$, the function*

$$(2.14) \qquad u(x, y, z; t) := \frac{\partial^2}{\partial x^2} \log \theta(x W_1 + y W_2 + t W_3 + z)$$

satisfies the so-called Kadomstev-Petriashvili equation *(briefly the K-P equation)*

$$(2.15) \qquad 3\, u_{yy} = (u_t - 3u u_x - 2u_{xxx})_x \, .$$

S. P. Novikov conjectured that $\tau \in \mathbb{H}_g$ is the period matrix of a curve if and only if the Riemann theta function corresponding to $\tau \in \mathbb{H}_g$ satisfies the K-P equation in the sense we just explained in Theorem 2.11. Shiota [136] proved that the Novikov conjecture is true, following the work of Mumford [103] and Mulase [102]. Arbarello and De Concini [10] gave another proof of the Novikov conjecture.

(E) The Approach via Geometry of the Kummer Variety

Definition 2.6. *The map is the embedding given by*

$$(2.16) \qquad \mathrm{Kum} : A_\tau/\pm 1 \longrightarrow \mathbb{P}^{2^g-1}(\mathbb{C}), \qquad \mathrm{Kum}(z) = \left\{ \Theta[\epsilon](\tau, z) \middle| \; \epsilon \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \right\}.$$

We call the image of Kum *the* Kummer *variety. Note that the involution* ± 1 *has* 2^{2g} *fixed points on* A_τ *which are precisely* $A_\tau[2]$, *and thus the Kummer variety singular at their images in* $\mathbb{P}^{2^g-1}(\mathbb{C})$.

Theorem 2.12. *For any points* p_1, p_2, p_3 *of a curve of genus* g, *the following three points on the Kummer variety are collinear:*

$$(2.17) \qquad \mathrm{Kum}(p + p_1 - p_2 - p_3), \; \mathrm{Kum}(p + p_2 - p_1 - p_3), \; \mathrm{Kum}(p + p_3 - p_1 - p_2).$$

Proof. See Fay [47] and Gunning [71]. $\qquad\qquad\square$

Theorem 2.13. *For any curve* $C \in \mathcal{M}_g$, *for any* $1 \le k \le g$ *and for any* $p_1, \cdots, p_{k+2}, q_1, \cdots, q_k \in C$ *the* $k+2$ *points of the Kummer variety*

$$(2.18) \qquad \mathrm{Kum}\left(2p_j + \sum_{i=1}^{k} q_i - \sum_{i=1}^{k=2} p_i \right), \quad j = 1, \cdots, k+2$$

are linearly dependent.

Proof. See Gunning [72]. $\qquad\qquad\square$

I. Krichever [85] gave a complete proof of a conjecture of Welters concerning a condition for an indecomposable principally polarized abelian variety to be the Jacobian of a curve:

Theorem 2.14. *Let* $\mathcal{A}_g^{\mathrm{ind}} := \mathcal{A}_g \backslash \mathcal{A}_g^{\mathrm{dec}}$ *be the locus of indecomposable ppavs of dimension* g. *For a ppav* $A \in \mathcal{A}_g^{\mathrm{ind}}$, *if* $\mathrm{Kum}(A) \subset \mathbb{P}^{2^g-1}$ *has one of the following*

- *(W1) a trisecant line*
- *(W2) a line tangent to it at one point, and intersecting it another point (this is a semi-degenerate trisecant, when two points of secancy coincides)*
- *(W3) a flex line (this is a most degenerate trisecant when all three points of secancy coincide)*

such that none of the points of intersection of this line with the Kummer variety are $A[2]$ *(where* $\mathrm{Kum}(A)$ *is singular), then* $A \in J_g$.

For the Prym-Schottky problem, it will be natural whether the Kummer varieties of Pryms have any special geometric properties. Indeed, Fay [48] and Beauville-Debarre [16] obtained the following.

Theorem 2.15. *Let* $C \in \mathcal{M}_g$. *For any* $p, p_1, p_2, p_3 \in \tilde{C} \longrightarrow Prym(\tilde{C} \longrightarrow C)$ *on the Abel-Prym curve the following four points of the Kummer variety*

$$\mathrm{Kum}(p + p_1 + p_2 + p_3), \qquad \mathrm{Kum}(p + p_1 - p_2 - p_3),$$
$$\mathrm{Kum}(p + p_2 - p_1 - p_3), \qquad \mathrm{Kum}(p + p_3 - p_1 - p_2)$$

lie on a 2-plane in $\mathbb{P}^{2^g-1}(\mathbb{C})$.

A suitable analog of the trisecant conjecture was found for Pryms using ideas of integrable systems by Grushevsky and Krichever [67]. They proved the following.

Theorem 2.16. *If for some $A \in \mathcal{A}_g^{\mathrm{ind}}$ and some $p, p_1, p_2, p_3 \in A$ the quadrisecant condition in Theorem 2.15 holds, and moreover there exists another quadrisecant given by Theorem 2.15 with p replaced by $-p$, then $A \in \tilde{\mathcal{P}}_g$.*

(F) The Approach via the Γ_{00} Conjecture

Definition 2.7. *Let $(A, \Theta) \in \mathcal{A}_g$. The linear system $\Gamma_{00} \subset |2\Theta|$ is defined to consist of those sections that vanish to order at least 4 at the origin:*

$$(2.19) \qquad \Gamma_{00} := \{ f \in H^0(A, 2\Theta) \,|\, \mathrm{mult}_0 f \geq 4 \,\}.$$

We define the base locus

$$F_A := \{ x \in A \,|\, s(x) = 0 \quad \text{for all } s \in \Gamma_{00} \,\}.$$

Theorem 2.17. *For any $g \geq 5$ and any $C \in \mathcal{M}_g$, we have on the Jacobian $J(C)$ of C the equality*

$$F_{J(C)} = C - C = \{ x - y \in J(C) \,|\, x, y \in C \,\}.$$

Proof. The above theorem was proved by Welters [153] set theoretically and also by Izadi scheme-theoretically. Originally Theorem 2.17 was conjectured by van Geeman and van der Geer [59]. $\qquad\square$

van Geeman and van der Geer [59] conjectured the following.

Γ_{00} **Conjecture.** Let $(A, \Theta) \in \mathcal{A}_g^{\mathrm{ind}}$. If $F_A \neq 0$, then $A \in J_g$.

Definition 2.8. *Let $(A, \Theta) \in \mathcal{A}_g$. For any curve Γ on A and any point $x \in A$, we define*

$$\varepsilon(A, \Gamma, x) := \frac{\Theta.\Gamma}{\mathrm{mult}_x \Gamma}, \qquad \varepsilon(A, x) := \inf_{\Gamma \ni x} \varepsilon(A, \Gamma, x).$$

We define the Seshadri constant *of (A, Θ) by*

$$\varepsilon(A) := \varepsilon(A, \Theta) := \inf_{x \in A} \varepsilon(A, x).$$

Theorem 2.18. *If the Γ_{00} conjecture holds, hyperelliptic Jacobians are characterized by the value of their Seshadri constants.*

Proof. See O. Debarre [33]. $\qquad\square$

Theorem 2.19. *If some $A \in \mathcal{A}_g^{\mathrm{ind}}$, the linear dependence*

$$(2.20) \qquad \Theta[\epsilon](\tau, z) = c\, \Theta[\epsilon](\tau, 0) + \sum_{1 \leq a \leq b \leq g} c_{ab} \frac{\partial \Theta[\epsilon](\tau, 0)}{\partial \tau_{ab}}$$

for some $c, c_{ab} \in \mathbb{C}$ ($1 \leq a \leq b \leq g$) and for all $\epsilon \in \frac{1}{2}\mathbb{Z}^g / \mathbb{Z}^g$ holds with $\mathrm{rank}(c_{ab}) = 1$, then $A \in J_g$.

Proof. See S. Grushevsky [64]. ☐

(G) Subvarieties of a ppav: Minimal Cohomology Classes

The existence of some *special* subvarieties of a ppav $(A, \Theta) \in \mathcal{A}_g$ gives a criterion that A is the Jacobian of a curve. We start by observing that for the Jacobian $J(C)$ of a curve $C \in \mathcal{M}_g$ we can map the symmetric product $\mathrm{Sym}^d(C)$ $(1 \le d < g)$ to $J(C) = \mathrm{Pic}^{g-1}(C)$ by fixing a divisor $D \in \mathrm{Pic}^{g-1-d}(C)$ and mapping

$$(2.21) \qquad \Phi_{(d)} : \mathrm{Sym}^d(C) \longrightarrow J(C), \qquad (p_1, \cdots, p_d) \mapsto D + \sum_{i=1}^{d} p_i.$$

The image $W^d(C)$ of the map $\Phi_{(d)}$ is independent of D up to translation, and we can compute its cohomology class

$$\left[W^d(C) \right] = \frac{[\Theta]^d}{(g-d)!} \in H^{2g-2d}(J(C)),$$

where $[\Theta]$ is the cohomology class of the polarization of $J(C)$. One can show that the cohomology class is indivisible in cohomology with $\mathbb{Z}$-coefficients, and we thus call this class minimal. We note that $W^1(C) \simeq C$. These subvarieties $W^d(C)$ $(1 \le d < g)$ are very *special*.

We have the following criterion.

Theorem 2.20. *A ppav $(A, \Theta) \in \mathcal{A}_g$ is a Jacobian if and only if there exists a curve $C \subset A$ with $[C] = \frac{[\Theta]^{g-1}}{(g-1)!} \in H^{2g-2}(J(C))$ in which case $(A, \Theta) = J(C)$.*

Proof. See Matsusaka [98] and Ran [123]. ☐

Debarre [32] proved that J_g is an irreducible component of the locus of ppavs for which there is a subvariety of the minimal cohomology class. He conjectured the following.

Conjecture 2.2. *If a ppav $(A, \Theta) \in \mathcal{A}_g$ has a d-dimensional subvariety of minimal class, then it is either the Jacobian of a curve or a five dimensional intermediate Jacobian of a cubic threefold.*

This approach to the Schottky problem gives a complete geometric solution to the weaker version of the problem: determining whether a given ppav is the Jacobian of a given curve.

3. Logarithmical Canonical Line Bundles on Toroidal Compactifications of the Siegel Modular Varieties

In this section, we review the interesting results obtained by S.-T. Yau and Y. Zhang [187] concerning the asymptotic behaviors of the logarithmic canonical line bundle on a toroidal compactification of the Siegel modular variety.

Let Γ be a neat arithmetic subgroup of Γ_g. Let $\mathcal{A}_{g,\Gamma} := \Gamma \backslash \mathbb{H}_g$ and $\overline{\mathcal{A}}_{g,\Gamma}$ be the toroidal compactification of $\mathcal{A}_{g,\Gamma}$ constructed by a $GL(g, \mathbb{Z})$-admissible family of polyhedral decompositions $\Sigma_{\mathcal{F}_0}$ of the cones. Here $\mathcal{F}_0$ denotes the standard minimal cusps of $\mathbb{H}_g$. $\overline{\mathcal{A}}_{g,\Gamma}$ is an algebraic space, but is not a projective variety in general. Y.-S. Tai proved that if Σ_{tor}^{Γ} is projective (see [6], Chapter IV, Corollary 2.3, p. 200), then $\overline{\mathcal{A}}_{g,\Gamma}$ is a projective variety.

It is known that $\overline{\mathcal{A}}_{g,\Gamma}$ is the unique Hausdorff analytic variety containing $\mathcal{A}_{g,\Gamma}$ as an open dense subset (cf. [6]).

Assume the boundary divisor $D_{\infty,\Gamma} := \overline{\mathcal{A}}_{g,\Gamma} \backslash \mathcal{A}_{g,\Gamma}$ is simple normal crossing. We put $N = g(g+1)/2$. For each irreducible component D_i of $D_{\infty,\Gamma} = \cup_j D_j$, let s_i a global section of the line bundle $[D_i]$ defining D_i. Let $\sigma_{\max}$ be an arbitrary top-dimensional cone in $\Sigma_{\mathcal{F}_0}$ and renumber all components $D_i's$ of $D_{\infty,\Gamma}$ such that $D_1, \cdots, D_N$ correspond to the edges of $\sigma_{\max}$ with marking order. Yau and Zhang [187, Theorem 3.2] showed that the volume form $\Phi_{g,\Gamma}$ on $\mathcal{A}_{g,\Gamma}$ may be written by

$$\Phi_{g,\Gamma} = \frac{2^{N-g} \operatorname{Vol}_\Gamma(\sigma_{\max})^2 \, d\mathcal{V}_g}{\left(\prod_{j=1}^N \|s_j\|^2\right) F_{\sigma_{\max}}^{g+1}(\log\|s_1\|_1, \cdots, \log\|s_N\|_N)},$$

where $d\mathcal{V}_g$ is a continuous volume form on a partial compactification $\mathcal{U}_{\sigma_{\max}}$ of $\mathcal{A}_{g,\Gamma}$ with $\mathcal{A}_{g,\Gamma} \subset \mathcal{U}_{\sigma_{\max}} \subset \overline{\mathcal{A}}_{g,\Gamma}$, and each $\|\cdot\|_j$ is a suitable Hermitian metric of the line bundle $[D_j]$ on $\overline{\mathcal{A}}_{g,\Gamma}$ ($1 \le j \le N$) and $F_{\sigma_{\max}} \in \mathbb{Z}[x_1, \cdots, x_N]$ is a homogeneous polynomial of degree g. Moreover the coefficients of $F_{\sigma_{\max}}$ depends only on both Γ and $\sigma_{\max}$ with marking order of edges. Using the above volume form formula they showed that the unique invariant Kähler-Einstein metric on $\mathcal{A}_{g,\Gamma}$ endows some restraint combinatorial conditions for all smooth toroidal compactifications of $\mathcal{A}_{g,\Gamma}$.

Let $E_1, \cdots, E_d$ be any different irreducible components of the boundary divisor $D_{\infty,\Gamma}$ such that $\cap_{k=1}^d E_k \ne \emptyset$. Let $K_{g,\Gamma}$ be the canonical line bundle on $\overline{\mathcal{A}}_{g,\Gamma}$. Yau and Zhang [187] also proved the following facts (a) and (b):

(a) Let $i_1, \cdots, i_d \in \mathbb{Z}^+$. If $d \ge g-1$ and $N - \sum_{k=1}^d i_k > 2$ (or if $d \ge g$ and $N - \sum_{k=1}^d i_k = 1$), then we have

$$(K_{g,\Gamma} + D_{\infty,\Gamma})^{N - \sum_{k=1}^d i_k} \cdot E_1^{i_1} \cdots E_d^{i_d} = 0.$$

(b) $K_{g,\Gamma} + D_{\infty,\Gamma}$ is not *ample* on $\overline{\mathcal{A}}_{g,\Gamma}$.

They also showed that if $d < g - 1$, then the intersection number

$$(K_{g,\Gamma} + D_{\infty,\Gamma})^{N-d} \cdot E_1 \cdots E_d$$

can be expressed explicitly using the above volume form formula. The proofs of (a) and (b) can be found in Theorem 4.15 in [187].

4. **Brief Review on the André-Oort Conjecture**

In this section we review recent progress on the André-Oort conjecture quite briefly.

Definition 4.1. *Let (G, X) be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_f)$. We let*

$$Sh_K(G, X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

be the Shimura variety associated to (G, X). An algebraic subvariety Z of the Shimura variety $Sh_K(G, X)$ is said to be weakly special *if there exist a Shimura sub-datum (H, X_H) of (G, X), and a decomposition*

$$(H^{\mathrm{ad}}, X_H^{\mathrm{ad}}) = (H_1, X_1) \times (H_2, X_2)$$

and $y_2 \in X_2$ such that Z is the image of $X_1 \times \{y_2\}$ in $Sh_K(G, X)$. Here $(H^{\mathrm{ad}}, X_H^{\mathrm{ad}})$ denotes the adjoint Shimura datum associated to (G, X) and (H_i, X_i) $(i = 1, 2)$ are Shimura data. In this definition, a weakly special subvariety is said to be special if it contains a special point and y_2 is special.

André [7] and Oort [115] made conjectures analogous to the Manin-Mumford conjecture where the ambient variety is a Shimura variety (the latter partially motivated by a conjecture of Coleman [30]). A combination of these has become known as the André-Oort conjecture (briefly the A-O conjecture).

A-O Conjecture. Let S be a Shimura variety and let Σ be a set of special points in S. Then every irreducible component of the Zariski closure of Σ is a special subvariety.

Definition 4.2. [118, 119] A pre-structure is a sequence $\Sigma = (\Sigma_n : n \geq 1)$ where each Σ_n is a collection of subsets of $\mathbb{R}^n$. A pre-structure Σ is called a structure over the real field if, for all $n, m \geq 1$ with $m \leq n$, the following conditions are satisfied:
   (1) Σ_n is a Boolean algebra (under the usual set-theoretic operations);
   (2) Σ_n contains every semi-algebraic subset of $\mathbb{R}^n$;
   (3) if $A \in \Sigma_m$ and $B \in \Sigma_n$, then $A \times B \in \Sigma_{m+n}$;
   (4) if $n \geq m$ and $A \in \Sigma_n$, then $\pi_{n,m}(A) \in \Sigma_m$, where $\pi_{n,m} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a coordinate projection on the first m coordinates.

If Σ is a structure, and, in addition,

   (5) the boundary of every set in Σ_1 is finite,

then Σ is called an o-minimal structure over the real field.

If Σ is a structure and $Z \subset \mathbb{R}^n$, then we say that Z is definable in Σ if $Z \in \Sigma_n$. A function $f : A \longrightarrow B$ is definable in in a structure Σ if its graph is definable, in which case the domain A of f and image $f(A)$ are also definable by the definition. If $A, \cdots, f, \cdots$ are sets or functions, then we denote by $\mathbb{R}_{A, \cdots, f, \dots}$ the smallest structure containing $A, \cdots, f, \cdots$. By a definable family of sets we mean a definable subset $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ which we view as a family of fibres $Z_y \subset \mathbb{R}^n$ as y varies over the projection of Z onto $\mathbb{R}^m$ which is definable, along with all the fibres Z_y. A family of functions is said to be definable if the family of their graphs is. A definable set usually means a definable set in some o-minimal structure over the real field.

Remark 4.1. The notion of a o-minimal structure grew out of work van den Dries [38, 39] on Tarski's problem concerning the decidability of the real ordered field with the exponential function, and was studied in the more general context of linearly ordered structures by Pillay and Steinhorn [122], to whom the term "o-minimal" ("order-minimal") is due.

In 2011 Pila gave a unconditional proof of the A-O conjecture for arbitrary products of modular curves using the theory of o-minimality.

Theorem 4.1. Let
$$X = Y_1 \times \cdots \times Y_n \times E_1 \times \cdots \times E_m \times \mathbb{G}_m^{\ell},$$

where $n, m, \ell \geq 0$, $Y_i = \Gamma_{(i)} \backslash \mathbb{H}_1 (1 \leq i \leq n)$ are modular curves corresponding to congruence subgroups $\Gamma_{(i)}$ of $SL(2, \mathbb{Z})$ and $E_j (1 \leq j \leq m)$ are elliptic curves defined over $\overline{\mathbb{Q}}$ and $\mathbb{G}_m$ is the multiplicative group. Suppose V is a subset of X. Then V contains only a finite number of maximal special subvarieties.

Proof. See Pila [118], Theorem 1.1. □

In 2013 Peterzil and Starchenko proved the following theorem using the theory of o-minimality.

Theorem 4.2. *The restriction of the uniformizing map $\pi : \mathbb{H}_g \longrightarrow \mathcal{A}_g$ to the classical fundamental domain for the Siegel modular group $Sp(2g, \mathbb{Z})$ is definable.*

Proof. See Peterzil and Starchenko [116, 117]. □

In 2014 Pila and Tsimerman gave a conditional proof of the A-O conjecture for the Siegel modular variety $\mathcal{A}_g$.

Theorem 4.3. *If $g \leq 6$, then the A-O conjecture holds for $\mathcal{A}_g$. If $g \geq 7$, the A-O conjecture holds for $\mathcal{A}_g$ under the assumption of the Generalized Riemann Hypothesis (GRH) for CM fields.*

Proof. See Pila-Tsimerman [120, 121]. □

Quite recently using Galois-theoretic techniques and geometric properties of Hecke correspondences, Klingler and Yafaev proved the A-O conjecture for a general Shimura variety, and independently using Galois-theoretic and ergodic techniques Ullmo and Yafaev proved the A-O conjecture for a general Shimura variety, under the assumption of the GRH for CM fields or another suitable assumption. The explicit statement is given as follows.

Theorem 4.4. *Let (G, X) be a Shimura datum and K a compact open subgroup of $G(\mathbb{A}_f)$. Let Σ be a set of special points in $Sh_K(G, X)$. We make one of the two following assumptions:*

(1) Assume the GRH for CM fields.

(2) Assume that there exists a faithful representation $G \hookrightarrow GL_n$ such that with respect to this representation, the Mumford-Tate group $MT(s)$ lie in one $GL_n(\mathbb{Q})$-conjugacy class as s ranges through Σ. Then every irreducible component of Σ in $Sh_K(G, X)$ is a special subvariety.

Proof. See Klingler-Yafaev [81] and Ullmo-Yafaev [144]. □

Remark 4.2. *We refer to [119] for the theory of o-minimality and the A-O conjecture. We also refer to [57] for the A-O conjecture for mixed Shimura varieties.*

5. **Okounkov Bodies associated to Divisors**

In this section, we briefly review the theory of Okounkov convex bodies associated to pseudoeffective divisors on a smooth projective variety. For more details of this theory, we refer to [86] and [22, 23, 24, 25].

Let X be a smooth projective variety of dimension d. We fix an admissible flag $Y_\bullet$ on X

$$Y_\bullet: \ X = Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset Y_{d-1} \supset Y_d = \{x\},$$

where each Y_k is a subvariety of X of codimension k which is nonsingular at x. We let $\mathbb{Z}_+$ denote the set of all non-negative integers. We first assume that D is a big Cartier divisor on X. For a section $s \in H^0(X, \mathcal{O}_X(D))\backslash\{0\}$, we define the function

$$\nu(s) = \nu_{Y_\bullet}(s) := (\nu_1(s), \cdots, \nu_d(s)) \in \mathbb{Z}_+^d$$

as follows:

First we set $\nu_1 = \nu_1(s) := \operatorname{ord}_{Y_1}(s)$. Using a local equation f_1 for Y_1 in X, we define naturally a section

$$s_1' = s \otimes f_1^{-\nu_1} \in H^0(X, \mathcal{O}_X(D - \nu_1 Y_1))$$

that does not vanish along Y_1, its restriction $s_1'|_{Y_1}$ defines a nonzero section

$$s_1 := s_1'|_{Y_1} \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1)).$$

We now take

$$\nu_2(s) := \operatorname{ord}_{Y_2}(s_1).$$

and continue in this manner to define the remaining $\nu_i(s)$.

Next we define

$$\operatorname{vect}(|D|) = \operatorname{Im}\left(\nu_{Y_\bullet}: (H^0(X, \mathcal{O}_X(D)) - \{0\}) \longrightarrow \mathbb{Z}^d\right)$$

be the set of valuation vectors of non-zero sections of $\mathcal{O}_X(D)$. Then we finally set

$$(5.1) \qquad \Delta(D) := \Delta_{Y_\bullet}(D) = \text{closed convex hull}\left(\bigcup_{m \geq 1} \frac{1}{m} \cdot \operatorname{vect}(|mD|)\right).$$

Therefore $\Delta(D)$ is a convex body in $\mathbb{R}^d$ that is called the **Okounkov body** of D with respect to the fixed flag $Y_\bullet$. We refer to [86, §1.2] for some properties and examples of $\Delta(D)$.

We recall that a **graded linear series** $W_\bullet(D) = \{W_m(D)\}_{m \geq 0}$ associated to D consists of subspaces

$$W_m := W_m(D) \subseteq H^0(X, \mathcal{O}_X(mD)), \qquad W_0 = \mathbb{C}$$

satisfying the inclusion

$$W_k \cdot W_l \subseteq W_{k+l} \qquad \text{for all } k, l \geq 0.$$

Here the product on the left denotes the image of $W_k \otimes W_l$ under the multiplication map $H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(lD)) \longrightarrow H^0(X, \mathcal{O}_X((k+l)D))$.

Definition 5.1. *([86, Definition 1.16]) Let $W_\bullet$ be a graded linear series on X belonging to a divisor D. The* **graded semigroup** *of $W_\bullet$ is defined to be*

$$\Gamma(W_\bullet) := \Gamma_{Y_\bullet}(W_\bullet) = \{(\nu_{Y_\bullet}(s), m) \mid 0 \neq s \in W_m, \ m \geq 0\} \subseteq \mathbb{Z}_+^d \times \mathbb{Z}_+ \subseteq \mathbb{Z}^{d+1}.$$

Under the above notations, we associate the convex body $\Delta_{Y_\bullet}(W_\bullet)$ of a graded linear series $W_\bullet$ with respect to $Y_\bullet$ on X as follows:

$$(5.2) \qquad \Delta_{Y_\bullet}(W_\bullet) := \sum(\Gamma(W_\bullet)) \cap \left(\mathbb{R}_+^d \times \{1\}\right),$$

where $\mathbb{R}_+$ denotes the set of all non-negative real numbers and $\sum(\Gamma(W_\bullet))$ denotes the closure of the convex cone in $\mathbb{R}_+^d \times \mathbb{R}_+$ spanned by $\Gamma(W_\bullet)$. $\Delta_{Y_\bullet}(W_\bullet)$ is called the **Okounkov**

body *of $W_\bullet$ with respect to $Y_\bullet$. If $W_\bullet$ is a complete graded linear series, that is, $W_m = H^0(X, \mathcal{O}(mD))$ for each m, then we define*

$$(5.3) \qquad\qquad\qquad \Delta_{Y_\bullet}(D) := \Delta_{Y_\bullet}(W_\bullet).$$

Remark 5.1. *$\Delta_{Y_\bullet}(D)$ depends on the choice of an admissible flag $Y_\bullet$. By the homogeneity of $\Delta_{Y_\bullet}(D)$ (see [86], Proposition 4.13), we can extend the construction of $\Delta_{Y_\bullet}(D)$ to $\mathbb{Q}$-divisors D and even to $\mathbb{R}$-divisors using the continuity of $\Delta_{Y_\bullet}(D)$.*

Definition 5.2. *([86, Definition 2.5 and 2.9])*
(I) We say that a graded linear series $W_\bullet$ satisfies Condition (B) *if $W_m \neq 0$ for all $m \gg 0$, and for all sufficiently large m, the rational map $\phi_m : X \dashrightarrow \mathbb{P}(W_m)$ defined by $|W_m|$ is birational on its image.*
(II) We say that a graded linear series $W_\bullet$ satisfies Condition (C) *if*
   (1) for any $m \gg 0$, there exists an effective divisor F_m such that $A_m := mD - F_m$ is ample, and
   (2) for all sufficiently large t, we have

$$H^0(X, \mathcal{O}_X(tA_m)) \subseteq W_{tm} \subseteq H^0(X, \mathcal{O}_X(tmA_m)).$$

If $W_\bullet$ is complete, that is, $W_m = H^0(X, \mathcal{O}_X(mA_m))$ for all $m \geq 0$ and D is big, then it satisfies Condition (C).

Lazarsfeld and Mustață [86] proved the following.

Theorem 5.1. *Let X be a smooth projective variety of dimension d. Suppose that a graded linear series $W_\bullet$ satisfies Condition (B) or Condition (C). Then for any admissible flag $Y_\bullet$ on X, we have*

$$\dim \Delta_{Y_\bullet}(W_\bullet) = \dim X = d$$

and

$$\mathrm{Vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(W_\bullet)) = \frac{1}{d!} \mathrm{Vol}_X(W_\bullet),$$

where

$$\mathrm{Vol}_X(W_\bullet) := \lim_{n \longrightarrow \infty} \frac{\dim W_m}{m^d/d!}.$$

Proof. See Theorem 2.13 in [86]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.2. *It is known by Lazarsfeld and Mustață ([86], Proposition 4.1) that for a fixed admissible flag $Y_\bullet$ on X, if D is big, then $\Delta_{Y_\bullet}(D)$ depends only on the numerical class of D. If D is not big, then it is not true (cf. Remark 3.13 in [22]).*

Definition 5.3. *For a divisor D on X, we let*

$$\mathbb{N}(D) := \left\{ m \in \mathbb{Z}^+ \,|\, |\lfloor mD \rfloor| \neq \emptyset \right\}.$$

For $m \in \mathbb{N}(D)$, we let

$$\Phi_{mD} : X \dashrightarrow \mathbb{P}^{\dim |\lfloor mD \rfloor|}$$

be the rational map defined by the linear system $|\lfloor mD \rfloor|$. We define the Iitaka dimension *of D as the following value*

$$\kappa(D) := \begin{cases} \max\{\dim \mathrm{Im}(\Phi_{mD}) \,|\, m \in \mathbb{N}(D)\} & \text{if } \mathbb{N}(D) \neq \emptyset \\ -\infty & \text{if } \mathbb{N}(D) = \emptyset. \end{cases}$$

Definition 5.4. *Let D be a divisor on X such that $\kappa(D) \geq 0$. A subset U of X is called a* Nakayama subvariety *of D if $\kappa(D) = \dim D$ and the natural map*

$$H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) \longrightarrow H^0(U, \mathcal{O}_U(\lfloor mD|_U \rfloor))$$

is injective for every non-negative integer m.

Definition 5.5. [22, Definition 3.8] *Let D be a divisor on X such that $\kappa(D) \geq 0$. The* valuative Okounkov body $\Delta_{Y_\bullet}^{\mathrm{val}}(D)$ *associated to D is defined to be*

$$\Delta_{Y_\bullet}^{\mathrm{val}}(D) := \Delta_{Y_\bullet}(D) \subset \mathbb{R}^n, \qquad n = \dim X.$$

For a divisor D with $\kappa(D) = -\infty$, we define $\Delta_{Y_\bullet}^{\mathrm{val}}(D) := \emptyset$.

Remark 5.3. *If D is big, then $\Delta_{Y_\bullet}^{\mathrm{val}}(D)$ coincides with $\Delta_{Y_\bullet}(D)$ for any admissible flag $Y_\bullet$ on X.*

Recently Choi, Hyun, Park and Won [22] showed the following.

Theorem 5.2. *Let D be a divisor with $\kappa(D) \geq 0$ on a smooth projective variety X of dimension n. Fix an admissible flag $Y_\bullet$ containing a Nakayama subvariety U of D such that $Y_n = \{x\}$ is a general point. Then we have*

$$\dim \Delta_{Y_\bullet}^{\mathrm{val}}(D) = \kappa(D)$$

and

$$\mathrm{Vol}_{\mathbb{R}^{\kappa(D)}}(\Delta_{Y_\bullet}^{\mathrm{val}}(D)) = \frac{1}{\kappa(D)!} \mathrm{Vol}_{X|U}(D).$$

Proof. See Theorem 3.12 in [22]. $\qquad\qquad\square$

Definition 5.6. [22, Definition 3.17] *Let D be a pseudo-effective divisor on a projective variety X of dimension n. The* limiting Okounkov body $\Delta_{Y_\bullet}^{\lim}(D)$ *of D with respect to an admissible flag $Y_\bullet$ is defined to be*

$$\Delta_{Y_\bullet}^{\lim}(D) := \lim_{\varepsilon \to 0^+} \Delta_{Y_\bullet}(D + \varepsilon A) \subset \mathbb{R}^n,$$

where A is any ample divisor on X. If D is not a pseudo-effective divisor, then we define $\Delta_{Y_\bullet}^{\lim}(D) := \emptyset$.

Definition 5.7. [22, Definition 2.11] *Let D be a divisor on a projective variety X of dimension d. We defne the* numerical litaka dimension $\kappa_\nu(D)$ *by*

$$\kappa_\nu(D) := \max \left\{ k \in \mathbb{Z}_+ \mid \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(|mD| + A))}{m^k} > 0 \right\}$$

for a fixed ample Cartier divisor A if $h^0(X, \mathcal{O}_X(|mD| + A)) \neq \emptyset$ for infinitely many $m > 0$, and we define $\kappa_\nu(D) = -\infty$ otherwise.

Let D be a pseudo-effective Cartier divisor on a projective variety X of dimension n. Let $V \subseteq X$ be a positive volume subvariety of D. Fix an admissible flag $V_\bullet$ on V

$$V_\bullet : \ V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_{n-1} \supset V_n = \{x\}.$$

let A be an ample Catier divisor on X. For each positive integer k, we consider the restricted graded linear series $W_\bullet^k := W_\bullet(kD + A|V)$ of $kD + A$ along V given by

$$W_m(kD + A|V) = H^0(X|V, m(kD + A)) \qquad \text{for } m \geq 0.$$

We define the restricted limiting Okounkov body of a Cartier divisor D with respect to a positive volume subvariety V of D as

$$\Delta_{V_\bullet}^{\lim}(D) := \lim_{k \to \infty} \frac{1}{k} \Delta_{V_\bullet}(W_\bullet^k) \subseteq \mathbb{R}^{\kappa_\nu(D)}.$$

By the continuity, we can extend this definition for any pseudo-effective $\mathbb{R}$-divisor.

Definition 5.8. *Let D be a pseudo-effective divisor on a projective variety X of dimension n with its positive volume subvariety $V \subseteq X$. We define the* restricted limiting Okounkov *body $\Delta_{V_\bullet}^{\lim}(D)$ of D with respect to an admissible flag $V_\bullet$ to be a closed convex subset*

$$\Delta_{V_\bullet}^{\lim}(D) := \lim_{\varepsilon \to 0^+} \Delta_{V_\bullet}(D + \varepsilon A) \subseteq \mathbb{R}^{\kappa_\nu(D)} \hookrightarrow \mathbb{R}^n,$$

where A is any ample divisor on X. If D is not a pseudo-effective divisor, then we define $\Delta_{V_\bullet}^{\lim}(D) := \emptyset$.

Recently Choi, Hyun, Park and Won [22] proved the following.

Theorem 5.3. *Let D be a pseudo-effective divisor on a projective variety X. Fix a positive volume subvariety $V \subseteq X$ of D (see [22, Definition 2.13]. For an admissible flag $V_\bullet$ of V, we have*

$$\dim \Delta_{V_\bullet}^{\lim}(D) = \kappa_\nu(D)$$

and

$$\mathrm{Vol}_{\mathbb{R}^{\kappa_\nu(D)}}(\Delta_{V_\bullet}^{\lim}(D)) = \frac{1}{\kappa_\nu(D)!} \mathrm{Vol}_{X|V}^+(D).$$

Here $\mathrm{Vol}_{X|V}^+(D)$ denotes the augmented restricted volume of D along V (see [22, Definition 2.2]) for the precise definition of $\mathrm{Vol}_{X|V}^+(D)$).

Proof. See Theorem 3.20 in [22]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

6. The Relations of the Schottky Problem to the André-Oort Conjecture, Okounkov Bodies and Coleman's Conjecture

In this section, we discuss the relations among logarithmical line bundles on toroidal compactifications, the André-Oort conjecture, Okounkov convex bodies, Coleman's conjecture and the Schottky problem.

For $\tau = (\tau_{ij}) \in \mathbb{H}_g$, we write $\tau = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real. We put $d\tau = (d\tau_{ij})$ and $d\bar{\tau} = (d\bar{\tau}_{ij})$. We also put

$$\frac{\partial}{\partial \tau} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \tau_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{\tau}} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \bar{\tau}_{ij}} \right).$$

C. L. Siegel [137] introduced the symplectic metric $ds_{g;A}^2$ on $\mathbb{H}_g$ invariant under the action (1.1) of $Sp(2g, \mathbb{R})$ that is given by

(6.1) $$ds_{g;A}^2 = A \operatorname{tr}(Y^{-1} d\tau \, Y^{-1} d\bar{\tau}), \qquad A \in \mathbb{R}^+$$

and H. Maass [94] proved that its Laplacian is given by

$$
(6.2) \qquad \Box_{g;A} = \frac{4}{A} \operatorname{tr}\left(Y \, {}^t\!\left(Y \frac{\partial}{\partial \overline{\tau}}\right) \frac{\partial}{\partial \tau}\right).
$$

Here $\operatorname{tr}(M)$ denotes the trace of a square matrix M. And

$$
(6.3) \qquad dv_g(\tau) = (\det Y)^{-(g+1)} \prod_{1 \le i \le j \le g} dx_{ij} \prod_{1 \le i \le j \le g} dy_{ij}
$$

is a $Sp(2g, \mathbb{R})$-invariant volume element on $\mathbb{H}_g$ (cf. [138], p. 130).

Siegel proved the following theorem for the Siegel space $(\mathbb{H}_g, ds_{g;1}^2)$.

Theorem 6.1. *(Siegel [137]). (1) There exists exactly one geodesic joining two arbitrary points τ_0, τ_1 in $\mathbb{H}_g$. Let $R(\tau_0, \tau_1)$ be the cross-ratio defined by*

$$
R(\tau_0, \tau_1) = (\tau_0 - \tau_1)(\tau_0 - \overline{\tau}_1)^{-1}(\overline{\tau}_0 - \overline{\tau}_1)(\overline{\tau}_0 - \tau_1)^{-1}.
$$

For brevity, we put $R_* = R(\tau_0, \tau_1)$. Then the symplectic length $\rho(\tau_0, \tau_1)$ of the geodesic joining τ_0 and τ_1 is given by

$$
\rho(\tau_0, \tau_1)^2 = \operatorname{tr}\left(\left(\log \frac{1 + R_*^{\frac{1}{2}}}{1 - R_*^{\frac{1}{2}}}\right)^2\right),
$$

where

$$
\left(\log \frac{1 + R_*^{\frac{1}{2}}}{1 - R_*^{\frac{1}{2}}}\right)^2 = 4 R_* \left(\sum_{k=0}^{\infty} \frac{R_*^k}{2k+1}\right)^2.
$$

(2) For $M \in Sp(2g, \mathbb{R})$, we set

$$
\tilde{\tau}_0 = M \cdot \tau_0 \quad and \quad \tilde{\tau}_1 = M \cdot \tau_1.
$$

Then $R(\tau_1, \tau_0)$ and $R(\tilde{\tau}_1, \tilde{\tau}_0)$ have the same eigenvalues.

(3) All geodesics are symplectic images of the special geodesics

$$
\alpha(t) = i \operatorname{diag}(a_1^t, a_2^t, \cdots, a_g^t),
$$

where $a_1, a_2, \cdots, a_g$ are arbitrary positive real numbers satisfying the condition

$$
\sum_{k=1}^{g} (\log a_k)^2 = 1.
$$

The proof of the above theorem can be found in [137, pp. 289-293] or [95].

Definition 6.1. *Let Z be an irreducible subvariety of a Shimura variety $Sh_K(G, X)$. Choose a connected component S of X and a class $\eta K \in G(\mathbb{A}_f)/K$ such that Z is contained in the image of S in $Sh_K(G, X)$. We say that Z is a* totally geodesic *subvariety if there is a totally geodesic subvariety $Y \subseteq S$ such that Z is the image of $Y \times \eta K$ in $Sh_K(G, X)$.*

B. Moonen [100] proved the following fact.

Theorem 6.2. *Let Z be an irreducible subvariety of a Shimura variety $Sh_K(G, X)$. Then Z is weakly special if and only if it is totally geodesic.*

Proof. See Theorem 4.3, pp. 553–554 in [100]. □

In the 1980s Coleman [30] proposed the following conjecture.

Coleman's Conjecture. *For a sufficiently large integer g, the Jacobian locus J_g contains only a finite number of special points in $\mathcal{A}_g$.*

We also have the following conjecture.

Conjecture 6.1. *For a sufficiently large integer g, the Jacobian locus J_g cannot contain a non-trivial totally geodesic subvariety.*

Remark 6.1. *Conjecture 6.1 is false for an integer $g \leq 6$.*

The stronger version of Conjecture 6.1 is given as follows:

Conjecture 6.2. *For a sufficiently large integer g, there does not exist a geodesic in $\mathcal{A}_g$ contained in $\overline{J}_g$ and intersecting J_g.*

Theorem 6.3. *Suppose the André-Oort conjecture and Conjecture 6.1 hold. Then Coleman's conjecture is true.*

Proof. Let g be a sufficiently large integer g. Suppose J_g contains an infinite set Σ of special points. Then

$$\Sigma \subset \overline{\Sigma} \subset \overline{J}_g \subset \mathcal{A}_g.$$

The truth of the André-Oort conjecture implies that $\overline{\Sigma}$ contains an irreducible special subvariety Y. According to Theorem 6.2, Y is a totally geodesic subvariety of $\overline{J}_g$. From the truth of Conjecture 6.1, we get a contradiction. Therefore J_g contains only finitely many special points. □

Now we propose the following problems.

Problem 6.1. Develop the spectral theory of the Laplace operator $\square_{g;A,B}$ on $\mathbb{H}_g$ and $\Gamma\backslash\mathbb{H}_g$ for a congruence subgroup Γ of Γ_g explicitly.

Problem 6.2. Construct all the geodesics contained in $\overline{J}_g$ with respect to the Siegel's metric $ds_{g;A}^2$.

Problem 6.3. Study variations of g-dimensional principally polarized abelian varieties along a geodesic inside J_g.

Problem 6.4. Prove the A-O conjecture for $\mathcal{A}_g$ unconditionally.

Recently Jacob Tsimerman [143] proved the André-Oort conjecture for $\mathcal{A}_g$ $(g \geq 1)$ unconditionally. His proof is based on the following facts (a) and (b):

(a) The size of the Galoia orbit of a special point is large.

(b) The averaged version of Clomez's conjecture for the Faltings heights of abelian varieties with complex multiplication solved by Andreatta-Goren-Howard-Madapusi Pera [5], and Yuan-Zhang [188] independently.

From now on, we will adopt the notations in Section 3.

Problem 6.5. Let $p_{4,\Gamma} : \mathcal{A}_{4,\Gamma} \longrightarrow \mathcal{A}_4$ be a covering map and let $J_{4,\Gamma} := p_{4,\Gamma}^{-1}(J_4)$. Let $\overline{\mathcal{A}}_{4,\Gamma}$ be a toroidal compactification of $\mathcal{A}_{4,\Gamma}$ which is projective. Then $J_{4,\Gamma}$ is a divisor on $\overline{\mathcal{A}}_{4,\Gamma}$. Compute the Okounkov bodies $\Delta_{Y_\bullet}(J_{4,\Gamma})$, $\Delta_{Y_\bullet}^{\mathrm{val}}(J_{4,\Gamma})$ and $\Delta_{Y_\bullet}^{\lim}(J_{4,\Gamma})$ explicitly. Describe the relations among J_4, $J_{4,\Gamma}$ and these Okounkov bodies explicitly. Describe the relations between these Okounkov bodies and the $GL(4,\mathbb{Z})$-admissible family of polyhedral decompositions defining the toroidal compactification $\overline{\mathcal{A}}_{4,\Gamma}$.

Problem 6.6. Assume that a toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ is a projective variety. Compute the Okoukov convex bodies $\Delta_{Y_\bullet}(K_{g,\Gamma})$, $\Delta_{Y_\bullet}^{\mathrm{val}}(K_{g,\Gamma})$, $\Delta_{Y_\bullet}^{\lim}(K_{g,\Gamma})$, $\Delta_{Y_\bullet}(D_{\infty,\Gamma})$, $\Delta_{Y_\bullet}^{\mathrm{val}}(D_{\infty,\Gamma})$, $\Delta_{Y_\bullet}^{\lim}(D_{\infty,\Gamma})$, $\Delta_{Y_\bullet}(K_{g,\Gamma}+D_{\infty,\Gamma})$, $\Delta_{Y_\bullet}^{\mathrm{val}}(K_{g,\Gamma}+D_{\infty,\Gamma})$, $\Delta_{Y_\bullet}^{\lim}(K_{g,\Gamma}+D_{\infty,\Gamma})$ explicitly. Describe the relations between these Okounkov bodies and the $GL(g,\mathbb{Z})$-admissible family of polyhedral decompositions defining the toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$.

Problem 6.7. Assume that $g \geq 5$. Let $p_{g,\Gamma} : \mathcal{A}_{g,\Gamma} \longrightarrow \mathcal{A}_g$ be a covering map and let $J_{g,\Gamma} := p_{g,\Gamma}^{-1}(J_g)$. Assume that $\overline{\mathcal{A}}_{g,\Gamma}$ is a toroidal compactification of $\mathcal{A}_{g,\Gamma}$ which is a projective variety. Let $D_{J,\Gamma}$ be a divisor on $\overline{\mathcal{A}}_{g,\Gamma}$ containing $J_{g,\Gamma}$. Describe the Okounkov bodies $\Delta_{Y_\bullet}(D_{J,\Gamma})$, $\Delta_{Y_\bullet}^{\mathrm{val}}(D_{J,\Gamma})$, $\Delta_{Y_\bullet}^{\lim}(D_{J,\Gamma})$. Study the relations between $J_{g,\Gamma}$, $D_{J,\Gamma}$ and these Okounkov bodies.

We have the following diagram:

$$
\begin{array}{ccccc}
J_{g,\Gamma} & \subset & \mathcal{A}_{g,\Gamma} & \subset & \overline{\mathcal{A}}_{g,\Gamma} = \mathcal{A}_{g,\Gamma}^{tor} \\
\downarrow & & \downarrow{\scriptstyle p_{g,\Gamma}} & & \\
J_g & \subset & \mathcal{A}_g & &
\end{array}
$$

Here $p_{g,\Gamma} : \mathcal{A}_{g,\Gamma} \longrightarrow \mathcal{A}_g$ is a covering map.

Finally we propose the following questions.

Question 6.1. Let Γ be a neat arithmetic subgroup of Γ_g. Does the closure $\overline{J}_{g,\Gamma}$ of $J_{g,\Gamma}$ intersect the infinity boundary divisor $D_{\infty,\Gamma}$? If g is sufficient large, it is probable that $\overline{J}_{g,\Gamma}$ will not intersect the boundary divisor $D_{\infty,\Gamma}$.

Question 6.2. Let Γ be a neat arithmetic subgroup of Γ_g. Does the closure $\overline{J}_{g,\Gamma}$ of $J_{g,\Gamma}$ intersect the canonical divisor $K_{g,\Gamma}$?

Question 6.3. Let Γ be a neat arithmetic subgroup of Γ_g. How curved is the closure $\overline{J}_{g,\Gamma}$ of $J_{g,\Gamma}$ along the boundary of $J_{g,\Gamma}$?

Quite recently using the good curvature properties of the moduli space $(\mathcal{M}_g, \omega_{\mathrm{WP}})$ endowed with the Weil-Petersson metric ω_{WP}, Liu, Sun, Yang and Yau [92] obtained interesting results related to Conjecture 6.2. Let us explain their results briefly. We consider the coarse moduli space $(\mathcal{M}_g, \omega_{\mathrm{WP}})$ endowed with the Weil-Petersson metric ω_{WP} and the Siegel modular variety $(\mathcal{A}_g, \omega_{\mathrm{H}})$ endowed with the Hodge metric ω_{H}. Let $T_g : \mathcal{M}_g \longrightarrow \mathcal{A}_g$ be the Torelli map (see (1.2)). Assume that V is a submanifold in $\mathcal{M}_g$ such that the image $T_g(V)$ is totally geodesic in $(\mathcal{A}_g, \omega_{\mathrm{H}})$, and also that $T_g(V)$ has finite volume. Under these two assumptions they proved that V must be a ball quotient. As a corollary of this fact,

it can be shown that there is no higher rank locally symmetric subspace in $\mathcal{M}_g$. A precise statement is as follows.

Theorem 6.4. *Let Ω be an irreducible bounded symmetric domain and let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion free cocompact lattice. We set $X = \Omega/\Gamma$. Let h be a canonical metric on X. If there exists a nonconstant holomorphic mapping*

$$f : (X, h) \longrightarrow (\mathcal{M}_g, \omega_{\mathrm{WP}}),$$

then Ω must be of rank 1, i.e., X must be a ball quotient.

Proof. The proof of the above theorem can be found in [92]. □

7. **Final Remarks and Open Problems**

In this final section we give some remarks and propose some open problems about the relations among the Schottky problem, the André-Oort conjecture, Okounkov convex bodies, stable Schottky-Siegel forms, stable Schottky-Jacobi forms and the geometry of the Siegel-Jacobi space. We define the notion of stable Schottky-Jacobi forms and the concept of stable Jacobi equations for the universal hyperelliptic locus.

For two positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} = \left\{ (\lambda, \mu; \kappa) \,|\, \lambda, \mu \in \mathbb{R}^{(h,g)}, \ \kappa \in \mathbb{R}^{(h,h)}, \ \kappa + \mu {}^t\lambda \text{ symmetric } \right\}$$

endowed with the following multiplication law

$$\left(\lambda, \mu; \kappa\right) \circ \left(\lambda', \mu'; \kappa'\right) = \left(\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda {}^t\mu' - \mu {}^t\lambda'\right)$$

with $\left(\lambda, \mu; \kappa\right), \left(\lambda', \mu'; \kappa'\right) \in H_{\mathbb{R}}^{(g,h)}$. We refer to [172, 176, 179, 180, 181, 182, 183, 184] for more details on the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$. We define the Jacobi group G^J of degree g and index h that is the semidirect product of $Sp(2g, \mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J = Sp(2g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\left(M, (\lambda, \mu; \kappa)\right) \cdot \left(M', (\lambda', \mu'; \kappa')\right) = \left(MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda} {}^t\mu' - \tilde{\mu} {}^t\lambda')\right)$$

with $M, M' \in Sp(2g, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(g,h)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(7.1) \qquad \left(M, (\lambda, \mu; \kappa)\right) \cdot (\tau, z) = \left((A\tau + B)(C\tau + D)^{-1}, (z + \lambda\tau + \mu)(C\tau + D)^{-1}\right),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}), \ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\tau, z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and the homogeneous space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ is not a symmetric space. From now on, for brevity we write $\mathbb{H}_{g,h} = \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. The homogeneous space $\mathbb{H}_{g,h}$ is called the Siegel-Jacobi space of degree g and index h.

For $\tau = (\tau_{ij}) \in \mathbb{H}_g$, we write $\tau = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real. We put $d\tau = (d\tau_{ij})$ and $d\bar{\tau} = (d\bar{\tau}_{ij})$. We also put

$$\frac{\partial}{\partial \tau} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \tau_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{\tau}} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \bar{\tau}_{ij}} \right).$$

For a coordinate $z \in \mathbb{C}^{(h,g)}$, we set

$$z = U + iV, \qquad U = (u_{kl}), \ V = (v_{kl}) \ \text{ real},$$
$$dz = (dz_{kl}), \qquad d\bar{z} = (d\bar{z}_{kl}),$$

$$\frac{\partial}{\partial z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1g}} & \cdots & \frac{\partial}{\partial z_{hg}} \end{pmatrix}, \qquad \frac{\partial}{\partial \bar{z}} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_{11}} & \cdots & \frac{\partial}{\partial \bar{z}_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \bar{z}_{1g}} & \cdots & \frac{\partial}{\partial \bar{z}_{hg}} \end{pmatrix}.$$

Yang proved the following theorems in [168].

Theorem 7.1. *For any two positive real numbers A and B,*

$$\begin{aligned} ds^2_{g,h;A,B} \ = \ & A \operatorname{tr}\left(Y^{-1} d\tau \, Y^{-1} d\bar{\tau} \right) \\ & + B \left\{ \operatorname{tr}\left(Y^{-1} \, {}^t V \, V \, Y^{-1} d\tau \, Y^{-1} d\bar{\tau} \right) + \operatorname{tr}\left(Y^{-1} \, {}^t(dz) \, d\bar{z} \right) \right. \\ & \left. - \operatorname{tr}\left(V \, Y^{-1} d\tau \, Y^{-1} \, {}^t(d\bar{z}) \right) - \operatorname{tr}\left(V \, Y^{-1} d\bar{\tau} \, Y^{-1} \, {}^t(dz) \right) \right\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{g,h}$ which is invariant under the action (7.1) of G^J. In fact, $ds^2_{g,h;A,B}$ is a Kähler metric of $\mathbb{H}_{g,h}$.

Proof. See Theorem 1.2 in [168]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Theorem 7.2. *The Laplacian $\Delta_{g,h;A,B}$ of the G^J-invariant metric $ds^2_{g,h;A,B}$ is given by*

$$\Delta_{g,h;A,B} = \frac{4}{A}\,\mathbb{M}_1 + \frac{4}{B}\mathbb{M}_2,$$

where

$$\begin{aligned} \mathbb{M}_1 \ = \ & \operatorname{tr}\left(Y \, {}^t\!\left(Y \frac{\partial}{\partial \bar{\tau}} \right) \frac{\partial}{\partial \tau} \right) + \operatorname{tr}\left(V Y^{-1} \, {}^t V \left(Y \frac{\partial}{\partial \bar{z}} \right)^t \frac{\partial}{\partial z} \right) \\ & + \operatorname{tr}\left(V \, {}^t\!\left(Y \frac{\partial}{\partial \bar{\tau}} \right) \frac{\partial}{\partial z} \right) + \operatorname{tr}\left({}^t V \, {}^t\!\left(Y \frac{\partial}{\partial \bar{z}} \right) \frac{\partial}{\partial \tau} \right) \end{aligned}$$

and

$$\mathbb{M}_2 = \operatorname{tr}\left(Y \frac{\partial}{\partial z} \, {}^t\!\left(\frac{\partial}{\partial \bar{z}} \right) \right).$$

Furthermore $\mathbb{M}_1$ and $\mathbb{M}_2$ are differential operators on $\mathbb{H}_{g,h}$ invariant under the action (7.1) of G^J.

Proof. See Theorem 1.2 in [168]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 7.1. *We refer to [169, 171, 173, 178] for topics related to $ds^2_{g,h;A,B}$ and $\square_{g,h;A,B}$.*

Remark 7.2. *Erik Balslev [13] developed the spectral theory of* $\square_{1,1;1,1}$ *on* $\mathbb{H}_1 \times \mathbb{C}$ *for certain arithmetic subgroups of the Jacobi modular group to prove that the set of all eigenvalues of* $\square_{1,1;1,1}$ *satisfies the Weyl law.*

Remark 7.3. *The sectional curvature of* $(\mathbb{H}_1 \times \mathbb{C}, ds^2_{1,1;A,B})$ *is* $-\frac{3}{A}$ *and hence is independent of the parameter B. We refer to [173] for more detail.*

Remark 7.4. *For an application of the invariant metric* $ds^2_{g,h;A,B}$ *we refer to [186].*

Definition 7.1. *Let* $D = \operatorname{diag}(d_1, d_2, \cdots, d_g)$ *be the* $g \times g$ *diagonal matrix with positive integers* $d_1, \cdots, d_g$ *satisfying* $d_1 | d_2 | \cdots | d_g$, *usually called a* polarization type. $D = I_g$ *is called the* principal *polarization type.*

For a fixed $\tau \in \mathbb{H}_g$ and a fixed polarization type $D = \operatorname{diag}(d_1, \cdots, d_g)$, we let $L^D_\tau := \mathbb{Z}^g \tau + \mathbb{Z}^g D$ be a lattice in $\mathbb{C}^g$ and $A^D_\tau := \mathbb{C}^g / L^D_\tau$ be a complex torus of a polarization type D. Let $\{c_0, \cdots, c_N\}$ be the set of representatives in $\mathbb{Z}^g D^{-1}$ whose components of each c_i $(0 \leq i \leq N)$ lie in the interval $[0, 1)$. Here $N = d_1 \cdots d_g - 1$.

We recall Lefschetz theorem (see [106] p. 128, Theorem 1.3).

Theorem 7.3. *Let* $D = \operatorname{diag}(d_1, \cdots, d_g)$ *be a polarization type and* $N = d_1 \cdots d_g - 1$.

(1) *Assume* $d_1 \geq 2$. *Then the functions* $\left\{ \theta \begin{bmatrix} c_0 \\ 0 \end{bmatrix} (\tau, z), \cdots, \theta \begin{bmatrix} c_N \\ 0 \end{bmatrix} (\tau, z) \right\}$ *have no zero in common, and the mapping* $\varphi^D : \mathbb{H}_g \times \mathbb{C}^g \longrightarrow \mathbb{P}^N(\mathbb{C})$ *defined by*

$$\varphi^D(\tau, z) := \left(\theta \begin{bmatrix} c_0 \\ 0 \end{bmatrix} (\tau, z) : \cdots \cdots : \theta \begin{bmatrix} c_N \\ 0 \end{bmatrix} (\tau, z) \right), \quad (\tau, z) \in \mathbb{H}_g \times \mathbb{C}^g$$

is a well-defined holomorphic mapping. For each $\tau \in \mathbb{H}_g$, *the map* $\varphi^D_\tau : \mathbb{C}^g \longrightarrow \mathbb{P}^N(\mathbb{C})$ *defined by*

$$\varphi^D_\tau(z) := \varphi^D(\tau, z), \qquad z \in \mathbb{C}^g$$

induces a holomorphic mapping from the complex torus A^D_τ *into* $\mathbb{P}^N(\mathbb{C})$.

(2) *If* $d_1 \geq 3$, *for each* $\tau \in \mathbb{H}_g$, *the map* $\varphi^D_\tau : \mathbb{C}^g \longrightarrow \mathbb{P}^N(\mathbb{C})$ *is an analytic embedding, whose image is an algebraic subvariety of* $\mathbb{P}^N(\mathbb{C})$.

Definition 7.2. *Let* $D = \operatorname{diag}(d_1, \cdots, d_g)$ *with* $d_1 \geq 2$ *be a polarization type, and* $N = d_1 \cdots d_g - 1$. *For each* $\tau \in \mathbb{H}_g$, *we define the map* $\Psi^D : \mathbb{H}_g \longrightarrow \mathbb{P}^N(\mathbb{C})$ *by*

$$\Psi^D(\tau) := \varphi^D(\tau, 0), \qquad \tau \in \mathbb{H}_g$$

and define the map $\Phi^D : \mathbb{H}_g \times \mathbb{C}^g \longrightarrow \mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$ *by*

$$\Phi^D(\tau, z) := \left(\varphi^D(\tau, z), \Psi^D(\tau) \right), \qquad (\tau, z) \in \mathbb{H}_g \times \mathbb{C}^g.$$

We have the following theorem proved by Baily [12].

Theorem 7.4. *Assume that* $d_1 \geq 4$ *and that* $2|d_1$ *or* $3|d_1$. *Then the image of* $\mathbb{H}_g \times \mathbb{C}^g$ *under* Φ^D *is a Zariski-open subset of an algebraic subvariety of* $\mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$.

Proof. See Section 5.1 in [12] or Theorem 8.11 in [117]. □

Let
$$\Gamma_g^J := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}$$
be the arithmetic subgroup of G^J, where
$$H_{\mathbb{Z}}^{(g,h)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \lambda, \mu, \kappa \text{ are integral} \right\}.$$

We let
$$(7.2) \qquad \mathcal{A}_{g,h} = \Gamma_g^J \backslash \mathbb{H}_{g,h}$$

be the universal family of principal polarized abelian varieties of dimension gh. Let $\pi_{g,h} : \mathcal{A}_{g,h} \longrightarrow \mathcal{A}_g$ be the natural projection. We define the *universal Jacobian locus*

$$(7.3) \qquad J_{g,h} := \pi_{g,h}^{-1}(J_g), \qquad J_g(\subset \mathcal{A}_g) := \text{the Jacobian locus}.$$

Problem 7.1. Characterize $J_{g,h} = \pi_{g,h}^{-1}(J_g)$. Describe $J_{g,h}$ in terms of Jacobi forms. We refer to [17, 42, 156, 157, 159, 160, 161, 163, 164, 165, 166, 167, 175, 178, 189] for more details about Jacobi forms.

Problem 7.2. Compute the geodesics, the distance between two points and curvatures explicitly in the Siegel-Jacobi space $(\mathbb{H}_{g,h}, ds^2_{g,h;A,B})$. See Theorem 6.1 for the Siegel space $\mathbb{H}_g$.

Problem 7.3. Find the analogue of the Hirzebruch-Mumford Proportionality Theorem for $\mathcal{A}_{g,h,\Gamma}$ (see (7.4) below).

Let us give some remarks for this problem. Before we describe the proportionality theorem for the Siegel modular variety, first of all we review the compact dual of the Siegel upper half plane $\mathbb{H}_g$. We note that $\mathbb{H}_g$ is biholomorphic to the generalized unit disk $\mathbb{D}_g$ of degree g through the Cayley transform. We suppose that $\Lambda = (\mathbb{Z}^{2g}, \langle \ , \ \rangle)$ is a symplectic lattice with a symplectic form $\langle \ , \ \rangle$. We extend scalars of the lattice Λ to $\mathbb{C}$. Let

$$\mathfrak{Y}_g := \left\{ L \subset \mathbb{C}^{2g} \mid \dim_{\mathbb{C}} L = g, \ \langle x, y \rangle = 0 \quad \text{for all } x, y \in L \right\}$$

be the complex Lagrangian Grassmannian variety parameterizing totally isotropic subspaces of complex dimension g. For the present time being, for brevity, we put $G = Sp(2g, \mathbb{R})$ and $K = U(g)$. The complexification $G_{\mathbb{C}} = Sp(2g, \mathbb{C})$ of G acts on $\mathfrak{Y}_g$ transitively. If H is the isotropy subgroup of $G_{\mathbb{C}}$ fixing the first summand $\mathbb{C}^g$, we can identify $\mathfrak{Y}_g$ with the compact homogeneous space $G_{\mathbb{C}}/H$. We let

$$\mathfrak{Y}_g^+ := \left\{ L \in \mathfrak{Y}_g \mid -i\langle x, \bar{x} \rangle > 0 \quad \text{for all } x(\neq 0) \in L \right\}$$

be an open subset of $\mathfrak{Y}_g$. We see that G acts on $\mathfrak{Y}_g^+$ transitively. It can be shown that $\mathfrak{Y}_g^+$ is biholomorphic to $G/K \cong \mathbb{H}_g$. A basis of a lattice $L \in \mathfrak{Y}_g^+$ is given by a unique $2g \times g$ matrix ${}^t(-I_g \ \tau)$ with $\tau \in \mathbb{H}_g$. Therefore we can identify L with τ in $\mathbb{H}_g$. In this way, we embed $\mathbb{H}_g$ into $\mathfrak{Y}_g$ as an open subset of $\mathfrak{Y}_g$. The complex projective variety $\mathfrak{Y}_g$ is called the *compact dual* of $\mathbb{H}_g$.

Let Γ be an arithmetic subgroup of Γ_g. Let E_0 be a G-equivariant holomorphic vector bundle over $\mathbb{H}_g = G/K$ of rank r. Then E_0 is defined by the representation $\tau : K \longrightarrow GL(r, \mathbb{C})$. That is, $E_0 \cong G \times_K \mathbb{C}^r$ is a homogeneous vector bundle over G/K. We naturally

obtain a holomorphic vector bundle E over $\mathcal{A}_{g,\Gamma} := \Gamma\backslash G/K$. E is often called an *automorphic* or *arithmetic* vector bundle over $\mathcal{A}_{g,\Gamma}$. Since K is compact, E_0 carries a G-equivariant Hermitian metric h_0 which induces a Hermitian metric h on E. According to Main Theorem in [104], E admits a *unique* extension $\tilde{E}$ to a smooth toroidal compactification $\tilde{\mathcal{A}}_{g,\Gamma}$ of $\mathcal{A}_{g,\Gamma}$ such that h is a singular Hermitian metric *good* on $\tilde{\mathcal{A}}_{g,\Gamma}$. For the precise definition of a *good metric* on $\mathcal{A}_{g,\Gamma}$ we refer to [104, p. 242]. According to Hirzebruch-Mumford's Proportionality Theorem (cf. [104, p. 262]), there is a natural metric on $G/K = \mathbb{H}_g$ such that the Chern numbers satisfy the following relation

$$c^\alpha\big(\tilde{E}\big) = (-1)^{\frac{1}{2}g(g+1)}\,\mathrm{vol}\,(\Gamma\backslash\mathbb{H}_g)\,c^\alpha\big(\check{E}_0\big)$$

for all $\alpha = (\alpha_1, \cdots, \alpha_r)$ with nonnegative integers $\alpha_i\,(1 \le i \le r)$ and $\sum_{i=1}^r \alpha_i = \frac{1}{2}g(g+1)$, where $\check{E}_0$ is the $G_\mathbb{C}$-equivariant holomorphic vector bundle on the compact dual $\mathfrak{Y}_g$ of $\mathbb{H}_g$ defined by a certain representation of the stabilizer $\mathrm{Stab}_{G_\mathbb{C}}(e)$ of a point e in $\mathfrak{Y}_g$. Here $\mathrm{vol}\,(\Gamma\backslash\mathbb{H}_g)$ is the volume of $\Gamma\backslash\mathbb{H}_g$ that can be computed (cf. [137]).

Problem 7.4. Compute the cohomology $H^\bullet(\mathcal{A}_{g,h}, *)$ of $\mathcal{A}_{g,h}$. Investigate the intersection cohomology of $\mathcal{A}_{g,h}$.

Problem 7.5. Generalize the trace formula on the Siegel modular variety obtained by Sophie Morel to the universal abelian variety. For her result on the trace formula on the Siegel modular variety, we refer to her paper [101, *Cohomologie d'intersection des variétés modulaires de Siegel, suite*].

Problem 7.6. Construct all the geodesics contained in $J_{g,h}$.

Problem 7.7. Develop the theory of variations of abelian varieties along the geodesic joining two points in $J_{g,h}$.

Problem 7.8. Discuss the André-Oort conjecture for $\mathcal{A}_{g,h}$. Gao claims that he proved the Ax-Lindemann-Weierstras theorem for $\mathcal{A}_{g,h}$, and using this theorem proved the André-Oort conjecture for $\mathcal{A}_{g,h}$ under the assumption of the Generalized Riemann Hypothesis for CM fields in his preprint [57].

Let Γ be a neat arithmetic subgroup of Γ_g. We put $\Gamma^J := \Gamma \ltimes H_\mathbb{Z}^{(g,h)}$. We let

$$(7.4) \qquad\qquad\qquad \mathcal{A}_{g,h,\Gamma} := \Gamma^J\backslash\mathbb{H}_{g,h}.$$

Let $\mathcal{A}_{g,h,\Gamma}^{tor}$ be a toroidal compactification of $\mathcal{A}_{g,h,\Gamma}$. Let $K_{g,h,\Gamma}$ be the canonical line bundle over $\mathcal{A}_{g,h,\Gamma}^{tor}$ and let

$$(7.5) \qquad\qquad\qquad D_{\infty,g,h,\Gamma} := \mathcal{A}_{g,h,\Gamma}^{tor}\backslash\mathcal{A}_{g,h,\Gamma}$$

be the infinity boundary divisor on $\mathcal{A}_{g,h,\Gamma}^{tor}$. Let $\pi_{g,h,\Gamma} : \mathcal{A}_{g,h,\Gamma} \longrightarrow \mathcal{A}_{g,\Gamma}$ be a projection and let $p_{g,\Gamma} : \mathcal{A}_{g,\Gamma} \longrightarrow \mathcal{A}_g$ be a covering map. We define

$$(7.6) \qquad\qquad\qquad J_{g,h,\Gamma} := (p_{g,\Gamma} \circ \pi_{g,h,\Gamma})^{-1}\,(J_g).$$
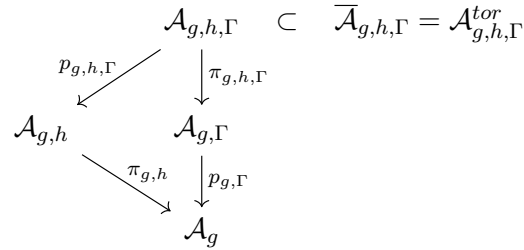
Problem 7.9. Assume that $\mathcal{A}_{4,h,\Gamma}^{tor}$ is a toroidal compactification of $\mathcal{A}_{4,h,\Gamma}$ which is projective. Compute the Okounkov bodies $\Delta_{Y_\bullet}(J_{4,h,\Gamma})$, $\Delta_{Y_\bullet}^{\mathrm{val}}(J_{4,h,\Gamma})$ and $\Delta_{Y_\bullet}^{\lim}(J_{4,h,\Gamma})$ explicitly. Describe the relations among J_4, $J_{4,h,\Gamma}$ and these Okounkov bodies explicitly. Describe the

relations between these Okounkov bodies and the $GL(4,\mathbb{Z})$-admissible family of polyhedral decompositions defining the toroidal compactification $\overline{\mathcal{A}}_{4,\Gamma}$.

Problem 7.10. Assume that a toroidal compactification $\mathcal{A}_{g,h,\Gamma}^{tor}$ is a projective variety. Let $K_{g,h,\Gamma}$ be the canonical line bundle over $\mathcal{A}_{g,h,\Gamma}^{tor}$ and $D_{\infty,g,h,\Gamma}$ be the infinity boundary divisor on $\mathcal{A}_{g,h,\Gamma}^{tor}$. Compute the Okoukov convex bodies $\Delta_{Y_\bullet}(K_{g,h,\Gamma})$, $\Delta_{Y_\bullet}^{val}(K_{g,h,\Gamma})$, $\Delta_{Y_\bullet}^{lim}(K_{g,h,\Gamma})$, $\Delta_{Y_\bullet}(D_{\infty,g,h,\Gamma})$, $\Delta_{Y_\bullet}^{val}(D_{\infty,g,h,\Gamma})$, $\Delta_{Y_\bullet}^{lim}(D_{\infty,g,h,\Gamma})$, $\Delta_{Y_\bullet}(K_{g,h,\Gamma}+D_{\infty,g,h,\Gamma})$, $\Delta_{Y_\bullet}^{val}(K_{g,h,\Gamma}+D_{\infty,g,h,\Gamma})$ and $\Delta_{Y_\bullet}^{lim}(K_{g,h,\Gamma}+D_{\infty,g,h,\Gamma})$ explicitly. Describe the relations between these Okounkov bodies and the $GL(g,\mathbb{Z})$-admissible family of polyhedral decompositions defining the toroidal compactification $\mathcal{A}_{g,h,\Gamma}^{tor}$.

Problem 7.11. Assume that a toroidal compactification $\mathcal{A}_{g,h,\Gamma}^{tor}$ of $\mathcal{A}_{g,h,\Gamma}$ is a projective variety. Let $D_{J,\Gamma}$ be a divisor on $\mathcal{A}_{g,h,\Gamma}^{tor}$ containing $J_{g,h,\Gamma}$. Describe the Okounkov bodies $\Delta_{Y_\bullet}(D_{J,\Gamma})$, $\Delta_{Y_\bullet}^{val}(D_{J,\Gamma})$ and $\Delta_{Y_\bullet}^{lim}(D_{J,\Gamma})$. Study the relations among $J_{g,\Gamma}$, $J_{g,h,\Gamma}$, $D_{J,\Gamma}$ and these Okounkov bodies.

We have the following diagram:

$$
\begin{array}{ccc}
& \mathcal{A}_{g,h,\Gamma} & \subset \quad \overline{\mathcal{A}}_{g,h,\Gamma} = \mathcal{A}_{g,h,\Gamma}^{tor} \\
{\scriptstyle p_{g,h,\Gamma}}\swarrow & \downarrow {\scriptstyle \pi_{g,h,\Gamma}} & \\
\mathcal{A}_{g,h} & \mathcal{A}_{g,\Gamma} & \\
& {\scriptstyle \pi_{g,h}}\searrow \quad \downarrow {\scriptstyle p_{g,\Gamma}} & \\
& \mathcal{A}_g &
\end{array}
$$

Here $p_{g,h,\Gamma} : \mathcal{A}_{g,h,\Gamma} \longrightarrow \mathcal{A}_{g,h}$ is a covering map.

We propose the following questions.

Question 7.1. Let Γ be a neat arithmetic subgroup of Γ_g. Does the closure $\overline{J}_{g,h,\Gamma}$ of $J_{g,h,\Gamma}$ intersect the infinity boundary divisor $D_{\infty,g,h,\Gamma}$? If g is sufficient large, it is probable that $\overline{J}_{g,h,\Gamma}$ will not intersect the boundary divisor $D_{\infty,g,h,\Gamma}^u$.

Question 7.2. Let Γ be a neat arithmetic subgroup of Γ_g. Does the closure $\overline{J}_{g,h,\Gamma}$ of $J_{g,h,\Gamma}$ intersect the canonical divisor $K_{g,h,\Gamma}$?

Question 7.3. Let Γ be a neat arithmetic subgroup of Γ_g. How curved is the closure $\overline{J}_{g,h,\Gamma}$ of $J_{g,h,\Gamma}$ along the boundary of $J_{g,h,\Gamma}$?

Now we make some conjectures.

Conjecture 7.1. *For a sufficiently large integer g, the locus $J_{g,h}$ contains only finitely many special points. This is an analogue (or generalization) of Coleman's conjecture.*

Conjecture 7.2. *For a sufficiently large integer g, the locus $J_{g,h}$ cannot contain a nontrivial totally geodesic subvariety inside $\mathcal{A}_{g,h}$ for the Riemannian metric $ds_{g,h;A,B}^2$.*

Conjecture 7.3. *For a sufficiently large integer g, there does not exist a geodesic that is contained in $J_{g,h}$ for the Riemannian metric $ds_{g,h;A,B}^2$.*

Finally we discuss the connection between the universal Jacobian locus $J_{g,h}$ and stable Jacobi forms. We refer to Appendix E in this article for more details on stable Jacobi forms. First we review the concept of *stable modular forms* introduced in [50]. The Siegel Φ-operator

$$(7.7) \qquad \Phi_{g,k} : [\Gamma_g, k] \longrightarrow [\Gamma_{g-1}, k], \qquad k \in \mathbb{Z}_+$$

defined by

$$(\Phi_{g,k}f)(\tau) := \lim_{t \to \infty} f\left(\begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}\right), \quad f \in [\Gamma_g, k], \, \tau \in \mathbb{H}_{g-1},$$

where $[\Gamma_g, k]$ denotes the vector space of all Siegel modular forms on $\mathbb{H}_g$ of weight k. Using the theory of Poincaré series, H. Maass [93] proved that if k is even and $k > 2g$, then $\Phi_{g,k}$ is a surjective linear map. In 1977, using the theory of singular modular forms, E. Freitag [50] proved the following facts (a) and (b) :

(a) for a fixed even integer k, $\Phi_{g,k}$ is an isomorphism if $g > 2k$;

(b) $[\Gamma_g, k] = 0$ if $g > 2k$, $k \not\equiv 0 \pmod 4$.

The fact (a) means that the vector spaces $[\Gamma_g, k]$ stabilize to the infinity vector space $[\Gamma_\infty, k]$ as g increases. In this sense, he introduced the notion of the stability of Siegel modular forms.

Definition 7.3. *A Siegel modular form $f \in [\Gamma_g, k]$ is said to be* stable *if there exists a nonnegative integer $m \in \mathbb{Z}_+$ satisfying the following conditions* (SM1) *and* (SM2) *:*

(SM1) $g + m > 2k$;

(SM2) $f = \Phi_{g+1,k} \circ \Phi_{g+2,k} \circ \cdots \circ \Phi_{g+m,k}(F)$ *for some $F \in [\Gamma_{g+m}, k]$.*

Scalar-valued Siegel modular forms on $\mathcal{A}_g$ vanishing on the Jacobian locus, equivalently, forms on the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ that vanish on the closure J_g^{Sat} of J_g in $\mathcal{A}_g^{\mathrm{Sat}}$ are called Schottky-Siegel forms. The normalization $\nu : \mathcal{A}_g^{\mathrm{Sat}} \longrightarrow \partial\mathcal{A}_{g+1}^{\mathrm{Sat}}$ gives a restriction map which coincides with the Siegel operator $\Phi_{g,k}$ $(k \in \mathbb{Z}_+)$.

We let

$$A(\Gamma_g) := \bigoplus_{k \geq 0} [\Gamma_g, k]$$

be the graded ring of Siegel modular forms on $\mathbb{H}_g$. It is known that $A(\Gamma_g)$ is a finitely generated $\mathbb{C}$-algebra and the field of modular functions $K(\Gamma_g)$ is an algebraic function field of transcendence degree $\frac{1}{2}g(g+1)$.

The ring

$$\mathbb{A} = \bigoplus_{k \geq 0} [\Gamma_\infty, k]$$

is an inverse limit in the category

$$(7.8) \qquad \mathbb{A} = \varprojlim_{g} A(\Gamma_g).$$

Freitag [50] proved that $\mathbb{A}$ is the polynomial ring over $\mathbb{C}$ on the set of theta series θ_S, where S runs over the set of equivalence classes of indecomposable positive definite unimodular even integral matrices. In general, $A(\Gamma_g)$ is not a polynomial ring (cf. [50, p. 204]

We define the stable Satake compactification $\mathcal{A}_\infty^{\mathrm{Sat}}$ by

(7.9)
$$\mathcal{A}_\infty^{\mathrm{Sat}} := \bigcup_g \mathcal{A}_g^{\mathrm{Sat}} = \varinjlim_g \mathcal{A}_g^{\mathrm{Sat}}$$

and the stable Jacobian locus J_∞^{Sat} by

(7.10)
$$J_\infty^{\mathrm{Sat}} := \bigcup_g J_g^{\mathrm{Sat}} = \varinjlim_g J_g^{\mathrm{Sat}}.$$

G. Codogni and N. I. Shepherd-Barron [29] proved the following theorem.

Theorem 7.5. *There are no stable Schottky-Siegel forms. That is, the homomorphism from*

(7.11)
$$\mathbb{A} = \varprojlim_g A(\Gamma_g) \longrightarrow \bigoplus_k H^0(J_\infty^{\mathrm{Sat}}, \omega_J^{\otimes k})$$

induced by the inclusion $J_\infty^{\mathrm{Sat}} \hookrightarrow \mathcal{A}_\infty^{\mathrm{Sat}}$ is injective, where ω_J is the restriction of the canonical line bundle ω on $\mathcal{A}_\infty^{\mathrm{Sat}}$ to J_∞^{Sat}.

Proof. See Theorem 1.3 and Corollary 1.4 in [29]. □

We refer to Appendix D in this paper for the definition of Jacobi forms.

Now we consider the special case $\rho = \det^k$ with $k \in \mathbb{Z}_+$. We define the Siegel-Jacobi operator
$$\Psi_{g,\mathcal{M}} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$$
by

(7.12)
$$(\Psi_{g,\mathcal{M}}F)(\tau, z) := \lim_{t \longrightarrow \infty} F\left(\begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, (z, 0)\right),$$

where $F \in J_{k,\mathcal{M}}(\Gamma_g)$, $\tau \in \mathbb{H}_{g-1}$ and $z \in \mathbb{C}^{(h,g-1)}$. We observe that the above limit exists and $\Psi_{g,\mathcal{M}}$ is a well-defined linear map (cf. [189]).

J.-H. Yang [159] proved the following theorems.

Theorem 7.6. *Let $2\mathcal{M}$ be a positive even unimodular symmetric integral matrix of degree h and let k be an even nonnegative integer. If $g + h > 2k$, then the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}}$ is injective.*

Proof. See Theorem 3.5 in [159]. □

Theorem 7.7. *Let $2\mathcal{M}$ be as above in Theorem 2.1 and let k be an even nonnegative integer. If $g + h > 2k + 1$, then the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}}$ is an isomorphism.*

Proof. See Theorem 3.6 in [159]. □

Theorem 7.8. *Let $2\mathcal{M}$ be as above in Theorem 2.1 and let k be an even nonnegative integer. If $2k > 4g + h$, then the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}}$ is surjective.*

Proof. See Theorem 3.7 in [159]. □

Remark 7.5. *Yang [159, Theorem 4.2] proved that the action of the Hecke operatos on Jacobi forms is compatible with that of the Siegel-Jacobi operator.*

Definition 7.4. *A collection* $(F_g)_{g\geq 0}$ *is called a* stable Jacobi form *of weight* k *and index* $\mathcal{M}$ *if it satisfies the following conditions* (SJ1) *and* (SJ2):

(SJ1) $F_g \in J_{k,\mathcal{M}}(\Gamma_g)$ *for all* $g \geq 0$.

(SJ2) $\Psi_{g,\mathcal{M}}F_g = F_{g-1}$ *for all* $g \geq 1$.

Remark 7.6. *The concept of a stable Jacobi forms was introduced by Yang* [158, 164].

Example. Let S be a positive even unimodular symmetric integral matrix of degree $2k$ and let $c \in \mathbb{Z}^{(2k,h)}$ be an integral matrix. We define the theta series $\vartheta_{S,c}^{(g)}$ by

$$\vartheta_{S,c}^{(g)}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^{(2k,g)}} e^{\pi i \left\{ \mathrm{tr}(S\lambda\tau\,^t\lambda) + 2\,\mathrm{tr}(^t c\, S\lambda\,^t z) \right\}}, \quad (\tau, z) \in \mathbb{H}_{g,h}.$$

It is easily seen that $\vartheta_{S,c}^{(g)} \in J_{k,\mathcal{M}}(\Gamma_g)$ with $\mathcal{M} := \frac{1}{2}\,^t c S c$ for all $g \geq 0$ and $\Psi_{g,\mathcal{M}}\vartheta_{S,c}^{(g)} = \vartheta_{S,c}^{(g-1)}$ for all $g \geq 1$. Thus the collection

$$\Theta_{S,c} := \left(\vartheta_{S,c}^{(g)} \right)_{g\geq 0}$$

is a stable Jacobi form of weight k and index $\mathcal{M}$.

Definition 7.5. *Let* $\mathcal{M}$ *be a half-integral semi-positive symmetric matrix of degree* h *and* $k \in \mathbb{Z}_+$. *A Jacobi form* $F \in J_{k,\mathcal{M}}(\Gamma_g)$ *is called a* Schottky-Jacobi form *of weight* k *and index* $\mathcal{M}$ *for the universal Jacobian locus if it vanishes along* $J_{g,h}$.

Definition 7.6. *Let* $\mathcal{M}$ *be a half-integral semi-positive symmetric matrix of degree* h *and* $k \in \mathbb{Z}_+$. *A collection* $(F_g)_{g\geq 0}$ *is called a* stable Schottky-Jacobi form *of weight* k *and index* $\mathcal{M}$ *for the universal Jacobian locus if it satisfies the following conditions* (1) *and* (2):

(1) $F_g \in J_{k,\mathcal{M}}(\Gamma_g)$ *is a Schottky-Jacobi form of weight* k *and index* $\mathcal{M}$ *for the universal Jacobian locus* $J_{g,h}$ *for all* $g \geq 0$.

(2) $\Psi_{g,\mathcal{M}}F_g = F_{g-1}$ *for all* $g \geq 1$.

We expect to prove the following claim :

Claim: *There are no stable Schottky-Jacobi forms for the universal Jacoban locus.*

J.-H. Yang [185] proved the following.

Theorem 7.9. *Let* $2\mathcal{M}$ *be a positive even unimodular symmetric integral matrix of degree* h. *Then there do not exist stable Schottky-Jacobi forms of index* $\mathcal{M}$ *for the universal Jacobian locus.*

Proof. See Theorem 4.1 in [185].   □

Let (Λ, Q) be an even unimodular positive definite quadratic form of rank m. That is, Λ is a finitely generated free group of rank m and Q is an integer-valued bilinear form on Λ such that Q is even and unimodular. For a positive integer g, the theta series $\theta_{Q,g}$ associated to (Λ, Q) is defined to be

$$\theta_{Q,g}(\tau) := \sum_{x_1,\cdots,x_g \in \Lambda} \exp\left(\pi i \sum_{p,q}^{g} Q(x_p, x_q)\tau_{pq} \right), \qquad \tau = (\tau_{pq}) \in \mathbb{H}_g.$$

It is well known that $\theta_{Q,g}(\tau)$ is a Siegel modular form on $\mathbb{H}_g$ of weight $\frac{m}{2}$. We easily see that
$$\Phi_{g+1,\frac{m}{2}}(\theta_{Q,g+1}) = \theta_{Q,g} \qquad \text{for all } g \in \mathbb{Z}_+.$$
Therefore the collection of all theta series associated to (Λ, Q)
$$\Theta_Q := (\theta_{Q,g})_{g \geq 0}$$
is a stable modular form.

Definition 7.7. *A* stable equation *for the hyperelliptic locus is a stable modular form* $(f_g)_{g \geq 0}$ *such that* f_g *vanishes along the hyperelliptic locus* Hyp_g *for every g.*

Recently G. Codogni [28] proved the following.

Theorem 7.10. *The ideal of stable equations of the hyperelliptic locus is generated by differences of theta series*
$$\Theta_P - \Theta_Q,$$
where P and Q are even unimodular positive definite quadratic forms of the same rank.

Proof. See Theorem 1.2 or Theorem 4.2 in [28]. $\qquad\square$

In a similar way we may define the concept of stable Jacobi equation.

Definition 7.8. *A* stable Jacobi equation *of index $\mathcal{M}$ for the universal hyperelliptic locus is a stable Jacobi form* $(F_{g,\mathcal{M}})_{g \geq 0}$ *of index $\mathcal{M}$ such that* $F_{g,\mathcal{M}}$ *vanishes along the universal hyperelliptic locus* $\text{Hyp}_{g,h} := \pi_{g,h}^{-1}(\text{Hyp}_g)$ *for every g.*

J.-H. Yang [185] proved the following.

Theorem 7.11. *Let $2\mathcal{M}$ be a positive even unimodular symmetric integral matrix of degree h. Then there exist non-trivial stable Schottky-Jacobi forms of index $\mathcal{M}$ for the universal hyperelliptic locus.*

Proof. See Theorem 4.2 in [185]. $\qquad\square$

Problem 7.12. Find the ideal of stable Jacobi equations of the universal hyperelliptic locus.

Remark 7.7. *We consider a half-integral semi-positive symmetric integral matrix $\mathcal{M}$ such that $2\mathcal{M}$ is not unimodular. The natural questions arise:*

Question 7.1. *Are there non-trivial stable Schottky-Jacobi forms of index $\mathcal{M}$ for the universal Jacobian locus ?*

Question 7.2. *Are there non-trivial stable Schottky-Jacobi forms of index $\mathcal{M}$ for the universal hyperelliptic locus ?*

Appendix A. Subvarieties of the Siegel Modular Variety

In this appendix A, we give a brief remark on subvarieties of the Siegel modular variety and present several problems. This appendix was written on the base of the review [128] of G. K. Sankaran for the paper [170]. In fact, Sankaran made a critical review on Section 10.

Subvarieties of the Siegel modular variety of the author's paper [170] and corrected some wrong statements and information given by the author. In this sense the author would like to give his deep thanks to the reviewer, Sankanran.

Here we assume that the ground field is the complex number field $\mathbb{C}$.

Definition A.1. A nonsingular variety X is said to be *rational* if X is birational to a projective space $\mathbb{P}^n(\mathbb{C})$ for some integer n. A nonsingular variety X is said to be *stably rational* if $X \times \mathbb{P}^k(\mathbb{C})$ is birational to $\mathbb{P}^N(\mathbb{C})$ for certain nonnegative integers k and N. A nonsingular variety X is called *unirational* if there exists a dominant rational map $\varphi : \mathbb{P}^n(\mathbb{C}) \longrightarrow X$ for a certain positive integer n, equivalently if the function field $\mathbb{C}(X)$ of X can be embedded in a purely transcendental extension $\mathbb{C}(z_1, \cdots, z_n)$ of $\mathbb{C}$.

Remarks A.2. (1) It is easy to see that the rationality implies the stably rationality and that the stably rationality implies the unirationality.

(2) If X is a Riemann surface or a complex surface, then the notions of rationality, stably rationality and unirationality are equivalent one another.

(3) H. Clemens and P. Griffiths [27] showed that most of cubic threefolds in $\mathbb{P}^4(\mathbb{C})$ are unirational but *not* rational.

The following natural questions arise :

QUESTION 1.   Is a stably rational variety *rational*?

QUESTION 2.   Is a general hypersurface $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \leq n+1$ *unirational*?

Question 1 is a famous one raised by O. Zariski (cf. B. Serge, *Algebra and Number Theory (French)*, CNRS, Paris (1950), 135–138; MR0041480). In [14], A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer gave counterexamples, e.g., the Châtelot surfaces $V_{d,P} \subset \mathbf{A}_{\mathbb{C}}^3$ defined by $y^2 - dz^2 = P(x)$, where $P \in \mathbb{C}[x]$ is an irreducible polynomial of degree 3, and d is the discriminant of P such that d is not a square and hence answered negatively to Question 1.

Definition A.3. Let X be a nonsingular variety of dimension n and let K_X be the canonical divisor of X. For each positive integer $m \in \mathbb{Z}^+$, we define the *m-genus* $P_m(X)$ of X by
$$P_m(X) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}(mK_X)).$$
The number $p_g(X) := P_1(X)$ is called the *geometric genus* of X. We let
$$N(X) := \left\{ m \in \mathbb{Z}^+ \,|\, P_m(X) \geq 1 \right\}.$$
For the present, we assume that $N(X)$ is nonempty. For each $m \in N(X)$, we let $\{\phi_0, \cdots, \phi_{N_m}\}$ be a basis of the vector space $H^0(X, \mathcal{O}(mK_X))$. Then we have the mapping $\Phi_{mK_X} : X \longrightarrow \mathbb{P}^{N_m}(\mathbb{C})$ by
$$\Phi_{mK_X}(z) := (\phi_0(z) : \cdots : \phi_{N^m}(z)), \quad z \in X.$$
We define the *Kodaira dimension* $\kappa(X)$ of X by
$$\kappa(X) := \max \left\{ \dim_{\mathbb{C}} \Phi_{mK_X}(X) \,|\, m \in N(X) \right\}.$$
If $N(X)$ is empty, we put $\kappa(X) := -\infty$. Obviously $\kappa(X) \leq \dim_{\mathbb{C}} X$. A nonsingular variety X is said to be *of general type* if $\kappa(X) = \dim_{\mathbb{C}} X$. A singular variety Y in general is said to

be rational, stably rational, unirational or of general type if any nonsingular model X of Y is rational, stably rational, unirational or of general type respectively. We define

$$P_m(Y) := P_m(X) \quad \text{and} \quad \kappa(Y) := \kappa(X).$$

A variety Y of dimension n is said to be *of logarithmic general type* if there exists a smooth compactification $\tilde{Y}$ of Y such that $D := \tilde{Y} - Y$ is a divisor with normal crossings only and the transcendence degree of the logarithmic canonical ring

$$\bigoplus_{m=0}^{\infty} H^0(\tilde{Y}, \, m(K_{\tilde{Y}} + [D]))$$

is $n + 1$, i.e., the *logarithmic Kodaira dimension* of Y is n. We observe that the notion of being of logarithmic general type is weaker than that of being of general type.

Let $\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g$ be the Siegel modular variety of degree g, that is, the moduli space of principally polarized abelian varieties of dimension g. So far it has been proved that $\mathcal{A}_g$ is of general type for $g \geq 7$. At first Freitag [49] proved this fact when g is a multiple of 24. Tai [140] proved this for $g \geq 9$ and Mumford [105] proved this fact for $g \geq 7$. On the other hand, $\mathcal{A}_g$ is known to be unirational for $g \leq 5$: Donagi [35] for $g = 5$, Clemens [26] for $g = 4$ and classical for $g \leq 3$. For $g = 3$, using the moduli theory of curves, Riemann [125], Weber [147] and Frobenius [56] showed that $\mathcal{A}_3(2) := \Gamma_3(2) \backslash \mathbb{H}_3$ is a rational variety and moreover gave 6 generators of the modular function field $K(\Gamma_3(2))$ written explicitly in terms of derivatives of odd theta functions at the origin. So $\mathcal{A}_3$ is a unirational variety with a Galois covering of a rational variety of degree $[\Gamma_3 : \Gamma_3(2)] = 1,451,520$. Here $\Gamma_3(2)$ denotes the principal congruence subgroup of Γ_3 of level 2. Furthermore it was shown that $\mathcal{A}_3$ is stably rational (cf. [82, 18]). For a positive integer k, we let $\Gamma_g(k)$ be the principal congruence subgroup of Γ_g of level k. Let $\mathcal{A}_g(k)$ be the moduli space of abelian varieties of dimension g with k-level structure. It is classically known that $\mathcal{A}_g(k)$ is of logarithmic general type for $k \geq 3$ (cf. [105]). Wang [145, 146] gave a different proof for the fact that $\mathcal{A}_2(k)$ is of general type for $k \geq 4$. On the other hand, the relation between the Burkhardt quartic and abelian surfaces with 3-level structure was established by H. Burkhardt [19] in 1890. We refer to [76, § IV.2, pp. 132–135] for more detail on the Burkhardt quartic. In 1936, J. A. Todd [141] proved that the Burkhardt quartic is rational. van der Geer [61] gave a modern proof for the rationality of $\mathcal{A}_2(3)$. The remaining unsolved problems are summarized as follows :

Problem 1. Are $\mathcal{A}_4$, $\mathcal{A}_5$ stably rational or rational ?

Problem 2. Discuss the (uni)rationality of $\mathcal{A}_6$.

We already mentioned that $\mathcal{A}_g$ is of general type if $g \geq 7$. It is natural to ask if the subvarieties of $\mathcal{A}_g$ ($g \geq 7$) are of general type, in particular the subvarieties of $\mathcal{A}_g$ of codimension one. Freitag [54] showed that there exists a certain bound g_0 such that for $g \geq g_0$, each irreducible subvariety of $\mathcal{A}_g$ of codimension one is of general type. Weissauer [149] proved that every irreducible divisor of $\mathcal{A}_g$ is of general type for $g \geq 10$. Moreover he proved that every subvariety of codimension $\leq g - 13$ in $\mathcal{A}_g$ is of general type for $g \geq 13$. We observe that the smallest known codimension for which there exist subvarieties of $\mathcal{A}_g$ for large g which are not of general type is $g - 1$. $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ is a subvariety of $\mathcal{A}_g$ of codimension $g - 1$ which is not of general type.

Remark A.4. Let $\mathcal{M}_g$ be the coarse moduli space of curves of genus g over $\mathbb{C}$. Then $\mathcal{M}_g$ is an analytic subvariety of $\mathcal{A}_g$ of dimension $3g - 3$. It is known that $\mathcal{M}_g$ is rational for $g = 2, 4, 5, 6$. In 1915 Severi proved that $\mathcal{M}_g$ is unirational for $g \leq 10$ (see E. Arbarello and E. Sernesi's paper [11] for a modern rigorous proof). The unirationality of $\mathcal{M}_{12}$ was proved by E. Sernesi [134] in 1981. Three years later the unirationality of $\mathcal{M}_{11}$ and $\mathcal{M}_{13}$ was proved by M. C. Chang and Z. Ran [21]. So the Kodaira dimension $\kappa(\mathcal{M}_g)$ of $\mathcal{M}_g$ is $-\infty$ for $g \leq 13$. In 1982 Harris and Mumford [75] proved that $\mathcal{M}_g$ is of general type for odd g with $g \geq 25$ and $\kappa(\mathcal{M}_{23}) \geq 0$. J. Harris [73] proved that if $g \geq 40$ and g is even, $\mathcal{M}_g$ is of general type. In 1987 D. Eisenbud and J. Harris [44] proved that $\mathcal{M}_g$ is of general type for all $g \geq 24$ and $\mathcal{M}_{23}$ has the Kodaira dimension at least one. In 1996 P. Katsylo [80] showed that $\mathcal{M}_3$ is rational and hence $\mathcal{A}_3$.

Remark A.5. For more details on the geometry and topology of $\mathcal{A}_g$ and compactifications of $\mathcal{A}_g$, we refer to [1, 45, 53, 60, 62, 63, 66, 77, 87, 96, 129, 130, 131, 145].

Appendix B. Extending of the Torelli Map to Toroidal Compactifications of the Siegel Modular Variety

Let $\mathcal{M}_g^{\mathrm{DM}}$ be the Deligne-Mumford compactification of $\mathcal{M}_g$ consisting of isomorphism classes of stable curves of genus g. We recall ([34, 109, 112]) that a complete curve C is said to be a stable curve of genus $g \geq 1$ if
  (S1) C is reduced;
  (S2) C has only ordinary double points as possible singularities;
  (S3) $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = 1$;
  (S4) each nonsingular rational component of C meets the other components at more than two points.

P. Deligne and D. Mumford [34] proved that the coarse moduli space $\mathcal{M}_g^{\mathrm{DM}}$ is an irreducible projective variety, and contains $\mathcal{M}_g$ as a Zariski open subset.

We have three standard explicit toroidal compactifications $\mathcal{A}_g^{\mathrm{VI}}$, $\mathcal{A}_g^{\mathrm{VII}}$ and $\mathcal{A}_g^{\mathrm{cent}}$ constructed from
  (VI) the 1st Voronoi (or perfect) cone decomposition;
  (VII) the 2nd Voronoi cone decomposition;
  (cent) the central cone decomposition
respectively. We refer to [99, 135] for more details on the perfect cone decomposition and the 2nd Voronoi cone decomposition. In 1973, Y. Namikawa [109] proposed a natural question if the Torelli map

$$T_g : \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

extends to a regular map

$$T_g^{\mathrm{cent}} : \mathcal{M}_g^{\mathrm{DM}} \longrightarrow \mathcal{A}_g^{\mathrm{cent}}.$$

In fact, $\mathcal{A}_g^{\mathrm{cent}}$ is the normalization of the Igusa blow-up of the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ along the boundary $\partial \mathcal{A}_g^{\mathrm{cent}}$. In the 1970s, Mumford and Namikawa [110, 111] showed that the Torelli map T_g extends to a regular map

$$T_g^{\mathrm{VII}} : \mathcal{M}_g^{\mathrm{DM}} \longrightarrow \mathcal{A}_g^{\mathrm{VII}}.$$

In 2012, V. Alexeev and A. Brunyate [3] proved that the Torelli map T_g can be extended to a regular map

$$T_g^{\text{VI}} : \mathcal{M}_g^{\text{DM}} \longrightarrow \mathcal{A}_g^{\text{VI}} = \mathcal{A}_g^{\text{perf}}$$

and that the extended Torelli map

$$T_g^{\text{cent}} : \mathcal{M}_g^{\text{DM}} \longrightarrow \mathcal{A}_g^{\text{cent}}$$

is regular for $g \leq 6$ but not regular for $g \geq 9$. Furthermore they also showed that the two compactifications $\mathcal{A}_g^{\text{VI}}$ and $\mathcal{A}_g^{\text{VII}}$ are equal near the closure of the Jacobian locus J_g. Almost at the same time the extended Torelli map T_g^{cent} is regular for $g \leq 8$ by Alexeev and et al.[4].

I would like to mention that K. Liu, X. Sun and S.-T. Yau [88, 89, 90, 91] showed the goodness of the Hermitian metrics on the logarithmic tangent bundle on $\mathcal{M}_g$ which are induced by the Ricci and the perturbed Ricci metrics on $\mathcal{M}_g$. They also showed that the Ricci metric on $\mathcal{M}_g$ extends naturally to the divisor $D_g := \mathcal{M}_g^{\text{DM}} \setminus \mathcal{M}_g$ and coincides with the Ricci metric on each component of D_g.

Liu, Sun and Yau [89] showed that the existence of Kähler-Einstein metric on $\mathcal{M}_g$ is related to the stability of the logarithmic cotangent bundle over $\mathcal{M}_g^{\text{DM}}$.

Let E be a holomorphic vector bundle over a complex manifold X of dimension n. Let $\Phi := \Phi_X$ be a Kähler class (or form) of X. Then Φ-degree of E is defined by

$$\deg(E) := \int_X c_1(E) \, \Phi^{n-1}$$

and the slope of E is defined to be

$$\mu(E) := \frac{\deg(E)}{\text{rank}(E)}.$$

A bundle E is said to be $\Phi-$stable if for any proper coherent subsheaf $\mathcal{F} \subset E$, we have

$$\mu(\mathcal{F}) < \mu(E).$$

Let U be a local chart of $\mathcal{M}_g$ near the boundary with pinching coordinates $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ such that $(t_1, \cdots, t_m)$ represent the degeneration direction. Let

$$F_i = \frac{dt_i}{t_i} \quad (1 \leq i \leq m), \qquad F_j = ds_j \quad (m+1 \leq j \leq n).$$

Then the logarithmic cotangent bundle $(T^*\mathcal{M}_g)^{\text{DM}}$ is the unique extension of the cotangent bundle $T^*\mathcal{M}_g$ over $\mathcal{M}_g$ to $\mathcal{M}_g^{\text{DM}}$ such that on U $F_1, F_2, \cdots, F_n$ is a local holomorphic frame of $(T^*\mathcal{M}_g)^{\text{DM}}$.

Liu, Sun and Yau [89] proved the following.

Theorem B.1. *The first Chern class $c_1\left((T^*\mathcal{M}_g)^{\text{DM}}\right)$ is positive and $(T^*\mathcal{M}_g)^{\text{DM}}$ is stable with respect to $c_1\left((T^*\mathcal{M}_g)^{\text{DM}}\right)$.*

Remark B.2. We refer to [20, 142, 154, 155] for some topics related to $\mathcal{M}_g$ and $\mathcal{M}_g^{\text{DM}}$.

Appendix C. Singular Modular Forms

Let ρ be a rational representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. A holomorphic function $f : \mathbb{H}_g \longrightarrow V_\rho$ with values in V_ρ is called a modular form of type ρ if it satisfies

$$f(M \cdot \tau) = \rho(C\tau + D)f(\tau)$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and $\tau \in \mathbb{H}_g$. We denote by $[\Gamma_g, \rho]$ the vector space of all modular forms of type ρ. A modular form $f \in [\Gamma_g, \rho]$ of type ρ has a Fourier series

$$f(\tau) = \sum_{T \geq 0} a(T) e^{2\pi i \, \mathrm{tr}(T\tau)}, \quad \tau \in \mathbb{H}_g,$$

where T runs over the set of all semipositive half-integral symmetric matrices of degree g. A modular form f of type ρ is said to be singular if a Fourier coefficient $a(T)$ vanishes unless $\det(T) = 0$.

For $\tau = (\tau_{ij}) \in \mathbb{H}_g$, we write $\tau = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real. We put

$$\frac{\partial}{\partial Y} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right).$$

H. Maass [95] introduced the following differential operator

(C.1) $$M_g := \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} \right)$$

characterizing singular modular forms. Using the differential operator M_g, Maass [95, pp. 202–204] proved that if a nonzero singular modular form of degree g and type $\rho :=$ $\det^k$ (or weight k) exists, then $gk \equiv 0 \pmod 2$ and $0 < 2k \leq g - 1$. The converse was proved by R. Weissauer [148].

Freitag [51] proved that every singular modular form can be written as a finite linear combination of theta series with harmonic coefficients and proposed the problem to characterize singular modular forms. Weissauer [148] gave the following criterion.

Theorem C.1. *Let ρ be an irreducible rational representation of $GL(g,\mathbb{C})$ with its highest weight $(\lambda_1, \cdots, \lambda_g)$. Let f be a modular form of type ρ. Then the following are equivalent:*

(a) f is singular.

(b) $2\lambda_g < g$.

Now we describe how the concept of singular modular forms is closely related to the geometry of the Siegel modular variety. Let $X := \mathcal{A}_g^{\mathrm{Sat}}$ be the Satake compactification of the Siegel modular variety $\mathcal{A}_g = \Gamma_g \backslash \mathbb{H}_g$. Then $\mathcal{A}_g$ is embedded in X as a quasiprojective algebraic subvariety of codimension g. Let X_s be the smooth part of $\mathcal{A}_g$ and $\tilde{X}$ the desingularization of X. Without loss of generality, we assume $X_s \subset \tilde{X}$. Let $\Omega^p(\tilde{X})$ (resp. $\Omega^p(X_s)$) be the space of holomorphic p-form on $\tilde{X}$ (resp. X_s). Freitag and Pommerening [55] showed that if $g > 1$, then the restriction map

$$\Omega^p(\tilde{X}) \longrightarrow \Omega^p(X_s)$$

is an isomorphism for $p < \dim_{\mathbb{C}} \tilde{X} = \frac{g(g+1)}{2}$. Since the singular part of $\mathcal{A}_g$ is at least codimension 2 for $g > 1$, we have an isomorphism

$$\Omega^p(\tilde{X}) \cong \Omega^p(\mathbb{H}_g)^{\Gamma_g}.$$

Here $\Omega^p(\mathbb{H}_g)^{\Gamma_g}$ denotes the space of Γ_g-invariant holomorphic p-forms on $\mathbb{H}_g$. Let $\mathrm{Sym}^2(\mathbb{C}^g)$ be the symmetric power of the canonical representation of $GL(g, \mathbb{C})$ on $\mathbb{C}^n$. Then we have an isomorphism

(C.2) $$\Omega^p(\mathbb{H}_g)^{\Gamma_g} \longrightarrow \left[\Gamma_g, \bigwedge^p \mathrm{Sym}^2(\mathbb{C}^g)\right].$$

Theorem C.2 [148] *Let ρ_α be the irreducible representation of $GL(g, \mathbb{C})$ with highest weight*

$$(g+1, \cdots, g+1, g-\alpha, \cdots, g-\alpha)$$

such that $\mathrm{corank}(\rho_\alpha) = \alpha$ for $1 \leq \alpha \leq g$. If $\alpha = -1$, we let $\rho_\alpha = (g+1, \cdots, g+1)$. Then

$$\Omega^p(\mathbb{H}_g)^{\Gamma_g} = \begin{cases} [\Gamma_g, \rho_\alpha], & \text{if } p = \frac{g(g+1)}{2} - \frac{\alpha(\alpha+1)}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Remark C.3. *If $2\alpha > g$, then any $f \in [\Gamma_g, \rho_\alpha]$ is singular. Thus if $p < \frac{g(3g+2)}{8}$, then any Γ_g-invariant holomorphic p-form on $\mathbb{H}_g$ can be expressed in terms of vector valued theta series with harmonic coefficients. It can be shown with a suitable modification that the just mentioned statement holds for a sufficiently small congruence subgroup of Γ_g.*

Thus the natural question is to ask how to determine the Γ_g-invariant holomorphic p-forms on $\mathbb{H}_g$ for the nonsingular range $\frac{g(3g+2)}{8} \leq p \leq \frac{g(g+1)}{2}$. Weissauer [150] answered the above question for $g = 2$. For $g > 2$, the above question is still open. It is well know that the vector space of vector valued modular forms of type ρ is finite dimensional. The computation or the estimate of the dimension of $\Omega^p(\mathbb{H}_g)^{\Gamma_g}$ is interesting because its dimension is finite even though the quotient space $\mathcal{A}_g$ is noncompact.

Finally we will mention the results due to Weisauer [149]. We let Γ be a congruence subgroup of Γ_2. According to Theorem C.2, Γ-invariant holomorphic forms in $\Omega^2(\mathbb{H}_2)^\Gamma$ are corresponded to modular forms of type (3,1). We note that these invariant holomorphic 2-forms are contained in the *nonsingular range*. And if these modular forms are not cusp forms, they are mapped under the Siegel Φ-operator to cusp forms of weight 3 with respect to some congruence subgroup (dependent on Γ) of the elliptic modular group. Since there are finitely many cusps, it is easy to deal with these modular forms in the adelic version. Observing these facts, he showed that any 2-holomorphic form on $\Gamma \backslash \mathbb{H}_2$ can be expressed in terms of theta series with harmonic coefficients associated to binary positive definite quadratic forms. Moreover he showed that $H^2(\Gamma \backslash \mathbb{H}_2, \mathbb{C})$ has a pure Hodge structure and that the Tate conjecture holds for a suitable compactification of $\Gamma \backslash \mathbb{H}_2$. If $g \geq 3$, for a congruence subgroup Γ of Γ_g it is difficult to compute the cohomology groups $H^*(\Gamma \backslash \mathbb{H}_g, \mathbb{C})$ because $\Gamma \backslash \mathbb{H}_g$ is noncompact and highly singular. Therefore in order to study their structure, it is natural to ask if they have pure Hodge structures or mixed Hodge structures.

Appendix D. Singular Jacobi Forms

In this section, we discuss the notion of singular Jacobi forms. First of all we define the concept of Jacobi forms.

Let ρ be a rational representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half-integral semi-positive definite matrix of degree h. The canonical automorphic factor

$$J_{\rho,\mathcal{M}} : G^J \times \mathbb{H}_{g,h} \longrightarrow GL(V_\rho)$$

for G^J on $\mathbb{H}_{g,h}$ is given as follows:

$$
\begin{aligned}
J_{\rho,\mathcal{M}}((g, (\lambda, \mu; \kappa)), (\tau, z)) &= e^{2\pi i \operatorname{tr}\left(\mathcal{M}(z+\lambda\tau+\mu)(C\tau+D)^{-1} C\,^t(z+\lambda\tau+\mu)\right)} \\
&\quad \times e^{-2\pi i \operatorname{tr}\left(\mathcal{M}(\lambda\tau\,^t\lambda + 2\lambda\,^t z + \kappa + \mu\,^t\lambda)\right)} \rho(C\tau + D),
\end{aligned}
$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R})$, $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\tau, z) \in \mathbb{H}_{g,h}$. We refer to [161] for a geometrical construction of $J_{\rho,\mathcal{M}}$.

Let $C^\infty(\mathbb{H}_{g,h}, V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{g,h}$ with values in V_ρ. For $f \in C^\infty(\mathbb{H}_{g,h}, V_\rho)$, we define

$$
\begin{aligned}
\left(f|_{\rho,\mathcal{M}}[(g, (\lambda, \mu; \kappa))]\right)(\tau, z) &= J_{\rho,\mathcal{M}}((g, (\lambda, \mu; \kappa)), (\tau, z))^{-1} \\
&\quad f\left(g\cdot\tau, (z + \lambda\tau + \mu)(C\tau + D)^{-1}\right),
\end{aligned}
$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R})$, $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\tau, z) \in \mathbb{H}_{g,h}$.

Definition D.1. *Let ρ and $\mathcal{M}$ be as above. Let*

$$H_{\mathbb{Z}}^{(g,h)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \lambda, \mu, \kappa \text{ integral} \right\}$$

*be the discrete subgroup of $H_{\mathbb{R}}^{(g,h)}$. A **Jacobi form** of index $\mathcal{M}$ with respect to ρ on a subgroup Γ of Γ_g of finite index is a holomorphic function $f \in C^\infty(\mathbb{H}_{g,h}, V_\rho)$ satisfying the following conditions* (A) *and* (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ *for all $\tilde{\gamma} \in \widetilde{\Gamma} := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$.*

(B) *For each $M \in \Gamma_g$, $f|_{\rho,\mathcal{M}}[M]$ has a Fourier expansion of the following form :*

$$(f|_{\rho,\mathcal{M}}[M])(\tau, z) = \sum_{\substack{T = \,^t T \geq 0 \\ half\text{-}integral}} \sum_{R \in \mathbb{Z}^{(g,h)}} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma} \operatorname{tr}(T\tau)} \cdot e^{2\pi i \operatorname{tr}(Rz)}$$

with $\lambda_\Gamma (\neq 0) \in \mathbb{Z}$ and $c(T, R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2}\,^t R & \mathcal{M} \end{pmatrix} \geq 0$.

If $g \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [189, Lemma 1.6]). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler (cf. [189, Theorem 1.8] or [42, Theorem 1.1]) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(g, \mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of the corresponding Jacobi forms. For more results about Jacobi forms with $g > 1$ and $h > 1$, we refer to

[156, 157, 159, 160, 161, 165] and [189]. Jacobi forms play an important role in lifting elliptic cusp forms to Siegel cusp forms of degree $2g$.

Without loss of generality we may assume that $\mathcal{M}$ is positive definite. For simplicity, we consider the case that Γ is the Siegel modular group Γ_g of degree g.

Let g and h be two positive integers. We recall that $\mathcal{M}$ is a symmetric positive definite, half-integral matrix of degree h. We let

$$\mathcal{P}_g := \{Y \in \mathbb{R}^{(g,g)} \,|\, Y = {}^t Y > 0\,\}$$

be the open convex cone of positive definite matrices of degree g in the Euclidean space $\mathbb{R}^{\frac{g(g+1)}{2}}$. We define the differential operator $M_{g,h,\mathcal{M}}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ defined by

(D.1)
$$M_{g,h,\mathcal{M}} := \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi} {}^t\left(\frac{\partial}{\partial V}\right) \mathcal{M}^{-1}\left(\frac{\partial}{\partial V}\right)\right),$$

where

$$Y = (y_{\mu\nu}) \in \mathcal{P}_g, \quad V = (v_{kl}) \in \mathbb{R}^{(h,g)}, \quad \frac{\partial}{\partial Y} = \left(\frac{1 + \delta_{\mu\nu}}{2}\frac{\partial}{\partial y_{\mu\nu}}\right)$$

and

$$\frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right).$$

We note that this differential operator $M_{g,h,\mathcal{M}}$ generalizes the Maass operator M_g (see Formula (C.1)).

Yang [160] characterized singular Jacobi forms as follows:

Theorem D.2. *Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form of index $\mathcal{M}$ with respect to a finite dimensional rational representation ρ of $GL(g,\mathbb{C})$. Then the following conditions are equivalent:*
(1) f is a singular Jacobi form.
(2) f satisfies the differential equation $M_{g,h,\mathcal{M}}f = 0$.

Theorem D.3. *Let ρ be an irreducible finite dimensional representation of $GL(g,\mathbb{C})$. Then there exists a nonvanishing singular Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ if and only if $2k(\rho) < g + h$. Here $k(\rho)$ denotes the weight of ρ.*

For the proofs of the above theorems we refer to [160], Theorem 4.1 and Theorem 4.5.

Exercise D.4. Compute the eigenfunctions and the eigenvalues of $M_{g,h,\mathcal{M}}$(cf. [160], pp. 2048–2049).

Now we consider the following group $GL(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ equipped with the multiplication law

$$(A, (\lambda, \mu, \kappa)) * (B, (\lambda', \mu', \kappa'))$$
$$= (AB, (\lambda B + \lambda', \mu\, {}^t B^{-1} + \mu', \kappa + \kappa' + \lambda B\, {}^t \mu' - \mu\, {}^t B^{-1}\, {}^t \lambda')),$$

where $A, B \in GL(g,\mathbb{R})$ and $(\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H_{\mathbb{R}}^{(g,h)}$. We observe that $GL(g,\mathbb{R})$ acts on $H_{\mathbb{R}}^{(g,h)}$ on the right as automorphisms. And we have the canonical action of $GL(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ defined by

(D.2)
$$(A, (\lambda, \mu, \kappa)) \circ (Y, V) := (AY\, {}^t A, (V + \lambda Y + \mu)\, {}^t A),$$

where $A \in GL(g, \mathbb{R}), (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(Y, V) \in \mathcal{P}_g \times \mathbb{R}^{(h,g)}$.

Lemma D.5. *The differential operator $M_{g,h,\mathcal{M}}$ defined by the formula* (D.1) *is invariant under the action* (D.2) *of $GL(g, \mathbb{R}) \ltimes \left\{ (0, \mu, 0) \,\middle|\, \mu \in \mathbb{R}^{(h,g)} \right\}$.*

Proof. It follows immediately from the direct calculation.

We have the following natural questions.

Problem D.6. Develop the invariant theory for the action of $GL(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$. We refer to [174, 177] for related topics.

Problem D.7. Discuss the application of the theory of singular Jacobi forms to the geometry of the universal abelian variety as that of singular modular forms to the geometry of the Siegel modular variety (see Appendix C).

Appendix E. Stable Jacobi Forms

Throughout this appendix we put

$$\Gamma_g := Sp(2g, \mathbb{Z}) \quad \text{and} \quad \Gamma_{g,h} := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}.$$

For a commutative ring R and an integer m, we denote by $S_m(R)$ the set of all $m \times m$ symmetric matrices with entries in R.

We know that the Siegel-Jacobi space

$$\mathbb{H}_{g,h} = G^J / K^J$$

is a non-symmetric homogeneous space. Here

$$K^J = \{(k, (0, 0; \kappa)) \mid k \in U(g), \ \kappa \in S_h(\mathbb{R}) \}$$

is a subgroup of G^J. Let $\mathfrak{g}^J$ be the Lie algebra of the Jacobi group G^J. Then $\mathfrak{g}^J$ has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\mathfrak{k}^J = \left\{ \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, (0, 0; \kappa) \right) \,\middle|\, a + {}^t a = 0, \ b \in S_g(\mathbb{R}), \ \kappa \in S_h(\mathbb{R}) \right\}$$

and

$$\mathfrak{p}^J = \left\{ \left(\begin{pmatrix} a & b \\ b & -a \end{pmatrix}, (P, Q; 0) \right) \,\middle|\, a, b \in S_g(\mathbb{R}), \ P, Q \in \mathbb{C}^{(h,g)} \right\}.$$

We observe that $\mathfrak{k}^J$ is the Lie algebra of K^J. The complexification $\mathfrak{p}_{\mathbb{C}}^J := \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{p}^J$ has a decomposition

$$\mathfrak{p}_{\mathbb{C}}^J = \mathfrak{p}_+^J + \mathfrak{p}_-^J,$$

where

$$\mathfrak{p}_+^J = \left\{ \left(\begin{pmatrix} X & -iX \\ -iX & -X \end{pmatrix}, (P, -iP; 0) \right) \,\middle|\, X \in S_g(\mathbb{C}), \ P \in \mathbb{C}^{(h,g)} \right\}.$$

and

$$\mathfrak{p}_-^J = \left\{ \left(\begin{pmatrix} X & -iX \\ -iX & -X \end{pmatrix}, (P, -iP; 0) \right) \,\middle|\, X \in S_g(\mathbb{C}), \ P \in \mathbb{C}^{(h,g)} \right\}.$$

We define a complex structure I^J on the tangent space $\mathfrak{p}^J$ of $\mathbb{H}_{g,h}$ at $(iI_g, 0)$ by

$$I^J\left(\begin{pmatrix} a & b \\ b & -a \end{pmatrix}, (P, Q; 0)\right) := \left(\begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}, (Q, -P; 0)\right).$$

Identifying $\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ with $\mathbb{C}^{(h,g)}$ via

$$(P, Q; 0) \longmapsto i\, P + Q, \quad P, Q \in \mathbb{R}^{(h,g)},$$

we may regard the complex structure I^J as a real linear map

$$I^J(X + i\, Y, Q + i\, P) = (-Y + i\, X, -P + i\, Q),$$

where $X + i\, Y \in S_g(\mathbb{R})$, $Q + i\, P \in \mathbb{C}^{(h,g)}$. I^J extends complex linearly on the complexification $\mathfrak{p}^J_{\mathbb{C}}$. With respect to this complex structure I^J, we may say that a function f on $\mathbb{H}_{g,h}$ is *holomorphic* if and only if $\xi f = 0$ for all $\xi \in \mathfrak{p}^J_-$.

Since the space $\mathbb{H}_{g,h}$ is diffeomorphic to the homogeneous space G^J/K^J, we may lift a function f on $\mathbb{H}_{g,h}$ with values in V_ρ to a function Φ_f on G^J with values in V_ρ in the following way. We define the lifting

(E.1) $$L_{\rho,\mathcal{M}} : \mathcal{F}(\mathbb{H}_{g,h}, V_\rho) \longrightarrow \mathcal{F}(G^J, V_\rho), \quad L_{\rho,\mathcal{M}}(f) := \Phi_f$$

by

$$\begin{aligned} \Phi_f(x) : \quad &= (f|_{\rho,\mathcal{M}}[x])(iI_g, 0) \\ &= J_{\rho,\mathcal{M}}(x, (iI_g, 0))\, f(x \cdot (iI_g, 0)), \end{aligned}$$

where $x \in G^J$ and $\mathcal{F}(\mathbb{H}_{g,h}, V_\rho)$ (resp. $\mathcal{F}(G^J, V_\rho)$) denotes the vector space consisting of functions on $\mathbb{H}_{g,h}$ (resp. G^J) with values in V_ρ.

We see easily that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ is isomorphic to the space $A_{\rho,\mathcal{M}}(\Gamma_{g,h})$ of smooth functions Φ on G^J with values in V_ρ satisfying the following conditions:

(1a) $\Phi(\gamma x) = \Phi(x)$ for all $\gamma \in \Gamma^J$ and $x \in G^J$.
(1b) $\Phi(x\, r(k, \kappa)) = e^{2\pi i\, \text{tr}(\mathcal{M}\kappa)} \rho(k)^{-1} \Phi(x)$ for all $x \in G^J$, $r(k, \kappa) \in K^J$.
(2) $Y^- \Phi = 0$ for all $Y^- \in \mathfrak{p}^J_-$.
(3) For all $M \in Sp(2g, \mathbb{R})$, the function $\psi : G^J \longrightarrow V_\rho$ defined by

$$\psi(x) := \rho(Y^{-\frac{1}{2}})\, \Phi(Mx), \quad x \in G^J$$

is bounded in the domain $Y \geq Y_0$. Here $x \cdot (iI_g, 0) = (\tau, z)$ with $\tau = X + iY$, $Y > 0$. Clearly $J^{\text{cusp}}_{\rho,\mathcal{M}}(\Gamma_g)$ is isomorphic to the subspace $A^0_{\rho,\mathcal{M}}(\Gamma_{g,h})$ of $A_{\rho,\mathcal{M}}(\Gamma_{g,h})$ with the condition (3+) that the function $g \longmapsto \Phi(g)$ is bounded.

Let $\mathcal{M}$ be a fixed positive definite symmetric half-integral matrix of degree h. Let $\rho_\infty := (\rho_n)$ be a stable representation of $GL(\infty, \mathbb{C})$. That is, for each $n \in \mathbb{Z}^+$, ρ_n is a finite dimensional rational representation of $GL(n, \mathbb{C})$ and ρ_∞ is compatible with the embeddings $\alpha_{kl} : GL(k, \mathbb{C}) \longrightarrow GL(l, \mathbb{C})\, (k < l)$ defined by

$$\alpha_{kl}(A) := \begin{pmatrix} A & 0 \\ 0 & I_{l-k} \end{pmatrix}, \quad A \in GL(k, \mathbb{C}), \quad k < l.$$

For two positive integers m and n, we put

$$G^J_{n,m} := Sp(2n, \mathbb{R}) \ltimes H^{(n,m)}_{\mathbb{R}}.$$

For $k, l \in \mathbb{Z}^+$ with $k < l$, we define the mapping $\Phi_{l,k,\mathcal{M}}$ of $A_{\rho_l,\mathcal{M}}(\Gamma_{l,m})$ into the functions on $G_{k,m}^J$ by

$$(\text{E.2}) \qquad (\Phi_{l,k,\mathcal{M}}F)(x) := J_{\mathcal{M},\rho_k}(x, (iI_k, 0)) \lim_{t \longrightarrow \infty} J_{\mathcal{M},\rho_l}(x_t, (iI_l, 0))^{-1} F(x_t),$$

where $F \in A_{\rho_l,\mathcal{M}}(\Gamma_{l,m}^J)$, $x = (M, (\lambda, \mu; \kappa)) \in G_{k,m}^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2k, \mathbb{R})$ and

$$x_t := \left(\begin{pmatrix} A & 0 & B & 0 \\ 0 & t^{1/2}I_{l-k} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & t^{-1/2}I_{l-k} \end{pmatrix}, ((\lambda, 0), (\mu, 0); \kappa) \right) \in G_{l,h}^J.$$

Proposition E.1. *The limit* (E.2) *always exists and the image of* $A_{\rho_l,\mathcal{M}}(\Gamma_{l,h})$ *under* $\Phi_{l,k,\mathcal{M}}$ *is contained in* $A_{\rho_k,\mathcal{M}}(\Gamma_{k,h})$. *Obviously the mapping*

$$\Phi_{l,k,\mathcal{M}} : A_{\rho_l,\mathcal{M}}(\Gamma_{l,h}^J) \longrightarrow A_{\rho_k,\mathcal{M}}(\Gamma_{k,h})$$

is a linear mapping.

The mapping $\Phi_{l,k,\mathcal{M}}$ is called the Siegel-Jacobi operator. For any $g \in \mathbb{Z}^+$, we put

$$(\text{E.3}) \qquad\qquad A_{g,\mathcal{M}} := \bigoplus_\rho A_{\rho,\mathcal{M}}(\Gamma_{g,h}),$$

where ρ runs over all isomorphism classes of irreducible rational representations of $GL(g, \mathbb{C})$. For $g = 0$, we set $A_{0,\mathcal{M}} := \mathbb{C}$.

For each $g \in \mathbb{Z}^+$, we put

$$(\text{E.4}) \qquad\qquad A_{g,\mathcal{M}}^* := \bigoplus_{\rho_*} A(\rho_*, \mathcal{M}),$$

where ρ_* runs over all isomorphism classes of irreducible rational representations of $GL(g, \mathbb{C})$ with highest weight $\lambda(\rho_*) \in (2\mathbb{Z})^g$. It is obvious that if $k < l$, then the Siegel-Jacobi operator $\Phi_{l,k,\mathcal{M}}$ maps $A_{l,\mathcal{M}}$ (resp. $A_{l,\mathcal{M}}^*$) into $A_{k,\mathcal{M}}$ (resp. $A_{k,\mathcal{M}}^*$).

We let

$$(\text{E.5}) \qquad\qquad A_{\infty,\mathcal{M}} := \varprojlim_k A_{k,\mathcal{M}} \quad \text{and} \quad A_{\infty,\mathcal{M}}^* := \varprojlim_k A_{k,\mathcal{M}}^*$$

be the inverse limits of $(A_{k,\mathcal{M}}, \Phi_{l,k,\mathcal{M}})$ and $(A_{k,\mathcal{M}}^*, \Phi_{l,k,\mathcal{M}})$ respectively.

Proposition E.2. $A_{\infty,\mathcal{M}}$ *has a commutative ring structure compatible with the Siegel-Jacobi operators. Obviously* $A_{\infty,\mathcal{M}}^*$ *is a subring of* $A_{\infty,\mathcal{M}}$.

For a stable irreducible representation $\rho_\infty = (\rho_g)$ of $GL(\infty, \mathbb{C})$, we define

$$(\text{E.6}) \qquad\qquad A_{\rho_\infty,\mathcal{M}} := \varprojlim_g A_{\rho_g,\mathcal{M}}(\Gamma_{g,h}).$$

Proposition E.3. *We have*

$$A_{\infty,\mathcal{M}} = \bigoplus_{\rho_\infty} A_{\rho_\infty,\mathcal{M}},$$

where ρ_∞ *runs over all isomorphism classes of stable irreducible representations of* $GL(\infty, \mathbb{C})$.

Definition E.**4.** *Elements in $A_{\infty,\mathcal{M}}$ are called* stable automorphic forms *on $G^J_{\infty,h}$ of index $\mathcal{M}$ and elements of $A^*_{\infty,\mathcal{M}}$ are called* even stable automorphic forms *on $G^J_{\infty,h}$ of index $\mathcal{M}$.*

For $g \geq 1$, we define

(E.7)
$$\mathbb{A}_g := \bigoplus_{\mathcal{M}} \bigoplus_{\rho} A_{\rho,\mathcal{M}}(\Gamma_{g,h}),$$

where $\mathcal{M}$ runs over all equivalence classes of positive definite symmetric, half-integral matrices of any degree ≥ 1 and ρ runs over all isomorphism classes of irreducible rational representations of $GL(g,\mathbb{C})$. We set $\mathbb{A}_0 := \mathbb{C}$.

For $g \geq 1$, we also define

(E.8)
$$\mathbb{A}^*_g := \bigoplus_{\mathcal{M}} \bigoplus_{\rho_*} A_{\rho_*,\mathcal{M}}(\Gamma_{g,h}),$$

where $\mathcal{M}$ runs over all equivalence classes of positive definite symmetric half-integral matrices of any degree ≥ 1 and ρ_* runs over all isomorphism classes of irreducible rational representations of $GL(g,\mathbb{C})$ with highest weight $\lambda(\rho_*) \in (2\mathbb{Z})^g$.

Let $\rho_\infty = (\rho_g)$ be a stable irreducible rational representation of $GL(\infty,\mathbb{C})$. For each irreducible rational representation ρ_g of $GL(g,\mathbb{C})$ appearing in ρ_∞, we put

(E.9)
$$A(\rho_g; \rho_\infty) := \bigoplus_{\mathcal{M}} A_{\rho_g,\mathcal{M}}(\Gamma_{g,h}),$$

where $\mathcal{M}$ runs over all equivalence classes of positive definite symmetric half-integral matrices of any degree ≥ 1. Clearly the Siegel-Jacobi operator $\Phi_{l,k} := \bigoplus_{\mathcal{M}} \Phi_{l,k,\mathcal{M}}$ ($k < l$) maps $A(\rho_l; \rho_\infty)$ into $A(\rho_k; \rho_\infty)$.

Using the Siegel-Jacobi operators, we can define the inverse limits

(E.10)
$$A(\rho_\infty) := \varprojlim_{g} A(\rho_g; \rho_\infty), \quad \mathbb{A}_\infty := \varprojlim_{g} \mathbb{A}_g \quad \text{and} \quad \mathbb{A}^*_\infty := \varprojlim_{g} \mathbb{A}^*_g.$$

Theorem E.**5.**
$$\mathbb{A}_\infty = \bigoplus_{\rho_\infty} A(\rho_\infty),$$

where ρ_∞ runs over all equivalence classes of stable irreducible representations of $GL(\infty,\mathbb{C})$.

Let ρ and $\mathcal{M}$ be the same as in the previous sections. For positive integers r and g with $r < g$, we let $\rho^{(r)} : GL(r,\mathbb{C}) \longrightarrow GL(V_\rho)$ be a rational representation of $GL(r,\mathbb{C})$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & I_{g-r} \end{pmatrix}\right)v, \quad a \in GL(r,\mathbb{C}), \quad v \in V_\rho.$$

The Siegel-Jacobi operator $\Psi_{g,r} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$ is defined by

(E.11)
$$(\Psi_{g,r}f)(\tau,z) := \lim_{t \to \infty} f\left(\begin{pmatrix} \tau & 0 \\ 0 & itI_{g-r} \end{pmatrix}, (z,0)\right),$$

where $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, $\tau \in \mathbb{H}_r$ and $z \in \mathbb{C}^{(h,r)}$. It is easy to check that the above limit always exists and the Siegel-Jacobi operator is a linear mapping. Let $V^{(r)}_\rho$ be the subspace of V_ρ

spanned by the values $\{\,(\Psi_{g,r}f)(\tau,z)\mid f\in J_{\rho,\mathcal{M}}(\Gamma_g),\ (\tau,z)\in\mathbb{H}_r\times\mathbb{C}^{(h,r)}\,\}$. Then $V_\rho^{(r)}$ is invariant under the action of the group

$$\left\{\begin{pmatrix} a & 0 \\ 0 & I_{g-r} \end{pmatrix} \ :\ a\in GL(r,\mathbb{C})\right\}\cong GL(r,\mathbb{C}).$$

We can show that if $V_\rho^{(r)}\neq 0$ and (ρ,V_ρ) is irreducible, then $(\rho^{(r)},V_\rho^{(r)})$ is also irreducible.

Theorem E.6. *The action of the Siegel-Jacobi operator is compatible with that of that of the Hecke operator.*

We refer to [159] for a precise detail on the Hecke operators and the proof of the above theorem.

Problem E.7. *Discuss the injectivity, surjectivity and bijectivity of the Siegel-Jacobi operator.*

This problem was partially discussed by Yang [159] and Kramer [83] in the special cases. For instance, Kramer [83] showed that if g is arbitrary, $h=1$ and $\rho:GL(g,\mathbb{C})\longrightarrow\mathbb{C}^\times$ is a one-dimensional representation of $GL(g,\mathbb{C})$ defined by $\rho(a):=(\det(a))^k$ for some $k\in\mathbb{Z}^+$, then the Siegel-Jacobi operator

$$\Psi_{g,g-1}:J_{k,m}(\Gamma_g)\longrightarrow J_{k,m}(\Gamma_{g-1})$$

is surjective for $k\gg m\gg 0$.

Theorem E.8. *Let $1\leq r\leq g-1$ and let ρ be an irreducible finite dimensional representation of $GL(g,\mathbb{C})$. Assume that $k(\rho)>g+r+\mathrm{rank}(\mathcal{M})+1$ and that k is even. Then*

$$J_{\rho^{(r)},\mathcal{M}}^{cusp}(\Gamma_r)\subset\Psi_{g,r}(J_{\rho,\mathcal{M}}(\Gamma_g)).$$

Here $J_{\rho^{(r)},\mathcal{M}}^{cusp}(\Gamma_r)$ denotes the subspace consisting of all cuspidal Jacobi forms in $J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$.

Idea of Proof. For each $f\in J_{\rho^{(r)},\mathcal{M}}^{\mathrm{cusp}}(\Gamma_r)$, we can show by a direct computation that

$$\Psi_{g,r}(E_{\rho,\mathcal{M}}^{(g)}(\tau,z;f))=f,$$

where $E_{\rho,\mathcal{M}}^{(g)}(\tau,z;f)$ is the Eisenstein series of Klingen's type associated with a cusp form f. For a precise detail, we refer to [189].

Remark E.9. Dulinski [40] decomposed the vector space $J_{k,\mathcal{M}}(\Gamma_g)\,(k\in\mathbb{Z}^+)$ into a direct sum of certain subspaces by calculating the action of the Siegel-Jacobi operator on Eisenstein series of Klingen's type explicitly.

For two positive integers r and g with $r\leq g-1$, we consider the bigraded ring

$$J_{*,*}^{(r)}(\ell):=\bigoplus_{\mathcal{M}}\bigoplus_{k=0}^{\infty}J_{k,\mathcal{M}}(\Gamma_r(\ell))$$

and

$$M_*^{(r)}(\ell):=\bigoplus_{k=0}^{\infty}J_{k,0}(\Gamma_r(\ell))=\bigoplus_{k=0}^{\infty}[\Gamma_r(\ell),k],$$

where $\Gamma_r(\ell)$ denotes the principal congruence subgroup of Γ_r of level ℓ and $\mathcal{M}$ runs over the set of all symmetric semi-positive half-integral matrices of degree h. Let

$$\Psi_{r,r-1,\ell} : J_{k,\mathcal{M}}(\Gamma_r(\ell)) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{r-1}(\ell))$$

be the Siegel-Jacobi operator defined by (E.11).

Problem E.10. Investigate $\operatorname{Proj} J^{(r)}_{*,*}(\ell)$ over $M^{(r)}_*(\ell)$ and the quotient space

$$Y_r(\ell) := (\Gamma_r(\ell) \ltimes (\ell\mathbb{Z})^2) \backslash (\mathbb{H}_r \times \mathbb{C}^r)$$

for $1 \le r \le g-1$.

The difficulty to this problem comes from the following facts (A) and (B):
  (A) $J^{(r)}_{*,*}(\ell)$ is not finitely generated over $M^{(r)}_*(\ell)$.
  (B) $J^{\mathrm{cusp}}_{k,\mathcal{M}}(\Gamma_r(\ell)) \neq \ker \Psi_{r,r-1,\ell}$ in general.

These are the facts different from the theory of Siegel modular forms. We remark that Runge (cf. [126], pp. 190–194) discussed some parts about the above problem.

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