Stability of surfaces with constant mean curvature in $\mathbb{R}^3$

by

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## § 0. Introduction

Let $x : M^n \longrightarrow \mathbb{R}^{n+1}$ be an immersion of an orientable n-dimensional connected manifold $M^n$ into the (n+1)-dimensional Euclidean space $\mathbb{R}^{n+1}$. Then it is well-known that the mean curvature of $x$ is constant if and only if $x$ is a critical point of the n-area for all compactly supported volume-preserving variations. We say that an immersion $x : M^n \longrightarrow \mathbb{R}^{n+1}$ with non-zero constant mean curvature is stable if the second variations of the n-area for all such variations as above are non-negative.

When $M$ is compact, Barbosa and do Carmo [2] proved that if the mean curvature of $x : M^n \longrightarrow \mathbb{R}^{n+1}$ is non-zero constant and $x$ is stable, then $x(M^n)$ is a round sphere $S^n \subset \mathbb{R}^{n+1}$.

On the other hand, they conjectured that there are no complete stable immersions $x : M^2 \longrightarrow \mathbb{R}^3$ with non-zero constant mean curvature. When $M^2$ is non-compact, $M^2$ is hyperbolic or parabolic with respect to the natural complex structure given by $x$. Under some additional condition about metric for the case that $M^2$ is parabolic, we prove the above conjecture.

Theorem 1. Let $M$ be a non-compact orientable 2-dimensional connected manifold. Then there is no complete stable immersion $x : M \longrightarrow \mathbb{R}^3$ with non-zero constant mean curvature which satisfies the following (i) or (ii).

(i) $M$ is hyperbolic.

(ii) $M$ is parabolic, and for the universal covering

$\pi : \mathbb{C} \longrightarrow M$ , the metric $ds^2 = \lambda^2 |dz|^2$ of $\mathbb{C}$ induced by $x \circ \pi$ satisfies the following inequality except some compact set.

$$(0\text{-}1) \qquad \lambda(z) \geq c_0 |z|^{-1} ,$$

where $c_0$ is a positive constant and $z$ is the canonical coordinate in $\mathbb{C}$ .

This theorem is proved in section 2 as a corollary of a more general result Theorem 2.

It should be remarked that in the case of zero mean curvature, do Carmo and Peng [4] proved that the plane is the only "stable" complete minimal surface in $\mathbb{R}^3$ . Of course, in their theorem "stable" means the usual stability of minimal surfaces, that is, the second variation of the area is non-negative for all compactly supported variations that need not be volume-preserving.

If we would employ the generalization of the usual stability of minimal surfaces as the definition of the stability of non-zero constant mean curvature surfaces, the statement of the above conjecture has already been proved by Mori [6]. However, we feel that our definition is more natural because even the sphere is not stable in the other definition of stability.

## § 1. Barbosa and do Carmo's formulation of stability

In this section, we recall Barbosa and do Carmo's formulation of stability of non-zero constant mean curvature hypersurfaces. In [2] we can find all definitions and formulas in this section with their proofs.

Let $x : M^n \longrightarrow \mathbb{R}^{n+1}$ be an immersion of an orientable n-dimensional differentiable manifold $M^n$ into $\mathbb{R}^{n+1}$, and let $D \subset M^n$ be a relatively compact domain with smooth boundary $\partial D$. Then the n-area of $D$ with respect to the induced metric by $x$ (which we denote by $A_D(x)$) and the volume of $D$ in $x$ (which we denote by $V_D(x)$) are defined as follows.

$$A_D(x) = \int_D dM , \qquad V_D(x) = \frac{1}{n+1} \int_D < x,N > dM ,$$

where $dM$ is the volume element of $M^n$ with respect to the induced metric by $x$, $N$ is the unit normal vector field along $x$, and $< , >$ is the inner product in $\mathbb{R}^{n+1}$.

Let $x_t : \bar{D} \longrightarrow \mathbb{R}^{n+1}$, $t \in (-\varepsilon,\varepsilon)$ $(\varepsilon>0)$, $x_0 = x$, be a variation of $x|_D$. We say that the variation $x_t$ is volume-preserving if $V_D(x_t) = V_D(x)$ for all $t$, and that $x_t$ fixes the boundary if $x_t|_{\partial D} = x|_{\partial D}$ for all $t$.

<u>Formula 1</u>. The mean curvature of $x$ is constant if and only if for any relatively compact domain $D$ with smooth boundary and for any volume-preserving variation $x_t : \bar{D} \longrightarrow \mathbb{R}^{n+1}$ that fixes the boundary,

$$\left.\frac{d\, A_D(x_t)}{dt}\right|_{t=0} = 0 \ .$$

Definition 1. Let $x : M^n \longrightarrow \mathbb{R}^{n+1}$ be an immersion with non-zero constant mean curvature. Then we say that $x$ is stable if and only if for any such $D$ and $x_t$ as in Formula 1,

$$\left.\frac{d^2 A_D(x_t)}{dt^2}\right|_{t=0} \geqq 0 \ .$$

Formula 2. Let $x : M^n \longrightarrow \mathbb{R}^{n+1}$ be an immersion with non-zero constant mean curvature. Then $x$ is stable, if and only if for any such $D$ as in Formula 1 and for any function $f$ belonging to the function space

$$(1\text{-}1) \quad F_D = \left\{ f : M^n \longrightarrow \mathbb{R} \;\middle|\; \text{support } f \subseteqq \bar{D} \ , \ f \ \text{ is piecewise-smooth, and } \int_{M^n} f \, dM = 0 \right\} ,$$

the integral $I(f)$ defined below is non-negative.

$$I(f) = -\int_{M^n} (\Delta_M f + \|B\|^2 f) f \, dM \ ,$$

where $\Delta_M f$ is the Laplacian of $f$ in the induced metric and $\|B\|^2$ is the square of the norm of the second fundamental form $B$ of $x$ .

Here we should remark about the sign of $\Delta_M$ . Let $p$ be a point in $M^n$ , and let $(u^1,\ldots,u^n)$ be coordinates in a neighbourhood of $p$ in $M^n$. Denote the induced metric in $M^n$ by $g = \sum_{i,j=1}^{n} g_{ij} du^i du^j$, and set

$$G = \det(g_{ij}) \quad \text{and} \quad (g^{ij}) = (g_{ij})^{-1} .$$

Then

$$\Delta_M = \frac{1}{\sqrt{G}} \sum_{i,j} \frac{\partial}{\partial u^i} (\sqrt{G}\, g^{ij} \frac{\partial}{\partial u^j}) .$$

## § 2. The main theorem and its proof

From now on $M$ is assumed to be a non-compact orientable 2-dimensional connected manifold. First we prove the following Theorem 2 which is more general than Theorem 1.

<u>Theorem 2</u>. Let $M$ be the same as in Theorem 1. Then there is no complete stable immersion $x : M \longrightarrow \mathbb{R}^3$ with non-zero constant mean curvature which satisfies the following (i) or (ii).

(i) $M$ is hyperbolic.

(ii) $M$ is parabolic, and for the universal covering $\pi : \mathbb{C} \longrightarrow M$, the metric $ds^2 = \lambda^2|dz|^2$ of $\mathbb{C}$ induced by $x \circ \pi$ satisfies the inequality

$$(2\text{-}1) \quad \int_{\rho_1}^{\rho_2}\left(\int_0^{2\pi} \lambda^4 \| B\|^2 d\theta\right)^{1/4} d\rho \geqq c \log \frac{\rho_2}{\rho_1}$$

for all $\rho_1$ and $\rho_2$ $(\rho_2 \geqq \rho_1 > \rho_0)$, where $c$ and $\rho_0$ are positive constants.

<u>Lemma 1</u>. Let $\pi : \tilde{M} \longrightarrow M$ be the universal covering of $M$. If $x \circ \pi$ is not stable, then $x$ is not stable also.

<u>Proof</u>. Let $\Omega$ be a relatively compact domain of $M^*$ $(= M$ or $\tilde{M})$ with smooth boundary. Consider $x$ and $x \circ \pi$ as critical points for the area functional with respect to compactly supported volume-preserving variations that fix the boundary. Then the corresponding Hessian form is

$$I(f) = -\int_\Omega (\Delta_{M^*}f + \|B\|^2 f) f \, dM^* ,$$

$$f \in F^*_\Omega = \{f \in H^1_0(\Omega) \ ; \ \int_\Omega f \, dM^* = 0\} ,$$

where we denote the second fundamental form of $x \circ \pi$ also by $B$ .

Consider the following eigenvalue problem associated with the quadratic form $I(f)$ .

$$(2\text{-}2) \qquad \Delta_{M^*}f + \|B\|^2 f - \frac{\int_\Omega (\Delta_{M^*}f + \|B\|^2 f)\, dM^*}{\int_\Omega dM^*} + \lambda f = 0 , \ f \in F^*_\Omega .$$

Denote the eigenvalues of (2-2) by

$$\lambda_1(\Omega) \leqq \lambda_2(\Omega) \leqq \lambda_3(\Omega) \leqq \ldots \longrightarrow +\infty .$$

Then it follows that

$$(2\text{-}3) \qquad \lambda_1(\Omega) = \inf_{f \in F^*_\Omega - \{0\}} \frac{I(f)}{\int_\Omega |f|^2 dM^*}$$

(c.f. Berger-Gauduchon-Mazet [3, p. 186]). Set

$$\text{index}\ (\Omega) = \#\ \{\lambda_j(\Omega) \ ; \ \lambda_j(\Omega) < 0\} ,$$

$$\text{nullity}\ (\Omega) = \#\ \{\lambda_j(\Omega) \ ; \ \lambda_j(\Omega) = 0\} .$$

Let $c_t : \Omega \longrightarrow \Omega$ , $t \geqq 0$ , be a smooth family of diffeomorphisms of $\Omega$ into $\Omega$ such that

(a) $c_0$ = identity ,

(b) $c_t(\Omega) \subsetneqq c_s(\Omega)$ for $t > s$ ,

(c) $\lim_{t\to\infty}$ Volume $(c_t(\Omega)) = 0$ .

Denote $c_t(\Omega)$ by $\Omega_t$ . Then from the Morse index theorem with constraints proved by Frid and Thayer [5],

$$\text{(2-4)} \qquad \text{index } (\Omega) = \sum_{t>0} \text{nullity } (\Omega_t) .$$

Assume that $x \circ \pi : \widetilde{M} \longrightarrow \mathbb{R}^3$ is not stable. Then there exists a relatively compact domain $G \subset \widetilde{M}$ with smooth boundary such that index $(G) > 0$ . Set $D = \pi(G)$ . Since $\pi$ is locally diffeomorphic, by enlarging $G$ if necessary, we can find a point $p_0 \in \partial G$ , a neighbourhood $U$ of $p_0$ in $\widetilde{M}$ , and a neighbourhood $V$ of $\pi(p_0)$ in $M$ such that

(i) $\pi(p_0) \in \partial D$ ,

(ii) $\pi^{-1}(V \cap D) \cap G = U \cap G$ ,

(iii) $\pi|_U : U \longrightarrow V$ is a diffeomorphism.

Let $\{G_t\}_{t \geq 0}$ be a smooth and strictly decreasing family of domains $G_t \subset \widetilde{M}$ such that

$$G_0 = G \quad \text{and} \quad \bigcap_t G_t = \{p_0\} .$$

By (2-4), there exist some $s > 0$ and $\tilde{f} \in F^*_{G_s} - \{0\}$ which satisfy

$$\Delta_{\tilde{M}}\tilde{f} + \|B\|^2\tilde{f} - \frac{\int_{G_s}(\Delta_{\tilde{M}}\tilde{f} + \|B\|^2\tilde{f})\,d\tilde{M}}{\int_{G_s}d\tilde{M}} = 0$$

in $G_s$. Set $D_s = \pi(G_s)$ and define a function $f$ on $M$ by $f(q) = \sum_{p\in\pi^{-1}(q)}\tilde{f}(p)$. Since $\tilde{f}$ is analytic in $G_s$ (c.f. Morrey [7, p. 166 Theorem 5.7.1]),

$$\tilde{f}(p_1) \neq 0$$

for some $p_1 \in G_s \cap U$. Therefore

$$f(\pi(p_1)) = \sum_{p\in\pi^{-1}(\pi(p_1))}\tilde{f}(p) = \tilde{f}(p_1) \neq 0 ,$$

by virtue of above (ii) and (iii). Moreover, $f \in F^*_{D_s}$ and we can show that $I(f) \leqq 0$ by the essentially same way as Barbosa and do Carmo [1, pp. 521-526]. Therefore, by (2-3),

$$\lambda_1(D_s) \leqq 0 .$$

Hence, by using (2-4), there exists a relatively compact domain $D' \supset D_s$ of $M$ such that

$$\text{index } D' > 0 ,$$

which implies that $x$ is not stable.

Q.E.D.

Proof of Theorem 2. By virtue of Lemma 1, it is sufficient to prove our theorem only for the case that $M$ is simply-connected.

Let $x : M \longrightarrow \mathbb{R}^3$ be a complete stable immersion with non-zero constant mean curvature. Denote by $K$ the Gaußian curvature of $x$. Consider $M$ with the natural complex structure given by $x$. We assume that $x$ satisfies (i) or (ii), and shall derive a contradiction. Let $z$ be the canonical coordinate in $\mathbb{C}$. Then the induced metric $ds^2$ in $M$ is given by

$$ds^2 = \lambda^2 |dz|^2 , \lambda > 0 .$$

Let $\Delta$ denote the Laplacian, $\nabla$ the gradient, and $dA$ the area element in the flat metric.

At first we assume (i), that is, $M$ is assumed to be conformally equivalent to $B = \{z \in \mathbb{C} ; |z| < 1\}$. Let $p$ be a point in $M$, and let $B_r(p)$ be the geodesic disks with $p$ as their center and $r$ as their radii which exhaust $M$. For any positive constant $\delta$, we define a piecewise-smooth function $f_{r,\delta} : M \longrightarrow \mathbb{R}$ as follows.

$$(2\text{-}5) \qquad f_{r,\delta}(q) = \begin{cases} 1 , & q \in B_r(p) , \\ 2-\mathrm{dist}(q,p)/r , & q \in B_{2r+\delta}(p) - B_r(p) , \\ \{\mathrm{dist}(q,p) - (2r+2\delta)\}/r , & q \in B_{2r+2\delta}(p) - B_{2r+\delta}(p) , \\ 0 , & q \in M - B_{2r+2\delta}(p) , \end{cases}$$

where $\mathrm{dist}(q,p)$ is the geodesic distance between $p$ and $q \in M$. Now we claim the following statement which will be proved in section 3 in order to avoid confusion.

Claim 1. For any $r > 0$, there exists a unique $\delta = \delta(r) > 0$ such that

$$\int_M \lambda^{-1} f_{r,\delta} dM = 0 .$$

Therefore, $\lambda^{-1} f_{r,\delta} \in F_{B_{2r+2\delta}}(p)$. Hence, by virtue of the stability of $x$ and Formula 2,

$$I(\lambda^{-1} f_{r,\delta}) \geq 0 . \tag{2-6}$$

Therefore,

$$\int_M f_{r,\delta}{}^2 |\nabla_M \lambda^{-1}|^2 dM < \int_M \lambda^{-2} |\nabla_M f_{r,\delta}|^2 dM , \tag{2-7}$$

where $\nabla_M$ is the gradient in the induced metric. In fact, if we set $\phi = \lambda^{-1}$ and $f = f_{r,\delta}$, by using the well-known formula $K = -\lambda^{-2} \Delta \log \lambda$ and the Stokes' theorem, we can achieve the following calculation.

The left hand side of (2-6)

$$\begin{aligned}
&= \int_M \left\{ |\nabla_M (\phi f)|^2 - (4H^2 - 2K)\phi^2 f^2 \right\} dM \\
&< \int_M \left\{ |\nabla_M (\phi f)|^2 + 2K\phi^2 f^2 \right\} dM \\
&= \int_M \left\{ \phi^2 |\nabla_M f|^2 - 3f^2 |\nabla_M \phi|^2 - 2\phi f (\nabla_M \phi, \nabla_M f) \right\} dM \\
&\leq 2\int_M (\phi^2 |\nabla_M f|^2 - f^2 |\nabla_M \phi|^2) dM .
\end{aligned}$$

It follows from (2-7) that

$$\int_{B_r(p)} |\nabla_M \lambda^{-1}|^2 dM < \int_M \lambda^{-2} |\nabla_M f_{r,\delta}|^2 dM$$

$$< r^{-2} \int_B dA$$

$$= \pi r^{-2} .$$

By letting $r \longrightarrow +\infty$, we know that $|\nabla_M \lambda^{-1}| \equiv 0$ on $M$, that is, $\lambda \equiv$ constant, which contradicts the completeness of the metric $ds^2 = \lambda^2 |dz|^2$ in $B$. (A similar inequality to (2-7) and a similar method to the above limitting process are found also in do Carmo and Peng [4] and Mori [6]).

Next, we assume the condition (ii). Set $\psi = \lambda^2 \| B \|^2$. Then there exists some constant $\beta > 0$ such that the inequality

$$\text{(2-8)} \qquad \int_{\mathbb{C}} \psi^3 f^6 dA \leq \beta \int_{\mathbb{C}} |\nabla f|^6 dA$$

follows for all compactly supported piecewise-smooth functions $f : \mathbb{C} \longrightarrow \mathbb{R}$ that satisfy $I(\psi f^3) \geq 0$, which is proved by Mori [6].

Let $n > 1$ be a constant. For any $r > 1$ and $\delta > 0$, we define a piecewise-smooth function $f_{r,\delta} : \mathbb{C} \longrightarrow \mathbb{R}$ as follows. Since $f_{r,\delta}$ is defined to be depending only on $\rho$, we write $f_{r,\delta}(z) = f_{r,\delta}(\rho)$, where $(\rho, \theta)$ is the polar coordinates in $\mathbb{C}$. If $0 < \delta \leq r$,

$$\text{(2-9)} \qquad f_{r,\delta}(\rho) = \begin{cases} 1 , & 0 \leq \rho \leq r , \\ 2 - \dfrac{\rho}{r} , & r \leq \rho \leq 2r+\delta , \\ \dfrac{\delta}{r}(2r+\delta)^{-k} \left\{ r^{-n}(\rho - 2r - \delta)^n - 1 \right\} \rho^k , & 2r+\delta \leq \rho \leq 3r+\delta , \\ 0 , & \rho \geq 3r+\delta , \end{cases}$$

and if $\delta \geqq r$ ,

$$
(2\text{-}10)\qquad f_{r,\delta}(\rho) = \begin{cases} 1 , & 0 \leqq \rho \leqq r , \\ 2 - \frac{\rho}{r} , & r \leqq \rho \leqq 3r , \\ (3r)^{-k}\left\{\delta^{-n}(\rho-3r)^{n}-1\right\}\rho^{k} , & 3r \leqq \rho \leqq 3r+\delta , \\ 0 , & \rho \geqq 3r+\delta , \end{cases}
$$

where $k > 0$ depends only on $r$ . Now we claim

<u>Claim 2</u>. There exists some constant $a_0$, $0 < a_0 < \frac{1}{2}$ , such that the following statement holds. Consider (2-9) and (2-10) substituted $k = \frac{2}{3} - a_0 r^{-3}$ . Then, for sufficiently large $r$ , there exists a unique $\delta = \delta(r) > 0$ such that

$$
(2\text{-}11)\qquad \int_M \psi\, f_{r,\delta}{}^3 dM = 0 ,
$$

which will be proved in section 3.

From the stability of $x$ , inequality (2-8) is satisfied for $f = f_{r,\delta}$ in Claim 2. Let us calculate the right hand side of (2-8) for $f = f_{r,\delta}$ . If $0 < \delta < r$ , then

$$
\int_{\mathbb{C}} |\nabla f_{r,\delta}|^6 dA = 2\pi\int_r^{2r+\delta} r^{-6}\rho\ d\rho + 2\pi\left\{\frac{\delta}{r}(2r+\delta)^{-k}\right\}^6 \int_{2r+\delta}^{3r+\delta}\left(\frac{d}{d\rho}\left[\left\{r^{-n}(\rho-2r-\delta)^{n}-1\right\}\rho^{k}\right]\right)^6 \rho\ d\rho
$$

$$
\leqq 2\pi r^{-6}\int_r^{2r+\delta} \rho\ d\rho + 2\pi(4n+k)^6(2r)^{-6k}\int_{2r+\delta}^{3r+\delta} \rho^{6k-5} d\rho
$$

$$
< 8\pi r^{-4} + 2\pi(4n+k)^6(2r)^{-6k}\int_{2r}^{+\infty} \rho^{6k-5} d\rho
$$

$$
= 8\pi r^{-4} + 2\pi(4n+k)^6(2r)^{-4}(4-6k)^{-1}
$$

$$(2\text{-}12) \qquad < 8\pi r^{-4} + \pi(4n+\tfrac{2}{3})^6 (48a_0)^{-1} r^{-1} .$$

And if $\delta \geqq r$ , then

$$\int_{\mathbb{C}} |\nabla f_{r,\delta}|^6 dA = 2\pi \int_r^{3r} r^{-6} \rho \, d\rho + 2\pi (3r)^{-6k} \int_{3r}^{3r+\delta} \left( \frac{d}{d\rho} \left[ \left\{ \delta^{-n} (\rho - 3r)^n - 1 \right\} \rho^k \right] \right)^6 \rho \, d\rho$$

$$\leqq 2\pi r^{-6} \int_r^{3r} \rho \, d\rho + 2\pi (4n+k)^6 (3r)^{-6k} \int_{3r}^{3r+\delta} \rho^{6k-5} d\rho$$

$$< 8\pi r^{-4} + 2\pi (4n+k)^6 (3r)^{-6k} \int_{3r}^{+\infty} \rho^{6k-5} d\rho$$

$$= 8\pi r^{-4} + 2\pi (4n+k)^6 (3r)^{-4} (4-6k)^{-1}$$

$$(2\text{-}13) \qquad < 8\pi r^{-4} + \pi(4n+\tfrac{2}{3})^6 (243a_0)^{-1} r^{-1} .$$

Both of (2-12) and (2-13) go to zero as $r$ goes to infinity. On the other hand, as for the left hand side of (2-8),

$$(2\text{-}14) \qquad \int_{\mathbb{C}} \psi^3 f_{r,\delta}{}^6 dA > \int_{0 \leqq |z| \leqq r} \psi^3 dA > \int_{0 \leqq |z| \leqq 1} (\lambda^2 \| B \|^2)^3 dA .$$

The most right term of (2-14) is a positive constant independent of $r$ , which contradicts the inequality (2-8).

Q.E.D.

<u>Proof of Theorem 1</u>. If an immersion $x : M \longrightarrow \mathbb{R}^3$ satisfies (0-1), then $x$ satisfies (2-1). In fact, if we choose a sufficiently large $\rho_0$ so that the exceptional compact set in Theorem 1 (ii) is

contained in $\{z \in \mathbb{C} \,\big|\, |z| \leq \rho_0\}$, then for $\rho_2 \geq \rho_1 > \rho_0$

$$\int_{\rho_1}^{\rho_2}\left(\int_0^{2\pi}\lambda^4\|B\|^2 d\theta\right)^{1/4} d\rho \geq \int_{\rho_1}^{\rho_2}\left(\int_0^{2\pi} c_0^4\ \rho^{-4}\cdot 2H^2 d\theta\right)^{1/4} d\rho$$

$$= c_0(4\pi H^2)^{1/4}\log\frac{\rho_2}{\rho_1}.$$

Q.E.D.

§ 3. Proof of Claim 1 and 2

Proof of Claim 1. Fix $r > 0$ , and set

$$J(\delta) = \int_{M-B_{2r}(p)} \lambda^{-1} f_{r,\delta} dM .$$

From the definition of $f_{r,\delta}$ , $\int_{B_{2r}(p)} \lambda^{-1} f_{r,\delta} dM > 0$ and $J(\delta) < 0$ for any $\delta > 0$ . If we can show the following (3-1), (3-2), and (3-3), then we know that Claim 1 is valid.

(3-1) $$\lim_{\delta \to +0} J(\delta) = 0 .$$

(3-2) $$\lim_{\delta \to +\infty} J(\delta) = -\infty .$$

(3-3) $J(\delta)$ is strictly decreasing with respect to $\delta > 0$ .

(3-1) is trivial. Let us prove (3-2). Let $\alpha > 0$ be a fixed positive constant. Then for any $\delta$ which is greater than $\alpha r$ ,

$$-J(\delta) > \alpha \int_{B_{2r+2\delta-\alpha r}(p) - B_{2r+\alpha r}(p)} \lambda^{-1} dM$$

(3-4) $$= \alpha \int_{B_{2r+2\delta-\alpha r}(p)} \lambda^{-1} dM - \alpha \int_{B_{2r+\alpha r}(p)} \lambda^{-1} dM$$

$$\longrightarrow + \infty , \text{ as } \delta \longrightarrow + \infty .$$

In fact, the first term of (3-4) goes to $+\infty$ as $\delta$ goes to

$+\infty$ by virtue of the completeness of the metric $ds^2 = \lambda^2|dz|^2$, and the second term of (3-4) is a finite constant which is independent of $\delta$. At last, let us prove (3-3). To do this, it is sufficient to show $J(\delta) - J(\delta+\varepsilon) > 0$ for $\delta > \varepsilon > 0$. Denote the geodesic distance between $p$ and $q$ by $d(q)$.

$$
\begin{aligned}
&J(\delta)-J(\delta+\varepsilon) \\
&= \int_{B_{2r+\delta+\varepsilon}(p)-B_{2r+\delta}(p)} \lambda^{-1}\cdot\frac{2\{d-(2r+\delta)\}}{r}\,dM \\
&+ \int_{B_{2r+2\delta}(p)-B_{2r+\delta+\varepsilon}(p)} \lambda^{-1}\cdot\frac{2\varepsilon}{r}\,dM \\
&+ \int_{B_{2r+2\delta+2\varepsilon}(p)-B_{2r+2\delta}(p)} \lambda^{-1}\cdot\frac{(2r+2\delta+2\varepsilon)-d}{r}\,dM \\
&> 0 .
\end{aligned}
$$

Q.E.D.

<u>Proof of Claim 2</u>. Let us fix $r > \max\{1, \rho_0/3\}$. $f_{r,\delta}(\rho)$ is continuous with respect to $\delta > 0$, $f_{r,\delta}(\rho) > 0$ for $0 \leqq \rho < 2r$, and $f_{r,\delta}(\rho) < 0$ for $2r < \rho < 3r + \delta$. Therefore, if we set

$$L_1(\delta) = \int_{\{\rho\leqq 2r\}} \psi f_{r,\delta}{}^3 dM \quad\text{and}\quad L_2(\delta) = \int_{\{\rho\geqq 2r\}} \psi f_{r,\delta}{}^3 dM ,$$

then $L_1(\delta) > 0$ and $L_2(\delta) < 0$ for any $\delta > 0$. Moreover, $L_1(\delta)$ is independent of $\delta$ from the definition of $f_{r,\delta}$. Hence we set

$$L_1 = L_1(\delta) .$$

On the other hand, $L_2(\delta)$ is strictly decreasing with respect to $\delta$. In fact, for any positive constant $\delta_0$, $f_{r,\delta}(\rho)$ is non-increasing everywhere and is strictly decreasing at least in $\{\rho;\ 3r < \rho < 3r + \delta_0\}$ with respect to $\delta$ $(\delta > \delta_0)$. Therefore, if there exists some $\delta = \delta(r)$ which satisfies (2-11), it is unique.

At first suppose that $L_1 + L_2(r) < 0$. Since $L_2(\delta)$ is continuous with respect to $\delta$, $\lim_{\delta \to +0} L_2(\delta) = 0$, and $L_1 > 0$, there exists some $\delta$ $(r > \delta > 0)$ such that $L_1 + L_2(\delta) = 0$ (i.e. (2-11)) is satisfied.

Next, we assume that

$$L_1 + L_2(r) \geqq 0 . \tag{3-5}$$

Let us find some $\delta \geqq r$ which satisfies the equality $L_1 + L_2(\delta) = 0$. Set

$$\varphi(\rho) = \int_0^{2\pi} \lambda^4 \| B \|^2 d\theta . \tag{3-6}$$

For some fixed $b$, $0 < b < 1/2$, we define subsets $E_j$ $(j = 1,2,3)$ of $\mathbb{R}^+ = \{\rho \in \mathbb{R};\ \rho > 0\}$ as follows.

$$\begin{cases} E_1 = \left\{ \rho \in \mathbb{R}^+ ;\ \varphi(\rho) \leqq \rho^{-4-4b} \right\} , \\ E_2 = \left\{ \rho \in \mathbb{R}^+ ;\ \rho^{-4-4b} \leqq \varphi(\rho) \leqq \rho^{-4+4b} \right\} , \\ E_3 = \left\{ \rho \in \mathbb{R}^+ ;\ \rho^{-4+4b} \leqq \varphi(\rho) \right\} . \end{cases} \tag{3-7}$$

Since $\|B\|^2 = 2H^2 + (2H^2-2K) \geqq 2H^2$,

$$\varphi(\rho)^{1/4} \geqq \left(2H^2\int_0^{2\pi}\lambda^4 d\theta\right)^{1/4} \geqq (2H^2)^{1/4}\left(\int_0^{2\pi} d\theta\right)^{-\frac{3}{4}}\int_0^{2\pi}\lambda \, d\theta$$

$$= (H^2/(4\pi^3))^{1/4}\int_0^{2\pi}\lambda \, d\theta .$$

Therefore, by virtue of the completeness of $M$,

$$\int_1^{+\infty}\varphi(\rho)^{1/4} d\rho \geqq (H^2/(4\pi^3))^{1/4}\int_1^{\infty}\left(\int_0^{2\pi}\lambda \, d\theta\right)d\rho$$

$$= (H^2/(4\pi^3))^{1/4}\lim_{R\to+\infty}\int_0^{2\pi}\left(\int_1^{R}\lambda \, d\rho\right)d\theta$$

$$= +\infty .$$

Hence,

(3-8) $$\int_{[1,\infty)\cap E_1}\rho^{1-b}d\rho+\int_{[1,\infty)\cap E_2}\varphi(\rho)^{1/4}d\rho+\int_{[1,\infty)\cap E_3}\varphi(\rho)\rho^{3-3b}d\rho\geqq\int_1^{\infty}\varphi(\rho)^{1/4}d\rho=+\infty .$$

Since the first term of the left hand side of (3-8) is finite,

(3-9) $$\int_{[1,\infty)\cap E_2}\varphi(\rho)^{1/4}d\rho + \int_{[1,\infty)\cap E_3}\varphi(\rho)\rho^{3-3b}d\rho = +\infty .$$

Now we separate our situation into two cases as follows.

<u>Case I</u>. $\int_{[1,\infty)\cap E_3}\varphi(\rho)\rho^{3-3b}d\rho = +\infty$ ,

<u>Case II</u>. $\int_{[1,\infty)\cap E_3}\varphi(\rho)\rho^{3-3b}d\rho < +\infty$ .

At first, we consider Case I. Let $a$ be a constant such that $0 < a \leqq b$. Assume that

(3-10) $\quad \delta \geqq \max\{r, \{1-(3r)^{-(b-a)}\}^{-1}\}$ .

If

(3-11) $\quad 3r \leqq \rho \leqq 3r + \delta^{1-\frac{1}{n}}$ ,

then

$$\{1-\delta^{-n}(\rho-3r)^n\}\rho^{b-a} \geqq (1-\delta^{-1})(3r)^{b-a} \quad [\text{because of } (3\text{-}11)]$$

(3-12) $$\geqq 1 \quad [\text{because of } (3\text{-}10)] .$$

Therefore, for $k = \frac{2}{3} - a$ ,

$$-L_2(\delta) > -\int_{\{3r\leqq\rho\leqq 3r+\delta\}} \psi f_{r,\delta}{}^3 dM = (3r)^{-3k}\int_{3r}^{3r+\delta} \varphi(\rho)\{1-\delta^{-n}(\rho-3r)^n\}^3 \rho^{3k+1} d\rho$$

$$\geqq (3r)^{-3k}\int_{[3r,3r+\delta^{1-\frac{1}{n}}]\cap E_3} \varphi(\rho)\rho^{3-3b}[\{1-\delta^{-n}(\rho-3r)^n\}\rho^{b-a}]^3 d\rho$$

$$\geqq (3r)^{-3k}\int_{[3r,3r+\delta^{1-\frac{1}{n}}]\cap E_3} \varphi(\rho)\rho^{3-3b} d\rho \quad [\text{because of } (3\text{-}12)]$$

$$\longrightarrow +\infty ,$$

as $\delta \longrightarrow +\infty$ by virtue of the assumption of Case I. Hence, there exists some $\delta = \delta(r) \geqq r$ such that

$$L_1 + L_2(\delta) = 0 \ .$$

If we take any constant $a_0$, $0 < a_0 < b$, then $0 < a_0 r^{-3} < a_0 < b < \frac{1}{2}$ for $r > 1$. By setting $a = a_0 r^{-3}$, we see that Claim 2 is valid for Case I.

Next, we consider Case II. For any $r > 1$ and $\delta > 0$,

$$L_1 = \int_{\{\rho \leq 2r\}} \psi f_{r,\delta}{}^3 dM < \int_{\{\rho \leq 2r\}} \psi dM = \int_0^{2r} \varphi(\rho)\rho \, d\rho$$

$$\leq \int_0^1 \varphi(\rho)\rho \, d\rho + \int_{[1,2r]\cap E_1} \rho^{-3-4b} d\rho + \int_{[1,2r]\cap E_2} \rho^{-3+4b} d\rho + \int_{[1,2r]\cap E_3} \varphi(\rho)\rho^{3-3b} d\rho$$

$$(3\text{-}13) \qquad < \int_0^1 \varphi(\rho)\rho \, d\rho + \int_1^{+\infty} \rho^{-3+4b} d\rho + \int_{[1,\infty)\cap E_3} \varphi(\rho)\rho^{3-3b} d\rho \ .$$

From the assumption of Case II and $b < \frac{1}{2}$, (3-13) is a finite constant, which we denote by $\alpha$. That is,

$$(3\text{-}14) \qquad L_1 < \alpha$$

for any $r > 1$.

For any $\delta > 0$, define $\tilde{f}_{r,\delta}(\rho)$ as the right hand side of (2-10). Then, for any fixed positive constant $\delta_0$, $\tilde{f}_{r,\delta}(\rho)$ is non-increasing everywhere and strictly decreasing in $\{\rho;\ 3r < \rho < 3r + \delta_0\}$ with respect to $\delta$, $\delta > \delta_0$. Therefore

$$\tilde{L}_2(\delta) = \int_{\{\rho \geq 2r\}} \psi \tilde{f}_{r,\delta}{}^3 dM$$

is strictly decreasing with respect to $\delta > 0$ . Moreover,

$$\tilde{L}_2(\delta) = L_2(\delta) \quad \text{for any } \delta \geq r ,$$

and

$$(3\text{-}5) \qquad L_1 + L_2(r) \geqq 0 .$$

Therefore, if we find some $\delta_1 = \delta_1(r) > 0$ such that

$$(3\text{-}15) \qquad L_1 + \tilde{L}_2(\delta_1) < 0 ,$$

then $\delta_1 > r$ , and there exists a unique $\delta = \delta(r)$ $(\delta_1 > \delta \geqq r)$ such that the equality

$$L_1 + L_2(\delta) = 0$$

holds, which proves Claim 2.

Set $k = \frac{2}{3} - a$ $(0 < a < b)$ . For $\varepsilon > 0$ , assume that

$$\delta \geqq \{1-(3r)^{-\varepsilon}\}^{-\frac{1}{\varepsilon}} .$$

Then if $3r \leqq \rho \leqq 3r + \delta^{1-\varepsilon/n}$ ,

$$(3\text{-}16) \qquad \{1-\delta^{-n}(\rho-3r)^n\}\rho^{\varepsilon} \geqq 1 .$$

Therefore,

$$-\tilde{L}_2(\delta) > -\int_{\{3r\le\rho\le 3r+\delta\}} \psi \tilde{f}_{r,\delta}^{\,3} dM = (3r)^{-2+3a}\int_{3r}^{3r+\delta} \varphi(\rho)\{1-\delta^{-n}(\rho-3r)^n\}^3 \rho^{3-3a} d\rho$$

$$> (3r)^{-2+3a}\int_{3r}^{3r+\delta^{1-\varepsilon/n}} \varphi(\rho)\{1-\delta^{-n}(\rho-3r)^n\}^3 \rho^{3-3a} d\rho$$

$$\geq (3r)^{-2+3a} \cdot \frac{\left(\int_{3r}^{3r+\delta^{1-\varepsilon/n}} \varphi(\rho)^{1/4} d\rho\right)^4}{\left(\int_{3r}^{3r+\delta^{1-\varepsilon/n}} \{1-\delta^{-n}(\rho-3r)^n\}^{-1} \rho^{-1+a} d\rho\right)^3}$$

$$\geq (3r)^{-2+3a} \cdot \frac{\left(\int_{3r}^{3r+\delta^{1-\varepsilon/n}} \varphi(\rho)^{1/4} d\rho\right)^4}{\left(\int_{3r}^{3r+\delta^{1-\varepsilon/n}} \rho^{-1+a+\varepsilon} d\rho\right)^3} \quad \text{[because of (3-16)]}$$

$$\text{(3-17)} \quad \geq c^4 (a+\varepsilon)^3 (3r)^{-2+3a} \cdot \frac{\left(\log \dfrac{3r+\delta^{1-\varepsilon/n}}{3r}\right)^4}{\left\{\left(3r+\delta^{1-\varepsilon/n}\right)^{a+\varepsilon} - (3r)^{a+\varepsilon}\right\}^3},$$

where the last inequality follows from the assumption (2-1).

Now we claim

<u>Claim 3</u>. There exists some constant $a_0$, $0 < a_0 < 1/2$, such that the following statement holds. For sufficiently large $r$, there exist some $\varepsilon = \varepsilon(r) > 0$ and $\delta = \delta(r) \geq \{1-(3r)^{-\varepsilon}\}^{-1/\varepsilon}$ such that

$$\text{(3-18)} \quad c^4 (a_0 r^{-3}+\varepsilon)^3 (3r)^{-2+3a_0 r^{-3}} \cdot \frac{\left(\log \dfrac{3r+\delta^{1-\varepsilon/n}}{3r}\right)^4}{\left\{\left(3r+\delta^{1-\varepsilon/n}\right)^{a_0 r^{-3}+\varepsilon} - (3r)^{a_0 r^{-3}+\varepsilon}\right\}^3} \geq \alpha .$$

If Claim 3 is valid, then, from (3-17) and (3-18), we have

$$-\widetilde{L}_2(\delta) > \alpha .$$

Therefore, by using (3-14), it follows that

$$L_1 + \widetilde{L}_2(\delta) < 0 ,$$

which is just the required inequality (3-15). Therefore, remaining thing is only to prove Claim 3.

Set

$$(3\text{-}19) \quad R = 3r,\ m = a_0 r^{-3}+\varepsilon,\ \text{and} \quad x = \delta^{1-\varepsilon/n} .$$

Then the left hand side of (3-18) becomes

$$c^4 m^3 R^{-2+3m-3\varepsilon} \cdot \frac{\left(\log \frac{R+x}{R}\right)^4}{\left\{(R+x)^m - R^m\right\}^3} ,$$

which we denote by $f(x)$. Then

$$(3\text{-}20) \quad f'(x) = \frac{c^4 m^3 R^{-2+3m-3\varepsilon}\left(\log \frac{R+x}{R}\right)^3}{(R+x)^{1-m}\{(R+x)^m - R^m\}^4} \left[4\left\{1-\left(\frac{R}{R+x}\right)^m\right\} - 3m \log \frac{R+x}{R}\right] .$$

Set

$$g(y) = 4(1-y^{-m}) - 3m \log y , \quad y > 1 ,$$

then

$$g'(y) = my^{-m-1}(4-3y^m) .$$

Therefore, $g(y)$ is strictly increasing in $1 < y < (4/3)^{1/m}$, and strictly decreasing in $y > (4/3)^{1/m}$. Moreover,

$$\lim_{y\to 1+0} g(y) = 0 \quad \text{and} \quad \lim_{y\to +\infty} g(y) = -\infty .$$

Hence there exists a unique $y_0 > 1$ such that $g(y_0) = 0$. Since $g((4/3)^{3/m}) = 4\{1-(3/4)^3\} - 9\log(4/3) = -0.276 < 0$,

$$(3\text{-}21) \qquad \left(\frac{4}{3}\right)^{1/m} < y_0 < \left(\frac{4}{3}\right)^{3/m} .$$

From (3-20) and the property of $g$ mentioned above, we know that $f(x)$ is strictly increasing in $0 < x < (y_0-1)R$, and strictly decreasing in $x > (y_0-1)R$. Moreover, since $g(y_0) = 0$,

$$\log y_0 = 4(1-y_0^{-m})/(3m) .$$

Therefore,

$$(3\text{-}22) \qquad \max_{x>0} f(x) = f((y_0-1)R) = \left(\frac{4}{3}\right)^4 c^4 m^{-1} R^{-2-3\varepsilon} y_0^{-3m}(1-y_0^{-m}) .$$

From (3-21) and (3-22),

$$(3\text{-}23) \qquad f((y_0-1)R) > \frac{3^5}{2^{12}} c^4 m^{-1} R^{-2-3\varepsilon} .$$

Choose $\varepsilon$ so that $0 < \varepsilon \leqq 1/3$. Then $R^{-2-3\varepsilon} \geqq R^{-3}$. Therefore, by (3-19) and (3-23),

$$(3\text{-}24) \qquad f(3r(y_0-1)) > \frac{9}{2^{12}} c^4 (a_0 r^{-3}+\varepsilon)^{-1} r^{-3} .$$

Hence, if there exists some $a_0 (0 < a_0 < \frac{1}{2})$ which is independent of $r$ and $\varepsilon = \varepsilon(r)$ $(0 < \varepsilon \leqq 1/3)$ such that

$$(3\text{-}25) \qquad \frac{9}{2^{12}} c^4 \alpha^{-1} r^{-3} \geqq a_0 r^{-3} + \varepsilon$$

holds, then (3-18) follows for $\delta = \{3r(y_0 - 1)\}^{n/(n-\varepsilon)}$. Let $a_0$ be a positive constant which satisfies the inequality

$$a_0 < \min \left\{ \frac{1}{2} , \frac{9c^4}{2^{13}\alpha} \right\} ,$$

and set

$$(3\text{-}26) \qquad \varepsilon = \frac{9c^4}{2^{13}\alpha} r^{-3} .$$

Then, for sufficiently large $r$, $\varepsilon \leqq 1/3$ and (3-25) is satisfied. Moreover, it follows that

$$(3\text{-}27) \qquad \{3r(y_0 - 1)\}^{n/(n-\varepsilon)} \geqq \{1 - (3r)^{-\varepsilon}\}^{-1/\varepsilon} ,$$

as we shall prove it below.

Set $c_1 = 9c^4/(2^{13}\alpha)$. Then, from (3-26),

$$\varepsilon = c_1 r^{-3} .$$

(3-27) is equivalent to

$$(3\text{-}28) \quad \left\{1 - (3r)^{-c_1 r^{-3}}\right\} \left\{3r(y_0 - 1)\right\}^{\frac{n}{n - c_1 r^{-3}} \cdot c_1 r^{-3}} \geqq 1 .$$

From (3-21),

(3-29) $$y_0 - 1 > \left(\frac{4}{3}\right)^{\frac{r^3}{a_0+c_1}} - 1 .$$

For sufficiently large $r$, the right hand side of (3-29) is greater than 1 and $n/(n-c_1r^{-3}) > 1$. Therefore,

the left hand side of (3-28)

$$> \left\{1-(3r)^{-c_1r^{-3}}\right\}\left[3r\left\{\left(\frac{4}{3}\right)^{r^3/(a_0+c_1)} - 1\right\}\right]^{c_1r^{-3}}$$

$$= \left\{(3r)^{c_1r^{-3}} - 1\right\}\left\{\left(\frac{4}{3}\right)^{r^3/(a_0+c_1)} - 1\right\}^{c_1r^{-3}} .$$

Therefore, for the purpose of proving (3-28), it is sufficient to prove the inequality

(3-30) $$\left\{(3r)^{c_1r^{-3}} - 1\right\}^{r^3}\left\{\left(\frac{4}{3}\right)^{r^3/(a_0+c_1)} - 1\right\}^{c_1} \geqq 1 .$$

Let us prove

(3-31) $$\lim_{r\to+\infty} \left\{(3r)^{c_1r^{-3}} - 1\right\}^{r^3} = +\infty .$$

(3-31) is equivalent to

(3-32) $$\lim_{r\to+\infty} r^3\log\left\{(3r)^{c_1r^{-3}} - 1\right\} = +\infty .$$

Here, the left hand side of (3-32) is equal to

$$\lim_{y\to+0} \frac{\log\left\{\left(\frac{3}{y}\right)^{c_1 y^3} - 1\right\}}{y^3} .$$

We see that

$$(3\text{-}33)\quad \lim_{y\to+0} \log\left\{\left(\frac{3}{y}\right)^{c_1 y^3} - 1\right\} = 0 \quad \text{and} \quad \lim_{y\to+0} y^3 = 0 .$$

In fact,

$$\lim_{y\to+0} \log\left(\frac{3}{y}\right)^{c_1 y^3} = \lim_{y\to+0} c_1 y^3 (\log 3 - \log y) = 0 ,$$

therefore

$$(3\text{-}34)\quad \lim_{y\to+0}\left(\frac{3}{y}\right)^{c_1 y^3} = 1 .$$

By virtue of (3-33), it follows that

the left hand side of (3-32)

$$= \lim_{y\to+0} \frac{\frac{d}{dy}\log\left\{\left(\frac{3}{y}\right)^{c_1 y^3} - 1\right\}}{\frac{d}{dy}\, y^3}$$

$$= \lim_{y\to+0} \frac{c_1\left(\frac{3}{y}\right)^{c_1 y^3}(3\log 3 - 1 - 3\log y)}{3\left\{\left(\frac{3}{y}\right)^{c_1 y^3} - 1\right\}}$$

$$= +\infty \qquad \text{[because of (3-34)]} ,$$

which proves (3-32) and assures (3-31). Moreover, for sufficiently large $r$ , $\left\{(4/3)^{r^3/(a_0+c_1)} - 1\right\}^{c_1} \geqq 1$ . Therefore (3-30) holds for sufficiently large $r$, which completes the proof of Claim 3 .

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