VIRTUAL OPERAD ALGEBRAS AND REALIZATION OF HOMOTOPY TYPES

VLADIMIR HINICH

1. Introduction

1.1. Let k be a base commutative ring, $C(k)$ be the category of complexes of k-modules. The category of operads $\mathtt{Op}(k)$ in $C(k)$ admits a closed model category (CMC) structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations (see [H], Sect. 6 and also Section 2 below).

Let now $\mathcal{O}$ be a cofibrant operad. The main result of this note (see Theorem 3.1) claims that the category of $\mathcal{O}$-algebras admits as well a CMC structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations. This allows one, following the pattern of [H], 5.4, to construct the homotopy category of *virtual* $\mathcal{O}$-algebras for any operad $\mathcal{O}$ over $C(k)$ as the homotopy category of $\mathcal{P}$-algebras for a cofibrant resolution $\mathcal{P} \to \mathcal{O}$ of the operad $\mathcal{O}$.

The main motivation of the note was to understand the following main result of Mandell's recent paper [Man].

1.2. **Theorem.** *The singular cochain functor with coefficients in $\overline{\mathbb{F}}_p$ induces a contravariant equivalence from the homotopy category of connected p-complete nilpotent spaces of finite p-type to a full subcategory of the homotopy category of E_∞ $\overline{\mathbb{F}}_p$-algebras.*

In his approach, Mandell realizes the homotopy category of E_∞-algebras as a localization of the category of algebras over a "particular but unspecified" operad $\mathcal{E}$. One of major technical problems was that the category of $\mathcal{E}$-algebras did not seem to admit a CMC structure.

We suggest to choose $\mathcal{E}$ to be a cofibrant resolution of the Eilenberg-Zilber operad. Then according to Theorem 3.1, the category of $\mathcal{E}$-algebras admits a CMC structure. This considerably simplifies the proof of Theorem 1.2.

1.3. **Content of Sections.** The main body of the note (Sections 2 – 4) can be considered as a complement to [H] where some general homology theory of operad algebras is presented.

In Section 2 we recall some results of [H] we need in the sequel. In Section 3 we prove the Main theorem 3.1. In Section 4 we present, using Theorem 3.1, a construction of the homotopy category $\mathtt{Viral}(\mathcal{O})$ of virtual $\mathcal{O}$-algebras.

In Section 5 we review the proof of Mandell's theorem [Man], stressing the simplifications due to our Theorem 3.1.

1.4. **Acknowledgement.** This work was made during my stay at the Max-Planck Institut für Mathematik at Bonn. I express my gratitude to the Institute for the hospitality. I am also grateful to P. Salvatore for a useful discussion.

2. Homotopical algebra of operads: a digest of [H]

In this Section we recall some results from [H] and give some definitions we will be using in the sequel.

2.1. **Category of operads.** Let k be a commutative ring and let $C(k)$ denote the category of complexes of k-modules.

Recall ([H], 6.1.1) that the category $\mathtt{Op}(k)$ of operads in $C(k)$ admits a closed model category (CMC) structure in which weak equivalences are componenwise quasi-isomorphisms and fibrations are componentwise surjective maps.

Cofibrations in $\mathtt{Op}(k)$ are retractions of *standard cofibrations*; a map $\mathcal{O} \to \mathcal{O}'$ is a standard cofibration if $\mathcal{O}' = \varinjlim_{s\in\mathbb{N}} \mathcal{O}_s$ with $\mathcal{O}_0 = \mathcal{O}$ and each $\mathcal{O}_{s+1}$ is obtained from $\mathcal{O}_s$ by adding a set of free generators g_i with prescribed values of $d(g_i) \in \mathcal{O}_s$.

2.2. **Algebras over an operad.** Let $\mathcal{O} \in \mathtt{Op}(k)$.

The category of $\mathcal{O}$-algebras is denoted by $\mathtt{Alg}(\mathcal{O})$. For $X \in C(k)$ we denote by $F(\mathcal{O}, X)$ the free $\mathcal{O}$-algebra generated by X.

For any $d \in \mathbb{Z}$ denote by $W_d \in C(k)$ the contractible complex

$$0 \to k \;=\; k \to 0$$

concentrated in degrees $d, d+1$.

2.2.1. **Definition.** An operad $\mathcal{O} \in \mathtt{Op}(k)$ is called H_1-operad if for any $A \in \mathtt{Alg}(\mathcal{O})$ the natural map

$$A \to A \sqcup F(\mathcal{O}, W_d)$$

is a quasi-isomorphism.

2.2.2. **Proposition.** *(see [H], Thm. 2.2.1) Let $\mathcal{O}$ be an H_1-operad. Then the category of $\mathcal{O}$-algebras admits a CMC structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations.*

2.3. **Examples.**

2.3.1. First of all, not all operads are H_1-operads. In fact, let $k = \mathbb{F}_p$, $\mathcal{O} = \mathtt{COM}$ (the operad of commutative algebras). Then the symmetric algebra of W_d fails to be contractible in degree p.

2.3.2. **Proposition.** *(see [H], Thm. 4.1.1) Any Σ-split operad (see [H], 4.2) is H_1-operad.*

In particular, all operads over $k \supseteq \mathbb{Q}$ are H_1-operads. Also, all operads of form $\mathcal{T}^\Sigma$ where $\mathcal{T}$ is an asymmetric operad, in particular, $\mathtt{ASS}$ (see [H], 4.2.5), are H_1-operads.

2.3.3. The main result of this note claims that any cofibrant operad is an H_1-operad.

2.4. **Base change and equivalence.** Let $f : \mathcal{O} \to \mathcal{O}'$ be a map of operads. Then a pair of adjoint functors

$$f^* : \mathtt{Alg}(\mathcal{O}) \to \mathtt{Alg}(\mathcal{O}') : f_* \tag{1}$$

is defined in a standard way.

2.4.1. **Proposition.** *(see* [H], *4.6.4.) Let $f : \mathcal{O} \to \mathcal{O}'$ be a map of H_1-operads. The inverse and direct image functors (1) induce the adjoint functors*

$$\mathbf{L}f^* : \mathtt{Hoalg}(\mathcal{O}) \to \mathtt{Hoalg}(\mathcal{O}') : \mathbf{R}f_* = f_* \tag{2}$$

between the corresponding homotopy categories.

2.4.2. **Definition.** A map $f : \mathcal{O} \to \mathcal{O}'$ of operads is called *strong equivalence* if for each $d = (d_1, \dots , d_n) \in \mathbb{N}^n$ the induced map

$$\mathcal{O}(|d|) \otimes_{\Sigma_d} k \to \mathcal{O}'(|d|) \otimes_{\Sigma_d} k$$

is a quasi-isomorphism.

Here $|d| = \sum d_i$ and $\Sigma_d = \Sigma_{d_1} \times \ldots \times \Sigma_{d_n} \subseteq \Sigma_{|d|}$.

2.4.3. **Proposition.** *Let $f : \mathcal{O} \to \mathcal{O}'$ be a strong equivalence of H_1-operads. Then the functors $\mathbf{L}f^*$, f_* are equivalences.*

In Section 5 we will be using the following version of Proposition 2.4.3.

2.4.4. **Proposition.** *Let $f : \mathcal{O} \to \mathcal{O}'$ be a strong equivalence of operads. Suppose $\mathcal{O}$ is H_1-operad. Then for each cofibrant $\mathcal{O}$-algebra A the natural map*

$$A \to f_*(f^*(A))$$

is an equivalence.

2.4.5. *Remark.* A quasi-isomorphism of Σ-split operads compatible with the Σ-splittings is necessarily a strong equivalence.

Theorem 4.7.4 of [H] actually proves Proposition 2.4.4 and Proposition 2.4.3 together with the last Remark.

3. Main theorem

3.1. **Theorem.** *Any cofibrant operad $\mathcal{O} \in \mathtt{Op}(k)$ is an H_1-operad.*

In particular, the category of algebras $\mathtt{Alg}(\mathcal{O})$ over a cofibrant operad $\mathcal{O}$ admits a CMC structure with quasi-isomorphisms as weak equivalences and epimorphisms as fibrations.

3.2. **Proof of the theorem.** First of all, we can easily reduce the claim to the case $\mathcal{O}$ is standard cofibrant. In fact, since $\mathcal{O}$ is cofibrant, it is a retraction of a standard cofibrant operad $\mathcal{O}'$. Let

$$\mathcal{O} \xrightarrow{\alpha} \mathcal{O}' \xrightarrow{\pi} \mathcal{O}$$

be a retraction. Let A be a $\mathcal{O}$-algebra. We can consider A as a $\mathcal{O}'$-algebra via π. Then the map $A \to A \sqcup F(\mathcal{O}, M)$ is a retraction of the map $A \to A \sqcup F(\mathcal{O}', M)$. This reduces the theorem to the case $\mathcal{O}$ is standard cofibrant.

3.3. **Standard cofibrant case.** Let $\mathcal{O} = \varinjlim_{s\in\mathbb{N}} \mathcal{O}_s$ (see notation of 2.1, $\mathcal{O}_0 = 0$) be a standard cofibrant operad. Let $\{g_i\}$, $i \in I$ be a set of free (homogeneous) generators of $\mathcal{O}$.

Let a function $s : I \to \mathbb{N}$ be given so that $\mathcal{O}_s$ is freely generated as a graded operad by g_i with $s(i) \leq s$ and, of course, $dg_i \in \mathcal{O}_{s(i)-1}$.

Let, finally, $\mathrm{val} : I \to \mathbb{N}$ and $d : I \to \mathbb{Z}$ be the valency and the degree functions defined by the condition $g_i \in \mathcal{O}(\mathrm{val}(i))^{d(i)}$.

The collection $\mathcal{I} = (I, s, \mathrm{val}, d)$ will be called *a type* of $\mathcal{O}$.

Since we deal with free operads and free algebras, it is worthwhile to have an appropriate notion of tree. Fix a type $\mathcal{I} = (I, s, \mathrm{val}, d)$.

Put $I^+ = I \cup \{a, m\}$ (a and m will be special marks on some terminal vertices of our trees) and extend the functions $\mathrm{val} : I \to \mathbb{N}$ and $d : I \to \mathbb{Z}$ to I^+ by setting $\mathrm{val}(a) = \mathrm{val}(m) = d(a) = d(m) = 0$.

3.3.1. **Definition.** *A $\mathcal{I}$-tree* is a finite connected directed graph such that any vertex has ≤ 1 ingoing arrows; each vertex is marked by an element $i \in I^+$ so that $\mathrm{val}(i)$ equals the number of outgoing arrows which are numbered by $1, \ldots, \mathrm{val}(i)$.

The set of vertices of a tree T will be denoted by $V(T)$. Terminal vertices of a $\mathcal{I}$-tree are the ones having no outgoing arrows. In particular, all vertices marked by a or by m are terminal.

3.3.2. **Definition.** A $\mathcal{I}$-tree T is called *proper* if the following property (P) is satisfied.

(P) For any vertex v of T one of the possibilities (a)–(c) below occurs:

(a) v is terminal;

(b) v admits an outgoing arrow to a non-terminal vertex;

(c) v admits an outgoing arrow to a vertex marked by m.

We denote by $\mathcal{P}(\mathcal{I})$ the set of isomorphism classes of proper $\mathcal{I}$-trees. The following obvious result justifies the notion of proper tree.

3.3.3. **Proposition.** *Let $\mathcal{O}$ be a standard cofibrant operad of type $\mathcal{I} = (I, s, \mathrm{val}, d)$, A be a $\mathcal{O}$-algebra and $M \in C(k)$. Then the coproduct $B := A \sqcup F(M)$ is given,* as a graded k-module, *by the formula*

$$B = \bigoplus_{T \in \mathcal{P}(\mathcal{I})} A^{\otimes a(T)} \otimes M^{\otimes m(T)}[d(T)] \tag{3}$$

where $a(T)$ (resp., $m(T)$) is the number of vertices of type a (resp., of type m) in T and $d(T) = \sum_{v\in V(T)} d(v)$.

3.3.4. Let $\mathcal{W}$ be the set of maps $\mathbb{N}\to\mathbb{N}$ having finite support. Endow $\mathcal{W}$ with the following lexicographic order. For $f,g\in\mathcal{W}$ we will say that $f>g$ if there exists a $s\in\mathbb{N}$ such that $f(s)>g(s)$ and $f(t)=g(t)$ for all $t>s$.

The set $\mathcal{W}$ well-ordered.

Our next step is to define a filtration of $B=A\sqcup F(M)$ indexed by $\mathcal{W}$.

3.3.5. **Definition.** Let $T\in\mathcal{P}(\mathcal{I})$. The weight of T, $w(T)\in\mathcal{W}$ is the function $\mathbb{N}\to\mathbb{N}$ which assigns to any $s\in\mathbb{N}$ the number of vertices v of T whose mark $i\in I$ satisfies $s(i)=s$.

Now we are able to define a filtration on B.

3.3.6. Let $A,M,B=A\sqcup F(M)$ be as above. For each $f\in\mathcal{W}$ define

$$\mathcal{F}_f(B)=\bigoplus_{T:w(T)\le f} A^{\otimes a(T)}\otimes M^{\otimes m(T)}[d(T)].$$

The homogeneous components of the associated graded complex are defined as

$$\mathrm{gr}^{\mathcal{F}}_f(B)=\mathcal{F}_f(B)/\sum_{g<f}\mathcal{F}_g(B).$$

3.3.7. **Proposition.** *1. For each $f\in\mathcal{W}$ the graded submodule $\mathcal{F}_f$ is a subcomplex of B.*

2. One has $\mathcal{F}_0=A$.

3. Suppose M is a contractible complex. Then for each $f>0$ the homogeneous components $\mathrm{gr}^{\mathcal{F}}_f$ are contractible.

Proof. Obvious. □

3.3.8. **Corollary.** *The natural map $A\to B=A\sqcup F(\mathcal{O},M)$ is a quasi-isomorphism of complexes. This implies Main Theorem 3.1.*

Proof. Obvious. □

4. Virtual algebras

4.1. Theorem 3.1 suggests the following definition.

Let $\mathcal{O}\in\mathtt{Op}(k)$. The homotopy category of virtual $\mathcal{O}$-algebras $\mathtt{Viral}(\mathcal{O})$ is defined as $\mathtt{Hoalg}(\mathcal{P})$ where $\mathcal{P}\to\mathcal{O}$ is a cofibrant resolution of $\mathcal{O}$ in the category of operads.

One should, however, do some work, to ensure the definition above makes sense.

4.2. **Base change.** Any morphism $f:\mathcal{P}\to\mathcal{Q}$ of operads induces a pair of adjoint functors

$$f^*:\mathtt{Alg}(\mathcal{P})\ \rightleftarrows\ \mathtt{Alg}(\mathcal{Q}):f_*. \tag{4}$$

Theorem 3.1 together with 2.4.1 give immediately the following

4.2.1. **Proposition.** *For any morphism $f:\mathcal{P}\to\mathcal{Q}$ of cofibrant operads the adjoint functors (4) induce a pair of adjoint functors*

$$\mathbf{L}f^*:\mathtt{Hoalg}(\mathcal{P})\ \rightleftarrows\ \mathtt{Hoalg}(\mathcal{Q}):\mathbf{R}f_*=f_* \tag{5}$$

between the homotopy categories.

4.2.2. **Proposition.** *1. Let $f:\mathcal{P}\to\mathcal{Q}$ be a weak equivalence of cofibrant operads. Then f is a strong equivalence. In paticular, the derived functors of inverse and direct image (5) establish an equivalence of the homotopy categories.*

2. Let $f,g:\mathcal{P}\to\mathcal{Q}$ be homotopic maps between cofibrant operads. Then there is an isomorphism of functors

$$f_*,g_*:\mathtt{Hoalg}(\mathcal{Q})\to\mathtt{Hoalg}(\mathcal{P}).$$

This isomorphism depends only on the homotopy class of the homotopy connecting f with g.

Proof. 1. Let $d=(d_1,\dots,d_n)$, $|d|=\sum d_i$ and let $\Sigma_d=\prod\Sigma_{d_i}\subseteq\Sigma_{|d|}$.
We have to check that the map

$$\mathcal{P}(|d|)\otimes_{\Sigma_d}k\to\mathcal{Q}(|d|)\otimes_{\Sigma_d}k,$$

induced by f, is a quasi-isomorphism.

Since $\mathcal{P}$ and $\mathcal{Q}$ are cofibrant operads, $\mathcal{P}(|d|)$ and $\mathcal{Q}(|d|)$ are cofibrant as complexes of $k(\Sigma_{|d|})$-modules. Therefore, their quasi-isomorphism is a homotopy equivalence of $k(\Sigma_{|d|})$-modules and therefore is preserved after tensoring by k.

2. We present here a proof which is identical to the proof of Lemma 5.4.3(2) of [H].

Let $\mathcal{Q}\xrightarrow{\alpha}\mathcal{Q}^I\overset{p_0,p_1}{\rightrightarrows}\mathcal{Q}$ be a path diagram for $\mathcal{Q}$ (see [Q], ch. 1) so that α is an acyclic cofibration. Since the functors p_{0*} and p_{1*} are both quasi-inverse to an equivalence $\alpha_*:\mathtt{Hoalg}(\mathcal{Q}^I)\to\mathtt{Hoalg}(\mathcal{Q})$, they are naturally isomorphic. Therefore, any homotopy $F:\mathcal{P}\to\mathcal{Q}^I$ between f and g defines an isomorphism θ_F between f_* and g_*. Let now $F_0,F_1:\mathcal{P}\to\mathcal{Q}^I$ be homotopic. The homotopy can be realized by a map $h:\mathcal{P}\to\mathcal{R}$ where $\mathcal{R}$ is taken from a path diagram

$$\mathcal{Q}^I\xrightarrow{\beta}\mathcal{R}\xrightarrow{q_0\times q_1}\mathcal{Q}^I\times_{\mathcal{Q}\times\mathcal{Q}}\mathcal{Q}^I \tag{6}$$

where β is an acyclic cofibration, $q_0\times q_1$ is a fibration, $q_i\circ h=F_i, i=0,1$. Passing to the corresponding homotopy categories we get the functors $q_{i*}\circ p_{j*}:\mathtt{Hoalg}(\mathcal{Q})\to\mathtt{Hoalg}(\mathcal{R})$ which are quasi-inverse to $\alpha_*\circ\beta_*:\mathtt{Hoalg}(\mathcal{R})\to\mathtt{Hoalg}(\mathcal{Q})$. This implies that $\theta_{F_0}=\theta_{F_1}$. ∎

4.3. **Virtual operad algebras.** Our construction of the category of virtual $\mathcal{O}$-algebras follows the construction of virtual modules in [H], 5.4.

Let $\mathtt{Op}^c(k)$ denote the category of cofibrant operads in $C(k)$. For each $\mathcal{P} \in \mathtt{Op}^c(k)$ let $\mathtt{Hoalg}(\mathcal{P})$ be the homotopy category of $\mathcal{P}$-algebras. These categories form a fibred category $\mathtt{Hoalg}$ over $\mathtt{Op}^c(k)$, with the functors $\mathbf{R}f_* = f_*$ playing the role of "inverse image functors".

Let $\mathcal{O} \in \mathtt{Op}(k)$. Let $\mathtt{Op}^c(k)/\mathcal{O}$ be the category of maps $\mathcal{P} \to \mathcal{O}$ of operads with cofibrant $\mathcal{P}$. The obvious functor

$$c_{\mathcal{O}} : \mathtt{Op}^c(k)/\mathcal{O} \to \mathtt{Op}^c(k)$$

assigns the cofibrant operad $\mathcal{P}$ to an arrow $\mathcal{P} \to \mathcal{O}$.

4.3.1. **Definition.** The (homotopy) category $\mathtt{Viral}(\mathcal{O})$ of virtual $\mathcal{O}$-algebras is the fibre of $\mathtt{Hoalg}$ at $c_{\mathcal{O}}$. In other words, an object of $\mathtt{Viral}(\mathcal{O})$ consists of a collection $A_a \in \mathtt{Hoalg}(\mathcal{P}_a)$ for each $a : \mathcal{P}_a \to \mathcal{O}$ in $\mathtt{Op}^c(k)/\mathcal{P}$ and of compatible collection of isomorphisms $\phi_f : A_a \to f_*(A_b)$ given for every $f : \mathcal{P}_a \to \mathcal{P}_b$ in $\mathtt{Op}^c(k)/\mathcal{O}$.

4.3.2. **Corollary.** *Let $\alpha : \mathcal{P} \to \mathcal{O}$ be a weak equivalence of operads with cofibrant $\mathcal{P}$. Then the obvious functor*

$$q_\alpha : \mathtt{Viral}(\mathcal{O}) \to \mathtt{Hoalg}(\mathcal{P})$$

is an equivalence of categories.

Proof. We will construct a quasi-inverse functor $q^\alpha : \mathtt{Hoalg}(\mathcal{P}) \to \mathtt{Viral}(\mathcal{O})$. For this choose for any map $\beta : \mathcal{Q} \to \mathcal{O}$ a map $f_\beta : \mathcal{Q} \to \mathcal{P}$ making the corresponding triangle homotopy commutative. Then, for any $A \in \mathtt{Hoalg}(\mathcal{P})$ we define $q^\alpha(A)$ to be the collection of $f_{\beta *}(A) \in \mathtt{Hoalg}(\mathcal{Q})$. According to Proposition 4.2.2, the definition does not depend on the choice of $f'_\beta s$. $\square$

The corollary means that the homotopy category of virtual $\mathcal{O}$-algebras is really the category of algebras over a cofibrant resolution of $\mathcal{O}$.

4.3.3. Any map $f : \mathcal{O} \to \mathcal{O}'$ defines an obvious functor $\mathtt{Op}^c(k)/\mathcal{O} \to \mathtt{Op}^c(k)/\mathcal{O}'$. This induces a direct image functor

$$f_* : \mathtt{Viral}(\mathcal{O}') \to \mathtt{Viral}(\mathcal{O}).$$

According to Corollary 4.3.2, this functor admits a left adjoint inverse image functor f^* which can be calculated using cofibrant resolutions for $\mathcal{O}$ and $\mathcal{O}'$.

4.4. **Comparing $\mathtt{Viral}(\mathcal{O})$ with $\mathtt{Hoalg}(\mathcal{O})$.**

4.4.1. Suppose $k \supseteq \mathbb{Q}$. Let $\mathcal{O} \in \mathtt{Op}(k)$ and let $f : \mathcal{P} \to \mathcal{O}$ be a cofibrant resolution of $\mathcal{O}$. Both $\mathcal{O}$ and $\mathcal{P}$ admit a Σ-splitting (see [H], 4.2.4 and 4.2.5.2.) Moreover, the quasi-isomorphism f preserves the Σ-splittings. Therefore, the categories $\mathtt{Viral}(\mathcal{O}) = \mathtt{Hoalg}(\mathcal{P})$ and $\mathtt{Hoalg}(\mathcal{O})$ are equivalent by [H], 4.7.4.

Thus, in the case $k \supseteq \mathbb{Q}$ virtual operad algebras give nothing new.

4.4.2. Let $\mathcal{T}$ be an "asymmetric operad" i.e. a collection of complexes $\mathcal{T}(n) \in C(k)$ (with no action of the symmetric group), associative multiplication

$$\mathcal{T}(n) \otimes \mathcal{T}(m_1) \otimes \ldots \otimes \mathcal{T}(m_n) \to \mathcal{T}(\sum m_i)$$

and unit element $1 \in \mathcal{T}(1)$ satisfying the standard properties.

Let $\mathcal{O} = \mathcal{T}^{\Sigma}$ be the operad induced by $\mathcal{T}$ (see [H], 4.2.1).

Lemma. *Suppose $\mathcal{T}(n)$ are cofibrant in $C(k)$ (for example, $\mathcal{T}(n) \in C^{-}(k)$ and consist of projective k-modules). Then the natural functor*

$$\mathtt{Viral}(\mathcal{O}) = \mathtt{Hoalg}(\mathcal{P}) \to \mathtt{Hoalg}(\mathcal{O})$$

induced by a(ny) resolution $\mathcal{P} \to \mathcal{O}$, is an equivalence of categories.

Proof. It is enough to check that the map $\mathcal{P}(n) \to \mathcal{O}(n)$ is a homotopy equivalence of $k(\Sigma_n)$-complexes for each n.

But $\mathcal{P}(n)$ is cofibrant over $k(\Sigma_n)$ since $\mathcal{P}$ is a cofibrant operad; $\mathcal{O}(n) = \mathcal{T}(n) \otimes k(\Sigma_n)$ is cofibrant over $k(\Sigma_n)$ since $\mathcal{T}(n)$ is cofibrant over k. This proves the claim. ☐

4.4.3. Although the categories $\mathtt{Viral}(\mathcal{O})$ and $\mathtt{Hoalg}(\mathcal{O})$ turn out to be equivalent in all examples of Σ-split operads we know ([H], 4.2.5), we do not see any reason why this should always be the case. No doubt, the category $\mathtt{Viral}(\mathcal{O})$ should always be used when it differs from $\mathtt{Hoalg}(\mathcal{O})$.

5. Application: realization of homotopy p-types

Mandell's theorem [Man] on the realization of homotopy p-types can be reformulated in terms of virtual commutative algebras. The advantage of this approach is that we can work with the category of operad algebras which has a CMC structure. This makes unnecessary a big part of [Man].

In this Section we review the proof Mandell's theorem 1.2.

5.1. Adjoint functors C^* and U.

5.1.1. Recall [HS] that the cochain complex $C^*(X)$ of an arbitrary simplicial set $X \in \Delta^{\mathrm{op}}\mathtt{Ens}$ admits a canonical structure of algebra over the Eilenberg-Zilber operad $\mathcal{Z}$ which is weakly equivalent to the operad $\mathtt{COM}$ of commutative algebras. Choose any cofibrant resolution $\mathcal{E}$ of $\mathcal{Z}$. The category of virtual commutative algebras $\mathtt{Viral}(\mathtt{COM})$ is canonically equivalent to $\mathtt{Hoalg}(\mathcal{E})$.

5.1.2. For each commutative ring k define

$$C^*(\_, k) : (\Delta^{\mathrm{op}}\mathtt{Ens})^{\mathrm{op}} \to \mathtt{Alg}(k \otimes \mathcal{E}) \tag{7}$$

(here and below $\otimes$ means tensoring over $\mathbb{Z}$) to be the functor of normalized k-valued cochains.

This functor admits an obvious left adjoint functor

$$U_k : \mathtt{Alg}(k \otimes \mathcal{E}) \to (\Delta^{\mathrm{op}}\mathtt{Ens})^{\mathrm{op}} \tag{8}$$

given by the formula

$$U_k(A)_n = \mathrm{Hom}(A, C^*(\Delta^n, k)) \tag{9}$$

The pair of functors $C^*(\_, k)$ and U_k satisfies the requirements of Quillen's theorem [Q], §4, Theorem 3.

Since the functor $C^*(\_, k)$ preserves weak equivalences, one therefore obtains a pair of derived adjoint functors

$$\mathbb{U}_k : \mathtt{Viral(COM)} = \mathtt{Hoalg}(k \otimes \mathcal{E}) \rightleftarrows \mathcal{H}o : C^*(\_, k), \tag{10}$$

$\mathcal{H}o$ being the homotopy category of simplicial sets.

5.2. Following [Man], we call $X \in \Delta^{\mathrm{op}}\mathtt{Ens}$ *k-resolvable* if the natural map

$$u_X : X \to \mathbb{U}_k C^*(X, k)$$

is a weak equivalence.

The following two lemmas allow one to construct resolvable spaces.

5.2.1. **Lemma.** *(*[Man]*, Thm. 1.1) Let X be the limit of a tower of Kan fibrations*

$$\ldots \to X_n \to \ldots \to X_0.$$

Assume that the canonical map from H^*X to colim *H^*X_n is an isomorphism. If each X_n is k-resolvable, then X is k-resolvable.*

5.2.2. **Lemma.** *(*[Man]*, Thm. 1.2) Let X, Y and Z be connected simplicial sets of finite type, and assume that Z is simply connected. Let $X \to Z$ and $Y \to Z$ be given, so that $Y \to Z$ is a Kan fibration. Then, if X, Y and Z are k-resolvable then so is the fibre product $X \times_Z Y$.*

Lemma 5.2.1 follows form the fact that the functor $\mathbb{U}$ carries homotopy colimits in $\mathtt{Alg}(\mathcal{E})$ into homotopy limits in $\Delta^{\mathrm{op}}\mathtt{Ens}$. The proof of Lemma 5.2.2 is similar, but needs in addition Proposition 5.2.3 below which can be also easily deduced from Theorem 3.1.

Using the CMC structure on $\mathtt{Op}(k)$, one can embed the obvious map of operads $\mathtt{ASS} \to \mathtt{COM}$ into the following commutative diagram

$$\begin{tikzcd}
\mathtt{ASS}_\infty \arrow[r, "\alpha"] \arrow[d] & \mathcal{E} \arrow[d, "\tau"'] & \\
\mathtt{ASS} \arrow[r, "\overline{\alpha}"] & \overline{\mathcal{E}} \arrow[r, "\pi"] & \mathtt{COM}
\end{tikzcd}$$

where $\mathtt{ASS}_\infty$ is the operad of A_∞-algebras, α is a cofibration, π is a weak equivalence and the square is cocartesian.

5.2.3. **Proposition.** *(compare to* [Man], *Lemma 5.2). Let $A \to B$ and $A \to C$ be cofibrations of cofibrant $\mathcal{E}$-algebras. Let $\overline{A} = \tau^*(A)$, and similarly for $\overline{B}, \overline{C}$. Then the natural maps*

$$B \sqcup^A C \xrightarrow{t} \overline{B} \sqcup^{\overline{A}} \overline{C} \xleftarrow{r} \overline{B} \otimes_{\overline{A}} \overline{C}$$

are quasi-isomorphisms in $C(k)$. Here t is induced by τ and r is induced by the composition

$$\overline{B} \otimes \overline{C} \to (\overline{B} \sqcup^{\overline{A}} \overline{C}) \otimes (\overline{B} \sqcup^{\overline{A}} \overline{C}) \xrightarrow{\text{mult.}} \overline{B} \sqcup^{\overline{A}} \overline{C}.$$

Proof. 1. *t is a quasi-isomorphism.* The functor τ^* commutes with colimits. Therefore, it is enough to prove that the natural map $A \to \tau_*\tau^*(A)$ is a weak equivalence for a cofibrant algebra A. According to 2.4.4, it is enough to check that $\tau : \mathcal{E} \to \overline{\mathcal{E}}$ is a strong equivalence of operads.

Since α is a cofibration, $\overline{\alpha}$ is a cofibration as well. Therefore, both $\mathcal{E}(n)$ and $\overline{\mathcal{E}}(n)$ are cofibrant over $k\Sigma_n$. Then the strong equivalence of $\mathcal{E}$ and $\overline{\mathcal{E}}$ follows from their weak equivalence.

2. *r is a quasi-isomorphism.*

Suppose A is standard cofibrant and the maps $A \to B$, $A \to C$ are standard cofibrations. Let $\{e_i, i \in I\}$, $\{e_j, j \in I \cup J\}$, $\{e_k, k \in I \cup K\}$, be graded free bases of A, B and C respectively (the index sets I, J, K are disjoint).

The sets I, J and K are well-ordered and the differential of e_i is expressed through $e_{i'}$ with $i' < i$.

Put $S = I \cup J \cup K$ with the order given by $i < j < k$ for $i \in I, j \in J, k \in K$. Let $\widetilde{S}$ be the set of maps $S \to \mathbb{N}$ with finite support and with the lexicographic order as in 3.3.4.

For $f \in \widetilde{S}$ denote $|f| = \sum_{s \in S} f(s)$.

The algebra $\overline{B} \sqcup^{\overline{A}} \overline{C}$ has an obvious increasing filtration by subcomplexes $\{F_f\}$ indexed by $f \in \widetilde{S}$.

The homogeneous component of the associated graded complex for $f \in \widetilde{S}$ takes form

$$\mathrm{gr}_f(F) = \overline{\mathcal{E}}(|f|) \otimes_{\Sigma_f} e^f$$

where $e^f = \prod_{s \in S} e_s^{f(s)}$ and $\Sigma_f = \prod_{s \in S} \Sigma_{f(s)}$.

Define a filtration $\{F'_f\}$ of $\overline{B} \otimes_{\overline{A}} \overline{C}$ indexed by the same set $\widetilde{S}$. It is given by the formula

$$F'_f = \bigoplus_{g < f} \overline{\mathcal{E}}(|g|_1) \otimes \overline{\mathcal{E}}(|g|_2) \otimes_{\Sigma_g} e^g$$

where $|g|_1 = \sum_{s \in I \cup J} g(s)$ and $|g|_2 = \sum_{s \in K} g(s)$. The homogeneous component for $f \in \widetilde{S}$ is given by

$$\mathrm{gr}_f(F') = \overline{\mathcal{E}}(|f|_1) \otimes \overline{\mathcal{E}}(|f|_2) \otimes_{\Sigma_f} e^f.$$

The map $r : \overline{B} \otimes_{\overline{A}} \overline{C} \to \overline{B} \sqcup^{\overline{A}} \overline{C}$ is compatible with the filtrations. The corresponding map of the homogeneous components

$$\mathrm{gr}_f(r) : \overline{\mathcal{E}}(|f|) \otimes_{\Sigma_f} e^f \to \overline{\mathcal{E}}(|f|_1) \otimes \overline{\mathcal{E}}(|f|_2) \otimes_{\Sigma_f} e^f$$

is induced by the map

$$\overline{\mathcal{E}}(|f|_1) \otimes \overline{\mathcal{E}}(|f|_2) \to \overline{\mathcal{E}}(|f|) \tag{11}$$

which is obviously quasi-isomorphism. The assertion then follows from the observation that both the left and the right hand side of (11) are cofibrant over $k(\Sigma_f)$. □

5.3. To construct k-resolvable spaces using 5.2.1 and 5.2.2 one needs a space "to start with". This is the Eilenberg-Maclane space $K(\mathbb{Z}/p, n)$. The key step in [Man] is the following

5.3.1. **Theorem.** *(cf.* [Man]*, Prop. A.7). The space $K(\mathbb{Z}/p, n)$ is k-resolvable iff $k \supseteq \mathbb{F}_p$ and the frobenius $F : k \to k$ gives rise to a short exact sequence of abelian groups*

$$0 \to \mathbb{F}_p \to k \xrightarrow{1-F} k \to 0. \tag{12}$$

Proof. This is the most important part of Mandell's result and we cannot simplify the original Mandell's proof.

1. The main step is to construct an explicit cofibrant resolution of $C := C^*(K(\mathbb{Z}/p, n), \mathbb{F}_p)$ over $k := \mathbb{F}_p$.

Let $k = \mathbb{F}_p$. Let $\mathcal{E}$ be a cofibrant resolution of the operad `COM` over $\mathbb{Z}$. Recall that $\mathcal{E}$-algebra structure on A gives rise to the action of the generalized Steenrod algebra $\mathfrak{B}$ on $H(A \otimes \mathbb{F}_p)$ — see [May].

Let A be a chain complex of a topological space and let the operad $\mathcal{E}$ be endowed with a weak equivalence $\mathcal{E} \to \mathcal{Z}$ to the Eilenberg-Zilber operad of [HS], so that A becomes an $\mathcal{E}$-algebra. Then the action of fB on $H(A \otimes \mathbb{F}_p$ induces an action of the (conventional) Steenrod algebra $\mathfrak{A}$ which is a quotient of $\mathfrak{B}$ by the ideal generated by $P^0 - 1$, P^0 being the degree zero generalized Steenrod operation.

Choose a fundamental cycle $e \in C^n$. This cycle defines a map $\phi : \mathcal{E}_{\mathbb{F}_p}\langle x \rangle \to C$ from the free $\mathbb{F}_p \otimes \mathcal{E}$-algebra with a generator x to C sending x to e. Since P^0 acts trivially on $H(C)$, the cohomology class $P^0([x]) - [x]$ of $\mathcal{E}_{\mathbb{F}_p}\langle x \rangle$ (here $[x]$ is the cohomology class of x), belongs to the kernel of $H(\phi)$. Choose a representative z of the cohomology class $P^0([x]) - [x]$ of $\mathcal{E}_{\mathbb{F}_p}\langle x \rangle$.

Finally, define $B = \mathcal{E}_{\mathbb{F}_p}\langle x, y; dy = z \rangle$. This is the $\mathbb{F}_p \otimes \mathcal{E}$-algebra obtained from the free algebra $\mathcal{E}_{\mathbb{F}_p}\langle x \rangle$ by adding a variable to kill the cycle z *see also* haha 2.2.2.

The map ϕ can be obviously extended to a map $\psi : B \to C$.

Theorem. *(see* [Man]*, Thm. 6.2). The map ψ is a quasi-isomorphism.*

The proof of the theorem given in [Man], Sect. 12, is based on a study of free unstable modules over $\mathfrak{B}$ and $\mathfrak{A}$.

2. Once we have found a cofibrant resolution B of the algebra C of cochains of $K(\mathbb{Z}/p, n)$, the life becomes very easy.

We have to study the map $u_X : X \to \mathbb{U}_k(C^*(X, k))$ for $X = K(\mathbb{Z}, n)$.

One has $\mathbb{U}_k(C^*(X,k)) = U_k(B_k)$ where $B_k = k \otimes_{\mathbb{F}_p} B = \mathcal{E}_k\langle x,y;dy=z\rangle$. Since the functor U_k carries cofibrations to Kan fibrations and colimits to limits, one has a cartesian diagram of spaces

$$\begin{array}{ccc} U_k(B_k) & \longrightarrow & U_k(\mathcal{E}_k\langle z,y;dy=z\rangle) \\ \downarrow & & \downarrow \\ U_k(\mathcal{E}_k\langle x\rangle) & \xrightarrow{p} & U_k(\mathcal{E}_k\langle t\rangle) \end{array}$$

The vertical maps are Kan fibrations and $U_k(\mathcal{E}_k\langle z,y;dy=z\rangle)$ is contractible since $\mathcal{E}_k\langle z,y;dy=z\rangle$ is a contractible $k\otimes\mathcal{E}$-algebra.

Furthermore, $U_k(\mathcal{E}_k\langle t\rangle)$ identifies easily with the Eilenberg-Mac Lane space $K(k,n)$ and the map p is induced by $1-F: k\to k$ where $F: k\to k$ is the frobenius ([Man], prop. 6.4, 6.5).

Then the long exact sequence of the homotopy groups for the fibration $U_k(B_k)\to U_k(\mathcal{E}_k\langle x\rangle)$ gives the long exact sequence

$$\ldots\to\pi_i(K(k,n+1))\to\pi_i(U_k(B_k))\to\pi_i(K(k,n))\xrightarrow{p}\pi_{i+1}(K(k,n+1))\to\ldots$$

where the map p is induced by $1-F$.

Now, if the condition on k is not fulfilled, $U_k(B_k)$ is not an Eilenberg-Mac Lane space. If the sequence (12) is exact, the natural map $K(\mathbb{Z}/p,n)\to U_k(B_k)$ induces isomorphism of homotopy groups and this proves the assertion. $\square$

References

[H] V. Hinich, Homological algebra of homotopy algebras, Comm. in algebra, **25(10)**(1997), 3291–3323.

[HS] V. Hinich, V. Schechtman, Homotopy limit of homotopy algebras, Lecture notes in math., **1289**(1987), 240–264.

[Man] M. Mandell, E_∞-algebras and p-adic homotopy theory, Hopf preprint server, October, 1998.

[May] P. May, A general algebraic approach to Steenrod operations, Lecture notes in math., **168**(1970), 153–231.

[Q] D. Quillen, Homotopical algebra, Lecture notes in math., **43** (1967).

Dept. of Mathematics, University of Haifa, Mount Carmel, Haifa 31905 Israel