BETTI NUMBERS OF 3-SASAKIAN QUOTIENTS OF SPHERES BY TORI

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ABSTRACT. We give a formula for the Betti numbers of 3-Sasakian manifolds or orbifolds which can be obtained as 3-Sasakian quotients of a sphere by a torus.

A $(4m+3)$-dimensional manifold is 3-Sasakian if it possesses a Riemannian metric with three orthonormal Killing fields defining a local $SU(2)$-action and satisfying a curvature condition. A complete 3-Sasakian manifold S is compact and its metric is Einstein with scalar curvature $2(2m+1)(2m+3)$. Moreover the local action extends to a global action of $SO(3)$ or $Sp(1)$ and the quotient of S is a quaternionic Kähler orbifold.

A large family of compact non-homogeneous 3-Sasakian manifolds was found by Boyer, Galicki and Mann in [BGM2]. They are obtained by the 3-Sasakian reduction procedure, analogous to the symplectic or hyperkähler quotient construction, from the standard $(4m+3)$-sphere. Recently, in [BGMR], Boyer, Galicki, Mann and Rees have calculated the second Betti number of a 7-dimensional 3-Sasakian quotient of the $(4q+7)$-sphere by a torus, as being equal to q. Using the ideas from [BD], we shall give a formula for the Betti numbers of 3-Sasakian quotients of spheres by tori, valid in arbitrary dimension.

Theorem 1. *Let S be a 3-Sasakian orbifold of dimension $4n-1$ which can be obtained as a 3-Sasakian quotient of the standard $(4n+4q-1)$-sphere by a q-dimensional torus $N \leq Sp(n+q)$. Then the Betti numbers of S depend only on n and q and are given by the following formula*

$$b_{2k} = \dim H^{2k}(S, \mathbb{Q}) = \binom{q+k-1}{k}$$

for $k \leq n-1$.

Remarks. 1. Galicki and Salamon [GS] have shown that the odd Betti numbers b_{2k+1} of any $(4n-1)$-dimensional 3-Sasakian manifold vanish for $0 \leq k \leq n-1$. Our proof reproduces this result for orbifolds satisfying the assumptions of Theorem 1. The Poincaré duality gives now the remaining Betti numbers b_p, $p \geq 2n$, of S.

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2. The quotient of S by any 1-PS of $SO(3)$ is a contact Fano orbifold Z. Theorem 1 in conjunction with Theorem 2.4 in [BG] gives the Betti numbers of Z.

3. For any $n > 2$, there is a bound on q ($q \leq 2^n - n - 1$) in order for S to be smooth (see Remark 2.3).

4.The formula of Theorem 1 gives also the Betti numbers of "generic" toric hyperkähler orbifolds; see section 3.

Let us discuss some consequences of Theorem 1.
A compact 3-Sasakian manifold is *regular* if its quotient by the $SO(3)$ or $Sp(1)$ action is a (quaternionic Kähler) manifold. At present the only known regular 3-Sasakian manifolds of dimension greater than 3 are homogeneous and in $1-1$ correspondence with simple Lie algebras [BGM2].

Galicki and Salamon [GS] have shown that the Betti numbers of a regular 3-Sasakian manifold of dimension $4n-1$ must satisfy the following relation:

$$(*) \qquad \sum_{k=1}^{n-1} k(n-k)(n-2k)b_{2k} = 0.$$

Theorem 1 shows that this relation is intimately related to S being regular:

Proposition 2. *Let S be a 3-Sasakian manifold satisfying the assumptions of Theorem 1 with $n \geq 3$. Then the Betti numbers of S satisfy the relation $(*)$ if and only if $q = 1$, i.e. S has Betti numbers of the homogeneous 3-Sasakian manifold of type A_n.*

Remark. There *are* smooth quotients with $q > 1$ - see Theorem 4.1 in [BD] (given as Theorem 2.2 below) or Theorem 2.14 in [BGMR].

Corollary 3. *Let S be a 3-Sasakian manifold satisfying the assumptions of Theorem 1 with $n > 1$. Then S is regular if and only if S is homogeneous.*

1. Hyperkähler and 3-Sasakian structures

A $4n$-dimensional manifold is hyperkähler if it possesses a Riemannian metric g which is Kähler with respect to three complex structures J_1, J_2, J_3 satisfying the quaternionic relations $J_1J_2 = -J_2J_1 = J_3$ etc. Such a manifold is automatically Ricci flat.
Instead of giving the intrinsic definition of a 3-Sasakian manifold, which can be found in [Bä,BGM1-2,GS], we simply recall that a Riemannian manifold (S, g) is 3-Sasakian if and only if the Riemannian cone $C(S) = (\mathbb{R}^+ \times S, dr^2 + r^2 g)$ is hyperkähler [Bä,BGM2]. The three Killing vector fields on S, defining the local $Sp(1)$ action, are then given by $\xi_i = J_i \frac{\partial}{\partial r}$ (we identify S with $S \times \{1\} \subset C(S)$).

To date the most powerful technique for constructing both hyperkähler and 3-Sasakian manifolds is the symplectic quotient construction, adapted to the hyperkähler setting by Hitchin, Karlhede, Lindström and Roček [HKLR], and to the 3-Sasakian setting by Boyer, Galicki and Mann [BGM2].

In the hyperkähler case we start with a hyperkähler manifold M with an isometric and triholomorphic action of a Lie group G. Each complex structure J_i gives a Kähler form ω_i and, in many cases, a moment map $\mu_i : M \to \mathfrak{g}^*$. We recall that an equivariant map μ from M to the dual of the Lie algebra of G is called a *moment map* if it satisfies $\langle d\mu(v), \rho\rangle = \omega(X_\rho, v)$, where $v \in TM$, $\rho \in \mathfrak{g}$ and X_ρ is the corresponding Hamiltonian vector field. If G is compact and acts freely on the common zero set of these moment maps, then the quotient by G of this zero set is a hyperkähler manifold.

If we start with a 3-Sasakian manifold S, whose structure is preserved by G, we can do the reduction for the hyperkähler manifold $C(S)$. The moment maps on $C(S)$ are defined only up to addition of elements in the center of $\mathfrak{g}^*$ and for a particular choice of of these elements we can obtain an induced $\mathbb{R}^+$-action on the hyperkähler quotient M of $C(S)$ by G. This means that M is is a Riemannian cone over a 3-Sasakian manifold.

More intrinsically, we can [BGM2] define the moment maps directly on S by the formula $\langle \mu_i(m), \rho\rangle = \frac{1}{2}\eta_i(X_\rho)$, where η_i is the 1-form dual to the Killing vector field ξ_i.

2. 3-Sasakian and hyperkähler quotients by tori

We shall now quickly review the hyperkähler and 3-Sasakian quotients by tori (see [BD] for more information). We consider the diagonal maximal torus T^d of the standard representation of $Sp(d)$ on $\mathbb{H}^d$. The three moment maps μ_1, μ_2, μ_3 corresponding to the complex structures of $\mathbb{H}^d$ can be written as

$$\mu_1(z, w) = \frac{1}{2}\sum_{k=1}^{d} \left(|z_k|^2 - |w_k|^2\right) e_k + c_1, \tag{2.1a}$$

$$(\mu_2 + \sqrt{-1}\mu_3)(z, w) = \sum_{k=1}^{d}(z_k w_k)e_k + c_2 + \sqrt{-1}c_3, \tag{2.1b}$$

where c_1, c_2, c_3 are arbitrary constant vectors in $\mathbb{R}^d$.

A rational subtorus N of T^d is determined by a collection of nonzero integer vectors $\{u_1, \dots, u_d\}$ (which we shall always take to be primitive) generating $\mathbb{R}^n$. For then we obtain exact sequences of vector spaces

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \longrightarrow 0, \tag{2.2}$$

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\beta^*} \mathbb{R}^d \xrightarrow{\iota^*} \mathfrak{n}^* \longrightarrow 0, \tag{2.3}$$

where the map β sends e_i to u_i. There is a corresponding exact sequence of groups

$$1 \to N \to T^d \to T^n \to 1. \tag{2.4}$$

The moment maps for the action of N are

$$\mu_1(z,w) = \frac{1}{2}\sum_{k=1}^{d} \left(|z_k|^2 - |w_k|^2\right)\alpha_k + c_1 \tag{2.5a}$$

$$(\mu_2 + \sqrt{-1}\mu_3)(z,w) = \sum_{k=1}^{d}(z_k w_k)\alpha_k + c_2 + \sqrt{-1}c_3\,. \tag{2.5b}$$

The constants c_j are of the form

$$c_j = \sum_{k=1}^{d} \lambda_k^j \alpha_k, \qquad (j = 1,2,3). \tag{2.5c}$$

where $\lambda_k^j \in \mathbb{R}$.

For our purposes it is enough to consider the case when $\lambda_k^2 = \lambda_k^3 = 0$ for $k = 1,\dots,d$. We then write $\lambda_k = \lambda_k^1$, $k = 1,\dots,d$, and we denote the hyperkähler quotient $\mu^{-1}(0)/N$ by $M(\underline{u},\underline{\lambda})$ or sometimes just M.
In [BD] necessary and sufficient conditions for $M(\underline{u},\underline{\lambda})$ to be a manifold or an orbifold were given. We shall only need the ones for an orbifold:

Theorem 2.1 [BD]. *Suppose we are given primitive integer vectors $u_1,\dots,u_d$ generating $\mathbb{R}^n$ and real scalars $\lambda_1,\dots,\lambda_d$ such that the hyperplanes $H_k = \{y \in \mathbb{R}^n; \langle y, u_k\rangle = \lambda_k\}$, $k = 1,\dots,d$, are distinct. Then the hyperkähler quotient $M(\underline{u},\underline{\lambda})$ is an orbifold if and only if every $n+1$ hyperplanes among the H_k have empty intersection.* □

If the condition of this theorem is satisfied we refer to $M = M(\underline{u},\underline{\lambda})$ as a *toric hyperkähler orbifold.*

If we set all λ_k equal to 0, then the hyperkähler quotient or $M(\underline{u},\underline{0})$ is the Riemannian cone over a (usually singular) 3-Sasakian space S. Equivalently S is the 3-Sasakian quotient of the unit sphere in $\mathbb{H}^d$ by the torus N. We have (see also [BGMR])

Theorem 2.2 [BD]. *Let $\underline{u} = (u_1, \dots, u_d) \in \mathbb{Z}^d$ be a primitive collection of vectors generating $\mathbb{R}^n$ and let N denote the corresponding torus defined by* (2.4). *Then the 3-Sasakian quotient S of the unit $(4d-1)$-sphere by N is manifold if and only if the following two conditions hold:*

(i) *every subset of $\underline{u}$ with n elements is linearly independent;*
(ii) *every subset of $\underline{u}$ with $n-1$ elements is a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Condition (i) is necessary and sufficient for S to be an orbifold. □

Remark 2.3. For $n = 2$, the vectors $u_1, \dots, u_d$ satisfy both conditions if each of them has relatively prime coordinates and each pair of vectors u_k is linearly independent. On the other hand, if $n \geq 3$ and the vectors $u_1, \dots, u_d$ satisfy both conditions, then $d < 2^n$. I am grateful to Krzysztof Galicki for informing me that Charles Boyer has found such a bound for $n = 3$ and to Gerd Mersmann for the following argument. Suppose there are 2^d such vectors. Then either a vector u_i has all coordinates equal to zero $\mod 2$ or two vectors u_i, u_j are equal $\mod 2$. In either case we obtain a subset ($\{u_i\}$ or $\{u_i, u_j\}$) which cannot be a part of a $\mathbb{Z}$-basis.

Finally we shall need some facts from [BD] about the topology of a toric hyperkähler orbifold $M = M(\underline{u}, \underline{\lambda})$. The hyperplanes H_k of Theorem 2.1 divide $\mathbb{R}^d$ into a finite family of closed convex polyhedra, some unbounded. We consider the polytopal complex $\mathcal{C}$ consisting of all bounded faces of these polyhedra. The support $|\mathcal{C}|$ of $\mathcal{C}$ is the union of all polyhedra in $\mathcal{C}$. If $\phi = (\phi_1, \phi_2, \phi_3) : M \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ is the induced moment map for the action of $T^n = T^d/N$ on M, then it is shown in [BD] that the compact variety

$$X = \phi^{-1}(|\mathcal{C}|, 0, 0) \tag{2.6}$$

is a deformation retract of M. The variety X is a union of toric varieties corresponding to maximal elements of $\mathcal{C}$ and intersecting along toric subvarieties (in other words X is the support of the complex of toric varieties corresponding to the polytopal complex $\mathcal{C}$).

3. Proof of Theorem 1

Let $d = n + q$. The idea is to consider a toric hyperkähler orbifold $M = M(\underline{u}, \underline{\lambda})$ where the vectors $u_1, \dots, u_d$ are the ones defining the torus N and to show that the infinity of M is homeomorphic to S. Observe that the condition of Theorem 2.1 is satisfied for generic choice of scalars λ_k if the vectors u_k satisfy the condition (i) of Theorem 2.2. We shall show that $M \cup S$ is a certain quotient of the closed unit ball $\bar{B}$ in $\mathbb{H}^d$.

Let s be a diffeomorphism between $[0,1]$ and $[0,+\infty]$ with $s'(0) = 1$, and let $f(r) = s(r)/r$.
We define a "moment map" $\nu : \bar{B} \to \mathfrak{n}^*$ by the formula

$$\nu(q) = \frac{1}{f^2(||q||)}\mu\left(f(||q||)q\right) \tag{3.1}$$

where μ is given by (2.5). We observe that

$$\nu(q) = \mu(q) + \frac{1}{f^2(||q||)}c \tag{3.2}$$

where c is given by (2.5c). In particular, restricted to the unit sphere, ν is just the 3-Sasakian moment map. We denote by Σ the 0-set of ν and by Σ^0 the intersection of Σ with the open unit ball $B \subset \bar{B}$.
We observe that Σ is T^d-invariant and that Σ^0 is T^d-equivariantly homeomorphic to the 0-set of μ. Therefore the quotient Σ^0/N is T^n-equivariantly homeomorphic to $M = M(\underline{u}, \underline{\lambda})$ and the compact Hausdorff space Σ/N can be identified with $M \cup S$. Moreover, it follows from the proof of Theorem 6.5 in [BD] that the deformation $h : M \times [0,1] \to M$, $h(m,1) = m$, $h(M,0) = X$, where X is given by (2.6), extends to S (it is important here that every n among the vectors u_k are independent, and, therefore, each of the unbounded n-dimensional polytopes in the complement of the hyperplanes H_k of Theorem 2.1 has an $(n-1)$-dimensional face at infinity). Therefore $\bar{M} = \Sigma/N$ is homotopy equivalent to X.

We have the long exact sequence of rational cohomology

$$\ldots \to H^k_c(M) \to H^k(\bar{M}) \to H^k(S) \to H^{k+1}_c(M) \to \ldots .$$

Since M is an orbifold, and so a rational homology manifold, we can apply Poincaré duality to M and obtain $H^k_c(M) \simeq H_{4n-k}(M) \simeq H_{4n-k}(X)$. If $k < 2n$, then $H_{4n-k}(X) = 0$ and so $H^k(S) \simeq H^k(\bar{M}) \simeq H^k(X)$ for $k < 2n-1$.

We shall now calculate the rational cohomology groups of X. In [BD] it was shown that if the complex $\mathcal{C}$ satisfies certain technical assumption, then the usual combinatorial formula for the Betti numbers of a toric variety (cf. [Fu]) holds for X (and so for M). We shall show now that this formula holds without any further assumptions for our toric hyperkähler orbifolds $M = M(\underline{u}, \underline{\lambda})$.

Theorem 3.1. *Let $M = M(\underline{u}, \underline{\lambda})$ be a toric hyperkähler orbifold with the vectors u_k satisfying the assumption (i) of Theorem 2.2. Then $H^j(M, \mathbb{Q}) = 0$ if j is odd and*

$$b_{2k} = dim H^{2k}(M, \mathbb{Q}) = \sum_{i=k}^{n}(-1)^{i-k}\binom{i}{k} d_i, \tag{3.3}$$

where d_i denotes the number of i-dimensional elements of the complex $\mathcal{C}$.

Proof. We observe first that both sides of (3.3) depend only on vectors u_k. Indeed, Theorem 6.1 in [BD] shows that it is so for the Betti numbers. On the other hand, since every n among of the vectors u_k are independent, the hyperplanes H_k are in general position and the number d_i depends only on d and n.

We proceed now by induction on n. Suppose that the formula (3.3) holds for $k < n$ (n may be 1). In dimension n we proceed by induction on the number d of hyperplanes H_k. The formula holds for n hyperplanes. Suppose that the formula holds for $q \leq d-1$ hyperplanes in $\mathbb{R}^n$ and let us consider a toric hyperkähler orbifold $M(\underline{u}, \underline{\lambda})$ corresponding to hyperplanes $H_1, \dots, H_d$. By the remark above we can move the hyperplane H_d until all of $|\mathcal{C}|$ lies to one side of H_d, say $|\mathcal{C}| \subset \{x; \langle x, u_d\rangle \geq \lambda_d\}$. The intersections of H_d with the H_k, $k < d$, determine a simple arrangement of hyperplanes in $H_d \simeq \mathbb{R}^{n-1}$ which gives a toric hyperkähler orbifold Y of quaternionic dimension $n-1$. Let us denote its polytopal complex by $\mathcal{E}$. On the other hand the hyperplanes $H_1, \dots, H_{d-1}$ also determine a toric hyperkähler orbifold W with a polytopal complex $\mathcal{F}$. By inductive assumptions, (3.3) holds both for Y and for W. We observe that, as the hyperplanes H_k are in general position and $d \geq n+1$, every maximal element of $\mathcal{C}$ has dimension n, and therefore every i-dimensional element of $\mathcal{E}$ is a face of an $(i+1)$-dimensional element of $\mathcal{C}$. This implies, that if e_k (resp. f_k) denotes the number of k-dimensional faces of $\mathcal{E}$ (resp. $\mathcal{F}$), then $d_0 = f_0 + e_0$ and $d_k = f_k + e_k + e_{k-1}$ for $k > 0$.

Let us now consider the neigbourhoods of $|\mathcal{E}|$ and $|\mathcal{F}|$ in $|\mathcal{C}|$ defined by $U_1 = |\mathcal{C}| \cap \{x \in \mathbb{R}^n; \langle x, u_d\rangle < \lambda_d + 2\epsilon\}$ and $U_2 = |\mathcal{C}| \cap \{x \in \mathbb{R}^n; \langle x, u_d\rangle > \lambda_d + \epsilon\}$. Then $U_1 \cap U_2$ is homeomorphic to $|\mathcal{E}| \times (0, \epsilon)$. We consider the deformation retract X of M given by (2.6). We have $X = V_1 \cup V_2$ where $V_1 = \phi_1^{-1}(U_1)$ and $V_2 = \phi_1^{-1}(U_2)$. Now, by the argument used in the proof of Theorem 6.5 in [BD], V_1 can be deformed onto the corresponding deformation retract of Y and so V_1 is homotopy equivalent to Y. Similarily V_2 is homotopy equivalent to W. Moreover $V_1 \cap V_2$ is homotopy equivalent to an S^1-bundle P over Y (the S^1 corresponds to the 1-PS of T^n determined by the vector u_d). We now use Mayer-Vietoris and Gysin sequences which, since the odd Betti numbers of Y and W vanish, split off at each even level as

$$0 \to H^{2k-1}(P) \to H^{2k}(M) \to H^{2k}(Y) \oplus H^{2k}(W) \to H^{2k}(P) \to H^{2k+1}(M) \to 0,$$

$$0 \to H^{2k-1}(P) \to H^{2k-2}(Y) \to H^{2k}(Y) \to H^{2k}(P) \to 0.$$

The Gysin sequence implies that $H^{2k}(Y) \to H^{2k}(P)$ is surjective and so the odd cohomology of M vanishes. Moreover the even Betti numbers satisfy the relation $b_{2k}(M) = b_{2k}(W) + b_{2k}(Y) + b_{2k-1}(P) - b_{2k}(P)$ and $b_{2k}(Y) = b_{2k-2}(Y) + b_{2k}(P) -$

$b_{2k-1}(P)$. From these we deduce that $b_{2k}(M) = b_{2k}(W) + b_{2k-2}(Y)$ for $k > 0$ and $b_0(M) = b_0(W)$. We now write down the left-hand side of (3.3) using these equalities and the corresponding formulas (3.3) for W and Y and we rewrite the right-hand side of (3.3) using the formula $d_k = f_k + e_k + e_{k-1}$ ($e_{-1} = 0$). Then the equality between the two sides reduces to the following equality $\binom{i}{k-1} = -\binom{i}{k} + \binom{i+1}{k}$ which is easily checked. □

Remark 3.2. We expect that the proof given here will carry to general toric hyperkähler orbifolds, proving Theorem 6.7 of [BD] in full generality. All there remains to be shown is that $\mathcal{C}$ is either contained in a single hyperplane or that every maximal element of $\mathcal{C}$ has dimension n.

In order to finish the proof of Theorem 1 we have to calculate the number d_i of i-dimensional elements of the complex $\mathcal{C}$ and to apply the formula (3.3). As noticed above, since every n among of the vectors u_k are independent, the number d_i depends only on d and n. We use the formula 18.1.3 in [Gr] giving the number $f_i(d,n)$ of i-dimensional faces of the simple (i.e. no more than n of the hyperplanes have a nonempty intersection) arrangement $\mathcal{A}$ of d hyperplanes in $\mathbb{R}P^n$:

$$f_i(d,n) = \binom{d}{n-i} \sum_{j=0}^{i} \binom{d-n-1+i}{j}.$$

The number of i-dimensional faces of the complex $\mathcal{C}$ is the number of i-dimensional faces in the arrangement $\mathcal{A}$ which do not meet the infinity in $\mathbb{R}P^n$. In other words

$$d_i = f_i(d,n) - f_{i-1}(d,n-1) = \binom{d}{n-i} \binom{d-n-1+i}{i}.$$

This yields

$$\dim H^{2k}(S,\mathbb{Q}) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \binom{q+n}{n-i} \binom{q+i-1}{i} \tag{3.4}$$

for $k \leq n-1$. We now use the simple identity

$$\binom{i}{k} \binom{q+i-1}{i} = \binom{q+i-1}{i-k} \binom{q+k-1}{k}$$

to rewrite the formula (3.4) as

$$\dim H^{2k}(S,\mathbb{Q}) = \binom{q+k-1}{k} \sum_{i=k}^{n} (-1)^{i-k} \binom{q+n}{n-i} \binom{q+i-1}{i-k}. \tag{3.5}$$

It remains to show that the summed expression is equal to 1 for $k \leq n-1$. Let us denote this expression by $G(q,n,k)$ and let $F(q,n,k,i) = (-1)^{i-k}\binom{q+n}{n-i}\binom{q+i-1}{i-k}$, so that $G(q,n,k) = \sum_{k=i}^{n} F(q,n,k,i)$. We observe that $F(q,n,k,i) = F(q-1,n+1,k+1,i+1)$ and therefore $G(q,n,k) = G(q-1,n+1,k+1)$. It follows that $G(q,n,k) = G(1,n+q-1,k+q-1)$. However for $q=1$ the right-hand side of (3.5) must be 1 for $k \leq n-1$, since the left-hand side is 1 for the homogeneous 3-Sasakian manifold of type A_n [GS]. Therefore $G(q,n,k) = G(1,n+q-1,k+q-1) = 1$ for all q and $k \leq n-1$. This proves Theorem 1.

4. Consequences

We shall now prove Proposition 2 and Corollary 3. The formula $(*)$ is invariant under the symmetry $k \mapsto n-k$ and we can write it as

$$\sum_{k=1}^{[(n-1)/2]} k(n-k)(n-2k)(b_{2k} - b_{2(n-k)}) = 0.$$

To prove Proposition 2 it is enough to show that, for $q > 1$, $b_{2k} - b_{2(n-k)} < 0$ for all $1 \leq k \leq [(n-1)/2]$. By Theorem 1 this is equivalent to $\frac{(q+n-k-1)!}{(n-k)!} > \frac{(q+k-1)!}{k!}$ for $1 \leq k \leq [(n-1)/2]$. We can write both expressions as products of $q-1$ terms such that each term on the left is greater than the respective term on the right. Proposition 2 follows. For Corollary 3 we observe that Proposition 2 implies that if $n \geq 3$, then $q = 1$, and so S is the 3-Sasakian quotient of a sphere by a circle. These were analyzed in detail by Boyer, Galicki and Mann in [BGM2] and the result follows in this case from their work. For $n = 2$ it is well-known that the only compact 4-dimensional self-dual Einstein manifolds are S^4 and $\mathbb{C}P^2$ [Hi]. This proves Corollary 3 in this case.

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