INEQUALITIES OF WILLMORE TYPE FOR SUBMANIFOLDS

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1) INTRODUCTION

The well-known Willmore conjecture [11] asserts for all immersed tori T^2 in $\mathbf{R}^3$ the following inequality:

(1) $$\int_{T^2} H^2\, dA \geqq 2\pi^2 \ .$$

Here H denotes the mean curvature. Equality is attained in (1) for stereographic projections of the Clifford torus in S^3. So far (1) has been established only for special types of immersed tori, e.g. tubes around closed space curves [8,12], tori of revolution [5], tori with a special intrinsic conformal structure [6]. It is however easy to prove for any compact surface M^2 in $\mathbf{R}^3$ the following inequality [11]:

(2) $$\int_{M^2} H^2\, dA \geqq 4\pi \ .$$

It has been noticed [3,6,10] that it is often useful to replace $\int H^2\, dA$ by the modified functional C, which for immersions $f : M^2 \longrightarrow \mathbf{R}^3$ is defined as

(3) $$C(f) = \frac{1}{2\pi} \int_{M^2} (H^2-K)\,dA = \frac{1}{8\pi} \int_{M^2} (k_1-k_2)^2\,dA \ .$$

Here k_1 and k_2 are the principal curvatures of f . The functional C was studied already in the 1920's by Blaschke and Thomsen [2,9], who called it the "conformal area". Because nowadays this term is used in a different manner [6], we call C the "total conformal curvature" or the "Willmore functional". Because of the Gauss-Bonnet theorem the study of C is equivalent to the study of $\int H^2 dA$, and the inequality (2) can also be written as

$$C(f) \geq \beta_1 \quad , \tag{4}$$

where β_1 is the first $\mathbb{Z}_2$-Betti number of M^2 .

In this paper we will define $C(f)$ for all immersions $f : M^n \longrightarrow \mathbb{R}^m$, where M^n is an arbitrary compact manifold. We will prove a generalization of (4) and state conjectures similar to (1).

2) THE FUNCTIONAL $C(f)$

Let M^n be a compact smooth manifold, $f : M^n \longrightarrow \mathbb{R}^m$ an immersion, $N(f)$ the unit normal bundle of f . To every $\xi \in N_p(f)$ there corresponds a shape operator $A_\xi : T_pM^n \longrightarrow T_pM^n$, whose eigenvalues $k_1(\xi),\ldots,k_n(\xi)$ are called the principal curvatures at ξ . Let $\sigma(A_\xi)$

denote the dispersion of the principal curvatures in the sense of probability theory:

(5)
$$\sigma(A_\xi)^2 = \frac{1}{n^2} \sum_{i<j} (k_i - k_j)^2 \quad .$$

Then the total conformal curvature $C(f)$ of the immersion f is defined as

(6)
$$C(f) = \frac{1}{\mathrm{vol}(S^m)} \int_{N(f)} \sigma(A_\xi)^n \, d\xi \quad .$$

Here $d\xi$ denotes the natural volume element on $N(f)$. The functional $C(f)$ has the following remarkable properties:

(i) $n = 2 \Rightarrow C(f) = \frac{1}{2\pi} \int_{M^2} (|H|^2 - K)\, dA$

(ii) $C(f) \geq 0$, $C(f) = 0 \Leftrightarrow f$ is totally umbillic

(iii) If $i : \mathbf{R}^m \longrightarrow \mathbf{R}^p$, $p \geq m$ denotes the canonical inclusion, then $C(i \circ f) = C(f)$.

(iv) If $\varphi : \mathbf{R}^m \cup \{\infty\} \longrightarrow \mathbf{R}^m \cup \{\infty\}$ is conformal, then $C(\varphi \circ f) = C(f)$.

The conformal invariance (iv) of $C(f)$ is proved in [1]. The verification of (i), (ii) and (iii) is left to the reader.

3) AN ESTIMATE IN TERMS OF THE BETTI NUMBERS

In this section we prove the following generalisation of inequality (4):

THEOREM 1: Let M^n be compact, $\mathbb{F}$ a field, $\beta_1,\dots,\beta_n$ the Betti numbers of M^n with respect to $\mathbb{F}$, $f : M^n \longrightarrow \mathbb{R}^m$ an immersion. Then

$$(7) \qquad C(f) \geq \sum_{k=1}^{n-1} a_k \beta_k \quad ,$$

where $a_k := \left(\frac{k}{n-k}\right)^{n/2-k}$.

PROOF: Let as above $N(f)$ denote the normal bundle of f and define

$$N_k = \{\xi \in N(f) \mid A_\xi \text{ has exactly } k \text{ negative eigenvalues}\}.$$

A standard argument from total absolute curvature theory [4] yields

$$(8) \qquad \int_{N_k} |\det A_\xi|\, d\xi \geq \beta_k \operatorname{vol}(S^{m-1}) \quad .$$

On the other hand we have

$$(9) \qquad C(f) = \frac{1}{\operatorname{vol}(S^{m-1})} \sum_{k=0}^{n} \int_{N_k} \sigma(A_\xi)^n d\xi \quad .$$

By (5), (8) and (9) the theorem now follows from the lemma below.

□

<u>LEMMA</u>: Let $k_1,\dots,k_n$ be real numbers, $r \neq 0,n$ such that

(10) $$k_1,\dots,k_r < 0 \quad , \qquad k_{r+1},\dots,k_n \geqq 0 \quad .$$

Then

(11) $$\left[\frac{1}{n^2} \sum_{i<j} (k_i-k_j)^2\right]^{n/2} \geqq a_r |k_1 \dots k_n| \quad .$$

<u>PROOF</u>: Since both sides of (11) are positively homogeneous of degree n we can restrict attention to the cylinder

(12) $$Z = \{(k_1,\dots,k_n) \in \mathbf{R}^n \mid \sum_{i<j} (k_i-k_j)^2 = b\} \ .$$

The constant b will be specified later on. The subset Z_r of Z defined by the sign conditions (10) is bounded, and the function $g : Z_r \longrightarrow \mathbf{R}$

(13) $$g(k_1,\dots,k_n) = |k_1 \dots k_n|$$

vanishes on the boundary of Z_r and is smooth in the interior of Z_r. Therefore g assumes its maximal value at some point $(x_1,\dots,x_n) \in \mathring{Z}_r$. There is a Lagrangian multiplier λ such that for $1 \leq i \leq n$ we have

$$(14) \qquad x_i - H = \lambda\, x_1 \ldots x_{i-1}\, x_{i+1} \ldots x_n \quad ,$$

where

$$(15) \qquad H = \frac{1}{n} \sum_{i=1}^{n} x_i \quad .$$

By (14) for all i,j we have

$$(16) \qquad x_i^2 - Hx_i = x_j^2 - Hx_j \quad .$$

Thus all x_i satisfy the same quadratic equation

$$(17) \qquad x_i^2 - Hx_i + \mu = 0 \ .$$

From (17) we conclude

$$(18) \qquad x_1 = \ldots = x_r =: \tilde{x} < 0 \quad , \quad x_{r+1} = \ldots = x_n =: x > 0$$

By (17) and Vietas theorem

$$(19) \qquad x + \tilde{x} = H \quad .$$

On the other hand (15) means

$$(20) \qquad px + q\tilde{x} = H \quad ,$$

where we have set $p = r/n$, $q = 1-p$. Subtracting (20) from (19) we obtain

$$(21) \qquad qx + p\tilde{x} = 0 \quad .$$

We now chose the free constant b such that $\tilde{x} = -1$. Then (21) yields

$$x = \frac{p}{q}$$

$$(22) \qquad x - \tilde{x} = 1 + \frac{p}{q} = \frac{1}{q}$$

$$\left[\frac{1}{n^2} \sum_{i<j} (x_i - x_j)^2\right]^{n/2} = \left(\frac{p}{q}\right)^{n/2-pn} |x_1 \ldots x_n| \quad .$$

Since $|k_1 \ldots k_n|$ was maximal at $(x_1, \ldots, x_n)$ the assertion of the lemma follows from (22).

□

4) WILLMORE PROBLEMS

Theorem 1 in the last section gives some information about the first of the following two types of "Willmore problems":

a) Given a compact smooth manifold M^n , determine (or estimate at least)

$$C(M^n) := \inf\{C(f) \,|\, f : M^n \longrightarrow \mathbf{R}^m \quad \text{an immersion}\} \,.$$

b) For $n \geq 2$ determine (or estimate)

$$C(n) := \inf\{C(M^n) \mid M^n \text{ not homeomorphic to } S^n\} .$$

In this section we study problem b). Surprisingly there is a complete answer for $n = 2$:

THEOREM 2: $$C(2) = C(\mathbb{R}P^2) = 2 .$$

Theorem 2 is an immediate consequence of (4) and Theorem 4 in [6]. For $n \geq 3$ we have the following estimate:

THEOREM 3: $$C(n) \geq 2(n-1)^{1-n/2} .$$

PROOF: Let M^n be compact, not homeomorphic to S^n, $f : M^n \longrightarrow \mathbb{R}^m$ an immersion. Then by the Morse inequalities for any height function $h = \ell \circ f$ ($\ell : \mathbb{R}^m \longrightarrow \mathbb{R}$ linear), which is a Morse function one of the following is true:

(i) h has at least two critical points of index 1 or $n-1$.

(ii) h has at least one critical point of index r, where $1 < r < n-1$.

(iii) h has only two critical points (one minimum and one maximum).

(iv) h or $-h$ has two minima, one critical point of index 1, one maximum and no other critical points.

Case (iii) cannot occur, because M^n would then be homeomorphic to S^n. Similarly (iv) is impossible, because here the critical point of index one can be "cancelled" against one of the minima [7], that means there is another Morse function $g : M^n \longrightarrow \mathbf{R}$ having only two critical points. Again M^n would be homeomorphic to S^n.

By the argument in the proof of theorem 1 this implies

$$\int_{N_1} |\det A_\xi| d\xi + \int_{N_{n-1}} |\det A_\xi| d\xi + 2 \sum_{k=2}^{n-2} \int_{N_k} |\det A_\xi| d\xi \geq 2 \operatorname{vol}(S^m) . \tag{23}$$

It is easy to check that for $2 \leq r \leq n-2$ we have

$$a_r \geq 3 a_1 = 3(n-1)^{1-n/2} . \tag{24}$$

The theorem now follows from (9), (11), (23) and (24).

□

We would like to state here the following conjecture, that might be regarded as a higher dimensional version of the original Willmore conjecture (the latter can be stated as $C(T^2) = \pi$) :

<u>CONJECTURE</u>: For $n \geq 3$ we have $C(n) = C(S^1 \times S^{n-1})$ and

(25) $$C(n) = \sqrt{\frac{n-1}{n^n}}\ \frac{\mathrm{vol}(S^{n-1})}{\mathrm{vol}(S^n)} \cdot 4\pi =: c_n .$$

The next theorem shows that at least $C(n)$ and c_n do not differ too much:

<u>THEOREM 4</u>: $0.64\, c_n \leq C(n) \leq c_n$.

<u>PROOF</u>: Let $S^{n-1}(R) \subset \mathbb{R}^n$ be a round sphere of radius R, $S^1(r) \subset \mathbb{R}^2$ a circle, $f : S^1 \times S^{n-1} \longrightarrow \mathbb{R}^{n+2} = \mathbb{R}^2 \times \mathbb{R}^n$ an embedding with $f(S^1 \times S^{n-1}) = S^1(r) \times S^{n-1}(R)$. Then for a suitable choice of the ratio r/R we obtain $C(f) = c_n$. This proves $C(f) \leq c_n$.

In Theorem 3 we established a lower bound

(26) $$e_n := 2(n-1)^{1-n/2}$$

for $C(n)$. Thus it suffices to show $q_n \leq 1/0.64$, where we have defined $q_n = c_n/e_n$. Explicitly

$$q_n = \begin{cases} \left(\dfrac{2\cdot 2\cdot 4\cdots(2m-2)\,(2m-2)}{1\cdot 3\cdot 3\cdots(2m-3)\,(2m-1)}\right)^{-1/2} \dfrac{\sqrt{2m-1}}{2} & \text{if } n = 2m \\[2ex] \left(\dfrac{2\cdot 2\cdot 4\cdots(2m)\ \ (2m)}{1\cdot 3\cdot 3\cdots(2m-1)\,(2m+1)}\right)^{1/2} \dfrac{\sqrt{2m+1}}{\pi} & \text{if } n = 2m+1 \end{cases}$$

The first two terms of the sequence q_n are

(27) $$q_3 \approx 1.540 \quad , \quad q_4 \approx 1.530 \quad ,$$

and using Wallis' product we find

(28) $$\lim_{n\to\infty} q_n = \sqrt{\frac{2\pi}{e}} \approx 1.520 \quad .$$

Since $q_3, q_4 \leq 1/0.64$ the proof will be finished once we have shown that the two subsequences (q_{2m}) and (q_{2m+1}) are monotonically decreasing, which means

(29) $$\sqrt{\frac{n^{n-1}(n+1)^{n+2}}{(n-1)^{n-1}(n+2)^{n+2}}} = \frac{q_{n+2}}{q_n} < 1$$

for all n . Taking the logarithm of both sides we see that (29) is equivalent to

(30) $$\begin{aligned} &(n-1)\log n + (n+2)\log(n+1) \\ &\leq (n-1)\log(n-1) + (n+2)\log(n+2) \quad . \end{aligned}$$

(30) is a consequence of the obvious inequalities

$$\begin{aligned} \log n - \log(n-1) &< 1/(n-1) \\ \log(n+2) - \log(n+1) &> 1/(n+2) \quad . \end{aligned}$$

□

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