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Monograph

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Abstract

This research program aims to build some new mathematical structures for the description of non-perturbative aspects of Quantum Field Theories whenever bare or running coupling constants are strong enough. Under a combinatorial setting, we apply infinite combinatorics to build a new topological Hopf algebra $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ of Feynman graphons which leads us to formulate a new Hopf algebraic renormalization theory for solutions of Dyson–Schwinger equations in the context of the theory of graphons and the Riemann–Hilbert problem. We then build a new parametric representation for infinite formal expansions of Feynman diagrams in the context of Tutte polynomial, Kirchhoff–Symanzik polynomial and other combinatorial tools. Furthermore, we formulate a multi-scale renormalization group theory on the collection $\mathcal{S}^{\Phi,g}$ of all Dyson–Schwinger equations in a gauge filed theory in terms of changing simultaneously the scales of momenta and bare coupling constant. This machinery enables us to describe the non-perturbative behavior of a given Dyson–Schwinger equation $\mathrm{DSE}(g)$ at strong coupling constant g in terms of a convergent sequence of Dyson–Schwinger equations at weaker couplings with respect to the cut-distance topology. In addition, this multi-scale renormalization machinery suggests a new way to study the complexity of non-perturbative computations in the context of the Kolmogorov complexity at the level of Feynman graphons where as the result, we will show that the BPHZ renormalization of Feynman graphons can encode the Halting problem of partial recursive maps on $\mathcal{S}^{\Phi,g}$. Under a geometric-analytic setting, we build a spectral triple model for the study of the geometry of Dyson–Schwinger equations. Then we explain a mathematical framework for the construction of a noncommutative geometry model for the description of the geometry of the renormalization Hopf algebra $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ of Feynman graphons. Thereafter, we work on the construction of a functional analysis theory for the study of large Feynman diagrams where we can formulate the Haar integration theory on $\mathcal{S}^{\Phi,g}$ as a modification of the classical Riemann–Lebesgue integration theory with respect to the Borel σ-algebra on real numbers. As some applications of this integration theory, at first, we obtain a new evolution method for the description of large Feynman diagrams in the language of Johnson–Lapidus Dyson series. At second, we work on the Banach algebra $L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$ where thanks to the Gelfand transform we can explain the formulation of a generalized version of the Fourier transformation which is useful for the evolution of large Feynman diagrams. Furthermore, we build the Gâteaux differential calculus machinery on the Banach space $\mathcal{S}^{\Phi,g}$ with respect to the cut-norm to study smooth functions on $\mathcal{S}^{\Phi,g}$ in the language of Taylor series of higher order Gâteaux differentiations and homomorphism densities. Under a lattice theoretic setting, we apply combinatorial Dyson–Schwinger equations, Feynman graphons and some topological treatments to explain the concept of quantum entanglement in Quantum Field Theory in the language of substructures organized in a lattice of topological Hopf subalgebras. We lift this story onto a categorical level to encode information flow among elementary particles on the basis of the representation theory of Lie groups and mixed Tate motives. Furthermore, in another direction, we explain the construction of a new topos model of presheaves to formulate logical propositions about non-perturbative aspects. We investigate that the strength of the couplings in gauge field theories can change the base category of the topos model of the physical theory. As the result, we obtain a new class of countable Heyting algebras which are capable to encode the evaluation of logical propositions about topological regions of Feynman diagrams.

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PREFACE

The strength of Mathematics is its ability to create models which are absolutely vital for producing physical parameters. A mathematician is like a surrealist painter who can design the purest portraits from known and unknown universes.

Recent discoveries in Science and Technology from the smallest to the largest scales have approved clearly the importance of advanced research activities in Basic Science which can bring a new package of fundamental knowledge for the analysis of complicated systems in natural phenomena. To obtain a comprehensive description of those complexities requires to build any possible interrelations among different fields in Mathematics (as the purest mental production of human beings). The resulting connections can lead us to achieve some new theoretical methodologies which are essential tools for scientists to build advanced practical models in dealing with complexities of the nature. The designed models together with some computational algorithms will lead scientists to solution procedures.

This research work has a multidisciplinary foundation in the context of Mathematics, High Energy Theoretical Physics and Theoretical Computer Science. It plans to discover some new knowledge about the most unknown parts of Quantum Field Theories whenever the coupling constants are strong enough on the basis of building advanced mathematical structures. The outstanding consequence of this research work is to provide a new mathematical interpretation of the phenomenology of Quantum Field Theory with strong coupling constants under discrete, analytic and logical settings. If we study simultaneously these different but related settings, then our mathematical outputs will be useful for the better understanding of the behavior of physical systems in non-perturbative situations. We apply diagrams, graph limits, combinatorial polynomials and some topological tools to address the discrete behavior of non-perturbative phenomenology, then we apply Noncommutative Geometry and Functional Analysis to address its analytic behavior and finally, we apply Category Theory and theory of ordered algebraic structures to address its quantum logical behavior.

The Lagrangian approach to Quantum Field Theory, which is on the basis of the Feynman path integral formalism, has made extraordinary theoretical and experimental progress for the study of elementary particles and their interactions at the highest level of energies and the smallest scales under a perturbative setting. This approach encodes physical information of a quantum system with infinite times degrees of freedom in terms of Green's functions as infinite formal expansions of ill-defined iterated integrals.

Quantum Electrodynamics (QED) concerns interactions among matter (electron, positron) and light (photons). Quantum Flavourdynamics (QFD) concerns weak interactions inside the nucleus of an atom which change the flavour or type of quarks to describe β^- decay and β^+ decay under W, Z bosons. Quantum Chromodynamics (QCD) concerns strong interactions of quarks and gluons inside the nucleus of an atom to build composite hadrons such as protons and neutrons. Standard Model, as the most successful experienced model, has provided a practical platform to collect quantum field theories corresponding to electromagnetic, weak and strong interactions into a united Quantum Field Theory model. The modified versions of the Standard Model in the context of Noncommutative Geometry have also provided a new updated (theoretical) model which is (minimally) coupled to gravity as the weakest fundamental force in the nature. The constructions of gauge field theories in Theoretical and Experimental High Energy Physics, as updated Quantum Field Theory models, are on the basis of the modified Standard Model of elementary particles which interprets electroweak and strong interactions of elementary particles in the scale of distances down to the order of 10^{-16} centimeters while neutrino masses have also been accounted. In addition, under a more theoretical setting, String Theory as other class of Quantum Field Theory models, which does not have ultraviolet divergencies, has been introduced and developed where the classical one-loop Feynman diagram should be replaced with its stringy counterpart which is a torus and more general Feynman diagrams should be replaced with Riemann surfaces and world sheets. This mathematical theory is capable of describing Quantum Gravity in Space-Time.

The first fundamental challenge in perturbative setting is the appearance of so complicated nested (sub-)divergencies which live in each term of Green's functions. These ill-defined terms, known as Feynman integrals, can be theoretically reduced to some finite values as the result of the renormalization machinery and many loop techniques where some extra parameters (i.e. counterterms) should be added to the original Lagrangian of the physical theory. The discovery of a comultiplication structure hidden inside of the (Bogoliubov)–Zimmermann's forest formula has led us to understand the Bogoliubov–Parasiuk–Hepp–Zimmermann perturbative renormalizaton in the language of the Connes–Kreimer Hopf algebra of Feynman diagrams and the Riemann–Hilbert problem. Thanks to this setting, a geometric interpretation of dimensional regularization on the basis of flat equi-singular connections has been formulated by Connes and Marcolli. This study had been lifted onto a universal categorical setting where we associated a category of Lie group representations to each renormalizable Quantum Field Theory. Thanks to this Hopf algebraic approach to Quantum Field Theory, nowadays we have a diverse spectrum of advanced mathematical techniques and tools to deal with ill-defined iterated Feynman integrals in physical theories to generate some computable values from infinities.

The second fundamental challenge in perturbative setting is dealing with complicated infinities originated from Green's functions which encode quantum motions in physical theories with strong couplings. The lack of a rigorous mathematical methodology for the study of aspects beyond perturbation boundary has made so many difficulties to understand completely Quantum Field Theory. In physical theories with strong (running or bare) couplings, it is already impossible to study the full behavior of quantum systems under perturbation series and in this situation, we need to concern non-perturbative methods such as numerical methods, Borel summation method, theory of instantons and lattice model. In addition, the self-similar nature of Green's functions makes an alternative way for us to study non-perturbative aspects in the context of fixed point equations of Green's functions. The resulting equations, which are known as Dyson–Schwinger equations, contain a collection of coupled integral equations depended on the coupling constant. In couplings more than or equal to 1, these equations behave non-perturbatively. In QCD with higher energies, we can expect the asymptotic freedom behavior which enables us to make computations via some perturbative tools such as many loop techniques but in QCD with relatively lower energies, the story is so complicated. Work on the phenomenology of running couplings in QCD has been considered under a physical setting to provide some computational methods in dealing with non-perturbative parameters. Thanks to the applications of the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams to Quantum Field Theory, we already have a combinatorial reformulation for Dyson–Schwinger equations in the language of Hochschild cohomology theory. The unique solution of each equation DSE determines a free commutative connected graded Hopf subalgebra of the renormalization Hopf algebra. This mathematical approach to Dyson–Schwinger equations has already provided some new combinatorial and geometric tools for the computation of some non-perturbative parameters where the foundations of a differential Galois theory and a Tannakian formalism for the study of non-perturbative aspects of Quantum Field Theories have been designed and developed (by the author) on the basis of the Connes–Marcolli universal category of flat equi-singular vector bundles. Furthermore, thanks to these investigations, some deep interrelationships between Dyson–Schwinger equations and some abstract mathematical structures in theory of motives and theory of computation have been found by the author.

This research work proposes some new applications of mathematical tools originated from Combinatorics, Functional Analysis, Noncommutative Geometry, Category Theory and Logic to deal with infinite graphs generated by solutions of Dyson–Schwinger equations. These new mathematical settings can provide some new techniques for the computation of non-perturbative parameters. In addition, they suggest a new methodology for the description of the intrinsic foundations of Quantum Field Theory such as quantum entanglement and quantum logic under a non-perturbative setting. These investigations will help us to understand the indeterministic nature of non-perturbative Quantum Field Theory.

Generally speaking, the achievements of this research work can improve our knowledge about the phenomenology of non-perturbative Quantum Field Theory under two different but related levels. The first level focuses on the mathematical foundations of Dyson–Schwinger equations to bring some new computational tools in dealing with non-perturbative parameters generated by large Feynman diagrams. At this level, we consider each Dyson–Schwinger equation as an individual object in the vector space $\mathcal{S}^{\Phi,g} = \bigcup \mathcal{S}^{\Phi}(\lambda g)$ of all Dyson–Schwinger equations derived from Green's functions of a given Quantum Field Theory Φ under different scales λg of the bare coupling constant g where $0 < \lambda \leq 1$. We equip this infinite dimensional vector space with a topological structure defined via the graphon representation of Feynman diagrams. Under a combinatorial setting, we discuss the structure of a new model for large Feynman diagrams in the language of combinatorial polynomials and random graphs. Furthermore, we discuss the complexity of non-perturbative parameters generated by Dyson–Schwinger equations in the context of theory of computation. In this direction we try to show the importance of a new multi-scale non-perturbative renormalization group for the description of the Kolmogorov complexity in dealing with Dyson–Schwinger equations. Under a geometric setting, we explain the dynamics of non-perturbative aspects in a physical theory with respect to the mathematical structures originated from Dyson–Schwinger equations. We build a noncommutative geometry model for each Dyson–Schwinger equation which leads us to interpret quantum motions in the context of theory of spectral triples and noncommutative differential forms. Under a functional analysis setting, we discuss the evolution of fixed point equations of Green's functions by defining a new generalized version of the Fourier transformation on the Banach algebra $L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$ defined on large Feynman graphs. The second level focuses on the mathematical foundations of non-perturbative Quantum Field Theory where we must deal with all possible Dyson–Schwinger equations and for this purpose we explain the construction of a new Hopf algebra structure $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ on the topological space of graphons which contribute to representations of Feynman diagrams and their finite or infinite formal expansions. The resulting topological Hopf algebra is capable to encode large Feynman diagrams generated by solutions of Dyson–Schwinger equations in different rescalings of the bare coupling constant g. Therefore we can embed the collection $\mathcal{S}^{\Phi,g}$

into $\mathcal{S}^{\Phi}_{\text{graphon}}$. This Hopf algebra leads us to formulate a Hopf algebraic renormalization theory for large Feynman diagrams which is the result of an adaption of the Connes–Kreimer BPHZ renormalization program. We then define a new multi-scale renormalization group on the collection $\mathcal{S}^{\Phi,g}$ where this renormalization group enables us to study the behavior of Dyson–Schwinger equations under changing the scales of running and bare coupling constants. As an application, we will enable to study an equation $\text{DSE}(g)$ in strong coupling g in terms of analyzing its approximations with respect to Dyson–Schwinger equations under weaker couplings. Under a calculus setting, we explain the foundations of a differential calculus theory on $\mathcal{S}^{\Phi,g}$ where thanks to graph function representation of Dyson–Schwinger equations and theory of Gâteaux derivative, a theory of differentiation and a theory of integration on $\mathcal{S}^{\Phi,g}$ will be provided. Under a categorical setting, we concern some foundations of Quantum Field Theory on the basis of a non-perturbative context. On the one hand, we explain mathematically the information flow among elementary (virtual) particles in QFT models via a new class of topological Hopf algebras generated by Dyson–Schwinger equations. This new perspective will lead us to explain quantum entanglement via a theory of lattices where we will show the importance of the universal Connes–Marcolli category of flat equi-singular vector bundles for the description of the geometry of quantum entanglement. Thanks to this investigation, we will discover the motivic nature of quantum entanglement in interacting gauge field theories with strong couplings. On the other hand, we have also plan to explain the original basics of a quantum logic theory for Quantum Field Theory where we will explain the construction of a topos of presheaves on a new base category. This base category enables us to encode topological regions of Feynman diagrams determined by objects in $\mathcal{S}^{\Phi,g}$. This new toposification method, which is depended upon the strength of the coupling constant in QFT models, provides a new logical formalism for the evaluation of logical propositions about Dyson–Schwinger equations.

Thanks to these two levels of observations, we expect to provide a new insight into the complicated problems of non-perturbative situations where the strength of the coupling constants do really change the mathematics and the logics of quantum theory models.

Chapter 1

Introduction

- ***Physical backgrounds***
- ***Mathematical backgrounds***
- ***Recent progress and objectives***

1.1 *Physical backgrounds*

Modern Theoretical and Experimental High Energy Physics have been established on the basis of Quantum Field Theory models under the Lagrangian setting which is (minimally) coupled to gravity via the incorporation of massive neutrinos. The foundations of Quantum Field Theory were initiated in terms of the interpretation of the quantized version of Electrodynamics in the language of the Feynman path integral formalism under a perturbation setting. The appearance of gauge field theories which include Quantum Electrodynamics (QED), Electroweak theory, Quantum Chromodynamics (QCD), Quantum Gravity have developed rigorously our theoretical knowledge about the fundamental properties of elementary particles before we could reach to appropriate empirical information. Thanks to these backgrounds, mathematicians and theoretical physicists have already made outstanding achievements for the description of interactions of elementary particles under different settings in the context of advanced mathematical models. For example, mathematical tools in Noncommutative Geometry, Algebraic Geometry, Combinatorics and Category Theory have been applied to formulate Standard Model and other extended theories which include supersymmetry, gravitational interactions or extended objects such as strings and brane theory. We can also address tensor models as higher dimensional generalizations of matrix models which aim to achieve a theory of random geometries in dimensions higher than two. This class of theories helps us for the construction of discrete approaches to quantizing gravity. [19, 20, 23, 30, 37, 44, 85, 86, 94, 99, 110, 120, 122, 129, 136, 137, 163, 169, 174]

The Lagrangian formalism enables us to understand Quantum Field Theory by working on Green's functions as infinite formal expansions of Feynman integrals or their corresponding diagrams where the amount of some fundamental parameters such as the strength of the bare coupling constants or the domain of momenta make the resulting series divergent or asymptotic free. In perturbative physical theories we expect to have some convergent series.

For example in ϕ^4 model the partition function is given by

$$Z[B] := \int \mathcal{D}\phi \ e^{-L(\phi)+\int B\phi} \tag{1.1}$$

such that

$$L(\phi) = \int d^4x \ \big(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2 + \frac{1}{4!}u_0\phi^4\big), \tag{1.2}$$

and B is an external field. If we set

$$Z_0 := \int \mathcal{D}\phi \ e^{-L_0(\phi)+\int B\phi}, \qquad L_0(\phi) := \int d^4x \ \big(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2\big), \tag{1.3}$$

then we can develop Z as a series in u_0 around Z_0 to achieve

$$Z = \int \mathcal{D}\phi \, \big(1 - \frac{u_0}{4!} \int_{x_1} \phi^4(x_1) + \frac{1}{2}(\frac{u_0}{4!})^2 \int_{x_1,x_2} \phi^4(x_1)\phi^4(x_2) + ...\big) e^{-L_0(\phi) + \int B\phi}. \tag{1.4}$$

This expansion can be represented in the language of Feynman diagrams which leads us to a combinatorial formulation for Green's functions. The fluctuations generated by the ϕ^4 term around the Gaussian integral Z_0 are large where they determine iterated integrals with the general form

$$\int^{\Lambda} d^d q_1 ... d^d q_L \prod_i (\text{propagator}(\mathrm{q_i})) \tag{1.5}$$

such that the ultra-violet regulator Λ provides a cut-off at the upper bound type of integral. The dependency of these integrals to the parameter Λ makes a rigorous challenge for the computation of universal quantities. Theory of perturbative renormalization provides the machinery to reparametrize the perturbative expansion in such a way that the sensitive dependence on Λ has been eliminated. In this situation, the renormalization group enables us to partially resum the perturbative expansions to achieve some universal computational results. [23, 30, 129, 171]

In general speaking, it is possible to investigate the situations beyond perturbation theory in terms of some expressions such as

$$P(g) = X_0 + X_1 g + X_2 g^2 + ... + X_n g^n + ... \tag{1.6}$$

such that g is the coupling constant and each term X_i represents the class of Feynman diagrams which contribute to the i-order of perturbative expansion. It is obvious that physical theories with very small g could be encoded by only some beginning finite number of terms from the above expansion while physical theories with strong coupling g produce infinite number of terms. These non-perturbative aspects have been concerned in Theoretical Physics via Dyson–Schwinger equations as a quantized version of the Euler–Lagrange equations of motion originated from the principal of the least action. These equations, which can be determined by fixed point equations of Green's functions, have been studied under analytic and numerical methods in Theoretical and Mathematical Physics such that we can address standard techniques such as Borel summation, theory of instantons, lattice models, etc in dealing with these equations to generate some estimations for non-perturbative parameters. In physical theories with strong couplings such as Hadron Physics, we should deal with hadrons such as protons and neutrons as the composite particles build up from quarks and gluons (as elementary particles under strong interaction). In general, QCD, as a nonabelian gauge theory with the symmetry group SU(3), has provided a modern understanding of the complicated nature of hadrons and nuclei where we study the strong interactions of quarks and gluons under confinement and chiral symmetry breaking. The appearance of nuclear weak force enables us to describe any change quark's flavor via W bosons. In fact, the weak force is not only responsible for interactions between particles, but it also allows heavy particles to decay by emitting or absorbing some of the force carriers. We can describe QCD as a matrix-valued modification of electromagnetic theory in terms of replacing photons by gluons and electrons by quarks while quantum fluctuations of the fields could determine the force law. The quantization of Chromodynamics involves the regularization and renormalization of ultraviolet divergencies which generate a mass-scale where mass-dimensionless quantities become dependent on a mass-scale. The current quark-masses are the only evident scales in QCD. The main experimental confirmations of QCD have been investigated at high energies and high momentum transfers (or short distances) where the QCD coupling is small and correspondingly the forces are weak. This situation, which is known as asymptotic freedom property, enables us to detect the composite structure of hadrons by scattering high energy electrons. The most difficult challenge in this model can be observed when perturbation theory fails to describe the short range static potential obtained from quenched lattice simulations where the difference between the non-perturbatively determined potential and perturbation theory at short distance has been parameterized by a linear term. In addition, there are also so many difficulties for the study of the asymptotic character of QCD perturbative series beyond the two-loop level where the original effort is to find a way to subtract perturbative contributions to a given physical process in order to isolate non-perturbative terms. In the domain of relatively low energies and momentum transfers such as $Q^2 \sim 1 - 5$ GeV^2 while the proton's mass is approximately 1 GeV, the QCD coupling constants are larger where many loops perturbative calculations should be applied. Because of nontrivial vacuum structure of QCD, in the domain of lower energies and momentum transfers (or large distances) such as $Q^2 \leq 1$ GeV^2, the QCD coupling constants are stronger than one where the analytic calculations do not useful but there are some methods such as the chiral effective theory, lattice calculations, large N limits and Dyson–Schwinger equations to provide some algorithmic computations. This situation, which is actually the failure to directly observe coloured excitations in a detector, is the origin of the concept of confinement as the fundamental fact that we do not see free quarks or gluons in nature but rather we only see colourless. The analytic description of confinement is one difficult challenge for the understanding of continuum QCD. The phenomenology of confinement can be studied in the context of Dyson–Schwinger equations. [7, 30, 38, 118, 119, 120, 137]

The situations beyond perturbation boundary deal with divergencies originated from strong coupling constants such as infinite many loops Feynman diagrams. What does this class of diagrams means and how can we deal with these extremely complicated concepts? Applying advanced mathematical structures is helpful for the better understanding of this class of divergencies. It is obvious that having strong mathematical modelings is the initial step to make fundamental progress in dealing with complicated behavior of elementary particles inside nuclei. We can address the mathematical foundations of the (modified) Standard Model as a good theoretical methodology which led scientists to obtain strong experimental investigations about elementary particles. String Theory is another powerful mathematical platform which aims to provide a unified theory for the study of elementary particles with respect to all fundamental forces. The basic philosophy of this research work is to build and develop the mathematical foundations of QFT models with strong couplings where as the consequence, we expect to provide some new mathematical tools for the better understanding of physical parameters beyond perturbation theory. Our study suggests a new interpretation from the phenomenology of strong couplings in the context of combinatorial, geometric and categorical settings.

1.2 *Mathematical backgrounds*

The contributions of mathematical tools to Quantum Field Theories have been extraordinary developed when the (Bogoliubov–)Zimmermann forest formula was reinterpreted by Kreimer in the context of (co)algebraic combinatorial tools. This reinterpretation had been concerned by Connes and Kreimer to build a new modern formulation for the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) perturbative renormalization in Quantum Field Theory on the basis of the theory of Hopf algebras and the Riemann–Hilbert problem. The Connes–Kreimer approach has become the main foundation in many research efforts for the study of complicated Feynman integrals, Green's functions and renormalization group where it has led the Theoretical Physics's community to achieve some new mathematical tools for the description of physical parameters in (renormalizable) gauge field theories under algebraic, combinatorial and geometric settings. It is now possible to encapsulate the machinery of perturbative renormalization in terms of a connected graded free commutative non-cocommutative (finite type) Hopf algebraic structure $H_{\rm FG}(\Phi)$ on Feynman diagrams of a physical theory Φ which has a Lie algebraic nature determined by the insertion operator. The compatibility of the fundamental identities such as Slavnov–Taylor and Ward identities in QCD and QED with the renormalization coproduct have been shown in the language of Hopf ideals. The phenomenology of counterterms has been concerned underlying a geometric treatment to provide some alternative methods for the computation of these physical values in the language of (singular) differential equations. In this setting, a new class of equi-singular flat connections governs the behavior of counterterms with respect to the β-function. This setting has been lifted onto a universal Tannakian formalism where a renormalizable Quantum Field Theory is studied via a category of geometric objects which can be recovered by a category of finite dimensional representations of the affine group scheme $\mathbb{G}_\Phi := \mathrm{Hom}(H_{\mathrm{FG}}(\Phi), -)$. [2, 3, 16, 18, 19, 20, 21, 24, 22, 26, 33, 34, 37, 65, 66, 87, 95, 96, 97, 98, 125, 155, 156, 168, 174]

Perhaps the most fundamental result in this direction would be the discovery of a very deep interrelationship between Feynman integrals and theory of motives in Algebraic Geometry where a motivic renormalization machinery has been formulated to deal with divergencies in the language of Picard–Fuchs equations and other powerful tools.

Theory of motives in Algebraic Geometry aims to concern the existence of a universal cohomology theory for algebraic varieties defined over a base field k while taking values into an abelian tensor category. The construction of a category of motives (mixed motives) related to general varieties is a difficult task where the noncommutative version of motivic objects provides the motivic cohomology applied in the construction of a universal cohomology theory. The structure of mixed Tate motives as elements of the subring $\mathbb{Z}[\mathbb{L}]$ of the Grothendieck group $K_0(\mathrm{Var}_k)$ of $k-$varieties has been considered where $\mathbb{L} := [\mathbb{A}^1]$ is the Grothendieck class of the affine line. The application of motives enables us to develop a unified setting underlying different cohomology theories such as Betti, de Rham, l-adic, crystalline and etale. For this purpose, the construction of an abelian tensor category that provides a linearization of the category of algebraic varieties has been studied to provide some fundamental requirements of standard conjectures of Grothendieck.

The importance of motives in Quantum Field Theory have been noted in different settings. The Bloch–Esnault–Kreimer approach which informs interesting applications of Hodge type structures in the calculation processes of Feynman integrals underlying graph polynomials [11, 12, 97, 169]. The Aluffi–Marcolli approach which builds the motivic version of Feynman rules characters where they have applied Kirchhoff–Symanzik polynomials to produce a new version of algebro-geometric (dimensionally regularized) Feynman rules characters. These abstract characters send classes in the Grothendieck ring of conical immersed affine varieties to the classes in the Grothendieck ring of varieties spanned by the classes $[X_\Gamma]$. This formalism, which is on the basis of the deletion–contraction operators and Tutte–Grothendieck polynomial, enables us to relate Feynman diagrams with periods of algebraic varieties. This framework provides a motivic treatment in the study of perturbative renormalization process at the level of the universal motivic Feynman rules character [5, 6]. The Connes–Marcolli approach deals with the geometric interpretation of counterterms on the basis of flat equisingular connections such that these geometric objects have been organized in a categorical structure $\mathcal{E}^\Phi$ which is recovered by the neutral Tannakian category of finite dimensional representations of the affine group scheme $\mathbb{G}_\Phi$. This category has been embedded (as a sub-category) inside of the universal category $\mathcal{E}^{\rm CM}$ of flat equi-singular vector bundles with the corresponding universal affine group scheme $\mathbb{U}$. Objects of this universal category, which has Tannakian nature, address mixed Tate motives which contribute to divergencies of renormalizable physical theories. In addition, $\mathcal{E}^{\rm CM}$ determines the universal singular frame as the unique loop with values in $\mathbb{U}$ which provides the universal counterterm. The Lie algebra of the universal affine group scheme leads us to formulate a particular shuffle type Hopf algebra. [37, 116, 117]

A single Feynman diagram reports only a small piece of information about a finite number of possible interactions among (virtual) elementary particles where its on-shell part (i.e. incoming and outgoing particles) obeys the mass-energy equation and conservation of momenta while its off-shell part (i.e. virtual particles) obeys no special rules or measurements. The iterated integral corresponding to a Feynman diagram might be so complicated and it contains nested or overlapping sub-divergencies but unfortunately, it does not restore much information about a physical system. Indeed infinite formal expansions of Feynman diagrams (as polynomials with respect to coupling constants) do play important roles because they are capable to encode various possible interactions which could or might happen among elementary particles in a physical theory. These expansions, which live in Green's functions, have been studied by using the self-similar nature of Green's functions which allows us to formulate fixed point equations known as Dyson–Schwinger equations. There are some numerical methods such as large N limit, Borel resummation, lattice models and theory of instantons to deal with these equations. [118, 119, 120]

In addition, these non-perturbative equations in physical theories with strong couplings have been considered in the context of the Connes–Kreimer renormalization Hopf algebra to provide some new advanced mathematical tools for the computation of their solutions. This Hopf algebraic formalism is one of the original motivations of this research program and it is necessary to review the structure of combinatorial Dyson–Schwinger equations.

The Hochschild cohomology of (commutative) bialgebras is formulated as the dual notion of the Hochschild cohomology of (commutative) algebras. For a given commutative Hopf algebra H, consider linear maps $T : H \longrightarrow H^{\otimes n}$ as n-cochains where the coboundary operator is defined by

$$\mathbf{b}T := (\mathrm{id} \otimes T)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i T + (-1)^{n+1} T \otimes \mathbb{I} \tag{1.7}$$

such that Δ_i is the coproduct Δ of H applied to the i-th factor in $H^{\otimes n}$. The Kreimer's renormalization coproduct on Feynman diagrams can be reformulated recursively in terms of a linear operator B^+ on Feynman diagrams known as the grafting operator as the following way

$$\Delta_{\rm FG} B^+ = (\mathrm{id} \otimes B^+)\Delta_{\rm FG} + B^+ \otimes \mathbb{I}. \tag{1.8}$$

The operator B^+, as a homogeneous linear endomorphism of degree one, replaces a vertex in a given Feynman diagram with a whole graph in terms of the type of the targeting vertex and the types of external edges of the second graph. Thanks to (1.7) and (1.8), the grafting operator is a generator of the first rank Hochschild cohomology group of the Connes–Kreimer Hopf algebra of Feynman diagrams. In other words, for each 1PI Feynman diagram γ, B^+_γ is a Hochschild one cocycle. [33, 54, 58, 90]

The first importance of the grafting operator is its role for the representation of Feynman diagrams in the langauge of decorated trees where we can encode the Hopf algebra $H_{\rm FG}(\Phi)$ by a decorated version of a combinatorial Hopf algebra $H_{\rm CK}$ of non-planar rooted trees. The grafting operator acts on each forest $t_1...t_n$ to deliver a rooted tree by adding a new vertex r as the root and n new edges which connect the roots of t_ns to r. Decorations on trees enable us to update $H_{\rm CK}$ with respect to each physical theory. Each 1PI Feynman graph, which is free of sub-divergencies, is a primitive element in the Hopf algebra $H_{\rm FG}$ and it can be encoded by a vertex in a tree. In this setting, we can show the existence of an injective Hopf algebraic homomorphism from $H_{\rm FG}(\Phi)$ to the Hopf algebra $H_{\rm CK}(\Phi)$ of decorated non-planar rooted trees. In addition, the pair $(H_{\rm CK}, B^+)$ enjoys a universal property with respect to the Hochschild cohomology theory where it plays the role of the initial object for a particular category of objects (H,T) consisting of a commutative Hopf algebra H and a Hochschild one cocycle T on H. The Hopf algebra homomorphisms which commute with the cocycles are morphisms of this category. Therefore we can have a Hopf algebra homomorphism from $H_{\rm CK}$ to $H_{\rm FG}(\Phi)$ for each physical theory Φ. [18, 19, 20, 56, 57, 74, 87, 89, 91]

The second importance of the grafting operator is its fundamental role in the reconstruction of Dyson–Schwinger equations under a combinatorial setting. For a given family $\{\gamma_n\}_{n\geq 1}$ of primitive (1PI) Feynman diagrams with the corresponding Hochschild one cocycles $\{B^+_{\gamma_n}\}_{n\geq 1}$, a class of combinatorial Dyson–Schwinger equations in $H[[g]]$ is defined by

$$X = \mathbb{I} + \sum_{n\geq 1} g^n \omega_n B^+_{\gamma_n}(X^{n+1}) \tag{1.9}$$

such that g is the coupling constant. This class of equations accepts a unique solution $X = \sum_{n\geq 0} g^n X_n$ as formal expansion of finite Feynman diagrams where for each $n > 0$, we have

$$X_n = \sum_{j=1}^{n} \omega_j B^+_{\gamma_j}\Big(\sum_{k_1+...+k_{j+1}=n-j,\ k_i\geq 0} X_{k_1}...X_{k_{j+1}}\Big). \tag{1.10}$$

We set X_0 as the empty tree and it can be seen that each X_n is an object in the Hopf algebra $H_{\rm FG}(\Phi)$ while the unique solution X lives in a completion of $H[[g]]$ with respect to the n-adic topology. Thanks to Cartier–Quillen–Milnor–Moore theorem, the unique solution of each Dyson–Schwinger equation DSE determines the generators of a Faa di Bruno type Hopf subalgebra H_{DSE} of the Connes–Kreimer renormalization Hopf algebra. H_{DSE} is a free commutative unitial counital non-cocommutative connected graded finite type Hopf subalgebra where its coproduct on generators X_n does not depend on the parameters ω_j. The Mellin transform allows us to deform these combinatorial type of equations to their original integral versions. It is possible to lift this formalism onto the level of systems of Dyson–Schwinger equations where we deal with a system (S) of a finite collection of equations with the general form

$$(S): \quad \forall i \in I, \quad x_i = \sum_{j \in J_i} B^+_{(i,j)}(f^{(i,j)}(x_k,\ k \in I)) \tag{1.11}$$

such that $I := \{1, ..., n\}$, J_i is a graded connected set, $B^+_{(i,j)}$s are Hochschild one cocycles and $f^{(i,j)}$s are formal series in $\mathbb{K}[[\alpha_1, ..., \alpha_n]]$. It is shown that the system (S) has a unique solution such that under some conditions it can determine the Hopf subalgebra $H_{(S)}$ as a consequence of the Hopf subalgebras $H_1, ..., H_n$ generated by combinatorial Dyson–Schwinger equations in the system. [9, 27, 53, 61, 62, 63, 90, 93]

The main skeleton of a combinatorial Dyson–Schwinger equation is actually a family of Hochschild one cocycles. There exists a surjective map from the first rank Hochschild cohomology group to the space of primitive Feynman diagrams of the renormalization Hopf algebra. It means that each family $\{\gamma_n\}_{n\geq 1}$ of primitive Feynman diagrams determines the corresponding family $\{B^+_{\gamma_n}\}_{n\geq 1}$ of Hochschild one cocycles. It is important to note that each 1PI Feynman diagram, which is free of sub-divergencies, is a primitive element but they are not the only primitives in the renormalization Hopf algebra. In other words, there are primitive Feynman diagrams in higher degrees which can determine Hochschild one cocycles. [9, 33, 91]

1.3 *Recent progress and objectives*

The combinatorial reformulation of Dyson–Schwinger equations in terms of the theory of Hopf algebras and Hochschild cohomology theory has played a central role for the creation of many interesting and rigorous mathematical constructions which can improve our knowledge about the phenomenology of non-perturbative parameters under different settings. [25, 53, 63, 64, 90, 92, 93, 100, 103, 124, 147, 150, 151, 152, 164, 165, 170]

The Milnor–Moore theorem ([126]) allows us to determine the infinite dimensional complex graded pro-unipotent Lie group $\mathbb{G}_\Phi(\mathbb{C})$ which is actually the complex points of the affine group scheme $\mathbb{G}_\Phi = \text{Hom}(H_{\text{FG}}(\Phi), -)$. This Lie group, which is the projective limit of linear algebraic groups G_n

embedded as Zariski closed subsets in some GL_{m_n}s, is rich enough to encode (dimensionally regularized) Feynman rules characters with respect to the scale and angle dependence of amplitudes [22]. In addition, we can also determine the infinite dimensional complex graded pro-unipotent Lie group

$$\mathbb{G}_{\mathrm{DSE}}(\mathbb{C}) := \mathrm{Hom}(H_{\mathrm{DSE}}, \mathbb{C}) \tag{1.12}$$

for each given Dyson–Schwinger equation DSE. There exists a natural injective Hopf algebra homomorphism $\rho : H_{\mathrm{DSE}} \longrightarrow H_{\mathrm{FG}}(\Phi)$. If we apply Spec as a contravariant functor, then we can obtain a surjective morphism $\tilde{\rho} : \mathrm{Spec}(H_{\mathrm{FG}}(\Phi)) \longrightarrow \mathrm{Spec}(H_{\mathrm{DSE}})$ between spaces of prime ideals in the commutative algebras $H_{\mathrm{FG}}(\Phi)$ and H_{DSE} equipped with the Zariski topology. This map can be lifted onto the surjective group homomorphism

$$\overline{\rho} : \mathbb{G}_{\Phi}(\mathbb{C}) \longrightarrow \mathbb{G}_{\mathrm{DSE}}(\mathbb{C}). \tag{1.13}$$

The existence of Lie subgroups $\mathbb{G}_{\mathrm{DSE}}(\mathbb{C})$ corresponding to Dyson–Schwinger equations have been applied to bring a new geometric setting for the study of non-perturbative parameters in the context of differential systems together with singularities. The construction of a category of flat equi-singular $\mathbb{G}_{\mathrm{DSE}}(\mathbb{C})$-connections with respect to each equation DSE has been addressed to encode the BPHZ renormalization of the unique solution X_{DSE} in the context of differential Galois theory. In other words, the Connes–Marcolli geometric interpretation of counterterms and the Connes–Marcolli universal Tannakian machinery in dealing with renormalizable physical theories have been developed for the study of Dyson–Schwinger equations where each equation DSE could be considered on the basis of representations of the Lie group $\mathbb{G}_{\mathrm{DSE}}(\mathbb{C})$ which is organized in a neutral Tannakian category $\mathrm{Rep}_{\mathbb{G}^*_{\mathrm{DSE}}}$ of finite dimensional representations. This class of categories has been embedded as subcategories into the Connes–Marcolli universal category $\mathcal{E}^{\mathrm{CM}}$. Thanks to these backgrounds, we already have the construction of a differential Galois theory for the computation of fundamental non-perturbative parameters such as global β-functions and non-perturbative counterterms in the language of Picard–Fuchs equations. In addition, it is now possible to identify a class of mixed Tate motives with respect to each Dyson–Schwinger equation. [140, 142, 143, 144]

On the one hand, under a Hopf algebraic setting, Dyson–Schwinger equations are important sources for the production of Hopf subalgebras. The concept of substructure is one of the fundamental tools in the theory of computation where the mathematics of Galois theory has been modified to deal with intermediate algorithms. On the other hand, the Manin's program for the interpretation of the Halting problem in the context of the Connes–Kreimer BPHZ renormalization machinery has initiated the foundations of a brilliant interrelationship between the amount of computability and the computation of counterterms originated from the renormalization Hopf algebra of Halting problem. Thanks to these investigations, now we have a mathematical machinery for the description of intermediate algorithms on the basis of Dyson–Schwinger equations where some new tools in Operad Theory and theory of Hall sets for the study of the amount of computability have been obtained. Results in this direction can also be applied in Theoretical Computer Science and Quantum Field Theory. [52, 113, 114, 115, 147, 172, 173]

Applications of the theory of graphons to Quantum Field Theory is another output of this Hopf algebraic formalism. The foundations of a new combinatorial interpretation of Feynman diagrams and their infinite formal expansions have been studied recently where we embed (large) Feynman diagrams into a compact topological space obtained by an enrichment of the Connes–Kreimer renormalization Hopf algebra with respect to the cut-distance topology. The immediate consequence of this new topological-combinatorial setting is the formulation of a generalization of the BPHZ renormalization for solutions of Dyson–Schwinger equations. In this direction, thanks to some tools in Measure Theory, a new differential calculus machinery on Feynman diagrams was built which has led us to study the evolution of Dyson–Schwinger equations in terms of their partial sums. [77, 107, 148, 149]

Applications of non-commutative differential graded algebras to Quantum Field Theory were also considered to study the geometry of quantum motions where some new models of gauge theories have been obtained. In addition, the structure of a non-perturbative version of the Connes–Kreimer renormalization group has been described in the language of integrable systems. [41, 42, 43, 45, 141, 146]

As the conclusion for this part, combinatorial Dyson–Schwinger equations and Connes–Kreimer–Marcolli Hopf algebraic renormalization are the main motivational objects for us in this research to study non-perturbative Quantum Field Theory. Our main attempt in this work is to develop mathematical structures originated from Dyson–Schwinger equations to discover some new information about complicated behavior of Quantum Field Theories in strong coupling constants. This work aims also to bring some new mathematical tools to deal with the computation of non-perturbative parameters. Under a combinatorial setting, we plan to apply the graphon representation of Feynman diagrams together with other combinatorial and topological tools to provide a Hopf algebraic renormalization machinery for objects in the topological space

$$\mathcal{S}^{\Phi,g} = \bigcup_{\lambda} \mathcal{S}^{\Phi}(\lambda g) \tag{1.14}$$

of all Dyson–Schwinger equations originated from Green's functions of a given Quantum Field Theory Φ under different scales λg of the bare coupling constant g where $0 < \lambda \leq 1$. The topology on this space, which is the result of the graphon representation of Feynman diagrams, can be determined by the metric

$$d(X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}) := d_{\mathrm{cut}}([W_{X_{\mathrm{DSE}_1}}], [W_{X_{\mathrm{DSE}_2}}]) \tag{1.15}$$

such that d_{cut} is the distance between two unlabeled graphon classes. We then address some new applications of combinatorial polynomials such as Kirchhoff–Symanzik and Tutte polynomials to formulate a parametric representation of large Feynman diagrams. This study is useful for the construction of algebro-geometric Feynman rules on the topological Hopf algebra $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$. Then we concern the concept of complexity for the description of non-perturbative parameters where we explain the construction of a new multi-scale renormalization group machinery on $\mathcal{S}^{\Phi,g}$ which is useful on two levels. Firstly, it can provide a mathematical machinery for the approximation of Dyson–Schwinger equations in strong couplings via equations in weaker couplings. Secondly, it helps us to initiate a new version of the Kolmogorov complexity in dealing with Dyson–Schwinger equations. Under a geometric setting, we show some new applications of Noncommutative Geometry, Measure Theory and Functional Analysis to describe the geometry of non-perturbative Quantum Field Theory in the language of spectral triples. This study enables us to bring the idea of a spectral geometry framework in dealing with Dyson–Schwinger equations. In addition, it leads us to define the concept of evolution on $\mathcal{S}^{\Phi,g}$ with respect to a generalized version of the Fourier transformation. Furthermore, this work considers some intrinsic foundations of Quantum Field Theory with strong couplings such as quantum entanglement and logical concepts. In this direction, we offer a new mathematical methodology for the description of quantum entanglement in interacting quantum physical theories in the context of the theory of lattices and intermediate structures in the theory of computation. This mathematical formalism enables us to explain information flow in physical theories with strong couplings on the basis of lattices of topological Hopf algebras and Lie subgroups. We lift this mathematical modeling onto a categorical setting to show that the universal category $\mathcal{E}^{\mathrm{CM}}$ is suitable to encode the quantum entanglement process. At this level, we expect to show a new application of motives in dealing with information flow. In addition, we put forward the construction of a new topos of presheaves on a particular base category which encodes the logic of topological regions of Feynman diagrams. We discuss that the topos model for non-perturbative parts of physical theories should respect the strength of couplings. Thanks to this setting, we expect to deal with logical propositions originated from Dyson–Schwinger equations.

Chapter 2

A theory of renormalization for Dyson–Schwinger equations

- *Quantum Field Theory*
- *Hochschild cohomology of the renormalization Hopf algebra*
- *Renormalization Hopf algebra of Feynman graphons and filtration of large Feynman diagrams*
- *The BPHZ renormalization of large Feynman diagrams via Feynman graphons*

Having no comprehensive physical description of infinite formal expansions of Feynman diagrams which contribute to polynomials with respect to strong coupling constants such as (1.6), it is indeed difficult to analyze a renormalization program for these infinite expansions. The major discourse in this situation is to find some meaningful mathematical insights deeply related to fixed point equations of Green's functions. Then these mathematical reasonings serve to compel technical terms and models for the explanation of the renormalization process for Dyson–Schwinger equations.

The original task in this chapter is to explain a renormalization program on the space $\mathcal{S}^{\Phi,g}$ which consists of Dyson–Schwinger equations with respect to running couplings λg in a given physical theory Φ with the bare coupling constant g. For this purpose, we apply theory of graphons to describe a graph function representation of (larga) Feynman diagrams which leads us to build a new graded Hopf algebraic structure $\mathcal{S}^{\Phi}_{\text{graphon}}$. Then we lift the Connes–Kreimer BPHZ formalism onto the level of this Hopf algebra with respect to the cut-distance topology where we can compute algebraically some physical parameters generated by Dyson–Schwinger equations such as non-perturbative counterterms and their corresponding renormalized values. It will be shown that the complex Lie group associated to the Hopf algebra $\mathcal{S}^{\Phi}_{\text{graphon}}$ is the central object for the formulation of a renormalization group at the level of large Feynman diagrams.

2.1 *Quantum Field Theory*

Suppose we have a quantized field theory with the Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ which is divided into two parts. The typical free Lagrangian density is given by

$$\mathcal{L}_{\text{free}} := \frac{1}{2}\big((\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2\big) \tag{2.1}$$

which leads us to the free Klein–Gordon equation of motion

$$\big(\partial_\mu \partial^\mu + m^2\big)\phi = 0. \tag{2.2}$$

The interaction part $\mathcal{L}_{\text{int}}$ encodes interactions of elementary particles in the physical theory. The transition amplitudes from initial states to all finite states is already studied under S-Matrix setting. These matrix elements can be calculated in terms of a class of correlation functions with the general form

$$G_n(x_1, ..., x_n) :=< 0|T\phi(x_1)...\phi(x_n)|0 > \tag{2.3}$$

such that $|0>$ is the vacuum ground state. These equations, known as Green's functions, allow us to formulate perturbative Quantum Field Theory in terms of formal expansions with the general form

$$G_n(x_1, ..., x_n) =$$

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int d^4y_1...d^4y_j < 0|T\phi_{\rm in}(x_1)...\phi_{\rm in}(x_n)\mathcal{L}_{\rm int}(y_1)...\mathcal{L}_{\rm int}(y_j)|0 > \quad (2.4)$$

such that $\phi_{\rm in}$ is the initial state of ϕ in the infinite past. If we apply Wick's Theorem and normal ordering, then the vacuum expectation value can be described as the integrals of propagators that typically depend on differences of space-time vectors. The rigorous challenge is the existence of divergencies in these integrals with respect to the domains of integrations where applying regularization machineries (such as dimensional regularization) help us to study these integrals in the context of Laurent series with finite pole parts. [23, 30]

Feynman diagrams in Quantum Field Theory enable us to represent combinatorially the summation over probability amplitudes corresponding to all possible exchanges of virtual particles compatible with a process at a given order. These decorated diagrams, as a set of edges and a set of vertices, aim to simplify the description of interactions of elementary particles with respect to the parameter of time in a quantum system. Their decorations, which are determined by fundamental parameters of the physical theory where we can interpret momentum and position as Fourier transforms of each other, are useful to translate diagrams with respect to momentum space to their corresponding iterated integrals via Feynman rules. As the fundamental rules, first, each closed loop associates to an integrate over the corresponding momentum and second, these graphs should obey the law of the conservation of momenta which tells us that the amount of momenta for input particles in an interaction procedure is the same as the amount of momenta for output particles.

For example, thanks to the Schwinger parameter t, we can consider

$$\frac{1}{p^2+m^2} = \int_0^{\infty} dt \ \exp(-t(p^2+m^2)) \quad (2.5)$$

as the propagator for each edge and

$$\int d^4x \ \exp(i\sum_j p_j x) \quad (2.6)$$

as the propagator for each vertex. In this setting, each edge has a factor

$$G(x,y;t) = \int \frac{d^4p}{(2\pi)^4} \exp(ip.(x-y) - t(p^2+m^2)). \quad (2.7)$$

Thanks to this machinery, it is possible to describe combinatorially Green's functions in the language of expansions of Feynman diagrams. It can be seen that the general formulation of Green's function enjoys a self-similar property which means that

$$\mathcal{G} := 1 + \int I_\gamma + \int\int I_\gamma I_\gamma + \int\int\int I_\gamma I_\gamma I_\gamma + ...$$

$$= 1 + \int I_\gamma (1 + \int I_\gamma + \int\int I_\gamma I_\gamma + ...) \tag{2.8}$$

such that I_γ is the Feynman integral corresponding to the primitive (1PI) Feynman diagram γ. Therefore we have

$$\mathcal{G} = 1 + \int I_\gamma \mathcal{G} \tag{2.9}$$

such that its fixed point equations determine Dyson–Schwinger equations. [23, 30, 129, 168]

This formulation of Quantum Field Theory is the result of the path integral method to Lagrangian formalism where we study the behavior of an elementary particle in a system with infinite degrees of freedom in terms of the sum over all possible situations (such as trajectories, interactions) which could be selected by particle. In terms of some conditions dictated by physical theory, each possible situation has a particular weight which should be considered in computational processes. For example in QED we deal with interactions of electron and positron (as matter) with photons (as electromagnetic waves with different quantized sizes of energies). There exists six fundamental interactions namely, the emission of photon from electron or positron, absorbing a photon via electron or positron, the creation of a photon via annihilation of the pair (electron, positron), creation of a pair (electron, positron) via the annihilation of a photon. All other Feynman diagrams in QED, which might contain complicated off-shell interactions of virtual particles, are built on the basis of those six fundamental interactions. There exists a class of elementary graphs which play the role of building blocks to make other Feynman diagrams in a physical theory. These graphs, which are called one particle irreducible Feynman diagrams, remain connected after removing one internal edge from each graph. By induction we can define n-particle irreducible Feynman diagrams which remain connected after removing n internal edges from each graph. It is easy to see that each n-particle irreducible graph is a $(n-1)$-particle irreducible graph.

The coupling constants in Quantum Field Theory aim to describe the strength of the interactions among elementary particles. The regularization of UV divergent integrals and the renormalization procedure results a scale dependence where the UV cut-off dependence of the coupling is eliminated by allowing the couplings and masses (which appear in the Lagrangian) to acquire a scale dependence. Then we normalize them to a measured value at a given scale. Generally speaking, there are two classes of couplings namely, the bare coupling constant as the original strength of a fundamental force and the running coupling constants or effective couplings as the result of renormalization procedures. Quantum Chromodynamics (QCD) is known as the most successful fundamental gauge theory of strong interactions. It studies the hadronic interactions involving quarks and gluons at both long and short distances. Its symmetry group is SU(3) where it includes N_f family of quarks ψ_f^i and gluons A_μ^i. Some experimental evidences inform us that at a critical temperature around $T_c \approx 170$ MeV QCD matter undergoes a deconfining phase transition into quark-gluon plasma. Perturbative QCD is a method based on expanding different physical quantities with respect to the gauge coupling constant g which is applied in the region $T \gg T_c$ where g is small. The phenomenology of the (bare and running) coupling constants have been discussed in terms of the uncertainties in their values at short distances which leads us to a total theoretical uncertainty in Physics at large hadron collider such as Higgs production via gluon fusion. In this situation we can still have hope to apply asymptotic freedom and perturbative calculations of renormalization group equations. However at high perturbative orders it becomes necessary to evaluate large numbers of multi-loop Feynman diagrams in the effective theory. [7, 40, 119]

But on the other hand, the behavior of coupling constants at long distances such as the scale of the proton mass in order to understand hadronic structure, quark confinement and hadronization processes should be analyzed under non-perturbative settings such as Dyson–Schwinger equations where the phenomenology of the bare and running coupling constants can be understandable via advanced mathematical methodologies. In this situation we can address recent theoretical progress for the computation of non-perturbative parameters in the context of Combinatorics, Geometry and Category Theory. [9, 25, 92, 118, 120, 137, 142, 143, 144, 165, 170]

The running of a coupling constant originates from the renormalization procedure while predictions for observables should be determined independent of the choice of renormalization map and regularization scheme. This invariance under the choice of renormalization program is encoded via a symmetry group. The running coupling is an expansion parameter in the perturbative series describing an observable and there exists the Landau pole as the point where the perturbative expression of the running coupling diverges. It means that this perturbative expression is a non-observable quantity. The observable is independent of the renormalization scheme but the series's coefficients and the running coupling will depend on the renormalization scheme. Under asymptotic freedom behavior at short distances, we can get the first coefficient series as an independent parameter but at very large distances dependency will play important role. This discussion tells us that the running couplings are not observables because they are strongly depended on the renormalization scheme at large distances. In short, the running couplings have weak scale dependence at distances smaller than 10^{-16} m such that this controllable weak behavior tends to a strong scale dependence larger than a tenth of a Fermi. This dependency on the scale is restored at larger distances due to the confinement of quarks and gluons. [7, 40, 118]

The Ward–Takahashi identities on Feynman diagrams tell us that the photon propagator is the only propagator in QED which contributes to the running of coupling constant. The Slavnov–Taylor identities on Feynman diagrams tell us that intermediate gauge-dependent quantities in non-abelian gauge theories provide final gauge-independent results for observables [99, 155]. Thanks to these facts, it is possible to rewrite Dyson–Schwinger equations in terms of some running couplings to achieve some intermediate quantities which are useful to simplify the original complicated non-perturbative type of equations by some approximations. In a general configuration, Dyson–Schwinger equations are polynomials with respect to bare or running coupling constants which means that any change in the amount of running couplings will make direct influence on the behavior of these equations. Therefore these non-perturbative type of equations are good tools for the study of the phenomenology of strong couplings.

Dimensional regularization was introduced by 't Hooft, Veltman, Bollini and Gambiagi as a method to regularize ultraviolet divergencies in a gauge invariant way to complete the proof of renormalizability. The method works in $D = 4 - 2\epsilon$ space-time dimensions where divergencies for $D \to 4$ appears as poles in $1/\epsilon$. This method also regulates infrared singularities where if we remove the auxiliary IR regulator, the IR divergencies appear as poles in $1/\epsilon$. For $\epsilon > 0$, we can obtain a well-defined result which we can be analytically extended to the whole complex D-plane. The only essential change in the structure of Feynman rules is to replace the couplings in the Lagrangian via the transformation $g \longmapsto g\mu^{\epsilon}$ such that μ is an arbitrary mass scale. Dimensional regularization together with minimal subtraction provide a practical renormalization program for Feynman integrals with nested sub-divergencies. These ill-defined parts can be eliminated step by step under a forest formula setting and the Bogoliubov–Parasiuk–Hepp preparation allows us to generate some finite values. This particular renormalization program was reconsidered by Connes and Kreimer under a Hopf algebraic setting to generate counterterms and renormalized values in terms of the Riemann–Hilbert problem and the Birkhoff factorization. In this context, dimensional regularization is encapsulated by the space of loops with the domain of a punctured infinitesimal disk around zero and with values in a pro-unipotent complex Lie group associated to the renormalization Hopf algebra of Feynman diagrams. [34, 35, 58, 66, 98, 102]

While working on the applications of the BPHZ procedure to the level of many-loop graphs is one of the interesting topics in Quantum Field Theory, the main information of a physical theory are encoded in (infinite) formal expansions of Feynman diagrams. Thanks to the Connes–Kreimer–Marcolli theory, the required mathematical tools for an extension of the BPHZ procedure to the level of Dyson–Schwinger equations has already been considered under geometric and algebraic settings. According to this new machinery, we consider each equation DSE with respect to its corresponding complex Lie group $\mathbb{G}_{\text{DSE}}(\mathbb{C})$ where the existence of the Hopf–Birkhoff factorization on this Lie group has led us to determine counterterms (which contribute to the unique solution of the equation DSE) in the language of differential systems together with irregular singularities. These differential systems, which are on the basis of equi-singular flat $\mathbb{G}_{\mathrm{DSE}}(\mathbb{C})$-connections, determine a new class of systems of Picard–Fuchs equations with regular singularities. [144, 147]

2.2 *Hochschild cohomology of the renormalization Hopf algebra*

The basic elements of the path integral method in Quantum Field Theory are (divergent) iterated Feynman integrals over the momentum space such that the integrands are determined from a definite collection of rules originated from the physical theory. We can encode these integrals in terms of a class of combinatorial decorated finite diagrams which are known as Feynman diagrams where sub-divergencies in the original integral are presented in terms of the existence of nested or overlapping loops in the main diagram. The Kreimer's coproduct, which highlights the combinatorics of removing sub-divergencies from integrals, enables us to factorize the original complicated Feynman diagram into its basic sub-divergencies (as sub-graphs). This factorization reduces several layers of complications in the computational processes of perturbative renormalization in terms of a certain graded commutative non-cocommutative Hopf algebra denoted by $H_{\mathrm{FG}}(\Phi)$. It is a graded Hopf algebra with respect to the first Betti number of Feynman diagrams which means that $H_{\mathrm{FG}}(\Phi) = \bigoplus_{n\geq 0} \mathcal{H}_n$ such that $\mathcal{H}_0 = \{\mathbb{I}\}$ and for each n, $\mathcal{H}_n$ is the vector space of divergent 1PI n-loop Feynman diagrams and products of Feynman diagrams with overall loop number n. There is also another graduation parameter to build a graded Hopf algebra. We can show that $H_{\mathrm{FG}}(\Phi)$ is a graded Hopf algebra with respect to the number of internal edges of Feynman graphs such that the components of this grading have finite dimensions as the vector spaces. [16, 37, 87, 95]

The original version of the Kreimer's coproduct was defined in the language of parenthesized words to characterize nested, independent or overlapped sub-divergencies via sequences of letters and their linear combinations. It encapsulates the Bogoliubov–Zimmermann forest formula based on the formal expansion

$$\Delta_{\mathrm{FG}}(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \tag{2.10}$$

for each Feynman diagram Γ such that the sum is over all disjoint unions of 1PI divergent proper subgraphs. [33, 65, 66, 87]

Generally speaking, for a unital algebra (A, m, e) and a counital coalgebra (C, Δ, ε) over a field $\mathbb{K}$ of characteristic zero, let $\mathrm{Hom}(C, A)$ be the vector space of all $\mathbb{K}$-linear maps from C to A. Equip this space with a convolution product defined in terms of the following composition

$$C \longrightarrow^{\Delta} C \otimes C \longrightarrow^{f*g} A \otimes A \longrightarrow^{m} A \tag{2.11}$$

to achieve an algebra with the unit $e \circ \varepsilon$. A bialgebra $(H, m, e, \Delta, \varepsilon)$ in which the identity map id_H is invertible under the convolution product is a Hopf algebra. This particular inverse S which obeys the following property

$$\mathrm{id}_H * S = S * \mathrm{id}_H = e \circ \varepsilon \tag{2.12}$$

is called the antipode such that it is a unital algebra counital coalgebra antihomomorphism. [27, 66]

There are natural graduation parameters on Feynman diagrams such as number of internal edges or number of independent loops. The graduation parameter and the coproduct (2.10) determine the required antipode for the construction of a free commutative connected graded finite type Hopf algebra on Feynman diagrams of a given physical theory Φ. The antipode deforms Feynman rules characters to obtain renormalized values. The Hopf algebra $H_{\mathrm{FG}}(\Phi)$ has a Lie algebraic origin and in addition, it can be simplified via decorated rooted trees to provide a universal model for perturbative renormalization. The rooted tree version of the renormalization coproduct (2.10) can be defined in terms of the notion of admissible cut on trees. [24, 54, 56, 57, 65, 125]

The factorization of a Feynman diagram into its primitive components can be reversed under some conditions via the insertion operator which enables us to glue sub-diagrams. It is important to note that in gauge field theories we should work on a quotient of the renormalization Hopf algebra with respect to Ward identities and Slavnov–Taylor identities to achieve a unique factorization for each Feynman diagram with respect to the insertion operator. The insertion operator provides a Lie algebraic structure on Feynman diagrams such that the graded dual of its universal enveloping algebra will be equivalent to the renormalization Hopf algebra. [66, 91, 125, 156]

Theory of Hochschild cohomology for bialgebras is useful for us to formulate the Hochschild equation on Feynman diagrams which results a recursive formulation for the renormalization coproduct. For a given bialgebra H such as $H_{\mathrm{FG}}(\Phi)$, the dual of the coalgebra (H, Δ, ε) is an algebra H^* such that the unit map $\mathbb{I}$ of H transposes to a character $\mathbb{I}^t$ of H^*. Therefore we can build Hochschild cohomology groups $H^n(H, H^*)$ such that n-cochains are linear maps such as $T : H \longrightarrow H^{\otimes n}$. We can transpose them to n-linear maps such as $\rho_T : (H^*)^n \longrightarrow H^*$ where we have

$$\rho_T(\Gamma_1, ..., \Gamma_n) := T^t(\Gamma_1 \otimes ... \otimes \Gamma_n). \tag{2.13}$$

In this setting, the Hochschild coboundary operator $\mathbf{b}$ can be determined by the relation

$$< \Gamma_1 \otimes ... \otimes \Gamma_{n+1}, \mathbf{b}T(\Gamma) > := < \mathbf{b}\rho_T(\Gamma_1, ..., \Gamma_{n+1}), \Gamma > \tag{2.14}$$

for each $\Gamma \in H$. Define $\Delta_j : H^{\otimes n} \longrightarrow H^{\otimes(n+1)}$ as the homomorphism which applies the coproduct Δ only on the jth factor. Now we can show that

$$< \rho_T(\Gamma_1, ..., \Gamma_j\Gamma_{j+1}, ..., \Gamma_{n+1}), \Gamma > = < \Gamma_1 \otimes ... \otimes \Gamma_{n+1}, \Delta_j(T(\Gamma)) > . \tag{2.15}$$

It leads us to rewrite the Hochschild coboundary operator as the following way

$$\mathbf{b}T(\Gamma) := (\mathrm{id} \otimes T)\Delta(\Gamma) + \sum_{j=1}^{n} (-1)^j \Delta_j(T(\Gamma)) + (-1)^{n+1} T(\Gamma) \otimes \mathbb{I}. \tag{2.16}$$

The resulting cohomology groups $H^n(H^*, H^*_{\mathbb{I}^t})$ are indeed the Hochschild cohomology theory of the bialgebra H. It is easy to check that linear forms on H are 0-cochains and one cocycles are linear maps such as $l : H \longrightarrow H$ which obeys the following relation

$$\Delta(l) = l \otimes \mathbb{I} + (\mathrm{id} \otimes l)\Delta. \tag{2.17}$$

The Hochschild cohomology with values in a H-bimodule A (such as the regularization algebra) is defined by working on n-cochains via the vector space $C^n := C^n(H, A)$ consisting of n-linear maps $\psi : H^n \longrightarrow A$ with the H-bimodule structure

$$(\gamma_1.\psi.\gamma_2)(\Gamma_1, ..., \Gamma_n) := \gamma_1.\psi(\Gamma_1, ..., \Gamma_n).\gamma_2. \tag{2.18}$$

The coboundary map $\mathbf{b} : C^n \longrightarrow C^{n+1}$ is given by

$$\mathbf{b}(\psi)(\Gamma_1, ..., \Gamma_{n+1}) = \Gamma_1.\psi(\Gamma_2, ..., \Gamma_{n+1})$$

$$+ \sum_{j=1}^{n} (-1)^j \psi(\Gamma_1, ..., \Gamma_j\Gamma_{j+1}, ..., \Gamma_{n+1}) + (-1)^{n+1} \psi(\Gamma_1, ..., \Gamma_n).\Gamma_{n+1}. \tag{2.19}$$

The resulting cohomology groups are denoted by $\mathcal{H}^n(H, A)$.

Let us consider the Hochschild equation for the algebra $\mathbb{K}[X]$ which is also equipped with a cocommutative coalgebra structure by considering the indeterminate X as the primitive object where $\varepsilon(X) = 0$. For all $k \geq 2$, by induction, we can show that

$$\Delta(X^k) = (\Delta X)^k = \sum_{j=0}^{k} \binom{k}{j} X^{k-j} \otimes X^j. \tag{2.20}$$

For any linear form ϱ on $\mathbb{K}[X]$, we have

$$\mathbf{b}\varrho(X^k) = (\mathrm{id} \otimes \varrho)\Delta(X^k) - \varrho(X^k) \otimes \mathbb{I} = \sum_{j=1}^{k} \binom{k}{j} \varrho(X^{k-j})X^j. \tag{2.21}$$

$\mathbf{b}\varrho$ is a linear transformation of polynomials which does not increase the degree. It shows that the integration map $T(X^k) := X^{k+1}/(k+1)$ is not a 1-coboundary but it is an one cocycle.

Thanks to this Hochschild cohomology theory, it is possible to define the renormalization coproduct under a recursive setting.

A graded bialgebra H over a field $\mathbb{K}$ is graded as an algebra and as a coalgebra. It is called connected if the degree zero component of the graduation structure consists of scalars (i.e. elements of the field $\mathbb{K}$). We have

$$H = \bigoplus_{n\geq 0} H_n, \quad H_m H_n \subset H_{m+n}, \quad \Delta(H_n) \subset \bigoplus_{p+q=n} H_p \otimes H_q. \tag{2.22}$$

The coproduct in a connected graded bialgebra can be presented in terms of the Sweedler notation such that for $\Gamma \in H^{(n)}$, we have

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum \Gamma'_1 \otimes \Gamma'_2 \tag{2.23}$$

where terms Γ'_1 and Γ'_2 all have degrees between 1 and $n-1$. The counit equations

$$\sum \varepsilon(\Gamma'_1)\Gamma'_2 = \sum \Gamma'_1 \varepsilon(\Gamma'_2) = \Gamma, \sum S(\Gamma'_1)\Gamma'_2 = \sum \Gamma'_1 S(\Gamma'_2) = \varepsilon(\Gamma)\mathbb{I} \tag{2.24}$$

tell us that $\Delta(\Gamma)$ must contain terms $\Gamma \otimes \mathbb{I} \in H^{(n)} \otimes H^{(0)}$ and $\mathbb{I} \otimes \Gamma \in H^{(0)} \otimes H^{(n)}$ and the remaining terms which have intermediate bidegrees. They address the equation

$$\Gamma = (\varepsilon \otimes \mathrm{id})(\Delta(\Gamma)) = \varepsilon(\Gamma)\mathbb{I} + \Gamma + \sum \varepsilon(\Gamma'_1)\Gamma'_2 \tag{2.25}$$

which leads us to the relation $\varepsilon(\Gamma) = 0$ for every non-trivial Feynman diagram. Therefore the augmentation ideal is given by

$$\mathrm{Ker}\varepsilon = \bigoplus_{n=1}^{\infty} H_n. \tag{2.26}$$

For $P := \mathrm{id} - \mathbb{I}\varepsilon$ as the projector onto the augmentation ideal, define $\mathrm{Aug}^m := (P \otimes ...^{m \text{ times}} ... \otimes P)\Delta^{m-1}$ and then set $H^m := \mathrm{Aug}^{m+1}/\mathrm{Aug}^m$ for all $m \geq 1$. This gives us the bigraded structure

$$H = \bigoplus_{n\geq 0} H_n = \bigoplus_{m\geq 0} H^m \tag{2.27}$$

such that for all $k \geq 1$

$$H_k \subset \bigoplus_{j=1}^{k} H^j, \quad H_0 \simeq H^0 \simeq \mathbb{K}. \tag{2.28}$$

In addition, the graded structure on the bialgebra of Feynman diagrams allows us to define a recursive formulation for the antipode

$$S(\Gamma) = -\Gamma - \sum S(\Gamma_1')\Gamma_2' \tag{2.29}$$

which can be applied to show the existence of a convolution inverse for the identity map. Number of internal edges or number of independent loops can be applied as the graduation parameters on the renormalization bialgebra which leads us to formulate its antipode inductively. [53, 65, 66, 90, 91, 174]

2.3 *Renormalization Hopf algebra of Feynman graphons and filtration of large Feynman diagrams*

In this section we present a new discovered interrelationship between theory of infinite graphs and the fundamental structure of Green's functions. We apply the rooted tree representation of Feynman diagrams to define a new graph function representation for these physical diagrams which leads us to make a new interpretation of solutions of Dyson–Schwinger equations with respect to partial sums and cut-distance topology. Thanks to this treatment, we will show the structure of a new Hopf algebra of graphons which contribute to Feynman diagrams as a result of the enrichment of the renormalization coproduct. In addition, we explain the construction of a filtration on large Feynman diagrams.

The representation of Feynman diagrams in the language of decorated rooted trees and the recursive nature of the renormalization coproduct are the key tools to build a new Hopf algebra $H_{\rm CK}$ of non-planar rooted trees. This combinatorial Hopf algebra together with the grafting operator B^+ has a universal property with respect to Hochschild cohomology theory in a category of commutative Hopf algebras.

On the one hand, Dyson–Schwinger equations are reformulated in terms of the renormalization Hopf algebra and the grafting operator. The unique solution of each equation DSE with the general form (1.9) is an infinite formal expansion of Feynman diagrams. In physical theories with weaker coupling constants, we can expect to achieve some finite values by applying many-loop computation techniques under perturbative setting. In physical theories with strong enough couplings, it is expected to deal with infinite expansions of finite Feynman diagrams such as $\sum_{n\geq 0}(\lambda g)^n X_n$ such that g is the bare coupling constant while λg is any running coupling or a re-scaled version of these values. These expansions are actually the solutions of equations of motion in Quantum Field Theory where we deal with Dyson–Schwinger equations which encode the self-similar nature of Green's functions. [92]

On the other hand, graph limits, as a modern branch in infinite combinatorics, study limits of finite combinatorial objects such as weighted or directed graphs, multi or hyper graphs, bipartite graphs and posets. Theory of graphons and random graphs is one of the recent progress in infinite combinatorics where we deal with (symmetric) measurable functions such as W defined on the probability space Ω. Actually, a graph limit, as the convergent limit of an infinite sequence of graphs, can be represented by a graphon which does not have necessarily the unique representation. The key tool which allows us to concern convergence and equivalence of graphons is the concept of cut-metric. [107]

Thanks to these information, it is reasonable to think about any relationship between solutions of combinatorial Dyson–Schwinger equations and theory of graphons. This idea has already been discussed in [148, 149] where these infinite complicated Feynman graphs are interpreted in the language of graph functions and cut-distance topology. Now we plan to develop our studies and build a new Hopf algebra structure on the space of all Dyson–Schwinger equations in a given physical theory Φ with strong bare coupling. As we know, Dyson–Schwinger equations are polynomials with respect to the coupling constants where by changing the running couplings or rescaling the bare coupling, the behavior of these equations can be changed. For a fixed bare coupling constant g, define $\mathcal{S}^{\Phi}(\lambda g)$ as the set of all Dyson–Schwinger equations with the general form $\mathrm{DSE}(\lambda g)$ which has the unique solution $X_{\mathrm{DSE}(\lambda g)} = \sum_{n\geq 0}(\lambda g)^n X_n$. The rescaling parameter λ has the values in $(0,1]$ which enables us to define the family $\mathcal{S}^{\Phi,g} = \bigcup_{\lambda} \mathcal{S}^{\Phi}(\lambda g)$. It contains all equations DSEs in different rescalings of the bare coupling in the physical theory.

Generally speaking, theory of graph limits aims to assign a limit to a sequence of finite graphs such as $\{G_n\}_{n\geq 0}$ when number of vertices of graphs in the sequence tends to infinity. There are some different approaches to define the concept of convergence at this level but the one approach which is based on random graphs and cut-distance topology is very useful. We can say that a sequence $\{G_n\}_{n\geq 0}$ of finite graphs is convergent when $|G_n|$ tends to infinity, if for each fixed value k, the distribution of the random graph $G_n[k]$ converges when n tends to infinity. In this setting, $G_n[k]$ is a labeled subgraph of G_n with vertices $1, ..., k$ obtained by selecting k distinct vertices $v_1, ..., v_k \in G_n$ under a uniformly random process. Graph limits can be also interpreted as equivalence classes of convergent sequences of finite graphs. There are abstract objects, known as graphons, which allow us to study graph limits. For a given probability space Ω, graphons are bounded measurable symmetric functions $W : \Omega \times \Omega \longrightarrow [0,1]$. The symmetric condition can be removed when we work on another class of graphons known as bigraphons. Since we plan to explain the graphon representation of Feynman diagrams in the language of rooted trees, in some situations we need to address the decorations of trees (such as direction from the root to other vertices) where it is useful to consider non-symmetric graphons, otherwise we can work on the symmertic version. In addition, the graphon representation of graph limits is not unique and it depends on selecting decorations and other presentation parameters. We use the notation "labeled graphons" in this situation. We can address pixel pictures as the most common examples of labeled graphons for the description of graph limits which are built by the adjacency matrix.

A map $\rho : \Omega_1 \longrightarrow \Omega_2$ between two probability spaces $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ is called measure preserving if it is measurable and $\mu_1(\rho^{-1}(A)) = \mu_2(A)$ for each measurable set $A \in \mathcal{F}_2$. ρ is called measure preserving bijection if it is a bijection map and ρ, ρ^{-1} are measure preserving. It is easy to check that for a given measure preserving map ρ, the map $\rho \otimes \rho : \Omega_1^2 \longrightarrow \Omega_2^2$ defined by $\rho \otimes \rho(x,y) := (\rho(x), \rho(y))$ is also a measure preserving map. If ρ is a bijection, then f^ρ, W^ρ are called rearrangements of f (as a function on Ω_2) and W (as a function on Ω_2^2). Actually, relabeling of labeled graphons can be understood as a kind of rearrangement. In other words, for a given measure preserving map ρ, the pull backs of f and W are defined by

$$f^\rho(x) := f(\rho(x)), \quad W^\rho(x,y) := W(\rho(x), \rho(y)). \tag{2.30}$$

If $f \in L^1(\Omega_2)$ and $W \in L^1(\Omega_2^2)$, then $\| f^\rho \|_1 = \| f \|_1$ and $\| W^\rho \|_1 = \| W \|_1$. [77, 107]

Definition 2.3.1. An unlabeled graphon is a graphon up to relabeling such that a relabeling is defined by an invertible measure preserving transformation of the unit interval.

For a given labeled graphon W, its corresponding unlabeled graphon class $[W]$ is given by

$$[W] := \{W^\rho : (x,y) \longmapsto W(\rho(x), \rho(y)) : \quad \rho \text{ is an arbitrary rearrangement}\}. \tag{2.31}$$

Set $\mathcal{W}(\Omega)$ as the set of all labeled graphons on a given probability space Ω. It is not difficult to see that if $[0,1]$ is the probability space, then $\mathcal{W}(\Omega)$ is the subspace of symmetric functions in $L_\infty([0,1]^2)$. By defining a suitable equivalence relation on labeled graphons, which encodes exchanging decorations, it is possible to associate a unique graphon class to each graph limit. This graphon class is called unlabeled graphon. Set $[\mathcal{W}](\Omega)$ as the family of all unlabeled graphons on a given probability space Ω.

Graphons, as edge weighted graphs on the vertex set $[0,1]$, provide a generalization of common graphs. We can show that each finite simple graph G defines naturally an unlabeled graphon class $[W_G]$. First we can build a labeled graphon W_G with respect to the information of the adjacency matrix. Consider $V(G)$ as a probability space such that each vertex has probability $\frac{1}{|G|}$. Define the map $W^1_G : V(G) \times V(G) \longrightarrow [0,1]$ as follows

$$W^1_G(u,v) := \{^{1, \text{ if u and v are adjacent}}_{0, \text{ otherwise}}. \tag{2.32}$$

It is easy to see that W^1_G is a symmetric measurable function. In an alternative setting, we can also consider $\Omega = (0,1]$ as the probability space which is equipped with a partition $\{I^n_i\}_{i=1,...,n}$ such that $I^n_i := (\frac{i-1}{n}, \frac{i}{n}]$. If the vertices of G is labeled by $1,..,n$, then the corresponding graph function W^2_G is given by

$$W^2_G(x,y) = W^1_G(i,j) = 1, \quad x \in I^n_i, y \in I^n_j. \tag{2.33}$$

Homomorphism density is one important concept in dealing with the probability of the existence of a subgraph in an extremely large graph. For a given finite graph G, the homomorphism density of each subgraph H into G is given by

$$t(H,G) := \frac{\hom(H,G)}{|V(G)|^{|V(H)|}} \tag{2.34}$$

such that $\hom(H,G)$ is the number of graph homomorphisms from H into G. It is possible to generalize this idea for the level of graph limits where this parameter informs the density of H as a subgraph in G asymptotically when the number of vertices of G tends to infinity. For a given graphon $W : \Omega^2 \longrightarrow [0,1]$ and a simple finite graph H, the homomorphism density is defined by

$$t(H,W) := \int_{\Omega^{|H|}} \prod_{ij \in E(H)} W(x_i,x_j) d\mu(x_1)...d\mu(x_{|H|}). \tag{2.35}$$

If the graphon W_G is a labeled graphon with respect to a given graph G, then we have $t(H,G) := t(H,W_G)$. The homomorphism density provides another alternative way to describe convergence. It is shown that the sequence $\{G_n\}_{n\geq 0}$ converges to the labeled graphon W iff the sequence $\{t(H,G_n)\}_{n\geq 0}$ of homomorphism densities converges to the homomorphism density $t(H,W)$ for each simple subgraph H. [77, 107]

The space of graphons provides the completion of the space of finite graphs with respect to a topology generated by a particular metric namely, cut-distance. The cut-distance between labeled graphons W,U is defined by

$$\delta_{\text{cut}}(W,U) := \inf_{\rho,\tau} \sup_{S,T} |\int_{S\times T} W(\rho(x),\rho(y)) - U(\tau(x),\tau(y)) dx dy| \tag{2.36}$$

such that the infimum is taken over all relabelings ρ of W and τ of U and the supremum is taken over all measurable subsets S,T of the closed interval.

The infimum over relabeling allows us to define the cut-distance on the space of unlabeled graphons.

Two graphons W, U are called weakly equivalent $U \approx W$ iff for all finite subgraphs H, we have $t(H, W) = t(H, U)$. It is equivalent to say that there exists a graphon V and measure-preserving maps ρ, τ (as relabelings) such that $V^{\rho} = W$ and $V^{\tau} = U$ almost everywhere. Therefore two weakly equivalent graphons have the same corresponding symmetric measurable functions almost everywhere. The distance (2.36) does not distinguish between weakly isomorphic graphons.

Theorem 2.3.2. *Each graphon is the cut-distance convergent limit of a sequence of finite graphs. In addition, the cut-distance* δ_{cut} *determines a compact topological structure on the space* $\mathcal{W}(\Omega)/\approx$ *of labeled graphons up to the weakly equivalent relation. [77, 107]*

It is the place to deal with a new application of graph limits to Quantum Field Theory where we aim to achieve a new interpretation of Feynman diagrams and their corresponding formal expansions. In general, a Feynman diagram, as a weighted graph decorated by some physical parameters, might have many nested or overlapping loops. Infinite series of many-loop Feynman diagrams, which appear in theories with strong couplings, are actually good examples of complicated dense graphs. Theory of graphons might be our best chance for the handling of these strange objects in the context of infinite combinatorial tools. For this purpose we apply rooted tree representations of Feynman diagrams to reach to a new graph function interpretation.

Lemma 2.3.3. *The algebraic combinatorics of each Feynman diagram can be encoded by a unique unlabeled graphon class.*

Proof. A rooted tree t is a finite, connected oriented graph without loops in which every vertex has exactly one incoming edge, except one namely, the root which has no incoming but only outgoing edges. We can put two classes of decorations on each tree namely, vertex-labeled and edge-labeled. The rooted tree representations of Feynman diagrams can be defined via the grafting operator. The free commutative algebra generated by isomorphism classes of non-planar rooted trees is actually the polynomial algebra generated by symbols t where each symbol represents one isomorphism class. The concatenation is the product and the empty tree is the unit for this polynomial algebra. In addition, this polynomial algebra can be equipped by a modified version of the renormalization coproduct given by

$$\Delta_{\text{CK}}(t) = \mathbb{I} \otimes t + t \otimes \mathbb{I} + \sum_{c} R_c(t) \otimes P_c(t) \tag{2.37}$$

such that the sum is taken over all admissible cuts c on t which divides the tree into two parts. The part $R_c(t)$ contains the original root of t and the part

$P_c(t)$ is a forest of subtrees. The resulting Hopf algebra H_{CK} of non-planar rooted trees is connected graded free commutative non-cocommutative finite type. Decorations enable us to adapt this combinatorial Hopf algebra with respect to physical theories. Each graph Γ, which might contain divergent subgraphs, is encoded by a decorated non-planar rooted tree t_Γ such that the root represents the full graph and each leaf is a divergent subgraph which has no further subdivergencies. If the original graph has overlapping subdivergencies, then we can replace the single rooted tree by a sum of decorated rooted trees after disentangling the overlaps. Thanks to these rules, we can embed the Hopf algebra $H_{\mathrm{FG}}(\Phi)$ of Feynman diagrams of Φ into the decorated Connes–Kreimer Hopf algebra $H_{\mathrm{CK}}(\Phi)$ as a closed Hopf subalgebra. This embedding is encapsulated by the injective Hopf algebra homomorphism

$$\Gamma \longmapsto \Xi(\Gamma) := \sum_{j=1}^{r} B^{+}_{\Gamma_j, G_{j,i}}\big(\prod_{i=1}^{k_j} \Xi(\gamma_{j,i})\big) \tag{2.38}$$

such that $\Gamma = \prod_{i=1}^{k_j} \Gamma_j \star_{j,i} \gamma_{j,i}$ and $G_{j,i}$'s are the gluing information. For a given decorated non-planar rooted tree t, if the longest path from the root to a leaf contains k edges, then the renormalization coproduct $\Delta_{\mathrm{CK}}(t)$ is a sum of at least $k+1$ terms. In other words, the decorated non-planar rooted tree t represents an iterated integral with k nested sub-divergencies while each vertex corresponds to a sub-integral without any sub-divergencies. [54, 56, 57, 65, 74]

Therefore each Feynman diagram with nested loops can be represented by a labeled rooted tree where the root is the symbol for the original graph and other vertices are symbols of nested loops. Edges among vertices determine the positions of nested loops with respect to each other. In addition, it is possible to represent Feynman diagrams with overlapping divergencies with rooted trees where we should deal with linear combinations of decorated rooted trees. [65, 66, 88, 91]

For a Feynman diagram Γ without overlapping sub-divergencies, the decorated tree $t_\Gamma := \Xi(\Gamma)$ is a simple finite weighted graph where thanks to its corresponding adjacency matrix we can determine the labeled graphons W_{t_Γ} of the form (2.32) or (2.33). Set $[W_{t_\Gamma}]$ as the unlabeled graphon class associated to t_Γ. The definition (2.31) of the class [.] guarantees the uniqueness of the unlabeled graphon $[W_{t_\Gamma}]$. Thanks to the embedding (2.38), $[W_{t_\Gamma}]$ is the unique unlabeld graphon class of the Feynman diagram Γ.

For a Feynman diagram Γ which has some overlapping sub-divergencies, $u_\Gamma := \Xi(\Gamma)$ is a linear combination of decorated non-planar rooted trees. In this situation, the labeled graphons such as W_{u_Γ} can be determined by normalizing the combination of labeled graphons $W_{t_1}, ..., W_{t_n}$ as the following

way
$$W_{u_\Gamma}(x,y) := \frac{W_{t_1} + ... + W_{t_n}}{|W_{t_1} + ... + W_{t_n}|}. \tag{2.39}$$

□

Definition 2.3.4. A sequence $\{\Gamma_n\}_{n\geq 0}$ of Feynman diagrams is called convergent when n tends to infinity, if the corresponding sequence $\{[W_{t_{\Gamma_n}}]\}_{n\geq 0}$ of unlabeled graphon classes is convergent with respect to the cut-distance topology when n tends to infinity.

Suppose the unlabeled graphon class $[W]$ is the convergent limit for the sequence $\{[W_{t_{\Gamma_n}}]\}_{n\geq 0}$. If we consider the pixel picture representation of the graphon $[W]$, then we can associate an infinite tree or forest t such that $W_t \in [W]$ and $W \in [W_t]$. Therefore $[W] = [W_t]$. Thanks to the homomorphism (2.38), it is possible to build an extremely large Feynman diagram Γ_t with respect to the infinite tree or forest t. This Γ_t can be described as the convergent limit of the sequence $\{\Gamma_n\}_{n\geq 0}$ with respect to the cut-distance topology. We can also show that this limit is unique up to the weakly equivalent relation.

Finding a new connection between random graphs and Feynman diagrams is the immediate consequence of the graph function representation of these physical theories.

The study of random graphs was begun by Erdos, Renyi and Gilbert when they were working on a probabilistic construction of a graph with large girth and large chromatic number. After a short period of time, work on random graphs $G_{n,m}$ has been concerned by many mathematicians in Combinatorics and Discrete Mathematics. Nowadays it is not difficult to observe various applications of these combinatorial objects in many fields in Mathematics and other applied sciences. Generally speaking, theory of random graphs aims to provide some results such as "a combinatorial property A almost always implies another combinatorial property B". Generally speaking, let n be an integer and $0 \leq p \leq 1$, a random graph $G(n,p)$ is defined by taking n nodes and connecting any two of them with the probability p, making an independent decision about each pair. There are alternative ways to achieve random graphs. As an example, consider $\mathcal{L}_{n,m}$ as the collection of all labeled graphs with vertex set $V = [n] = \{1, 2, ..., n\}$ and m edges such that $0 \leq m \leq \binom{n}{2}$. To each $G \in \mathcal{L}_{n,m}$, assign a probability

$$\mathbb{P}(G) = \frac{1}{\binom{\binom{n}{2}}{m}}. \tag{2.40}$$

In other words, start with an empty graph on the set $[n]$ and insert m edges in such a way that all possible $\binom{\binom{n}{2}}{m}$ choices are equally likely. The resulting graph $G_{n,m} := ([n], E_{n,m})$ is known as the uniform random graph.

As other example, fix $0 \leq p \leq 1$ and for each graph G with vertex set $[n]$ and $0 \leq m \leq \binom{n}{2}$ edges, assign the following probability

$$\mathbb{P}(G) = p^m (1-p)^{\binom{n}{2}-m}. \tag{2.41}$$

In other words, start with an empty graph with the vertex set $[n]$ and consider $\binom{n}{2}$ to insert edges independently with probability p. The resulting graph $G_{n,p} := ([n], E_{n,p})$ is known as the binomial random graph.

It is shown that the random graph $G_{n,p}$ with $0 \leq m \leq \binom{n}{2}$ edges is the same as one of the $\binom{\binom{n}{2}}{m}$ graphs that have m edges. For enough large n, random graphs $G_{n,m}$ and $G_{n,p}$ have the same behavior whenever the number of edges m in $G_{n,m}$ is very close to the expected number of edges of $G_{n,p}$ in the following means

$$m = \binom{n}{2} p \approx \frac{n^2 p}{2}. \tag{2.42}$$

It is equivalent to say that the edge probability in $G_{n,p}$ should be $p \approx \frac{2m}{n^2}$. [67]

Lemma 2.3.5. *Each labeled graphon determines a class of random graphs.*

Proof. If we have a simple weighted graph G, then we can build a random simple graph $R(G)$ by including the edge with probability equal to its weight. Thanks to this idea, suppose we have a labeled graphon W and finite subset $S := \{s_1, ..., s_n\}$ in $[0,1]$. We can make a weighted graph $G(S,W)$ with $|S| = n$ nodes such that the edge $s_i s_j$ has the weight $W(s_i, s_j)$. In general, the random graph $R(n,W) := R(G(S,W))$ with respect to the weighted graph $G(S,W)$ is our promising graph such that S is a set of n points which are selected independently from the closed interval. □

The random graphs $R(n,W)$ have the ability to approximate graphons W associated to large numbers of points in the closed interval. It is shown that with probability 1, the sequence $\{R(n,W)\}_{n\geq 0}$ is convergent to the graphon W with respect to the cut-distance topology when n tends to infinity. [107]

Thanks to the discussed topics, it is time to observe some new applications of the graph function representations of Feynman diagrams in dealing with expansions of these physical graphs in Quantum Field Theory. At the first step we address a new interpretation of Dyson–Schwinger equations in the context of random graphs.

Theorem 2.3.6. *The unique solution of each combinatorial Dyson–Schwinger equation can be described as the cut-distance convergent limit of a sequence of finite Feynman diagrams.*

Proof. The full proof is given in [149] and here we only address the main idea. Suppose DSE be a combinatorial Dyson–Schwinger equation with the general form (1.9) such that its unique solution is given by

$$X_{\mathrm{DSE}} = \sum_{n \geq 0} (\lambda g)^n X_n \tag{2.43}$$

such that g is the bare coupling constant and generators X_n are determined by the recursive relations (1.10). Make the new sequence $\{Y_m\}_{m \geq 1}$ of partial sums of the expansion $\sum_{n \geq 0} g^n X_n$ such that we have

$$Y_m := (\lambda g)^1 X_1 + ... + (\lambda g)^m X_m. \tag{2.44}$$

It is shown in [9, 61] that the expression X_{DSE} belongs to a completion of $H_{\mathrm{FG}}[[g]]$ with respect to the n-adic topology. We claim that the sequence $\{Y_m\}_{m \geq 1}$ of finite graphs converges to the large Feynman diagram X_{DSE} with respect to the cut-distance topology. For this purpose, we can apply the n-adic metric and the graphon representations of the components X_n to build a random graph with respect to each graph Y_m. It leads us to associate a sequence $\{R(Y_m)\}_{m \geq 1}$ of random graphs with respect to the sequence $\{Y_m\}_{m \geq 1}$ which is cut-distance convergent to the large graph X_{DSE}. □

The structure of a modification of the Connes–Kreimer BPHZ renormalization for large Feynman diagrams has been formulated in [149] where we worked on a topological completion of the renormalization Hopf algebra of Feynman diagrams with respect to the cut-distance topology. As the second application of the theory of graphons, we are going to develop this formalism and build a renormalization program on the collection $\mathcal{S}^{\Phi,g}$ under a Hopf algebraic setting. This new approach enables us to proceed our knowledge about non-perturbative versions of Feynman rules which act on large Feynman diagrams. For this purpose we explain the structure of a new Hopf algebra derived from the renormalization coproduct on graphons.

Theorem 2.3.7. *Thanks to the renormalization coproduct, there exists a topological Hopf algebraic structure on the collection $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ of all unlabeled graphons which contribute to represent (large) Feynman diagrams of a physical theory Φ.*

Proof. We plan to equip $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ with an enriched version of the renormalization Hopf algebra which is completed with respect to the cut-distance topology.

Thanks to Lemma 2.3.3, for each finite Feynman diagram Γ, we associate the unlabeled graphon class $[W_\Gamma]$. In addition, the unique solution of each combinatorial Dyson–Schwinger equation DSE in $\mathcal{S}^{\Phi,g}$ determines a unique large Feynman diagram X_{DSE} such that thanks to Theorem 2.3.6, this infinite graph can be interpreted as the convergent limit of the sequence of partial sums with respect to the cut-distance topology. Therefore it does make sense to replace objects of $\mathcal{S}^{\Phi,g}$ with large Feynman diagrams such as X_{DSE} as the unique solution of the equation DSE. Thanks to Lemma 2.3.3, we associate a unique unlabeled graphon class $[W_{t_{X_{\mathrm{DSE}}}}]$ to the large Feynman diagram X_{DSE} via the rooted tree representation of Feynman diagrams. For simplicity in the presentation, from now we use the notation $[W_{X_{\mathrm{DSE}}}]$ for this graphon class.

It is possible to lift the renormalization coproduct (2.10) onto the level of unlabeled graphons which contribute to the description of (large) Feynamn diagrams. For a given finite Feynman diagram Γ with the corresponding unlabeled graphon $[W_\Gamma]$, define

$$\Delta_{\mathrm{graphon}}([W_\Gamma]) := \sum [W_\gamma] \otimes [W_{\Gamma/\gamma}] \tag{2.45}$$

such that the sum is taken over all unlabeled graphon classes such as $[W_\gamma]$ associated to γ as the disjoint union of 1PI superficially divergent subgraphs of Γ.

Thanks to Theorem 2.3.6, for the unlabeled graphon class $[W_{X_{\mathrm{DSE}}}]$ corresponding to the large Feynman diagram X_{DSE}, define its coproduct as the convergent limit of the sequence $\{\Delta_{\mathrm{graphon}}([W_{Y_m}])\}_{m\geq 1}$ of the coproducts of the finite partial sums with respect to the cut-distance topology.

Now we can adapt (2.45) for the level of large Feynman diagrams and define

$$\Delta_{\mathrm{graphon}}([W_{X_{\mathrm{DSE}}}]) := \sum [W_\Upsilon] \otimes [W_{X_{\mathrm{DSE}}/\Upsilon}] \tag{2.46}$$

such that the sum is taken over all unlabeled graphon classes such as $[W_\Upsilon]$ associated to Υ as the disjoint union of 1PI superficially divergent subgraphs of X_{DSE}.

If we consider objects of $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ as generators of a free commutative algebra, then thanks to (2.45) we obtain a bialgebra structure on Feynman graphons which is graded in terms of the number of independent loops of the corresponding Feynman diagrams. The unlabeled graphon class $[W_{\mathbb{I}}]$ corresponding to the empty graph is the unit for this bialgebra. The counit is also defined by

$$\tilde{\varepsilon}([W_\Gamma]) = \begin{cases} 1, & [W_\Gamma]=[W_{\mathbb{I}}] \\ 0, & \text{else} \end{cases}. \tag{2.47}$$

The existence of the graduation parameter is the key tool to define an antipode map. For each finite Feynman diagram Γ, we have

$$S_{\mathrm{graphon}}([W_\Gamma]) = -[W_\Gamma] - \sum S([W_{\gamma_{(1)}}])[W_{\gamma_{(2)}}] \tag{2.48}$$

such that $\Delta_{\mathrm{graphon}}([W_\Gamma]) = \sum [W_{\gamma_{(1)}}] \otimes [W_{\gamma_{(2)}}]$.

Thanks to Theorem 2.3.6, for the unlabeled graphon class $[W_{X_{\mathrm{DSE}}}]$ corresponding to the large Feynman diagram X_{DSE}, define its antipode as the convergent limit of the sequence $\{S_{\mathrm{graphon}}([W_{Y_m}])\}_{m\geq 1}$ of unlabeled graphons of finite partial sums Y_m with respect to the cut-distance topology. Since partial sums are finite graphs, their corresponding graphon type antipodes $S_{\text{graphon}}([W_{Y_m}])$ can be obtained inductively by the coproduct Δ_{graphon} where we have

$$S_{\text{graphon}}([W_{Y_m}]) = -[W_{Y_m}] - \sum S([W_{\Gamma_{(1)}}])[W_{\Gamma_{(2)}}] \tag{2.49}$$

such that $\Delta_{\text{graphon}}([W_{Y_m}]) = \sum [W_{\Gamma_{(1)}}] \otimes [W_{\Gamma_{(2)}}]$.

Now we can adapt the antipode (2.48) for the level of large Feynman diagrams and define

$$S_{\text{graphon}}([W_{\text{DSE}}]) = -[W_{\text{DSE}}] - \sum S([W_{\Upsilon_{(1)}}])[W_{\Upsilon_{(2)}}] \tag{2.50}$$

such that $\Delta_{\text{graphon}}([W_{\text{DSE}}]) = \sum [W_{\Upsilon_{(1)}}] \otimes [W_{\Upsilon_{(2)}}]$.

Therefore $\mathcal{S}^{\Phi}_{\text{graphon}}$ becomes a connected graded free commutative non-cocommutative (not necessarily finite type) Hopf algebra. In addition, the constructions of the coproduct (2.46) and the antipode (2.50) guarantee the compatibility of this Hopf algebraic structure with the cut-distance topology. In addition, $\mathcal{S}^{\Phi}_{\text{graphon}}$ is completed with respect to this topology. □

We use the phrase "Feynman graphons" to address the objects of $\mathcal{S}^{\Phi}_{\text{graphon}}$.

Corollary 2.3.8. *Let V be a complex vector space with a basis labeled by coupling constants of a given Quantum Field Theory Φ, and suppose* $\text{Diff}(V)$ *be the group of formal diffeomorphisms of V tangent to the identity at $0 \in V$ and $H_{\text{diff}}(V)$ be its corresponding Hopf algebra. The complex Lie group $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$ of characters on Feynman graphons can be represented by* $\text{Diff}(V)$.

Proof. The Hopf algebra $H_{\text{diff}}(\mathbb{C})$ of formal diffeomorphisms of $\mathbb{C}$ tangent to the identity has generators such as a_n which play the role of coordinates of

$$\phi(x) = x + \sum_{n \geq 2} a_n(\phi) x^n \tag{2.51}$$

such that ϕ is a formal diffeomorphism satisfying $\phi(0) = 0$, $\phi'(0) = \text{id}$. Its coproduct is given by

$$\Delta(a_n)(\phi_1 \otimes \phi_2) = a_n(\phi_2 \circ \phi_1). \tag{2.52}$$

We can define a Hopf algebra homomorphism $\Psi : H_{\text{diff}}(V) \longrightarrow H_{\text{FG}}(\Phi)$ with the corresponding dual group homomorphism $\hat{\Psi} : \mathbb{G}_{\Phi}(\mathbb{C}) \longrightarrow \text{Diff}(V)$. The map Ψ maps the coefficients of the expansion of formal diffeomorphisms to the coefficients in the renormalization Hopf algebra of the expansion of the effective coupling constants of theory as formal power series in the bare coupling constants. As the consequence, for each Dyson–Schwinger equation DSE with the corresponding Hopf subalgebra H_{DSE} and Lie subgroup

$\mathbb{G}_{\rm DSE}(\mathbb{C})$, we can define a group homomorphism $\hat{\Psi}_{\rm DSE}$ from $\mathbb{G}_{\rm DSE}(\mathbb{C})$ to $\mathrm{Diff}(V)$. [37, 144]

Thanks to Lemma 2.3.3, we can embed $H_{\rm FG}(\Phi)$ into the renormalization Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$ of Feynman graphons. This allows us to lift the map Ψ onto the level of Feynman graphons and build a new Hopf algebra homomorphism $\overline{\Psi} : H_{\rm diff}(V) \longrightarrow \mathcal{S}^{\Phi}_{\rm graphon}$ with the corresponding dual group homomorphism $\hat{\overline{\Psi}} : \mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C}) \longrightarrow \mathrm{Diff}(V)$. □

The construction of a canonical filtration on terms X_ns of the unique solution of a Dyson–Schwinger equation has been explained in [103] where each filtered term maps to a certain power of L in the log-expansion. The original idea is to filter images of Feynman diagrams in a particular universal enveloping algebra which generates a quasi-shuffle type Hopf algebra. Thanks to Theorem 2.3.6 and Theorem 2.3.7, we aim to adapt this filtration for large Feynman diagrams.

Theorem 2.3.9. *Renormalized Feynman rules characters of the Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$ filtrate large Feynman diagrams.*

Proof. Set $H_{\rm word}$ as the vector space of words which contains $H_{\rm letter}$ as the subspace of letters. Set a commutative associative map $\Theta : H_{\rm letter} \times H_{\rm letter} \longrightarrow H_{\rm letter}$ as the Hoffman pairing which sends two generators a, b to another generator $\Theta(a, b)$ and adds degrees. Define the generalized quasi-shuffle product $\ominus_\Theta$ on $H_{\rm word}$ as follows

$$au \ominus_\Theta bv := a(u \ominus_\Theta bv) + b(au \ominus_\Theta v) + \Theta(a, b)(u \ominus_\Theta v) \tag{2.53}$$

which builds a commutative associative algebra with empty word $\mathbb{I}$ as the unit. We can equip this algebra with the following coproduct structure

$$\Delta_{\rm word}(w) = \sum_{vu=w} u \otimes v \tag{2.54}$$

which gives us a bialgebra structure on $H_{\rm word}$ with the counit $\hat{\mathbb{I}}_{\rm word}$. The length of each word determines a natural graduation parameter on this bialgebra which leads us to define an antipode recursively. As the consequence, $(H_{\rm word}, \ominus_\Theta, \mathbb{I}, \Delta_{\rm word}, \hat{\mathbb{I}}_{\rm word}, S_{\rm word})$ is a graded connected commutative unital non-cocommutative counital Hopf algebra [55, 73]. We have

$$\ominus_\Theta \circ (S_{\rm word} \otimes \mathrm{id}) \circ \Delta_{\rm word} = \ominus_\Theta \circ (\mathrm{id} \otimes S_{\rm word}) \circ \Delta_{\rm word} = \mathbb{I}_{\rm word} \circ \hat{\mathbb{I}}_{\rm word}. \tag{2.55}$$

In this setting, the grafting operator on words allows us to add a letter to the first place

$$B^+_a(u) := au. \tag{2.56}$$

We can check that for each a, the grafting operators are Hochschild one-cocycles. It is possible to embed the renormalization Hopf algebra of Feynman diagrams into the Hopf algebra of words. This embedding is defined in terms of the following homomorphism $\nu : H_{\rm FG}(\Phi) \longrightarrow H_{\rm word}$

$$\nu(\mathbb{I}) = \mathbb{I}_{\rm word}, \quad \ominus_\Theta \circ (\nu \otimes \nu) = \nu \circ m, \quad \hat{\mathbb{I}}_{\rm word} \circ \nu = \nu \circ \hat{\mathbb{I}},$$

$$\Delta_{\rm word} \circ \nu = (\nu \otimes \nu) \circ \Delta_{\rm FG}, \quad S_{\rm word} \circ \nu = \nu \circ S, \quad B^+_{a_n} \circ \nu = \nu \circ B^+_{\gamma_n}. \tag{2.57}$$

The morphism ν sends each primitive Feynman graph γ_n to a letter a_n. Thanks to Theorem 2.3.7, it is possible to lift the embedding ν onto a new homomorphism $\overline{\nu}$ which embeds the renormalization Hopf algebra of graphons $\mathcal{S}^{\Phi}_{\rm graphon}$ into the Hopf algebra of words. It is enough to replace each Feynman diagram Γ with its corresponding unlabeled graphon class $[W_\Gamma]$.

Consider Dyson–Schownger equations for 1PI Green's functions with the general form

$$\Gamma^{\bar{n}} = 1 + \sum_{\gamma, {\rm res}(\gamma) = \bar{n}} \frac{g^{|\gamma|}}{{\rm Sym}(\gamma)} B^+_\gamma (X^\gamma_{\mathcal{R}}) \tag{2.58}$$

such that B^+_γ are Hochschild closed one-cocycles of the Hopf algebra of Feynman diagrams indexed by Hopf algebra primitives γ with external legs $\bar{n}$, $X^\gamma_{\mathcal{R}}$ is a monomial in superficially divergent Green's functions which dress the internal vertices and edges of γ. If we apply the renormalized Feynman rules character ϕ_r to a Feynman graph which contributes to this class of equations, then we can obtain a polynomial in a suitable external scale parameter $L = \log S/S_0$ such that S_0 fixes a reference scale for the renormalization process. At the end of the day, we can get a renormalized version $G_r(g, L, \theta)$ of Green's functions. Lemma 2.3.3 and Theorem 2.3.7 are useful to reformulate the equation (2.58) in the language of graphons as an equation in the Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$. The embedding $\overline{\nu}$ enables us to lift this graphon model Dyson–Schwinger equations onto their corresponding equations in the Hopf algebra of words. We have

$$X_{\rm DSE, word} = \overline{\nu}([W_{X_{\rm DSE}}]) = \mathbb{I}_{\rm word} + \sum_{n \geq 1} g^n B^+_{l_n}(X^{\ominus_\Theta (n+1)}_{\rm DSE, word}) \tag{2.59}$$

such that $X_{\rm DSE, word}$ is the word representation of the unlabeled graphon class $[W_{X_{\rm DSE}}]$ with respect to the large graph $X_{\rm DSE}$. We have $X_{\rm DSE, word} = \sum_{n \geq 0} (\lambda g)^n z_n$ such that each $z_n = \nu(X_n)$ is determined recursively by the relations

$$z_n = \sum_{m=1}^{n} B^+_{l_m}\Big(\sum_{k_1 + ... + k_{m+1} = n-m, \quad k_i \geq 0} z_{k_1} \ominus_\Theta ... \ominus_\Theta z_{k_{m+1}} \Big). \tag{2.60}$$

We plan to explain the filtration structure on words and then by applying the inverse of the embedding ν, we can adapt it for the level of (large) Feynman diagrams.

The canonical candidate for the filtration on words is built in terms of the lower central series at the Lie algebra level where we need to apply theory of Hall sets and Hall basis. The Milnor–Moore theorem ([126]) allows us to build the graded dual Hopf algebra to H_{word} in terms of the universal algebra of a particular Lie algebra.

A bilinear anti-symmetric map $[.,.]$ on a vector space $\mathcal{L}$ over the field $\mathbb{K}$ with characteristic zero defines a Lie algebra structure if it obeys the following conditions

- $\forall x \in \mathcal{L}: \quad [x,x] = 0$,
- $\forall x,y,z \in \mathcal{L}: \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

The lexicographical ordering enables us to build the Hall basis for the Lie algebra $\mathcal{L}$ [70, 71]. For a given ordering $x_1 < x_2 < ... < [x_1,x_2] < ...$ on $\mathcal{L}$, define $[x,x']$ as an element of a Hall basis for $\mathcal{L}$ iff

- $x, x' \in \mathcal{L}$ are Hall basis elements with $x < x'$,
- if $x' = [x_1,x_2]$, then $x' \geq x_2$.

The unique universal enveloping algebra associated to $\mathcal{L}$ is defined in terms of the tensor algebra $(T(\mathcal{L}), \otimes, 1)$ such that

$$T(\mathcal{L}) := \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}. \tag{2.61}$$

Set

$$I := \{s \otimes (x \otimes y - y \otimes x - [x,y]) \otimes t : \quad x,y \in \mathcal{L}; \quad s,t \in T(\mathcal{L})\} \tag{2.62}$$

as a two sided ideal and then define the equivalent classes of the form

$$[t] := \{s \in T(\mathcal{L}) : \quad s - t \in I\}. \tag{2.63}$$

$T(\mathcal{L})/I$ is actually the unique universal enveloping algebra $\mathcal{U}(\mathcal{L})$ generated by $\mathcal{L}$ where the product of this algebra is given by

$$m_{\mathcal{U}(\mathcal{L})}([s] \otimes [t]) := [s \otimes t] \tag{2.64}$$

and $[1]$ is the unit of this algebra. In addition, we can equip $\mathcal{U}(\mathcal{L})$ by a graded Hopf algebra structure with the coproduct

$$\Delta_{\mathcal{U}(\mathcal{L})}([x]) = [x] \otimes \mathbb{I} + \mathbb{I} \otimes [x] \tag{2.65}$$

Set $\mathcal{L}_{\text{word}}$ as the Lie algebra corresponding to the Hopf algebra H_{word}. Define the decreasing sequence

$$\mathcal{L}_{\text{word}} = \mathcal{L}_1 \geq \mathcal{L}_2 \geq \mathcal{L}_3 \geq ... \tag{2.66}$$

such that $\mathcal{L}_{n+1}$ is generated by all objects $[x, y]$ with $x \in \mathcal{L}$ and $y \in \mathcal{L}_n$. For letters $a_1, a_2, ..., \Theta(a_1, a_2), ... \in H_{\text{letter}}$, set $x_1, x_2, ..., \Theta(x_1, x_2), ... \in \mathcal{L}/\mathcal{L}_2$. The duality between H_{word} and $\mathcal{U}(\mathcal{L}_{\text{word}})$ can be determined by the unique linear invertible map ν such that

$$\nu(a_i) = [x_i], \quad \nu(\Theta(a_i, a_j)) = [\Theta(x_i, x_j)], \quad \nu(a_i a_j) = [x_i \otimes x_j], ... \tag{2.67}$$

The universal enveloping algebra $\mathcal{U}(\mathcal{L}_{\text{word}})$ is a filtered bialgebra.

Thanks to the structure of the quasi-shuffle product, we can build a filtration algorithm where it requires to consider all words with length k into the lexicographical order in terms of the concatenation commutator with respect to the Hall basis which generates words with the length $k-1$. This procedure, which starts with the maximal length of words, should be repeated for the full quasi-shuffle products of the k corresponding letters and then insert them into the expression [103]. Now if we apply the inverse of the embedding $\overline{\nu}$, then this filtration can be defined on Feynman graphons and large Feynman diagrams which live in $\mathcal{S}^{\Phi}_{\text{graphon}}$.

Let us now apply renormalized Feynman rules characters on large Feynman diagrams. The Hopf algebra homomorphism ν sends the renormalized Feynman rules character ϕ_r to

$$\psi_r = \phi_r \circ \nu^{-1}. \tag{2.68}$$

In [103], it is shown that for each word $w \in H_{\text{word}}$ with the corresponding $[x] \in \mathcal{U}(\mathcal{L}_{\text{word}})$, if $x \in T(\mathcal{L}_{\text{word}})$ is also an element of the Lie algebra $\mathcal{L}_{\text{word}}$, then $\psi_r(u)$ maps to the L-linear part of the log-expansion of the renormalized Green's functions. In addition, we have

$$\psi_r(u \ominus_\Theta v) = \psi_r(u).\psi_r(v). \tag{2.69}$$

For a given Feynman diagram Γ with the coradical degree r_Γ, we have

$$\phi_r(\Gamma) = \sum_{j=1}^{r_\Gamma} c_j^\Gamma(\theta) L^j \tag{2.70}$$

such that

$$c_j^\Gamma = c_1^{\otimes j} \tilde{\Delta}^{j-1}_{\text{FG}}(\Gamma) \tag{2.71}$$

while $c_1^{\otimes j} : H_{\text{FG}}(\Phi) \otimes ...^{j \text{ times}} \otimes H_{\text{FG}}(\Phi) \longrightarrow \mathbb{C}$ is a symmetric function. Thanks to the Hopf algebra homomorphism ν which preserves the co-radical degree, for any word $u \in H_{\text{word}}$, we have

$$\psi_r(u) = \sum_{j=1}^{r_u} d_j^u L^j \tag{2.72}$$

such that $d_j^u = c_j^{\nu^{-1}(u)}$.

Thanks to the graphon representations of Dyson–Schwinger equations (Theorem 2.3.6 and Theorem 2.3.7), we want to lift the Feynman rules character (2.70) onto the level of large Feynman diagrams.

In [103], it is shown that ψ_r maps the shuffle product $u_1 \ominus_\Theta ... \ominus_\Theta u_n$ to the L^n-term in the log expansion such that as the result, this process filtrates coefficients X_n in the unique solution of each Dyson–Schwinger equation. We lift this story onto the level of Feynman graphons where the renormalized character $\tilde{\psi}_r := \tilde{\phi}_r \circ \overline{\nu}^{-1}$ maps the formal expansions $\sum_1^m u_{i_1} \ominus_\Theta ... \ominus_\Theta u_{i_k}$ of shuffle products of words corresponding to the partial sums Y_m of $X_{\rm DSE}$ to a certain term in the expansion

$$\tilde{\phi}_r(Y_m) = \sum_{j=1}^{r_{Y_m}} c_j^{Y_m}(\theta) L^j \tag{2.73}$$

such that

$$c_j^{Y_m} = c_1^{\otimes j} \tilde{\Delta}^{j-1}_{\rm graphon}(Y_m). \tag{2.74}$$

When m tends to infinity the sequence $\{c_j^{Y_m}\}_{m\geq 1}$ of coefficients converges to $c_j^{X_{\rm DSE}}$ (for each j) with respect to the cut-distance topology. In addition, Feynman rules characters are linear homomorphisms which means that when m tends to infinity, the sequence $\{\tilde{\phi}_r(Y_m)\}_{m\geq 1}$ is cut-distance convergent to $\tilde{\phi}_r(X_{\rm DSE})$.

Therefore for the infinite graph $X_{\rm DSE}$, we can obtain the following formal expansion as the result of the application of the renormalized Feynman rules character $\tilde{\phi}_r$.

$$\tilde{\phi}_r(X_{\rm DSE}) = \sum_{j=1}^{r_{X_{\rm DSE}}} c_j^{X_{\rm DSE}}(\theta) L^j \tag{2.75}$$

such that

$$c_j^{X_{\rm DSE}} = c_1^{\otimes j} \widetilde{\Delta}^{j-1}_{\rm graphon}(X_{\rm DSE}). \tag{2.76}$$

Suppose $\mathcal{S}^{\Phi}_{{\rm graphon},(i)}$ is the vector space generated by some Feynman graphons derived from Dyson–Schwinger equations such that these graphons are filtered in terms of the canonical filtration on their corresponding words. Namely, the filtration (i) can be defined by applying $\overline{\nu}$ and $\tilde{\psi}_r$ while the associated words map to a similar term i in the log-expansion (2.75). Set

$$\mathcal{S}^{\Phi}_{{\rm graphon},(0)} \preceq \mathcal{S}^{\Phi}_{{\rm graphon},(1)} \preceq ... \preceq \mathcal{S}^{\Phi}_{{\rm graphon},(i)} \preceq ... \preceq \mathcal{S}^{\Phi,g}_{\rm graphon} \tag{2.77}$$

as the resulting filtration on all Feynman graphons which contribute to solutions of Dyson–Schwinger equations such that $\mathcal{S}^{\Phi,g}_{\rm graphon} := \bigcup_{i\geq 0} \mathcal{S}^{\Phi}_{{\rm graphon},(i)} \subset \mathcal{S}^{\Phi}_{\rm graphon}$. It defines the graded vector space $\mathcal{G}^{\Phi}$ given by

$$\mathcal{G}^{\Phi}_{[0]} = \mathcal{S}^{\Phi}_{{\rm graphon},(0)}$$

$$\mathcal{G}^{\Phi}_{[i]} := \mathcal{S}^{\Phi}_{\text{graphon},(i)}/\mathcal{S}^{\Phi}_{\text{graphon},(i-1)}, \quad \forall i > 0 \tag{2.78}$$

where we have

$$\mathcal{G}^{\Phi} = \bigoplus_{i \geq 0} \mathcal{G}^{\Phi}_{[i]}. \tag{2.79}$$

We can show that $\mathcal{G}^{\Phi}$ and $\mathcal{S}^{\Phi,g}_{\text{graphon}}$ are isomorphic as vector spaces. □

Theory of words and quasi-shuffle products were studied in [73] where the existence of Hopf algebra structures on words have been addressed. The applications of shuffle type of products to Hopf algebraic renormalization have been addressed in different settings [55, 91, 140, 145]. There is also another alternative machinery ([140]) to lift Dyson–Schwinger equations in $\mathcal{S}^{\Phi,g}$ onto their corresponding equations in the Hopf algebra of words. According to this approach, we apply the rooted tree representation of the Connes–Marcolli shuffle type renormalization Hopf algebra $H_{\mathbb{U}}$ and then embed $H_{\mathbb{U}}$ into an adapted version of the Hopf algebra H_{CK} decorated by a particular Hall set. In this setting, we can address the question about the existence of another filtration on Dyson–Schwinger equations originated from Hopf algebra of words.

Thanks to the explained Hopf algebraic formalism we are ready to formulate a generalization of the BPHZ renormalization machinery for non-perturbative QFT in the context of Feynman graphons which will be discussed in the next part.

2.4 *The BPHZ renormalization of large Feynman diagrams via Feynman graphons*

In gauge field theories with strong couplings such as QCD, the size of the coupling constant even at rather large values of the exchanged momentum is such that the convergence of the perturbative expansion is slow. Although in higher energy levels, the theory enjoys the asymptotic freedom property, several orders of perturbation theory should be concerned to provide a greater accuracy where we need to deal with the evaluation of a large class of higher order Feynman diagrams. We can address the corrections to the quark self-energy as a complicated example in this setting. The situation goes stranger when we deal with QCD in relatively lower energy levels where non-perturbative aspects do appear. This is the level that we need to build a powerful theoretical model for the study of interactions of elementary particles. Thanks to the Hopf algebra structure $\mathcal{S}^{\Phi}_{\text{graphon}}$, which encodes Dyson–Schwinger equations of a given physical theory Φ, in this part we plan to build a Hopf algebraic renormalization program for large Feynman diagrams in the context of the Riemann–Hilbert problem. We describe the construction of the Connes–Kreimer renormalization group at the level of Feynman graphons.

The renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ of Feynman diagrams of the physical theory Φ encodes enough mathematical tools to explain the step by step removal of sub-divergencies. The graded dual of this Hopf algebra identifies an infinite dimensional complex pro-unipotent Lie group denoted by $\mathbb{G}_\Phi(\mathbb{C})$. Feynman rules, which allow us to replace Feynman diagrams with their corresponding Feynman integrals, are encoded by some elements of $\mathbb{G}_\Phi(\mathbb{C})$. This Lie group has been applied by Connes and Kreimer to describe perturbative renormalization as a special instance of a general mathematical procedure of multiplicative extraction of finite values in the context of the Riemann–Hilbert problem. According to this approach, the BPHZ perturbative renormalization can be described as the existence of the Birkhoff factorization for loops such as γ_μ defined on an analytic curve $C \subset \mathbb{C}P^1$ (as the domain) which has values in $\mathbb{G}_\Phi(\mathbb{C})$. It is shown that

$$\gamma_\mu(z) = \gamma_-(z)^{-1}\gamma_{\mu,+}(z) \tag{2.80}$$

such that $\gamma_{\mu,+}(z)$ is the boundary value of a holomorphic map from the inner domain of C to the group $\mathbb{G}_\Phi(\mathbb{C})$ and $\gamma_-(z)$ is the boundary value of an outer domain of C to the group $\mathbb{G}_\Phi(\mathbb{C})$. In addition, γ_- is normalized by $\gamma_-(\infty) = 1$. The renormalized theory is the evaluation of the holomorphic part $\gamma_{\mu,+}$ of γ_μ as a product of two holomorphic maps $\gamma_\pm$ from the connected components $C_\pm$ of the complement of the circle C in the Riemann sphere $\mathbb{C}P^1$. For dimensional regularization, we are interested in an infinitesimal disk around $z = 0$ and C as the boundary of this disk. In this situation we have $0 \in C_+$ and $\infty \in C_-$. [34, 35]

Each regularized Feynman integral $U^z(\Gamma(p_1, ..., p_N))$ defines a loop $\gamma_\mu(z)$ which allows us to lift the analytic formulation of the Birkhoff factorization onto the level of affine group schemes. Set

$$K = \mathbb{C}\{z\}[z^{-1}], \quad O_1 = \mathbb{C}\{z\}, \quad O_2 = \mathbb{C}[z^{-1}]. \tag{2.81}$$

It is shown that each character $\phi \in \mathbb{G}_\Phi(K)$ has a unique Birkhoff factorization $\phi = (\phi_- \circ S) * \phi_+$ such that $\phi_+ \in \mathbb{G}_\Phi(O_1)$, $\phi_- \in \mathbb{G}_\Phi(O_2)$ and $\varepsilon_- \circ \phi_- = \varepsilon$. The BPHZ renormalization procedure has been interpreted by

$$\Gamma \longmapsto S^\phi_{R_{ms}} * \phi(\Gamma) \tag{2.82}$$

such that

$$S^\phi_{R_{ms}}(\Gamma) = -R_{ms}(\phi(\Gamma)) - R_{ms}(\sum_{\gamma \subset \Gamma} S^\phi_{R_{ms}}(\gamma)\phi(\Gamma/\gamma)). \tag{2.83}$$

Therefore we have

$$S^\phi_{R_{ms}} * \phi(\Gamma) = \overline{R}(\Gamma) + S^\phi_{R_{ms}}(\Gamma) \tag{2.84}$$

such that

$$\overline{R}(\Gamma) = U^z_\mu(\Gamma) + \sum_{\gamma \subset \Gamma} c(\gamma) U^z_\mu(\Gamma/\gamma) = \phi(\Gamma) + \sum_{\gamma \subset \Gamma} S^\phi_{R_{ms}}(\gamma)\phi(\Gamma/\gamma) \qquad (2.85)$$

is the Bogoliubov's operation. For the given Feynman integral $U_\mu(\Gamma)$, the mathematical term $S^\phi_{R_{ms}}(\Gamma)$ generates the counterterm and the mathematical term $S^\phi_{R_{ms}} * \phi(\Gamma)$ generates the corresponding renormalized value. [34, 35, 54, 66]

Each Dyson–Schwinger equation determines a commutative graded Hopf subalgebra H_{DSE} of $H_{\mathrm{FG}}(\Phi)$ where the morphism (1.13) embeds the complex Lie group $\mathbb{G}_{\mathrm{DSE}}(\mathbb{C})$ into $\mathbb{G}_\Phi(\mathbb{C})$. This Lie group has been applied to formulate an algebro-geometric setting for the computation of a class of counterterms which contribute to the renormalization of Feynman diagrams in the equation DSE. In addition, this approach has determined global β-functions with respect to DSEs where they play the central role for the transformation of the information between different regularization schemes. Furthermore, this study has been lifted onto a categorical setting to associate the category $\mathrm{Rep}_{\mathbb{G}^*_{\mathrm{DSE}}}$ of finite dimensional representations of the affine group scheme $\mathbb{G}_{\mathrm{DSE}}$ to each equation DSE. This categorical formalism has led us to encode some non-perturbative physical information in the Connes–Marcolli universal category $\mathcal{E}^{\mathrm{CM}}$ in the context of differential systems with singularities. [140, 142, 143, 144]

In this part we plan to provide a new interpretation of the renormalization of (large) Feynman diagrams in the context of the Hopf algebra $\mathcal{S}^\Phi_{\mathrm{graphon}}$. We will then provide a new class of differential equations which contribute in the computation of non-perturbative counterterms.

Theorem 2.4.1. *The Hopf–Birkhoff factorization process provides a renormalization program for each large Feynman diagram in $\mathcal{S}^{\Phi,g}$.*

Proof. We first build a renormalization program for Feynman graphons which belong to the Hopf algebra $\mathcal{S}^\Phi_{\mathrm{graphon}}$ and then thanks to the graph function representation of Feynman diagrams, we will enable to pull back the results to the level of Feynman diagrams.

Thanks to Milnor–Moore Theorem ([126]), the commutative graded Hopf algebra $\mathcal{S}^\Phi_{\mathrm{graphon}}$ (Theorem 2.3.7) determines the complex infinite dimensional pro-unipotent Lie group $\mathbb{G}^\Phi_{\mathrm{graphon}}(\mathbb{C})$. Choose dimensional regularization and minimal subtraction as the renormalization program where the commutative algebra A_{dr} of Laurent series with finite pole parts encodes the regularization scheme and the linear map R_{ms}, which projects series onto their pole parts, encodes the renormalization scheme. Set $\mathrm{Loop}(\mathbb{G}^\Phi_{\mathrm{graphon}}(\mathbb{C}), \mu)$ as the space of loops γ_μ on the infinitesimal punctured disk $\boldsymbol{\Delta}^*$ around the origin in the complex plane with values in $\mathbb{G}^\Phi_{\mathrm{graphon}}(\mathbb{C})$. These loops describe unrenormalized regularized Feynman rules characters in $\mathbb{G}^\Phi_{\mathrm{graphon}}(\mathbb{C})$ which act on Feynman graphons. The Rota–Baxter property of $(A_{\rm dr}, R_{\rm ms})$ supports the existence of a unique Birkhoff factorization (γ_-, γ_+) which can be lifted onto the level of Feynman rules characters to achieve the factorization $(\tilde{\phi}_-, \tilde{\phi}_+)$ for Feynman rules character $\tilde{\phi}$. In addition, we have

$$\tilde{\phi}^z([W_\Gamma]) := \phi^z(\Gamma) \tag{2.86}$$

as the modified version of the regularized Feynman rules character ϕ which acts on Feynman graphons.

For a given Feynman graphon $[W_X]$ corresponding to the unique solution of an equation DSE, set $[W_{Y_m}]$ (for each m) as the unlabeled graphon classes with respect to the partial sums Y_m of the infinite graph X.

Now if we apply the renormalization coproduct formulas (2.45), (2.46) on Feynman graphons, the renormalization antipode formulas (2.49), (2.50) on Feynman graphons and also the Hopf algebraic BPHZ process given by (2.83), (2.84), (2.85), then we can build the sequence $\{S^{\tilde{\phi}}_{R_{\rm ms}}([W_{Y_m}])\}_{m\geq 1}$ of Feynman graphons which is convergent with respect to the cut-distance topology. We have

$$S^{\tilde{\phi}}_{R_{\rm ms}}([W_X]) = \lim_{m\to\infty} S^{\tilde{\phi}}_{R_{\rm ms}}([W_{Y_m}]) = \lim_{m\to\infty} \sum_{i=1}^{m} S^{\tilde{\phi}}_{R_{\rm ms}}([W_{X_i}])$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m} (-R_{\rm ms}(\tilde{\phi}([W_{X_i}]) - R_{\rm ms}(\sum S^{\tilde{\phi}}_{R_{\rm ms}}([W_\gamma])\tilde{\phi}([W_{X_i/\gamma}]))). \tag{2.87}$$

The functional $S^{\tilde{\phi}}_{R_{\rm ms}}$ is the negative component of the Birkhoff factorization of $\tilde{\phi}$. In addition, the expression $S^{\tilde{\phi}}_{R_{\rm ms}}([W_X])$ addresses the counterterm with respect to the Feynman graphon $[W_X]$.

Furthermore, we can build the sequence $\{S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}([W_{Y_m}])\}_{m\geq 1}$ of Feynman graphons which is convergent with respect to the cut-distance topology. We have

$$S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}([W_X]) = \lim_{m\to\infty} S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}([W_{Y_m}]) = \lim_{m\to\infty} \sum_{i=1}^{m} S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}([W_{X_i}]) \tag{2.88}$$

such that $*$ is the convolution product of the Lie group $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ determined by the coproduct $\Delta_{\rm graphon}$. The functional $S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}$ is the positive component of the Birkhoff factorization of $\tilde{\phi}$. In addition, the expression $S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}([W_X])$ addresses the renormalized value with respect to the Feynman graphon $[W_X]$. □

Thanks to the filtration parameter on Feynman graphons on the basis of the Hopf algebra of words given by Theorem 2.3.9, consider a new one-parameter group $\{\theta_t\}_t$ of automorphisms on $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ with the infinitesimal generator

$$\frac{d}{dt}|_{t=0}\theta_t = Y \tag{2.89}$$

such that Y sends each Feynman graphon $[W_\Gamma]$ to its corresponding filtration rank $n_{[W_\Gamma]}$. In other words, for each character $\tilde{\phi}$, we have

$$< \theta_t(\tilde{\phi}), [W_\Gamma] > := < \tilde{\phi}, \theta_t([W_\Gamma]) > \tag{2.90}$$

Lemma 2.4.2. *Suppose $\gamma_\mu \in {\rm Loop}(\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C}), \mu)$ encodes the regularized unrenormalized Feynman rules character $\tilde{\phi}$. Then we have*

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)).$$

In addition, the limit

$$F_t = \lim_{z\to 0}\gamma_-(z)\theta_{tz}(\gamma_-(z)^{-1})$$

defines a new one-parameter subgroup of $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ such that for each $t \in \mathbb{R}$

$$\gamma_{e^t\mu^+}(0) = F_t\gamma_{\mu^+}(0).$$

Proof. Thanks to the construction of the renormalization Hopf algebra of Feynman graphons (Theorem 2.3.7) and Proposition 1.47 in [37], we have the proof. □

It is possible to lift the Connes–Marcolli geometric approach onto the level of the renormalization Hopf algebra of Feynman graphons. For this purpose we need to adapt the regularization bundle and then classify equi-singular flat connections, which encode counterterms, in terms of the renormalization of Feynman graphons.

Proposition 2.4.3. *There exists a bijective correspondence (independent of the choice of a local regular section $\sigma : \mathbf{\Delta} \longrightarrow B$) between equivalence classes of flat equi-singular $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$-connections on the regularization bundle and elements of the Lie algebra $\mathfrak{g}^{\Phi}_{\rm graphon}(\mathbb{C})$.*

Proof. The regularization parameter in dimensional regularization can be encoded by the punctured version of an infinitesimal disk $\mathbf{\Delta}$ around the origin $z = 0$. Set $P_{\rm graphon} := (\mathbf{\Delta} \times \mathbb{C}^*) \times \mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ as the trivial principal bundle over the base space $\mathbf{\Delta} \times \mathbb{C}^*$. Remove the fiber over $z = 0$ to obtain the bundle $P^0_{\rm graphon} = \mathbf{\Delta} \times \mathbb{C}^* - \pi^{-1}(\{0\}) \times \mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ as the regularization bundle modified with respect to the renormalization Hopf algebra of Feynman graphons.

A flat $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$-connection ϖ on P^{0}_{graphon} is a $\mathfrak{g}^{\Phi}_{\text{graphon}}(\mathbb{C})$-valued one form such that $\mathfrak{g}^{\Phi}_{\text{graphon}}(\mathbb{C})$ is the Lie algebra of the Lie group $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$ which contains all linear maps $l : \mathcal{S}^{\Phi}_{\text{graphon}} \longrightarrow \mathbb{C}$ with the property

$$l([W_{\Gamma_1}][W_{\Gamma_2}]) = l([W_{\Gamma_1}])\varepsilon([W_{\Gamma_2}]) + \varepsilon([W_{\Gamma_1}])l([W_{\Gamma_2}]). \tag{2.91}$$

The Lie bracket is given by the formula

$$[l_1, l_2]([W_\Gamma]) =< l_1 \otimes l_2 - l_2 \otimes l_1, \Delta_{\text{graphon}}([W_\Gamma]) > . \tag{2.92}$$

The one-parameter group $\{\theta_t\}_{t\in\mathbb{C}}$ of automorphisms of $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$ can be lifted onto the level of Lie algebra.

A flat $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$-connection ϖ on P^{0}_{graphon} is called equi-singular if it is $\mathbb{G}_m$-invariant and for any solution f of the differential equation $\mathbf{D}f = \varpi$ with respect to the logarithmic derivative, the restrictions of f to sections $\sigma : \mathbf{\Delta} \longrightarrow \mathbf{\Delta} \times \mathbb{C}^*$ have the same type of singularity.

Thanks to [37], the negative component of the Birkhoff factorization of each $\gamma_\mu \in \text{Loop}(\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C}), \mu)$ determines a unique element β in $\mathfrak{g}^{\Phi}_{\text{graphon}}(\mathbb{C})$ where we have

$$\gamma_-(z) = T\text{exp}(\frac{-1}{z}\int_0^\infty \theta_{-t}(\beta)dt) \tag{2.93}$$

formulated in terms of the time order exponential.

We can show that for each $\varpi \in \mathfrak{g}^{\Phi}_{\text{graphon}}(K)$ with the trivial monodromy, there exists a solution $\tilde{\psi} \in \mathbb{G}^{\Phi}_{\text{graphon}}(K)$ for the differential equation $\mathbf{D}\tilde{\psi} = \varpi$.

Two connections ϖ_1, ϖ_2 with the trivial monodromy are called equivalent if they are gauge conjugate by an element regular at $z = 0$. It leads us to show that for equivalent connections ϖ_1, ϖ_2, the solutions of the differential equations $\mathbf{D}\tilde{\psi}^{\varpi_1} = \varpi_1$ and $\mathbf{D}\tilde{\psi}^{\varpi_2} = \varpi_2$ have the same negative components of the Birkhoff factorization

$$\tilde{\psi}^{\varpi_1}_- = \tilde{\psi}^{\varpi_2}_-. \tag{2.94}$$

Thanks to (2.93), (2.94) and the Connes–Marcolli Classification Theorem (Theorem 1.67 in [37]), each element $\beta \in \mathfrak{g}^{\Phi}_{\text{graphon}}(\mathbb{C})$ determines a unique class ϖ of flat equi-singular $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$-connections on P^{0}_{graphon} in terms of a differential equation with the general form $\mathbf{D}\gamma_\mu = \varpi$ such that

$$\gamma_\mu(z, v) = T\text{exp}(\frac{-1}{z}\int_0^v u^Y(\beta)\frac{du}{u}) \tag{2.95}$$

where $u = tv,\ t \in [0, 1]$ and u^Y is the action of $\mathbb{G}_m$ on $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$. □

Proposition 2.4.4. *The category $\mathcal{E}^{\text{CM}}$ encodes the renormalization group corresponding to the BPHZ renormalization of large Feynman diagrams.*

Proof. Thanks to Proposition 2.4.3, for the Hopf algebra $\mathcal{S}^{\Phi}_{\text{graphon}}$, we can determine a family of flat equi-singular $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$-connections which encode counterterms on the basis of the β-functions. Thanks to [37], these geometric objects form a new category $\mathcal{E}^{\Phi}_{\text{graphon}}$ which is recovered by the category $\text{Rep}_{\mathbb{G}^{\Phi,*}_{\text{graphon}}}$ of finite dimensional representations of the affine group scheme $\mathbb{G}^{\Phi,*}_{\text{graphon}}$. In addition, the renormalization Hopf algebra $H_{\text{FG}}(\Phi)$ of Feynman diagrams of the physical theory Φ determines the category $\mathcal{E}^{\Phi}$ of geometric objects recovered by the category $\text{Rep}_{\mathbb{G}^{*}_{\Phi}}$ of finite dimensional representations of the affine group scheme $\mathbb{G}^{*}_{\Phi}$. Thanks to the explained categorical formalism in [37], we can embed $\text{Rep}_{\mathbb{G}^{*}_{\Phi}}$ as a sub-category of $\mathcal{E}^{\text{CM}}$. It is shown that $\mathcal{E}^{\text{CM}}$ is isomorphic to the category $\text{Rep}_{\mathbb{U}^{*}}$ such that the complex Lie group $\mathbb{U}(\mathbb{C})$ can be described in terms of its Lie algebra $L_{\mathbb{U}}$ generated by elements e_{-n} of degree $-n$ for each $n > 0$ such that the sum $e = \sum e_{-n}$ is an element of this Lie algebra. We can lift e onto the morphism $\text{rg} : \mathbb{G}_a \longrightarrow \mathbb{U}$. The universality of $\mathcal{E}^{\text{CM}}$ supports the existence of a new class of graded representations such as

$$\vartheta : \mathbb{U}(\mathbb{C}) \longrightarrow \mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C}). \tag{2.96}$$

Now the composition $\vartheta \circ \text{rg}$ determines the renormalization group $\{F_t\}_{t\in\mathbb{C}}$ at the level of Feynman graphons (i.e. Lemma 2.4.2). □

Lemma 2.3.3 and Theorem 2.3.7 enable us to embed $H_{\text{FG}}(\Phi)$ into $\mathcal{S}^{\Phi}_{\text{graphon}}$ which leads us to define an epimorphism of affine group schemes from $\mathbb{G}^{\Phi,*}_{\text{graphon}}$ to $\mathbb{G}^{*}_{\Phi}$. In addition, the renormalization Hopf algebra of Feynman graphons includes solutions of all Dyson–Schwinger equations in the physical theory Φ. It means that the category $\mathcal{E}^{\Phi}_{\text{graphon}}$ restores some more physical information than the category $\mathcal{E}^{\Phi}$ (or $\text{Rep}_{\mathbb{G}^{*}_{\Phi}}$).

As the summary, we have shown that the renormalization topological Hopf algebra of Feynman graphons is capable to encode the renormalization of Feynman diagrams and solutions of Dyson–Schwinger equations. In addition, we have embedded this graphon model of renormalization into the universal Connes–Marcolli categorical setting. These achievements suggest the existence of a new unexplored interconnection between motivic renormalization and Dyson–Schwinger equations in the context of the theory of graphons.

Chapter 3

The complexity of non-perturbative computations under a combinatorial setting

• ***A parametric representation for large Feynman diagrams: a computational machinery***

• ***The optimization of non-perturbative complexity via a multi-scale renormalization group***

– ***A renormalization group program on*** $\mathcal{S}^{\Phi,g}$

– ***Kolmogorov complexity of Dyson–Schwinger equations***

The major motivation for the introduction of Feynman graphons is to clarify and study infinities originated from non-perturbative aspects in Quantum Field Theory with strong couplings. From a physicist's perspective, these infinities do not acceptable and applying some approximation methods are useful for the production of some intermediate values such as running couplings, N large methods, etc. Then the Physics of elementary particles and its phenomenology shall be described in terms of those approximations. From a mathematician's perspective, we have a different story where it is possible to deal with infinities under different settings instead of only removing them. The Cartier's cosmic Galois group as a universal group of symmetries aims to organize the structure of the divergencies in perturbative gauge field theories. The motivic Galois group associated to the renormalization Hopf algebra $\mathcal{S}^{\Phi}_{\text{graphon}}$ of Feynman graphons can be considered as a suitable candidate to organize the structure divergencies originated from large Feynman diagrams. In this chapter we plan to apply Feynman graphons for the discovery of more entities about solutions of Dyson–Schwinger equations via graph polynomials.

As we have seen in the previous chapter, graphons were applied for the renormalization of large Feynman diagrams where some new interpretations from counterterms and renormalized values originated from Dyson–Schwinger equations have been achieved. In this chapter we are going to investigate some other combinatorial applications of graphons to Quantum Field Theory where we build a new parametric representation of large Feynman diagrams in the language of graph polynomials. Then we focus on the concept of complexity for the description of non-perturbative parameters where we explain the construction of a new multi-scale renormalization group on the space $\mathcal{S}^{\Phi,g}$ of all Dyson–Schwinger equations in a physical theory with the bare coupling constant g equipped with the cut-distance topology. We try to show that this renormalization group machinery optimizes the complexity of non-perturbative computations. In addition, we lift the concept of Kolmogorov complexity onto $\mathcal{S}^{\Phi,g}$ to discuss more about the role of the defined multi-scale renormalization group in optimizing non-perturbative computations. This study suggests a new contextualization for the description of non-perturbative situations and their complexity.

3.1 *A parametric representation for large Feynman diagrams: a computational machinery*

The original task in Quantum Field Theory is to compute correlations (Green's functions) in a (non-)perturbative expansion setting whose terms are decorated by Feynman diagrams. Each term in this class of expansions consists of a multiple ill-defined integral such that the integrand is codified by the combinatorial information of its corresponding Feynman diagram.

Generally speaking, we can work in momentum space of D dimensions such that a preliminary count of the powers of the momenta in the integrands determines a superficially divergent integral. In this situation, the renormalization program associates a counterterm to each superficially divergent subgraph to finally produce a finite result by subtraction. All superficially divergent subgraphs should be considered under a recursive setting to assign a finite value to the full graph. Studying Feynman diagrams via tree representations enables us to formulate perturbative renormalization theory under a simplified universal setting. Furthermore, it provides also a combinatorial reformulation of Dyson–Schwinger equations where we can study solutions of these non-perturbative type of equations in the context of partial sums of decorated non-planar rooted trees. [9, 61, 62, 92, 97, 164, 174]

In the previous sections we have shown that the unique solution of each Dyson–Schwinger equation is described as the convergent limit of a sequence of Feynman graphons with respect to the cut-distance topology. This formalism has been applied to lift the BPHZ renormalization program onto the level of large Feynman graphs to generate some new expressions for the description of counterterms and renormalized values associated to fixed point equations of Green's functions. Using graph polynomials for the study of Feynman integrals has played an important role in the computational processes where this class of combinatorial polynomials can bring some powerful algorithms for the analysis of the behavior of these divergent integrals ([6, 11, 83, 84, 101, 117, 130, 162, 168]). In this section we show another application of this graphon representation of non-perturbative parameters where we deal with the concept of parametric representation of large Feynman diagrams. We study solutions of Dyson–Schwinger equations in the language of Tutte polynomial and Kirchhoff–Symanzik polynomials.

The Tutte Polynomial, as a two variables graph polynomial, enjoys a universal property which enables us to evaluate any multiplicative graph invariant with a deletion/contraction reduction machinery [158, 159, 160, 161, 166]. This fundamental property provides the opportunity to demonstrate how graph polynomials can be specialized or generalized. The Aluffi–Marcolli approach has clarified the practical importance of Tutte Polynomials in dealing with Feynman rules characters and Feynman integrals under an algebro-geometric setting where a motivic perspective on perturbative renormalization program has been formulated very nicely. [5, 6, 116, 117]

In this part we plan to review the basic structure of Tutte polynomials on finite graphs, its different reformulations and its universal property ([117, 159, 160, 161, 166]) and then we will formulate graph polynomials for large Feynman diagrams.

A given finite graph G has a set $V(G)$ of vertices and a set $E(G)$ of edges. For two isomorphic graphs G_1, G_2, we should have a bijection such as ρ between the sets $V(G_1)$ and $V(G_2)$ such that for each edge uv in G_1, $\rho(uv)$ is an edge in G_2 and vice versa. For any subset $A \subset E(G)$ of edges, the rank $r(A)$ and the nullity $n(A)$ are defined by the relations

$$r(A) := |V(G)| - \kappa(A), \quad n(A) := |A| - r(A) \tag{3.1}$$

such that $\kappa(A)$ is the number of connected components of the graph. In general, finite graphs can be classified in terms of their numbers of non-trivial connected components where a graph is called n-connected, if we should remove at least n edges from the graph to obtain a disconnected graph. Rooted trees, as fundamental tools for us, are connected graphs which have no cycles or loops. They can be applied as non-trivial connected components of forests (as more complicated graphs). Sometimes working on subgraphs of a given complicated (finite) graph enables us to determine some fundamental properties of the original graph. In general, subsets of the set of vertices or the set of edges can give us subgraphs. Spanning subgraphs are applied as one important class of subgraphs for the construction of graph polynomials. A spanning subgraph covers all vertices of the original graph with the optimum number of edges.

The notion of "dual" in Graph Theory enables us to build the algebraic combinatorics of graphs. If we can embed a graph into the plane without any crossing in edges, then the graph is called planar. Each planar graph can separate the plane into regions known as faces. Faces are key tools for the construction of the dual of a graph. For a given planar graph G, its corresponding connected dual graph is built by assigning a vertex to each face where there exists m edges between two vertices in the dual graph if the corresponding faces of the original graph have m edges in their boundaries. We denote G^* as the dual of the connected planar graph G and it can be seen that

$$(G^*)^* = G. \tag{3.2}$$

There are two fundamental commutative operations on graphs namely, deletion and contraction which enable us to build the algebraic combinatorics of graphs. For a given finite graph G, we can build a new graph $G \setminus e$ as the result of deleting an edge $e \in E(G)$. This new graph has the same set of vertices $V(G)$ and the set of edges $E(G) - \{e\}$. We can also build another new graph G/e as the result of contracting an edge e in terms of identifying the endpoints of the edge e by shrinking this edge. It is easy to check that the deletion and the insertion on a self-loop edge determine the same resulting graph.

Lemma 3.1.1. *(i) For any given different edges e_1, e_2 of a given planar graph G, the graph $(G \setminus e_1)/e_2$ is isomorphic to the graph $(G/e_2) \setminus e_1$.*

(ii) A planar graph and its dual have the same numbers of spanning trees.

(iii) The rank of a dual graph is well-defined. [159, 166]

The deletion or contraction of an edge determines a minor of a graph. In more general setting, if a graph H is isomorphic to $G \setminus A/A'$ for some choice of disjoint subsets A_1, A_2 of $E(G)$, then it is called a minor graph. In this setting, a class of graphs is called minor closed if whenever the graph G is in the class, then any minor of G is also in the class.

Graph invariants allow us to characterize graphs in terms of particular properties. In general, a graph invariant is a function on the class of all graphs such that it has the same output on isomorphic graphs. Graph polynomials (such as Tutte polynomial) are indeed some graph invariants such that the images belong to some polynomial rings.

There are several different (but equivalent) (re)formulations for Tutte polynomials such as rank–nullity generating function method, linear recursion machinery and spanning tree expansion method which was originally applied by Tutte. The linear recursion form can be described as a collection of reduction rules to rewrite a graph as a weighted formal sum of graphs that are less complicated than the original graph. As the output of this formalism, we can identify a collection of simplest or irreducible graphs. [117, 159, 160, 161, 166]

Definition 3.1.2. The Tutte polynomial $T(G;x,y)$ of a given graph G as a two variables polynomial with respect to the independent variables x, y is defined by the following recursive machinery.
- If G has no edges, then $T(G;x,y) = 1$; otherwise, for any edge $e \in E(G)$,
- $T(G;x,y) = T(G \setminus e;x,y) + T(G/e;x,y)$,
- $T(G;x,y) = xT(G/e;x,y)$, if e is a coloop,
- $T(G;x,y) = yT(G \setminus e;x,y)$, if e is a loop.

In general, if G has i bridges and j loops, then its corresponding Tutte polynomial is given by

$$T(G;x,y) = x^i y^j. \tag{3.3}$$

In addition, thanks to Definition 3.1.2, the Tutte polynomial of the disjoint union of two graphs is determined by

$$T(G \cup H) = T(G)T(H). \tag{3.4}$$

We can redefine Tutte polynomials in the language of the rank–nullity generating functions known as (infinite) polynomials with coefficients which can count structures that are encoded by the exponents of variables. In this setting we have

$$T(G;x,y) = \sum_{A \subset E(G)} (x-1)^{r(E(G))-r(A)}(y-1)^{n(A)}. \tag{3.5}$$

The Tutte polynomials of a planar graph G and its dual graph G^* can determine each other in the sense that

$$T(G;x,y) = T(G^*;y,x). \tag{3.6}$$

We can also redefine Tutte polynomials in the language of spanning trees. In this setting, we need to define a total order $\prec$ on the set of edges $E(G) = \{v_1, ..., v_n\}$ of a given graph G such as

$$v_i \prec v_j \Longleftrightarrow i > j. \tag{3.7}$$

For a given tree t, an edge e is called internally active if e is an edge of t and it is the smallest edge in the cut defined by e. We can lift this concept onto the dual level where an edge u is called externally active if $e \notin t$ and it is the smallest edge in the cycle defined by u. Now the Tutte polynomial of the totally ordered graph G can be defined (independent of the chosen total order) by the formal expansion

$$T(G; x, y) = \sum_{i,j} t_{ij} x^i y^j \tag{3.8}$$

such that t_{ij} counts spanning trees with internal activity i and external activity j. [6, 83, 117, 158, 159, 166]

The most fundamental property of the Tutte polynomial is its universality under a graph invariant setting. This means that any multiplicative graph invariant on disjoint unions and one-point joins of graphs which is formulated via a deletion/contraction reduction can be described as an evaluation of the Tutte polynomial. There are different notions for the generalization of the Tutte polynomials and here we address the one which is useful for us. [6, 117]

Definition 3.1.3. Let $\mathfrak{G}$ be the set of isomorphism classes of finite graphs. A graph invariant F from $\mathfrak{G}$ to a commutative ring such as the polynomial ring $\mathbb{C}[\alpha, \beta, \eta, x, y]$ is called Tutte–Grothendieck invariant of graphs, if it has the following properties:
- $F(G) = \eta^{\# \ V(G)}$ if the set of edges is empty,
- $F(G) = xF(G/e)$ if the edge $e \in E(G)$ is a bridge,
- $F(G) = yF(G \setminus e)$ if the edge $e \in E(G)$ is a looping edge,
- For any ordinary edge, which is not a bridge nor a looping edge,

$$F(G) = \alpha F(G/e) + \beta F(G \setminus e). \tag{3.9}$$

- For every $G, H \in \mathfrak{G}$, if $G \cup H \in \mathfrak{G}$ or $G \bullet H \in \mathfrak{G}$, then $F(G \cup H) = F(G)F(H)$ and $F(G \bullet H) = F(G)F(H)$ such that the one-point join $G \bullet H$ is defined by identifying a vertex of G and a vertex of H into a new single vertex of $G \bullet H$.

We can apply induction machinery to show that

$$T(G \bullet H) = T(G)T(H). \tag{3.10}$$

Therefore the Tutte polynomial does not distinguish between the one-point join of two graphs and their disjoint union. In fact, the Tutte polynomial is a Tutte–Grothendieck invariant which is independent of the choice of any ordering of edges of the graph. We can show that for any given map $f : \mathfrak{G} \longrightarrow R$, if there exist $a, b \in R$ such that f is a Tutte–Grothendieck invariant, then f can be presented in terms of the Tutte polynomial and we have

$$f(G) = a^{|E(G)|-r(E(G))} b^{r(E(G))} T(G; \frac{x_0}{b}, \frac{y_0}{a}). \tag{3.11}$$

Thanks to this background and the graphon representation of large Feynman diagrams (discussed in the previous parts), now we can build a new class of Tutte polynomials which contribute to the fundamental structures of Dyson–Schwinger equations.

Theorem 3.1.4. *There exists a new class of Tutte polynomials with respect to large Feynman diagrams.*

Proof. Let the large Feynman diagram $X_{\mathrm{DSE}} = \sum_{n \geq 0} X_n$ be the unique solution of a given Dyson–Schwinger equation DSE. Thanks to Theorem 2.3.6, the sequence $\{Y_m\}_{m \geq 1}$ of partial sums is convergent to X_{DSE} with respect to the cut-distance topology. For each $m \geq 1$, the Tutte polynomial $T(Y_m; x, y)$ with respect to the finite graph Y_m can be determined by Definition 3.1.2 which leads us to the formulation (3.8). Our idea is to implement an efficient algorithm for the computation of the Tutte polynomial $T(X_{\mathrm{DSE}}; x, y)$ in terms of handling intermediate graphs (i.e. partial sums) and their corresponding Tutte polynomials to avoid unnecessary recomputations.

On the first side, for each $m \geq 1$, the Tutte polynomial of the disjoint union $Y_m = X_1 + ... + X_m$ is given by

$$T(Y_m; x, y) = \prod_{s=1}^{m} T(X_s; x, y) = \prod_{s=1}^{m} \sum_{i_s, j_s} t_{i_s j_s} x^{i_s} y^{j_s} \tag{3.12}$$

such that $t_{i_s j_s}$ is the number of spanning trees in X_s with internal activity i_s and external activity j_s.

On the second side, $\lim_{m \to \infty} Y_m = X_{\mathrm{DSE}}$ with respect to the cut-distance topology. It means that for each $\epsilon > 0$, there exists N_ϵ such that for each $m_1, m_2 \geq N_\epsilon$, we have

$$d(Y_{m_1}, Y_{m_2}) = d_{\mathrm{cut}}([W_{Y_{m_1}}], [W_{Y_{m_2}}]) < \epsilon. \tag{3.13}$$

Therefore

$$d_{\mathrm{cut}}([W_{Y_{m_1}}], [W_{Y_{m_2}}]) = 0 \Leftrightarrow [W_{Y_{m_1}}] = [W_{Y_{m_2}}]. \tag{3.14}$$

As we know for each m, the class $[W_{Y_m}]$ is determined in terms of the rooted tree representations of Feynman diagrams $X_1, ..., X_m$ where decorated rooted trees $t_{X_1}, ..., t_{X_m}$ are the only spanning trees in themselves. Thanks to the relation (3.14), for enough large orders, unlabeled graphon classes corresponding to partial sums are weakly equivalent and actually they converge to the unique graphon class $[W_{X_{\mathrm{DSE}}}]$. It means that spanning forests of partial sums for enough large orders tend to the spanning forest $t_{X_{\mathrm{DSE}}}$ of the unique graph limit X_{DSE}. In addition, the Tutte polynomial for each arbitrary rooted tree t is given by

$$T(t;x,y) = \sum_{s \in R(t)} x^{|E(s)|}(y+1)^{|E(s)|-|L(s)|} \tag{3.15}$$

such that $R(t)$ is the set of all subtrees of t, $|E(s)|$ is the number of edges of a subtree s and $|L(s)|$ is the number of leaves of a subtree s.

For the collection $\{\prod_{s=1}^{m} T(t_{X_m};x,y)\}_{m\geq 1}$ of Tutte polynomials, we can define a collection $\{p_m : \prod_{s=1}^{\infty} T(t_{X_s};x,y) \longrightarrow \prod_{s=1}^{m} T(t_{X_m};x,y)\}_{m\geq 1}$ of projections. Thanks to the universal property of the Tutte polynomial, for any graph invariant T (which enjoys the properties in Definition 3.1.2) together with the collection $\{f_m : T \longrightarrow \prod_{s=1}^{m} T(t_{X_m};x,y)\}_{m\geq 1}$, we can define the unique map

$$F : T \longrightarrow \prod_{s=1}^{\infty} T(t_{X_s};x,y) \tag{3.16}$$

such that $f_m = p_m \circ F$. As the consequence, we can consider the direct product $\prod_{s=1}^{\infty} T(t_{X_s};x,y)$ as the Tutte polynomial for the infinite tree (or forest) $t_{X_{\mathrm{DSE}}}$.

If we replace rooted tree representations with the original Feynman diagrams, then we can build the Tutte polynomial for the large Feynman diagram X_{DSE} in terms of the direct product over the Tutte polynomials for simpler finite graphs (i.e. partial sums) $\{T(X_s;x,y)\}_{s\geq 1}$. We have

$$T(X_{\mathrm{DSE}};x,y) = \prod_{s=1}^{\infty} T(X_s;x,y). \tag{3.17}$$

□

We can address here some interesting applications of this class of Tutte polynomials in dealing with the complexity of non-perturbative parameters.

As the first application, it is possible to describe the complexity of a large Feynman diagram X_{DSE} in terms of the complexity of finite Feynman diagrams which live in partial sums Y_m, $(m \geq 1)$. The complexity of Y_m is interpreted in terms of the number of different spanning trees which live in the graph. We can compute the complexity of Y_m under a recursive algorithm where at each stage of the algorithm, only an edge belonging to the proper cycle is chosen. The algorithm starts with a given graph and produces two graphs at the end of the first stage. By applying the elementary contraction to a multiple edge, the resulting graph can have a loop and therefore the procedure can be still continued. At each subsequent stage one proper cyclic edge from each graph is chosen (if it exists) for applying the recurrence. On termination of the algorithm, we get a set of graphs (or general graphs) none of which have a proper cycle. The complexity of Y_m is the sum of the number of these graphs. If we perform this recursive algorithm for each Y_m when m tends to infinity, then we can get a sequence which presents the behavior of complexities when the partial sums converge to X_{DSE}.

As the second application, we can interpret Feynman rules characters of the renormalization Hopf algebra of Feynman graphons in the context of deletion and contraction operators. This approach leads us to formulate a universal motivic Feynman rule character on large Feynman diagrams.

Corollary 3.1.5. *The Tutte polynomial invariant defines an abstract version of Feynman rules on the renormalization Hopf algebra of Feynman graphons.*

Proof. Graphon classes in $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ recover finite Feynman diagrams and their finite or infinite formal expansions which contribute to Dyson–Schwinger equations.

For each unlabeled graphon class $[W_\Gamma]$ corresponding to a finite Feynman diagram Γ, the Tutte polynomial $T([W_\Gamma]; x, y)$ can be defined as follows

$$T([W_\Gamma]; x, y) := T(\Gamma; x, y). \tag{3.18}$$

Thanks to Proposition 2.2 in [6], the Tutte polynomial is multiplicative over disjoint unions of finite (Feynman) diagrams. To see this property requires to describe each connected Feynman diagram Γ as a tree t_Γ with 1PI graphs inserted at the vertices of that tree. Now we can compute the Tutte polynomials of the resulting trees (i.e. formula (3.15)) to show that the Tutte polynomial of disjoint union of Feynman graphons $[W_{\Gamma_1}], [W_{\Gamma_2}]$ corresponding to finite Feynman diagrams Γ_1, Γ_2 can be determined by the Tutte polynomial of disjoint union of decorated rooted trees t_{Γ_1} and t_{Γ_2} which is multiplicative. As the result, we have

$$T([W_{\Gamma_1}] \sqcup [W_{\Gamma_2}]; x, y) = T(\Gamma_1 \sqcup \Gamma_2; x, y) \tag{3.19}$$

$$\sum_{s=(s_1,s_2)} (x-1)^{b_0(s_1)+b_0(s_2)-b_0(\Gamma_1 \sqcup \Gamma_2)} (y-1)^{b_1(s_1)+b_1(s_2)}$$

$$= T(\Gamma_1; x, y) T(\Gamma_2; x, y) = T([W_{\Gamma_1}]; x, y) T([W_{\Gamma_2}]; x, y)$$

such that the sum is taken over all pairs $s = (s_1, s_2)$ of subgraphs of Γ_1 and Γ_2, respectively where $V(s_i) \subseteq V(\Gamma_i)$, $E(s_i) \subset E(\Gamma_i)$, $b_0(s) = b_0(s_1) + b_0(s_2)$.

Furthermore, for a finite connected Feynman diagram Γ, we have

$$\Gamma = \bigcup_{v \in V(t_\Gamma)} \Gamma_v \tag{3.20}$$

such that Γ_vs are 1PI Feynman diagrams inserted at the vertices of the tree t_Γ. The internal edges of the tree t_Γ are all bridges in the resulting graph and thus

$$T(\Gamma; x, y) = x^{|E_{\rm int}(t_\Gamma)|} T(\Gamma / \cup_{e \in E_{\rm int}(t_\Gamma)} e; x, y). \tag{3.21}$$

It is possible to lift this property onto the level of Feynman graphons where the decomposition (3.20) can be described by the disjoint unions of Feynman graphons. In other words, for each $v_1, ..., v_r \in V(t_\Gamma)$, set $[W_{\Gamma_v}]$ as the unlabeled graphon class with respect to the graph Γ_v. Then we have

$$W_{\Gamma_{v_1} \sqcup ... \sqcup \Gamma_{v_r}} = \frac{\sum_{j=1}^r W_{\Gamma_{v_j}}}{|\sum_{j=1}^r W_{\Gamma_{v_j}}|}, \tag{3.22}$$

$$[W_\Gamma] = [W_{\Gamma_{v_1}}] \sqcup ... \sqcup [W_{\Gamma_{v_r}}]. \tag{3.23}$$

Thanks to (3.18), the Tutte polynomial of each $[W_{\Gamma_{v_j}}]$ is defined in terms of the Tutte polynomial of the graph Γ_{v_j}. Then we have

$$T([W_\Gamma]; x, y) = \prod_{j=1}^r T([W_{\Gamma_{v_j}}]; x, y) \tag{3.24}$$

which leads us to a Feynman graphon version of the relation (3.21).

Theorem 3.1.4 describes the Tutte polynomial of a large Feynman diagram $X_{\rm DSE}$ on the basis of the Tutte polynomials of the partial sums $\{Y_m\}_{m \geq 1}$. The cut-distance convergence of the sequence of partial sums to $X_{\rm DSE}$ and the universality of the Tutte polynomial enable us to lift the properties (3.19) and (3.21) onto the Feynman graphon $[W_{X_{\rm DSE}}]$. As the consequence, we can define the abstract Feynman rules characters on large Feynman diagrams in terms of the Tutte polynomial where we have

$$\tilde{U}([W_{X_{\rm DSE}}]) := T([W_{X_{\rm DSE}}]; x, y) = T(X_{\rm DSE}; x, y). \tag{3.25}$$

□

Now we explain the construction of another important class of combinatorial polynomials namely, the first Kirchhoff–Symanzik polynomials for large Feynman diagrams.

The Feynman parametric representation of a Feynman integral $U(\Gamma)$ can be described by the integration theory over a topological simplex such as σ_n with respect to Feynman parameters $w = (w_1, ..., w_n) \in \sigma_n$ such that n is the number of internal edges of the corresponding Feynman diagram Γ. If $l = b_1(\Gamma)$ be the first Betti number of Γ (as the maximum number of independent loops in the graph) and an orientation has been fixed on the graph, then we can define the circuit matrix $\hat{\eta} = (\eta_{ik})_{ik}$ such that $i \in E(\Gamma)$ and k ranges over the chosen basis of loops. If an edge e_i belongs to a loop l_k with the same/reverse orientations, then $\eta_{ik} = 1$, $\eta_{ik} = -1$, respectively.

If the edge e_i does not belong to a loop l_k, then $\eta_{ik} = 0$. The arrays of the corresponding $l \times l$ Kirchhoff–Symanzik matrix $M_\Gamma(w)$ are given by

$$(M_\Gamma(w))_{kr} = \sum_{i=1}^{n} w_i \eta_{ik} \eta_{ir} \tag{3.26}$$

which defines a function $M_\Gamma : \mathbb{A}^n \longrightarrow \mathbb{A}^{l^2}$, $w = (w_1, ..., w_n) \longmapsto M_\Gamma(w)$ over higher dimensional affine spaces. The first Kirchhoff–Symanzik polynomial of the graph Γ is then defined by the equation

$$\Psi_\Gamma(w) = \det(M_\Gamma(w)) \tag{3.27}$$

which is independent of the choice of an orientation on the graph and the basis of loops. This function on $\mathbb{A}^n$, which is a homogeneous polynomial of degree l, can be formulated in the language of spanning trees where we have

$$\Psi_\Gamma(w) = \sum_{T \subset \Gamma} \prod_{e \notin E(T)} w_e \tag{3.28}$$

such that the sum is over all spanning trees T of the graph Γ and for each spanning tree the product is over all edges of Γ that are not in the selected spanning tree. We can show that this product is multiplicative over connected components.

Now consider a large Feynman diagram X with the corresponding sequence $\{Y_m\}_{m \geq 1}$ of partial sums. For each m, we know that the first Kirchhoff–Symanzik polynomial of Y_m is the product of the polynomials of each of its components which means that

$$\Psi_{Y_m}(w) = \prod_{j=1}^{m} \Psi_{X_j} \tag{3.29}$$

where

$$\Psi_{X_j}(w) = \sum_{T_j \subset X_j} \prod_{e \notin E(T_j)} w_e \tag{3.30}$$

such that the sum is taken over all the spanning forests T_j of X_j and for each spanning forest the product is taken over all edges of X_j that are not in that spanning forest.

Thanks to the cut-distance divergence of the sequence $\{Y_m\}_{m \geq 1}$ to X, for each $\epsilon > 0$, there exists N_ϵ such that for each $m_1, m_2 \geq N_\epsilon$, we have

$$d(Y_{m_1}, Y_{m_2}) = d_{\text{cut}}([W_{Y_{m_1}}], [W_{Y_{m_2}}]) < \epsilon. \tag{3.31}$$

It means that

$$d_{\text{cut}}([W_{Y_{m_1}}], [W_{Y_{m_2}}]) = 0 \Leftrightarrow [W_{Y_{m_1}}] = [W_{Y_{m_2}}]. \tag{3.32}$$

For enough large m, spanning forests of Y_m tend to the spanning forests of the unique graph limit X.

Definition 3.1.6. The first Kirchhoff–Symanzik polynomial $\Psi_X(w)$ of the large Feynman diagram X is defined as the convergent limit of the sequence $\{\Psi_{Y_m}(w)\}_{m\geq 1}$ of the first Kirchhoff–Symanzik polynomials of finite graphs $Y_m := X_1 \sqcup ... \sqcup X_m$ with respect to the cut-distance topology.

We can present this polynomial by the expansion

$$\Psi_X(w) = \prod_{j=1}^{\infty} \Psi_{X_j} = \sum_{T\subset X} \prod_{e\notin E(T)} w_e \tag{3.33}$$

such that the sum is taken over all the spanning forests T of X and for each spanning forest the product is taken over all edges of X that are not in that spanning forest.

Lemma 3.1.7. *The first Kirchhoff–Symanzik polynomial $\Psi_X(w)$ of the large Feynman diagram X can be defined recursively in terms of the deletion and the contraction operators.*

Proof. Set

$$F := \frac{\partial \Psi_X}{\partial w_n} = \Psi_X \setminus e \tag{3.34}$$

as the deletion operator which is the result of deleting the edge $e = e_n$ from the original graph. In addition, set

$$G := \Psi_X|_{w_n=0} = \Psi_X / e \tag{3.35}$$

as the contraction operator which is the result of contracting the edge $e = e_n$ to a point in the original graph.

For each edge e which is not a bridge or self-loop in the large Feynman diagram X, we can show that

$$\Psi_X = w_e F + G \tag{3.36}$$

such that $w_e F$ collects the monomials corresponding to spanning forests that do not include e. □

At the end of this section, we address a new application of the first Kirchhoff–Symanzik polynomial for the study of polynomial invariants of large Feynman diagrams and Feynman rules characters which act on Feynman graphons.

For a given large Feynman diagram X with the corresponding first Kirchhoff–Symanzik polynomial $\Psi_X(w)$, define

$$\hat{V}_X = \{w \in \mathbb{A}^{\infty} := \prod_{i=1}^{\infty} \mathbb{A}^{n_i} : \Psi_X(w) = 0\} \tag{3.37}$$

such that the affine hypersurface complement $\mathbb{A}^\infty \setminus \hat{V}_X$ enjoys the multiplicative property. We have

$$\mathbb{A}^\infty \setminus \hat{V}_X = \prod_{i=1}^{\infty} \mathbb{A}^{n_i} \setminus \hat{V}_{X_i} \tag{3.38}$$

such that n_i is the number of internal edges of the Feynman diagram X_i.

Consider the Grothendieck ring $\mathcal{F}$ of immersed conical varieties generated by the equivalence classes $[\hat{V}]$ up to linear changes of coordinates of varieties $\hat{V} \subset \mathbb{A}^\infty$ embedded in some affine space, that are defined by homogeneous ideals, with the usual inclusion-exclusion relation

$$[\hat{V}] = [\hat{R}] + [\hat{V} \setminus \hat{R}] \tag{3.39}$$

for closed embedding. Now we can define an algebro-geometric Feynman rules character on the renormalization Hopf algebra of Feynman graphons. It is an abstract Feynman rules character $\hat{U} : \mathcal{S}^{\Phi}_{\text{graphon}} \longrightarrow A_{\text{dr}}$ with the general form

$$\hat{U}([W_X]) = I([\mathbb{A}^\infty \setminus \hat{V}_X]) \tag{3.40}$$

such that $[\mathbb{A}^\infty \setminus \hat{V}_X]$ is the class in $\mathcal{F}$ and $I : \mathcal{F} \longrightarrow A_{\text{dr}}$ is a ring homomorphism.

We can also define a new invariant of infinite Feynman diagrams in terms of a generalization of the Chern–Schwartz–MacPherson (CSM) characteristic classes of singular varieties. The algebro-geometric Feynman rules have been constructed in terms of a polynomial invariant originated from the CSM characteristic classes ([5, 6, 117]) and here we plan to lift that study onto the level of Feynman graphons.

Corollary 3.1.8. *There exists an extension of the CSM homomorphism for the level of large Feynman diagrams generated by Dyson–Schwinger equations.*

Proof. The existence of the CSM-homomorphism I^{∞}_{CSM} is another consequence of the cut-distance topology and graphon representation of Feynman diagrams.

For a given large Feynman diagram X_{DSE} as the unique solution of an equation DSE, suppose $\Psi_{X_{\text{DSE}}}(w)$ is the first Kirchhoff–Symanzik polynomial and $\hat{V}_{X_{\text{DSE}}}$ is its associated hypersurface. In addition, suppose $1_{\hat{V}_{X_{\text{DSE}}}}$ is the function for $\hat{V}_{X_{\text{DSE}}} \subset \mathbb{A}^\infty$ and $A(\mathbb{P}^\infty)$ is the associated Chow group. The natural transformation

$$1_{\hat{V}_{X_{\text{DSE}}}} \longmapsto a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + a_2[\mathbb{P}^2] + ... \in A(\mathbb{P}^\infty) \tag{3.41}$$

allows us to define

$$G_{\hat{V}_{X_{\text{DSE}}}}(T) := a_0 + a_1 T + a_2 T^2 + ... + a_N T^N + ... \tag{3.42}$$

Now define

$$I^{\infty}_{\mathrm{CSM}} : \mathcal{F} \longrightarrow \mathbb{Z}[T], \quad [\hat{V}_{X_{\mathrm{DSE}}}] \longmapsto G_{\hat{V}_{X_{\mathrm{DSE}}}}(T) \tag{3.43}$$

and extend it by linearity to achieve a group homomorphism. □

3.2 *The optimization of non-perturbative complexity via a multi-scale renormalization group*

In complexity theory, the efficiency of an algorithm against a problem is judged in terms of the algorithm's capability in dealing with computational demands about quantities originated from the intrinsic complexity of that problem. An algorithm is known as feasible if it has a polynomial-time asymptotic scaling and it is known as infeasible if it has a super-polynomial (typically, exponential) scaling. The calculations of quantum field-theoretical scattering amplitudes at high precision or strong couplings are infeasible on classical computers but recently, there are some research efforts which aim to show that these calculations can be feasible on quantum computers. Traditional calculations of scattering amplitudes in Quantum Field Theory is on the basis of a series expansion in powers of the coupling constant (i.e. the coefficients of the interaction terms) such that the running coupling constant is taken to be small. Feynman diagrams provide an intuitive way to organize this class of perturbative expansions where the number of loops is associated with the power of the coupling constant. The number of this class of combinatorial diagrams gives us a reasonable measure to evaluate the computational complexity of perturbative calculations. This measure increases factorially with the number of loops and the number of external particles. Furthermore, if the amount of the coupling constant is insufficiently small, then the perturbative machinery can not provide correct results while the series expansions are divergent or asymptotic even at weak coupling constants. Indeed, if we include higher-order terms beyond a certain point, then the approximations can be inappropriate. In fact, by increasing the coupling constant, one eventually reaches a quantum phase transition at some critical couplings such that in the parameter space near this phase transition perturbative methods become unreliable. This region can be studied under strong-coupling regimes.

Generally speaking, limits of computations and the efficiently computing of things are the most important topics in information theory where people deal with the Halting problem as an undecidable type of problem which determines whether the program will finish running or continue to run forever. Thanks to rooted trees decorated by primitive recursive functions, Manin discovered a new reinterpretation of the Halting problem in the context of the BPHZ perturbative renormalization where the amount of (non-)computability have been encapsulated via the existence of the Birkhoff factorization at the level of the renormalization Hopf algebra of the Halting problem [52, 113, 114, 115].

Algorithms belong to the intermediate steps between programs and functions which means that they are classified as substructures in the context of Galois theory. This fundamental fact has already been applied to describe the foundations of a new categorical-geometric setting for the study of (systems) of Dyson–Schwinger equations (as the generators of intermediate steps) in the renormalization Hopf algebra of the Halting problem under dimensional regularization and the global β-functions. As the consequence of this treatment, we already have the construction of a new class of neutral Tannakian subcategories of the universal Connes–Marcolli category $\mathcal{E}^{\rm CM}$ which encode intermediate algorithms in the context of systems of differential equations together with singularities. In addition, these subcategories can address the existence of a new interrelationship between mixed Tate motives and theory of computation. Furthermore, thanks to the combinatorial reformulation of the universal counterterm, some new computational techniques for the study of the amount of non-computability in the language of the theory of Hall words have been obtained. As the big picture, infinities in the Computation Theory can be dealt via a renormalization theory on (systems) of Dyson–Schwinger equations and vice versa. [52, 113, 147]

It is so difficult to have an optimal solution when we want to consider a complex problem under a limited period of time. In this situation we work on the construction of anytime algorithms by computing an initial potentially highly suboptimal solution and then we improve the computed suboptimal solution as time allows.

The Kolmogorov complexity, as an uncomputable concept, aims to determine the length of the shortest algorithm which produces an object as the output of a procedure. Let Σ as the set of alphabets or letters and f be a computable function on the set of all possible strings generated by elements in Σ. A description of a string σ is some string τ with $f(\tau) = \sigma$. The Kolmogorov complexity K_f is defined by

$$K_f(\sigma) := \{^{\min\{|\tau|:f(\tau)=\sigma\}}_{\infty, \quad \text{otherwise}}. \tag{3.44}$$

It is possible to modify this definition independent of choosing f where we need to apply a universal Turing machine. In fact, there exists a Turing machine U such that for all partial computable functions f, there exists a program p such that for all y, we have $U(p, y) = f(y)$. It enables us to define $K(\sigma)$ as the Kolmogorov complexity of σ. It is shown that for all n, there exists some σ with $|\sigma| = n$ such that $K(\sigma) \geq n$. Such σ are called Kolmogorov random.

In this section, we plan to apply the graphon representation of (large) Feynman diagrams to study the Kolmogorov complexity of non-perturbative parameters. We will show that optimal algorithms in dealing with non-perturbative parameters can be achieved by working on a multi-scale non-perturbative Wilsonian renormalization group defined on the space $\mathcal{S}^{\Phi,g}$.

3.2.1 *A renormalization group program on* $\mathcal{S}^{\Phi,g}$

One important method for the study of the dynamics of quantum systems is changing the scales of fundamental parameters of the physical theory such as momentum, energy and mass. Theory of Renormalization Group aims to describe the behavior of quantum systems under this class of re-scalings where the possibility of exchanging information from scale to scale is considered under the fundamental principles of Quantum Mechanics. The interpretation of the concept of mass in the context of time and distance by using the Planck constant and the interpretation of the concept of time in the context of distance by using the speed of light enable us to study the dynamics of relativistic quantum systems under the re-scaling of the distance parameter. In this situation, small distances and times are equivalent to large momenta, energies and masses which produce divergencies in Quantum Field Theory.

There is another important parameter in Quantum Field Theory which encodes the strength of the fundamental forces. This parameter, which is known as the coupling constant, appears in the interaction part of the Lagrangian where we encode information of physical theory in the language of Green's functions and Feynman integrals. The amount of the coupling constant has direct influence on the complexity of Green's functions. As the basic fact, in QED we deal with couplings smaller than 1 while in QCD we deal with couplings at the size of 1 or larger than 1. In theoretical and experimental studies we study coupling constants under two settings namely, the bare couplings and the running couplings. Running coupling constants are the outputs of (dimensional) regularization and renormalization schemes and they have been applied in high energy levels to generate some intermediate quantities which are useful for the approximation of non-perturbative parameters. Running couplings guide us to deal with changing the scale of the momentum parameter where the Wilsonian type of the renormalization group has been formulated.

Generally speaking, there are two different well-known approaches for the formulation of non-perturbative renormalization group in Theoretical Physics namely, Wilson–Polchinski framework and effective average action. In Wilson–Polchinski framework, Physics at very small scale corresponds to a scale Λ in momentum space which is actually the inverse of a microscopic length where the partition function is given by

$$Z[B] = \int d\mu_{C_\Lambda}(\phi)\exp\big(- \int V(\phi) + \int B\phi\big) \tag{3.45}$$

such that $d\mu_{C_\Lambda}$ is a functional Gaussian measure with a cut-off at scale Λ. Now if we separate the field $\phi_p = \phi(p)$ into rapid and slow modes $\phi_{p,<}, \phi_{p,>}$

with respect to a scale k, then we can rewrite the partition function in terms of these components which lead us to define a running potential V_k at scale k via performing the integration on $\phi_>$. As the output we have

$$Z = \int d\mu_{C_k}(\phi_<)\exp(-\int V_k(\phi_<)) \tag{3.46}$$

such that when $k \leq \Lambda$, V_k involves derivative terms with any power of the derivatives of $\phi_<$. The Wilson–Polchinski equation is indeed a differential equation for the evolution of V_k with k such that the flow of potentials $V_k(\phi_<)$ do not contain all information on the initial theory and in addition, $V_k(\phi)$ involves infinitely many couplings contrarily to perturbation theory that involves only the renormalizable ones. In this method of non-perturbative renormalization group there is no general achievement about the convergence of the series of approximations that are used. In addition, the anomalous dimension is depended on the choice of cut-off parameters that separate the rapid and the slow modes whereas it should be independent of it. In effective average action method, the basic idea is to build the 1-parameter family of models for which a momentum depended mass term is added to the original Hamiltonian where we have

$$Z_k[B] = \int \mathcal{D}\phi(x)\exp\big(-H[\phi] - \Delta H_k[\phi] + \int B\phi\big) \tag{3.47}$$

$$\Delta H_k[\phi] = \frac{1}{2}\int_q R_k(q)\phi_q\phi_{-q}. \tag{3.48}$$

For $0 < k < \Lambda$, the rapid modes are almost unaffected by the cut-off function $R_k(q)$ (as a homogeneous to a mass square) which means that $R_k(|q| > k) \simeq 0$. Set

$$W_k[B] := \log Z_k[B] \tag{3.49}$$

with the corresponding Legendre transformation $\Gamma_k[M(x)] = \Gamma_k[\frac{\delta W_k}{\delta B(x)}]$. The renormalization group equation on Γ_k is the differential equation of the type

$$\partial_k\Gamma_k = f(\Gamma_k). \tag{3.50}$$

It is shown that by working on dimensionless and renormalized quantities, the resulting non-perturbative renormalization group can be written independently of the scales k and Λ. The geometry of the resulting renormalization group flow from this framework supports the universality of self-similarity and decoupling of massive modes. [38, 85, 118, 127, 128, 131, 132, 134, 167]

In this part we address an alternative Renormalization Group treatment for the study of non-perturbative parameters in terms of changing the scales of Dyson–Schwinger equations via re-scaling bare and running couplings. Our study provides a mathematical procedure to exchange information among non-perturbative aspects under different scales. For this purpose, we plan to build a new multi-scale Renormalization Group machinery on the space $\mathcal{S}^{\Phi,g}$ of all Dyson–Schwinger equations in the physical theory Φ under different scales λg of the bare coupling constant g.

The bare couplings are independent of any regularization and renormalization schemes and therefore their re-scalings can be helpful for us to approximate non-perturbative parameters originated from Dyson–Schwinger equations under a universal setting. There are two fundamental challenges in Theoretical High Energy Physics about running coupling constants. On the one hand, running couplings are unobervable. On the other hand, the most important mission of the renormalization group is to show that the predictions for the observables do not depend on theoretical conventions such as renormalization or regularization schemes, the initial state, the choice of effective charge or the choice of running coupling constants. Therefore different choices of these couplings should be related to each other which means that search for an optimal choice is very important. Our promising non-perturbative renormalization group enables us to study Dyson–Schwinger equations under changing the scales of bare couplings and running couplings not simultaneously but related to each other. We expect that this alternative machinery is helpful to provide a theoretical algorithm for the determination of effective couplings where the complexity of the corresponding Dyson–Schwinger equations will be controlled in terms of changing the scale of the bare coupling constant.

Let $\mathcal{X}^g$ be the collection of all interacting Lagrangians with coefficients in the ring $\mathbb{R}[[g]]$, invariant under the change $\phi \longrightarrow -\phi$, and interaction parts with the general form

$$I(\phi) := \sum_{k\geq 2} I_k(\phi) \tag{3.51}$$

such that for all k, $I_k = O(g)$ with respect to the bare coupling constant g. Changing the scale of g allows us to obtain an effective Lagrangian at the scale $\tau \leq \lambda$ of a Lagrangian L at the initial scale λ. The interaction part is the original source of Dyson–Schwinger equations and therefore the re-scaling of the coupling constants lead us to re-scale these non-perturbative type of equations. The renormalization group with respect to this class of momentum type re-scaling enables us to discuss about the possibility of exchanging information among re-scaled Dyson–Schwinger equations.

For each k, set F_k as the set of all smooth functions on the hyperplane $\sum_{i=1}^{k} v_i = 0$ in $(V^*)^{\oplus k}$ such that V is the Euclidean 4-dimensional space-time. Define $F := \prod_{i=1}^{\infty} F_{2i}$ to formulate Green's functions $\mathcal{G}$ given by

$$\mathcal{G} : \mathcal{X}^g \times M_m \longrightarrow F, \ \ \mathcal{G} := (\mathcal{G}_2, \mathcal{G}_4, ...) \tag{3.52}$$

such that
- M_m is the set of scales of the momentum parameter,

- the value $\mathcal{G}_k$ at (L, λ) is called the k-point correlation function of the Lagrangian L at the scale λ,
- for each k, $\mathcal{G}_k$ is the formal expansion of amplitudes of all Feynman diagrams with k external edges.

Dyson–Schwinger equations are actually formulated as the fixed point equations of $\mathcal{G}$ with the general form

$$\mathcal{G} = 1 + \int I_\gamma \mathcal{G} \tag{3.53}$$

such that I_γ is the integral kernel with respect to the (IPI) primitive Feynman diagram γ.

Definition 3.2.1. An equation in $\mathcal{S}^{\Phi,g}$ is called effective at the scale τ of the original Dyson–Schwinger equation DSE at the initial scale λ, if the fixed point equation of the Green's function $\mathcal{G}(L^\Phi, \lambda)$ coincides with the fixed point equation of the Green's function $\mathcal{G}(L^\Phi, \tau)$.

It is shown in [150] that we can build a unique effective equation at the scale τ for any equation DSE in $\mathcal{S}^{\Phi,g}$ at the original scale λ of the momentum parameter. It enables us to change the scale of the momenta of internal edges of each term in the formal expansion of the solution of DSE.

In higher orders in perturbation theory we should deal with a large number of Feynman diagrams which cost us exponentially growing of the momentum scale. Therefore all orders in perturbation theory do not accessible for any scale of the momentum parameter. The asymptotic freedom behavior of QCD at very high energies enables us to study the physics of hadrons under perturbative setting but at a relatively low energy scale, the amount of the coupling constant becomes too large where non-perturbative situations do happen. Running coupling constants, as the functions of the momentum parameter, describe the strength of the interactions among quarks and gluons. The determination of this class of couplings has very uncertainty nature which makes so many computational and phenomenological difficulties. Dimensional regularization allows us to replace the bare coupling constant with a class of scaled depended couplings. The ultraviolet divergencies are eliminated by normalizing the coupling at a specific momentum scale. In addition, the ultraviolet cut-off dependency is removed by allowing the couplings and masses, which appear in the Lagrangian, to have a scale dependency where we can produce running couplings on the basis of normalizing them to a measured value at a given scale. This normalization of the coupling to a measured value makes the running coupling to not have sensitivity to the ultraviolet cut-off. The scale dependency of the strong coupling can be controlled by β-function as the infinitesimal generator of the renormalization group. [40, 119, 120]

Thanks to this background, it is possible to formulate a new multi-scale renormalization group on $\mathcal{S}^{\Phi,g}$ to analyze the behavior of Dyson–Schwinger equations.

Theorem 3.2.2. *There exists a renormalization group machinery on $\mathcal{S}^{\Phi,g}$ which encodes the dynamics of Dyson–Schwinger equations under changing the scales of the bare and running coupling constants.*

Proof. Set M_{running} as the set of scales of the running couplings. For scales $\Lambda_1, \Lambda_2, \Lambda_3 \in M_{\text{running}}$ such that $\Lambda_1 < \Lambda_2 < \Lambda_3$, define the scale map R^{running}_{--} on $\mathcal{S}^{\Phi,g}$ which satisfies the property

$$R^{\text{running}}_{\Lambda_1\Lambda_2} R^{\text{running}}_{\Lambda_2\Lambda_3} = R^{\text{running}}_{\Lambda_1\Lambda_3}. \tag{3.54}$$

For each equation DSE, $R^{\text{running}}_{\Lambda_1\Lambda_2}$DSE is the effective Dyson–Schwinger equation at the scale Λ_2 of the equation DSE at the original scale Λ_1. Now define an action of the semigroup $\mathbb{R}^+_{\leq 1}$ on the space $\mathcal{S}^{\Phi,g} \times M_{\text{running}}$ given by

$$r \circ (\text{DSE}, \Lambda) := (R^{\text{running}}_{\Lambda,r\Lambda}\text{DSE}, r\Lambda). \tag{3.55}$$

The resulting Renormalization Group allows us to study the dynamics of Dyson–Schwinger equations under the re-scaling of the running couplings.

Set M_{bare} as the set of scales of the bare coupling constant g. For scales $\tau_1, \tau_2, \tau_3 \in M_{\text{bare}}$ such that $\tau_1 < \tau_2 < \tau_3$, define the scale map R^{bare}_{--} on $\mathcal{S}^{\Phi,g}$ which satisfies the property

$$R^{\text{bare}}_{\tau_1\tau_2} R^{\text{bare}}_{\tau_2\tau_3} = R^{\text{bare}}_{\tau_1\tau_3}. \tag{3.56}$$

For each equation DSE in $\mathcal{S}^{\Phi,g}$ define a new Dyson–Schwinger equation $R^{\text{bare}}_{\tau_1\tau_2}$DSE which is a re-scaled equation at the scale τ_2 of the equation DSE at the initial scale τ_1. Now define an action of the semigroup $\mathbb{R}^+_{\leq 1}$ on the space $\mathcal{S}^{\Phi,g} \times M_{\text{bare}}$ given by

$$r \circ (\text{DSE}, \tau) := (R^{\text{bare}}_{\tau,r\tau}\text{DSE}, r\tau). \tag{3.57}$$

The resulting Renormalization Group allows us to study the dynamics of Dyson–Schwinger equations under the re-scaling of the bare coupling constant g.

Thanks to (3.55) and (3.57), we can define a new multi-scale renormalization group on $\mathcal{S}^{\Phi,g}$ where it is possible to re-scale the bare coupling constant $g \longmapsto \tau g$ before the application of regularization schemes.

Each equation $(\text{DSE}, \tau g, \Lambda_\tau)$ in $\mathcal{S}^{\Phi,g} \times M_{\text{bare}} \times M_{\text{running}}$ presents a Dyson–Schwinger equation DSE as a polynomial on the re-scaled bare and running coupling constants which is an infinite formal expansion of Feynman integrals with respect to the re-scaled bare coupling constant τg (as the initial scale) such that each Feynman integral in the expansion is defined based on the momentum parameter at the initial scale Λ_τ. Now define a new action of the semi-group $\mathbb{R}^+_{\leq 1}$ on $\mathcal{S}^\Phi \times M_{\text{bare}} \times M_{\text{running}}$ as the following way

$$\lambda \circ (\text{DSE}, \tau g, \Lambda_\tau) := (R^{\text{multi}}_{(\tau g, \Lambda_\tau),(\lambda\tau g, \lambda\Lambda_\tau)}\text{DSE}, (\lambda\tau g, \lambda\Lambda_\tau)). \tag{3.58}$$

The equation $R^{\text{multi}}_{(\tau g, \Lambda_\tau),(\lambda\tau g, \lambda\Lambda_\tau)}\text{DSE}$ is the unique effective Dyson–Schwinger equation with respect to the re-scaled bare coupling $\lambda\tau g$ and re-scaled momentum parameter $\lambda\Lambda_\tau$ which lives in $\mathcal{S}^{\Phi,g}$. □

Roughly speaking, the renormalization machinery enables us to redefine the unrenormalized constants which exist in the Lagrangian in such a way that the observable quantities remain finite when the ultraviolet cut-off is removed. This machinery requires a new quantity μ with the dimension of a mass where all intermediate quantities are depended on μ. The confinement in QCD does not allow us to determine a natural scale for μ. The μ dependence of the coupling constant and various quark masses in QCD force us to define running coupling constants and running masses where the renormalization group equations can control the μ dependence of the resulting renormalized quantities. The running coupling constant $g(\mu^2)$ can be studied in terms of the equation

$$\mu^2 \frac{dg(\mu^2)}{d\mu^2} = \beta(g(\mu^2)) \tag{3.59}$$

which leads us to

$$g(\mu^2) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} \tag{3.60}$$

such that the dimensional scale Λ is the scale at which the coupling diverges and perturbation theory becomes meaningless. When the cut-off parameter tends to infinity, $\beta(g(\mu^2))$ remains finite such that in perturbation theory we have

$$\beta(g(\mu^2)) = -g(\mu^2)^2(\beta_0 + \beta_1 g(\mu^2) + \beta_2 g(\mu^2)^2 + ...). \tag{3.61}$$

The Renormalization Group machinery defined by Theorem 3.2.2 is non-commutative because the scale of the momentum parameter is completely depended on the chosen re-scaling of the bare coupling. It encodes the dynamics of non-perturbative aspects of quantum systems by collections of Dyson–Schwinger equations derived from changing the scales of couplings.

Dimensional regularization or other regularization schemes changes the nature of the bare couplings to describe QFT under a perturbative setting but it fails to be functional in enough high orders. Theorem 3.2.2 enables us to generate a new class of running couplings in terms of the re-scaled bare coupling which are independent of any regularization process. Therefore the resulting running couplings preserve the nature of the bare coupling which means that they have physical meanings. As one practical consequence, this multi-scale Renormalization Group provides a new alternative machinery for the description of each Dyson–Schwinger equation at the strong coupling constant g in terms of a sequence of Dyson–Schwinger equations at the weaker couplings λg in the space $\mathcal{S}^{\Phi,g}$ equipped with the cut-distance topology. This approach enables us to apply Feynman graphons for the description of the complexity of an equation DSE in terms of less complicated equations with respect to our given running couplings. In other words, we can compute the unique solution $X(\lambda g)$ of the equation DSE(λg) for $\lambda g < 1$ as intermediate values for the approximation of the large Feynman diagram $X(g)$.

Corollary 3.2.3. *For each large Feynman diagram $X(g) = \sum_{m=0}^{\infty} g^m X_m$ derived from an equation DSE in $\mathcal{S}^{\Phi,g}$ at the strong bare coupling constant $g \geq 1$, there exists a sequence of large Feynman diagrams at weaker effective couplings which converges to $X(g)$ with respect to the cut-distance topology.*

Proof. Thanks to Theorem 3.2.2, we can build the sequence $\{R^{\text{bare}}_{g,\frac{n}{n+1}g}\text{DSE}\}_{n\geq 1}$ of Dyson–Schwinger equations with respect to the re-scaled bare coupling constants $\frac{n}{n+1}g$ for each $n \geq 1$ where the initial scale of the equation DSE is at least 1.

For each n, $R^{\text{bare}}_{g,\frac{n}{n+1}g}\text{DSE}$ is an equation in $\mathcal{S}^{\Phi,g}$ which has the unique solution

$$Y(\frac{n}{n+1}g) = \sum_{m=0}^{\infty} (\frac{n}{n+1}g)^m X_m. \tag{3.62}$$

The scales $\frac{n}{n+1}$ for each $n \geq 1$ provide an increasing sequence of effective couplings derived from the bare coupling constant g where $\frac{n}{n+1}g < g$. Therefore for each n, the solution $Y(\frac{n}{n+1}g)$ of the equation $R^{\text{bare}}_{g,\frac{n}{n+1}g}\text{DSE}$ is actually a disjoint union of multi-loop Feynman diagrams which can be handled by higher order perturbation methods. It remains to show that the sequence $\{Y(\frac{n}{n+1}g)\}_{n\geq 1}$ is convergent to $X(g)$ with respect to the cut-distance topology. Thanks to Lemma 2.3.3, for each $n \geq 1$, we can associate a unique unlabeled graphon class $[W_{Y(\frac{n}{n+1}g)}]$ with respect to each large Feynman diagram $Y(\frac{n}{n+1}g)$. Thanks to Definition 2.3.4, it is enough to show that the sequence $\{[W_{Y(\frac{n}{n+1}g)}]\}_{n\geq 1}$ is convergent to the unlabeled graphon class $[W_{X(g)}]$.

For a fixed $n \geq 1$, we can check that the labeled grahons $W_{(\frac{n}{n+1}g)^m X_m}$ and $W_{g^m X_m}$ belong to the same unlabeled graphon class which means that they are weakly equivalent (for each $m \geq 0$). Therefore when m tends to infinity, labeled graphons $W_{Y(\frac{n}{n+1}g)}$ and $W_{X(g)}$ are also weakly equivalent. □

Corollary 3.2.4. *For any Dyson–Schwinger DSE in $\mathcal{S}^{\Phi,g}$, there exists a sequence of many-loop Feynman diagrams such that their corresponding BPHZ* *counterterms and renormalized values converge to the counterterm and renormalized value generated by the renormalization of the equation DSE.*

Proof. Thanks to Corollary 3.2.3, there exists a sequence $\{\Gamma_n\}_{n\geq 1}$ of many-loop Feynman diagrams in $\mathcal{S}^{\Phi,g}$ at the re-scaled bare coupling constants $\lambda_n g$ and running couplings Λ_{λ_n} which converges to the unique solution $X_{\mathrm{DSE}}(g)$ with respect to the cut-distance topology. Now apply Theorem 2.4.1 to each graph Γ_n to build sequences

$$\{S^{\tilde{\phi}}_{R_{\mathrm{ms}}}([W_{\Gamma_n}])\}_{n\geq 1} \tag{3.63}$$

and

$$\{S^{\tilde{\phi}}_{R_{\mathrm{ms}}} * \tilde{\phi}([W_{\Gamma_n}])\}_{n\geq 1} \tag{3.64}$$

which are cut-distance convergent to $S^{\tilde{\phi}}_{R_{\mathrm{ms}}}([W_{X_{\mathrm{DSE}}(g)}])$ and $S^{\tilde{\phi}}_{R_{\mathrm{ms}}}*\tilde{\phi}([W_{X_{\mathrm{DSE}}(g)}])$, respectively □

Thanks to these investigations, we expect that the multi-scale renormalization group defined by Theorem 3.2.2 plays a practical role to optimize the computational procedures in dealing with non-perturbative parameters. We deal with this topic in the next section.

3.2.2 *Kolmogorov complexity of Dyson–Schwinger equations*

In this part, we plan to address the complexity of computations in non-perturbative parameters in the context of the multi-scale renormalization group machinery $R^{\mathrm{multi}}_{(.,.),\lambda(.,.)}$ and the Halting problem to study non-perturbative Feynman rules characters on Feynman graphons.

We define the Kolmogorov complexity of each Dyson–Schwinger equation DSE in $\mathcal{S}^{\Phi,g}$ in terms of changing the scale of the bare coupling constant where exchanging information among equations at different scales have been encoded by the non-perturbative multi-scale renormalization group $R^{\mathrm{multi}}_{(.,.),\lambda(.,.)}$ (i.e. Theorem 3.2.2).

For our framework we need to see $\mathcal{S}^{\Phi,g}$ as the constructive world which covers all non-perturbative situations in Quantum Field Theory Φ with strong bare coupling constant g. Thanks to Corollary 3.2.3, define

$$u^g : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}, \quad (n, \mathrm{DSE}(g)) \longmapsto \mathrm{DSE}(\frac{n}{n+1}g) \tag{3.65}$$

as a semi-computable function in the sense that there exists an algorithm which encodes the application of u^g on Dyson–Schwinger equations.

Definition 3.2.5. The Kolmogorov complexity of an equation $\mathrm{DSE}(\lambda g)$ at the scale λg with respect to the function u^g is determined by the relation

$$K_{u^g}(\mathrm{DSE}(\lambda g)) := \min\{n \in \mathbb{Z}_+ : \quad u^g(n, \mathrm{DSE}(g)) \subseteq \mathrm{DSE}(\lambda g)\} \tag{3.66}$$

such that the inclusion means that the unique solution $X_{\mathrm{DSE}(\frac{n}{n+1}g)}$ as a large graph can be embedded as a subgraph into the large Feynman diagram $X_{\mathrm{DSE}(\lambda g)}$.

Lemma 3.2.6. *There exists the Kolmogorov total order on* $\mathcal{S}^{\Phi,g}$.

Proof. The Kolmogorov order of $\mathcal{S}^{\Phi,g}$ is defined as a bijection $\mathbf{K}_{u^g} : \mathcal{S}^{\Phi,g} \longrightarrow \mathbb{Z}_+$ which arranges all Dyson–Schwinger equations of the physical theory Φ in the increasing order of their complexities $\mathbf{K}_{u^g}(\mathrm{DSE}(\lambda g))$. Define

$$\mathrm{DSE}(\lambda_1 g) < \mathrm{DSE}(\lambda_2 g) \Longleftrightarrow K_{u^g}(\mathrm{DSE}(\lambda_1 g)) < K_{u^g}(\mathrm{DSE}(\lambda_2 g)). \quad (3.67)$$

It is possible to determine some constants $c_0 > 0$ such that for all Dyson–Schwinger equations such as $\mathrm{DSE}(\lambda g)$,

$$c_0 K_{u^g}(\mathrm{DSE}(\lambda g)) \leq \mathbf{K}_{u^g}(\mathrm{DSE}(\lambda g)) \leq K_{u^g}(\mathrm{DSE}(\lambda g)). \quad (3.68)$$

□

Thanks to Definition 3.2.5 and Definition 3.2.6, it is now possible to consider $\mathcal{S}^{\Phi,g}$ as a poset such that for any given partial recursive map $\sigma : \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}$ which generates a permutation, we can define a new map

$$\sigma_{\mathbf{K}_{u^g}} := \mathbf{K}_{u^g} \circ \sigma \circ \mathbf{K}_{u^g}^{-1} \quad (3.69)$$

where it provides a permutation of the subset

$$\mathrm{D}(\sigma_{\mathbf{K}_{u^g}}) := \mathbf{K}_{u^g}(\mathrm{Dom}(\sigma)) \subseteq \mathbb{Z}_+. \quad (3.70)$$

Now suppose equation $\mathrm{DSE}(\lambda g) \in \mathrm{Dom}(\sigma)$ such that its corresponding orbit $\sigma^{\mathbb{Z}}(\mathrm{DSE}(\lambda g))$ is infinite. Set

$$\mathbf{K}_{u^g}(\mathrm{DSE}(\lambda g)) := k^{\lambda}_{\mathrm{DSE}} \quad (3.71)$$

such that for each $n > 0$, we have

$$\sigma^n_{\mathbf{K}_{u^g}}(k^{\lambda}_{\mathrm{DSE}}) = \mathbf{K}_{u^g}(\sigma^n(\mathrm{DSE}(\lambda g))) \leq c\mathbf{K}_{u^g}(n) \quad (3.72)$$

In [112] it is discussed that for any partial recursive function $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ and $x \in \mathrm{Dom}(f)$ we have

$$K(f(x)) \leq c_f K(x) \leq c_f^{'} x. \quad (3.73)$$

We want to apply the inequality (3.73) for the Kolmogorov complexity of Dyson–Schwinger equations defined by Definition 3.2.5 and bijection $\mathbf{K}_{u^g}$. Define

$$Y := \{\sigma^n(\mathrm{DSE}(\lambda g)) : \quad n \in \mathbb{Z}_+\} \quad (3.74)$$

as a recursively enumerable subset of $\mathcal{S}^{\Phi,g}$ which plays the role of the domain for a partial recursive function $A : \mathcal{S}^{\Phi,g} \longrightarrow \mathbb{Z}_+$ given by

$$A(\mathrm{DSE}(\tau g)) = n, \quad \text{if } \sigma^n(\mathrm{DSE}(\lambda g)) = \mathrm{DSE}(\tau g). \tag{3.75}$$

We have

$$\mathbf{K}_{u^g}^{-1}(n) = \mathbf{K}_{u^g}^{-1}(A(\mathrm{DSE}(\tau g))) \leq c^{'}\mathbf{K}_{u^g}(\mathrm{DSE}(\tau g)) = c^{'}\mathbf{K}_{u^g}(\sigma^n(\mathrm{DSE}(\lambda g))). \tag{3.76}$$

As the consequence, we can obtain the following upper and lower boundaries for the permutation $\sigma_{\mathbf{K}_{u^g}}$,

$$c_1\mathbf{K}_{u^g}^{-1}(n) \leq \sigma^n_{\mathbf{K}_{u^g}}(k^\lambda_{\mathrm{DSE}}) \leq c_2\mathbf{K}_{u^g}^{-1}(n). \tag{3.77}$$

Lemma 3.2.7. *Consider $\mathcal{S}^{\Phi,g}$ as the constructive world and equip this collection with a total recursive structure of additive group without torsion with the zero element $\mathbf{0}$. The Halting problem for any partial recursive function $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}$ can be described in the language of fixed point equations.*

Proof. Extend f to a new function

$$g_f : \mathbb{Z}_+ \times (\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\}) \longrightarrow (\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\}) \tag{3.78}$$

such that

$$g_f((n,X)) := \mathbf{0}, \quad \text{if } \ (n,X) \notin \mathrm{Dom}(f). \tag{3.79}$$

Now define a new permutation

$$\tau_f : \mathbb{Z}_+ \times (\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\}) \longrightarrow (\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\}),$$

$$\tau_f(n,(X,Y)) := (X + g_f((n,Y)), Y). \tag{3.80}$$

We can check that finite orbits of τ_f are fixed points. It leads us to build a new partial recursive permutation σ_f with the domain

$$\mathrm{Dom}(\sigma_f) := (\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\}) \times \mathrm{Dom}(f). \tag{3.81}$$

Thanks to [112, 114] and the definition of g_f, we can show that the complement to $\mathrm{Dom}(\sigma_f)$ in the constructive world $(\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g}\coprod\{\mathbf{0}\})$ covers the fixed points of τ_f. This process reduces the Halting problem for f to the determination of the fixed points of τ_f. □

For the constructive world $\mathcal{S}^{\Phi,g}$, the map u^g given by (3.65), the map $\sigma_{\mathbf{K}_{u^g}}$ given by (3.69) and (3.70), the integer value k^λ_{DSE} given by (3.71), define

$$\Psi(k^\lambda_{\mathrm{DSE}}, \sigma, u^g, z) := \frac{1}{(k^\lambda_{\mathrm{DSE}})^2} + \sum_{n\geq 1} \frac{z^{\mathbf{K}_{u^g}(\mathrm{DSE}(\lambda\frac{n}{n+1}g))}}{(\sigma^n_{\mathbf{K}_{u^g}}(k^\lambda_{\mathrm{DSE}}))^2}. \tag{3.82}$$

Corollary 3.2.8. *- If the σ-orbit of the equation* $\mathrm{DSE}(\lambda g) \in \mathrm{Dom}(\sigma)$ *is finite, then* $\Psi(k^{\lambda}_{\mathrm{DSE}}, \sigma, u^g, z)$ *is a rational function in the complex variable* z*. All poles of this formal series, which are of the first order, live at roots of unity.*

- If the σ-orbit of the equation $\mathrm{DSE}(\lambda g) \in \mathrm{Dom}(\sigma)$ *is infinite, then* $\Psi(k^{\lambda}_{\mathrm{DSE}}, \sigma, u^g, z)$ *is the Taylor series of an analytic function on the region* $|z| < 1$ *which is continuous at the boundary of this region.*

Proof. It is a direct result of the discussions in [112, 114] where we need to replace the constructive world $\mathbb{Z}_+$ with $\mathcal{S}^{\Phi,g}$. □

Thanks to the Manin's reconstruction of the Halting problem in the language of the BPHZ renormalization program ([113, 114, 115]) and the explained machinery with respect to the constructive world $\mathcal{S}^{\Phi,g}$, now it is possible to deal with the Halting problem for a given partial recursive map $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}$. We can reduce f to a partial recursive permutation

$$\sigma_f : \mathrm{Dom}(\sigma_f) \subset \mathcal{S}^{\Phi,g} \longrightarrow \mathrm{Dom}(\sigma_f) \subset \mathcal{S}^{\Phi,g} \tag{3.83}$$

to interpret the problem of recognizing whether a positive integer number k belongs to the domain $\mathrm{Dom}(\sigma_f)$ to the problem of whether the corresponding analytic function $\Psi(k, \sigma_f, u^g, z)$ of a complex parameter z has a pole at $z = 1$.

Theorem 3.2.9. *The BPHZ renormalization of Feynman graphons encodes the Halting problem for a given partial recursive map* $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}$.

Proof. Thanks to the construction of the renormalization Hopf algebra of Feynman graphons $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ and the BPHZ renormalization of large Feynman diagrams, consider the character

$$\varphi_k : \mathcal{S}^{\Phi}_{\mathrm{graphon}} \longrightarrow A_{\mathrm{dr}}, \quad \varphi_k([W_{X_{\mathrm{DSE}}}]) := \Psi(k^{\lambda}_{\mathrm{DSE}}, \sigma_f, u^g, z). \tag{3.84}$$

Thanks to the Birkhoff factorization on the regularization algebra A_{dr}, we have $A_{\mathrm{dr}} = \mathcal{A}_+ \oplus \mathcal{A}_-$ such that $\mathcal{A}_+$ is the unital algebra of analytic functions in the region $|z| < 1$ which are continuous on the boundary $|z| = 1$ and $\mathcal{A}_- := (1-z)^{-1}\mathbb{C}[(1-z)^{-1}]$.

By applying Lemma 3.2.7 and Corollary 3.2.8, discussion about the existence of a pole at $z = 1$ for the analytic function $\Psi(k^{\lambda}_{\mathrm{DSE}}, \sigma_f, u^g, z)$ enables us to determine whether $k^{\lambda}_{\mathrm{DSE}}$ lives in $D(\sigma_f)$. □

The main reason of this important result is the existence of a class of semi-computable maps such as u^g (for a given strong coupling g) which has led us to define a modified version of the Kolmogorov complexity for Dyson–Schwinger equations (i.e. Definition 3.2.5). The dynamics of the well-defined map u^g (3.65) can be studied by the multi-scale Renormalization Group machinery which is defined on $\mathcal{S}^{\Phi,g}$. We can define the Kolmogorov complexity

K_w on Dyson–Schwinger equations with respect to other arbitrary elements w of the set of Kolmogorov optimal functions where the optimality means that for any partial recursive $v : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}$ there exists a constant $c_{v,w} > 0$ such that for each Dyson–Schwinger equation $\mathrm{DSE}(\lambda g)$,

$$K_w((n, \mathrm{DSE}(\lambda g))) \leq c_{v,w} K_v((n, \mathrm{DSE}(\lambda g))). \tag{3.85}$$

Thanks to Corollary 3.2.3, relations (3.68) and (3.77) and Theorem 3.2.9, which determines the amount of non-computability via the Halting problem at the level of Feynman graphons, those semi-computable maps defined in terms of the map R_{--}^{multi} can be considered as the truth candidate for this optimality.

Chapter 4

The dynamics of non-perturbative QFT in the language of noncommutative geometry

- *A spectral triple model for quantum motions*
- *A noncommutative symplectic geometry model for* $\mathcal{S}^{\Phi}_{\text{graphon}}$

Noncommutative geometry studies geometric properties of singular spaces on the basis of suitable coordinate algebras where point spaces are replaced by (noncommutative) function algebras. The standard differential and integral calculi have been adapted to a more general setting in the way compatible with the interpretation of variable quantities in Quantum Mechanics as operators on the Hilbert space of states and spectral analysis. The interplay between Algebra and Topology has been studied conceptually and contextually under two general settings on the basis of the theory of Hopf algebras (or quantum groups) and the theory of C*-algebras. In the resulting dictionary, noncommutative C*-algebras, which are interpreted as the algebras of continuous functions on some virtual noncommutative spaces, are the dual arena for noncommutative topology. As the important consequence of this interrelationship, the theory of spectral triples and the theory of noncommutative differential graded algebras enable us to build the foundations of differential and integral calculi in noncommutative geometry. [36, 43]

The idea of applying noncommutative geometry to Quantum Field Theory has already been considered and developed by different groups of mathematicians and mathematical/theoretical physicists where we can address new models of gauge field theories or the mathematical foundations of Standard Model and its modified versions in dealing with elementary particles [29, 37, 41, 44, 45, 109, 122]. Furthermore, thanks to the renormalization Hopf algebra, some new applications of noncommutative geometry tools in dealing with Dyson–Schwinger equations were found where two classes of differential graded algebras had been formulated to describe the geometry of quantum motions. The first class of differential graded algebras was built in the way to determine a family of connections which encode quantum motions independent of the chosen regularization scheme [141]. The second class of differential graded algebras was built in the way to encode regularization and renormalization processes of Feynman diagrams which contribute to solutions of Dyson–Schwinger equations in the language of noncommutative differential forms. This setting, which applies shuffle products and Rota–Baxter algebras ([68]), has provided a new geometric interpretation of the Connes–Kreimer renormalization group in the context of integrable systems under a non-perturbative setting [146].

In this chapter, we plan to continue our search for some new applications of noncommutative geometry to non-perturbative aspects ([150]). At the first step, we explain the construction of a new class of spectral triples which encodes the geometry of Dyson–Schwinger equations under an operator theoretic setting. This study provides the foundations of a theory of spectral geometry for the description of large Feynman diagrams. At the second step, we search for a new class of differential graded algebras on Feynman graphons to build a noncommutative differential geometry machinery for the description of physical parameters generated by large Feynman diagrams.

4.1 *A spectral triple model for quantum motions*

Geometric objects associated to any n-dimensional C^∞ manifold M such as vector fields, differential forms, general tensor fields, vector bundles, Riemannian metric, connections, curvature tensor, etc are encoded via the commutative algebra $C^\infty(M)$ (i.e. infinite times differentiable functions on M) and some extra operators on this algebra. If we replace the algebra $C^\infty(M)$ with a noncommutative algebra A (such as the algebra generated by some deformation procedures on $C^\infty(M)$), then we can achieve the basic elements of noncommutative geometry as a generalization of the standard commutative geometry of manifolds. The basic pedagogical example of a noncommutative space is given via Gelfand–Naimark Theorem where studying commutative C*-algebras is translated to studying compact topological (Hausdorff) spaces and vice versa. It leads us to a general idea that studying noncommutative C*-algebras becomes to studying "noncommutative" compact topological spaces. [36]

Classical Mechanics can be interpreted as the fundamental example of a commutative geometry where the phase space of a system of N non-relativistic particles is a $6N$ dimensional symplectic manifold M and the physical observables, energy, angular momentum, etc are functions in $C^\infty(M)$. Quantum Mechanics can be interpreted as the fundamental example of a noncommutative geometry where we should deal with a noncommutative algebra of quantum observables consisting of operators on the Hilbert space of states. The position operator Q and the momentum operator P (as unbounded self-adjoint operators) satisfy the canonical Heisenberg's commutation relation

$$PQ - QP = -i\hbar I. \tag{4.1}$$

The physical observables are represented by hermitian operators. If we apply one-parameter unitary groups $U_s = e^{isP}, \ \ V_t = e^{itQ}$, then we have the Weyl form of the commutation relation namely,

$$U_s V_t = e^{-i\hbar st} V_t U_s. \tag{4.2}$$

Set $s = t = 1, \ \ \lambda = -2\pi\hbar$ to obtain unitary bounded operators U, V on the same Hilbert space which enjoy the property $UV = e^{2\pi i\lambda} VU$. The noncommutative polynomial algebra A_λ generated by U, V and their corresponding adjoint operators and equipped with the operator norm is actually a noncommutative C^*-algebra derived from Quantum Mechanics.

Deformation quantization focuses on the construction of a noncommutative algebra of quantum observables in terms of defining some new noncommutative type of products on the vector space $C^\infty(M)$. The deformation of the coordinates of space-time with respect to relations such as $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ is another machinery in this setting to build a noncommutative geometry model.

In an alternative approach, Connes developed a formulation of differential geometry in terms of commutative algebras to build a noncommutative generalization where we can consider a compact manifold of arbitrary dimension with a well-defined Riemannian structure which gives rise to a first order differential operator known as the Dirac operator. It is shown that the manifold, including the metric tensor, can be completely reconstructed from the discrete eigenvalues of this operator such that the properties of the spectrum can be encoded by a spectral triple which contains some algebraic information. In summary, an ordinary compact Riemannian manifold M is reinterpreted in terms of the spectral triple $(A = C^\infty(M), \mathbb{H} = L^2(.), D = i\gamma_\mu \partial x^\mu)$ which is called a commutative spectral triple. Thanks to this setting, Connes achieved a new modified version of the Gelfand–Naimark Theorem for compact Riemannian manifolds and spectral triples. The generalization of this approach has led us to the concept of noncommutative spectral triples where some new applications of noncommutative geometry to the description of relativistic quantum theory, elementary particles and space-time at the micro-scale Physics have been discovered by mathematicians and mathematical/theoretical physicists. As an example we can address the mathematical description of Standard Model and its modified versions in the language of noncommutative geometry. [36, 37]

Here we plan to explain the structure of a new class of spectral triples originated from solutions of Dyson–Schwinger equations. The resulting spectral triples encode the geometry of those parts of Quantum Field Theories with strong coupling constants where quantum motions have complicated non-perturbative behaviors.

In general, a spectral triple is a collection $(A, \mathbb{H}, D)$ of related mathematical structures such that A is a (unital) involutive algebra which is faithfully represented on a Hilbert space $\mathbb{H}$ via a representation π. D is a self-adjoint operator acting on $\mathbb{H}$ with compact resolvent where for any $a \in A$, $\pi(a)$ maps $\text{dom}(D)$ into itself and also, $[D, \pi(a)]$ extends to a bounded operator on $\mathbb{H}$.

Theory of Clifford algebras and spin structures have provided the foundations of the algebraic reconstruction of the geometry of smooth (compact) Riemannian manifolds in the context of theory of spectral triples. For a given n-dimensional (locally) compact C^∞-Riemannian manifold M without boundary, set $A^1(M) := \Gamma(M, T^*_{\mathbb{C}}M)$ as the space of sections of the complex cotangent bundle, which are differentiable 1-forms on M, with the corresponding dual space $\aleph(M) := \Gamma(M, T_{\mathbb{C}}M)$ as the space of sections of the tangent bundle, which are differentiable vector fields on M. The metric g is therefore a $C^\infty(M)$-valued symmetric bilinear positive definite form on $A^1(M)$ (or $\aleph(M)$). Application of the Cech cohomology theory to the algebra of Clifford sections enables us to define spin^c structures and then determine the corresponding spin structures under Morita equivalent relation. A spin^c connection on a spinor module $\Gamma(M, S)$ is defined (compatible with the action of the algebra of Clifford sections) as a Hermitian connection

$$\nabla^S : \Gamma(M,S) \longrightarrow A^1(M) \otimes_{C^\infty(M)} \Gamma(M,S). \tag{4.3}$$

It is called a spin connection, if it commutes with the anti-linear charge conjugation c for each real vector field. The Riemannian distance on the manifold M is determined in terms of the Dirac operator as a complex linear operator such as $D : \Gamma(M,S) \longrightarrow \Gamma(M,S)$ defined by the composition $-i\widehat{c} \circ \nabla^S$ such that

$$\widehat{c} \in \mathrm{Hom}_{C^\infty(M)}(B \otimes \Gamma(M,S), \Gamma(M,S)) \tag{4.4}$$

is given by $\widehat{c}(\rho_1,\rho_2) := c(\rho_1)\rho_2$ while B is the Clifford algebra bundle. It is also possible to present this operator under a local setting in terms of the spaces of vector fields and 1-forms. This explains the Dirac operator as an essentially self-adjoint operator on its original domain, where we can see that $[D,f] = -ic(df)$ for any smooth function f. Thanks to this treatment

$$d(x,y) = \sup\{|f(y)-f(x)| : f \in C^\infty(M), ||[D,f]|| \leq 1\} \tag{4.5}$$

describes the geodesic distance in terms of an unbounded Fredholm module over the C*-algebra $C^\infty(M)$ [36, 37]. Therefore all geometric information of the manifold M can be encapsulated by the spectral triple

$$(C^\infty(M), L^2(M,S), D). \tag{4.6}$$

Theorem 4.1.1. *Consider $\{(A_m, \mathbb{H}_m, D_m)\}_{m\geq 1}$ as a countable family of spectral triples with the corresponding family of representations $\{\pi_m\}_{m\geq 1}$. For each m, let $||.||_m$ be the norm on $\mathbb{H}_m$ and then choose $\{\alpha_m\}_{m\geq 1}$ as a sequence of non-zero real numbers such that the sequence $\{||(1+\alpha_m^2 D_m^2)^{\frac{-1}{2}}||_m\}_{m\geq 1}$ converges to zero when m goes to infinity. There exists a spectral triple*

$$(A^\oplus, \mathbb{H}^\oplus, \overline{D^\oplus}) \tag{4.7}$$

such that $\mathbb{H}^\oplus := \bigoplus_{m\geq 1} \mathbb{H}_m$, $D^\oplus := \bigoplus_{m\geq 1} \alpha_m D_m$ with the corresponding self-adjoint extension $\overline{D^\oplus}$. In addition,

$$A^\oplus := \{(a_m)_{m\geq 1} \in \prod_m A_m :$$

$$\sup_{m\geq 1}||\pi_m(a_m)||_m < +\infty, \quad \sup_{m\geq 1}||[\alpha_m D_m, \pi_m(a_m)]||_m < +\infty\}$$

such that for each $a^\oplus \in A^\oplus$, $\pi^\oplus(a^\oplus) := \bigoplus_{m\geq 1} \pi_m(a_m)$. [59]

The graduation parameter on the renormalization Hopf algebra and Hopf subalgebras generated by Dyson–Schwinger equations enable us to describe the corresponding complex Lie groups $\mathbb{G}_\Phi(\mathbb{C})$ and $\mathbb{G}_{\mathrm{DSE}}(\mathbb{C})$ under projective limits of Lie subgroups.

Generally speaking, for a given commutative (graded) Hopf algebra H, let $\mathrm{Spec}(H)$ be the set of all prime ideals of H equipped with the Zariski topology and the structure sheaf. This topological space accepts a group structure generated by the coproduct of H. Under a categorical setting, the functional Spec is a contravariant functor from the category of commutative algebras to the category of topological spaces which leads us to define another functional $\mathbb{G}_H = \mathrm{Spec}(H)$ as a covariant representable functor from the category of commutative algebras to the category of groups. For each commutative algebra A, the Lie group $\mathbb{G}_H(A) = \mathrm{Spec}(H)(A)$ is the set of morphisms with the general form

$$\varphi : H \longrightarrow A, \quad \varphi(h_1h_2) = \varphi(h_1)\varphi(h_2), \quad \varphi(1_h) = 1_A, \tag{4.8}$$

which is equipped with the convolution product

$$\varphi_1 * \varphi_2(h) := m \circ (\varphi_1 \otimes \varphi_2) \circ \Delta_H(h). \tag{4.9}$$

Thanks to Milnor–Moore Theorem ([126]), the finite dimensional complex Lie group GL_n of $n\times n$ matrices with non-zero determinants corresponds to the Hopf algebra

$$H_{\mathrm{GL}_n} = k[x_{i,j}, t]_{i,j=1,\dots,n}/\det(x_{i,j})t - 1 \tag{4.10}$$

with the coproduct

$$\Delta(x_{i,j}) = \sum_s x_{i,s} \otimes x_{s,j}. \tag{4.11}$$

It is shown that if the Hopf algebra H is finitely generated as an algebra, then its corresponding affine group scheme is a linear algebraic group which can be embedded as a Zariski closed subset of some GL_n. [121]

If we have a graduation parameter on the commutative Hopf algebra H, then there exists a family $\{H_n\}_{n\geq 0}$ of commutative Hopf subalgebras such that $H = \bigcup_{n\geq 0} H_n$ and for all n and m, we can find some k where $H_n \cup H_m \subset H_k$. It is called a finite type graded Hopf algebra if each component of the grading structure is finitely generated as an algebra which means that for each n, there exists the corresponding linear algebraic group of the form

$$\mathbb{G}_n(\mathbb{C}) = \mathrm{Spec}(H_n)(\mathbb{C}) < \mathrm{GL}_{m_n}(\mathbb{C}) \tag{4.12}$$

for some m_n. These algebraic groups generate the affine group scheme $\mathbb{G}_H$ corresponding to the Hopf algebra H via the projective limit

$$\mathbb{G}_H = \lim_{\longleftarrow n} \mathbb{G}_n. \tag{4.13}$$

Theorem 4.1.2. *There exists a class of infinite dimensional spectral triples which describes the geometry of quantum motions in physical theories with strong coupling constants.*

Proof. We are going to build a spectral triple with respect to each Dyson–Schwinger equation in $\mathcal{S}^{\Phi,g}$ such that the bare coupling constant g is strong enough to produce non-perturbative situations. Suppose the large Feynman diagram $X_{\mathrm{DSE}} = \sum_{n \geq 0} X_n$ is the unique solution of an equation DSE. It is discussed that terms X_n are generators of the free graded connected commutative finite type Hopf subalgebra $H_{\mathrm{DSE}}(\Phi)$ of the Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ of Feynman diagrams graded by number of internal edges. Present $H_{\mathrm{DSE}}(\Phi)$ in terms of its graded components as follows

$$H_{\mathrm{DSE}}(\Phi) = \bigcup_{n \geq 0} H^{(n)}_{\mathrm{DSE}}(\Phi). \tag{4.14}$$

For each n, the finite dimensional Hopf subalgebra $H^{(n)}_{\mathrm{DSE}}(\Phi)$ determines the finite dimensional complex Lie subgroup $\mathbb{G}^{(n)}_{\mathrm{DSE}}(\mathbb{C})$ which is embedded as a closed subset of the linear algebraic group $\mathrm{GL}_{m_n}(\mathbb{C})$ for some m_n with respect to the Zariski topology. Thanks to (4.13), the complex pro-unipotent graded Lie group $\mathbb{G}_{\mathrm{DSE}}(\mathbb{C})$ is the projective limit of $\mathbb{G}^{(n)}_{\mathrm{DSE}}(\mathbb{C})$s as closed subsets of $\mathrm{GL}_{m_n}(\mathbb{C})$s.

For each m_n, $\mathrm{GL}_{m_n}(\mathbb{C})$ is a finite dimensional Riemannian manifold with the corresponding spectral triple

$$\mathcal{S}^{(m_n)} := (C^{\infty}(\mathrm{GL}_{m_n}(\mathbb{C})), L^2(\mathrm{GL}_{m_n}(\mathbb{C}), S), D_{\mathrm{GL}_{m_n}(\mathbb{C})}). \tag{4.15}$$

A restriction of this spectral triple enables us to build the spectral triple corresponding to the complex Lie group $\mathbb{G}^{(n)}_{\mathrm{DSE}}(\mathbb{C})$. We present it by

$$\mathcal{S}^{(n)}_{\mathrm{DSE}} = (A^{(n)}_{\mathrm{DSE}}, \mathbb{H}^{(n)}_{\mathrm{DSE}}, D^{(n)}_{\mathrm{DSE}}). \tag{4.16}$$

Now consider the family $\{\mathcal{S}^{(n)}_{\mathrm{DSE}}\}_{n \geq 0}$ of countable number of spectral triples derived from components of the graduation structure of the Hopf subalgebra $H_{\mathrm{DSE}}(\Phi)$ generated by the equation DSE. Let $\{\alpha_n\}_{n \geq 1}$ be a sequence of non-zero real numbers such that

$$\{||(1 + \alpha_n^2 (D^{(n)}_{\mathrm{DSE}})^2)^{\frac{-1}{2}}||_n\}_{n \geq 1} \tag{4.17}$$

converges to zero when n tends to infinity where $||.||_n$ is the norm on $\mathbb{H}^{(n)}_{\mathrm{DSE}}$. Apply Theorem 4.1.1 to achieve the infinite dimensional spectral triple

$$\mathcal{S}^{\oplus}_{\mathrm{DSE}} := (A^{\oplus}_{\mathrm{DSE}}, \mathbb{H}^{\oplus}_{\mathrm{DSE}}, \overline{D^{\oplus}_{\mathrm{DSE}}}) \tag{4.18}$$

originated from the five-tuples $(A^{(n)}_{\mathrm{DSE}}, \mathbb{H}^{(n)}_{\mathrm{DSE}}, D^{(n)}_{\mathrm{DSE}}, \pi^{(n)}_{\mathrm{DSE}}, \alpha_n)$ for each n. The norm of the Hilbert space $\mathbb{H}^{\oplus}_{\mathrm{DSE}}$ is given by

$$||.||^{\oplus} := \sup_n ||.||_n. \tag{4.19}$$

In addition, we can check that the representation $\pi^{\oplus}_{\rm DSE}$ and the commutator $[D^{\oplus}_{\rm DSE}, \pi^{\oplus}_{\rm DSE}(A^{\oplus}_{\rm DSE})]$ are bounded where the sequence $\{\alpha_n\}_{n\geq 1}$ controls the behavior of the sequence $\{D^{(n)}_{\rm DSE}\}_{n\geq 1}$. It means that

$$\sum_n \dim({\rm Ker} D^{(n)}_{\rm DSE}) < \infty. \tag{4.20}$$

□

It is reasonable to name $\mathcal{S}^{\oplus}_{\rm DSE}$ as the non-perturbative spectral triple with respect to the Dyson–Schwinger equation DSE.

Remark 4.1.3. If the coupling constant of a physical theory is weak enough where (many-loop) perturbation methods can be applied to solve Dyson–Schwinger equations, then we can describe the geometry of this class of quantum motions in terms of summing a finite number of finite dimensional spectral triples.

Corollary 4.1.4. *Each non-perturbative spectral triple has a graphon representation.*

Proof. For a given spectral triple $\mathcal{S}^{\oplus}_{\rm DSE}$ with respect to the equation DSE, we can associate the unlabeled graphon class $[W_{t_{X_{\rm DSE}}}]$ determined by the labeled graph functions of the infinite tree (or forest) $t_{X_{\rm DSE}}$. □

The geometry of the underlying manifold determines the spectrum but the main challenge is the possibility of recovering geometrical information from the spectrum to determine completely the metric or the shape of the boundary. While the answer to this challenge is negative but noncommutative geometry gives us an advanced machinery to deal with the theory of spectral geometry on the basis of an operator theoretic setting. The fundamental integral in noncommutative geometry is described as the Dixmier trace which extends the Wodzicki residue from pseudodifferential operators on a manifold to a general framework which concern spectral triples [36]. For a given spectral triple, we have

$$⨍ T := {\rm Res}_{s=0}{\rm Tr}(T|D|^{-s}). \tag{4.21}$$

It is possible to adapt this integral to deal with the geometry of Dyson–Schwinger equations. The construction of the non-perturbative spectral triple $\mathcal{S}^{\oplus}_{\rm DSE}$ (Theorem 4.1.2) ensures that for each n, $\mathcal{S}^{(n)}_{\rm DSE}$ is a finite dimensional spectral triple. Actually, $\mathcal{S}^{(n)}_{\rm DSE}$ is the result of the restriction of the spectral triple associated to the complex Lie group ${\rm Gl}_{m_n}(\mathbb{C})$ for some m_n. Therefore for each n, the functional

$$a \longmapsto {\rm Tr}^{+}(a|D^{(n)}_{\rm DSE}|^{-m_n}) \tag{4.22}$$

determines a differential calculus theory and spectral geometry with respect to the Riemannian volume form for $\mathcal{S}^{(n)}_{\mathrm{DSE}}$. This differential calculus is describing the geometric behavior of a quantum motion in terms of its approximation with respect to partial sums of the unique solution X_{DSE} of the corresponding equation DSE. Thanks to this interpretation, we may have chance to search for the existence of a noncommutative integral with the general form

$$a^{\oplus} \longmapsto \mathrm{Tr}_{\omega}(a^{\oplus}|D^{\oplus}_{\mathrm{DSE}}|^{-p}) \tag{4.23}$$

for some $p \geq 1$ and state ω. This noncommutative integral, which is on the basis of the Connes–Dixmier traces, can lead us to build a theory of spectral geometry for large Feynman diagrams.

4.2 *A noncommutative symplectic geometry model for* $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$

We have discussed that for a given smooth manifold M with the corresponding complex commutative unital *-algebra $C^{\infty}(M)$, it is possible to reconstruct M together with its smooth structure and the objects attached to the manifold (such as smooth vector fields) in terms of the spaces of characters and derivations of the algebra $C^{\infty}(M)$. The choice of the generalization method for the notion of module over a commutative algebra when this algebra is replaced by a noncommutative algebra is related to the choice of the noncommutative generalization of the classical commutative case. There are some approaches to build the algebraic generalizations of differential geometry such as Koszul framework. This framework is on the basis of $\mathrm{Der}(A)$ as the space of all derivations of a commutative associative algebra A where a graded differential algebra (as the generalization of the algebra of differential forms) determines another graded differential algebra $C_{\wedge}(\mathrm{Der}(A), A)$ of A-valued Chevalley–Eilenberg cochains of the Lie algebra $\mathrm{Der}(A)$. The Koszul framework admits a generalization to the noncommutative setting via derivation-based differential calculus. It is actually the suitable differential calculus for Quantum Mechanics. In this setting, an algebraic version of differential geometry in terms of a commutative associative algebra A, A-modules and connections on these modules have been designed. If we replace the commutativity of the algebra with non-commutativity, then different classes of generalizations of the notion of a module over a noncommutative algebra can be resulted such as the notions of left or right A-modules and left or right $Z(A)$-modules. [41, 42, 43, 45, 109]

The algebraic interpretation of classical geometry requires a commutative setting where we have two options to fix the algebra. The first one is the real commutative algebra $A_{\mathbb{R}}$ of smooth real valued functions where its complexified extension is canonically a complex commutative *-algebra.

The second one is the complex commutative *-algebra $A_{\mathbb{C}}$ of smooth complex valued functions where the set $A^{\text{hermitian}}$ of its hermitian elements is a real commutative algebra and thus $A_{\mathbb{C}}$ will be the complexification of $A^{\text{hermitian}}$.

The algebraic interpretation of Quantum Physics requires a noncommutative setting where we already have two classes of generalizations of the algebra of real valued functions. The first one is the real Jordan algebra $A^{\text{hermitian}}$ of all hermitian elements of a complex noncommutative associative *-algebra A. The second one is a real associative noncommutative algebra. The most important challenge at this level is the choice of the mathematical machinery to build a differential calculus theory. One generalization approach has been formulated by Connes in terms of theory of cyclic cohomology of an algebra where the generalization of the cohomology of a manifold in noncommutative geometry is actually the reduced cyclic homology of an algebra which replaces the standard algebra of smooth functions. As we know the computation of cohomology theory of classical manifolds is not a unique way and furthermore, we can expect the construction of noncommutative generalizations of differential geometry for which the generalization of de Rham theorem fails to be true. These facts show that any cochain complex, which has the reduced cyclic homology as cohomology, can not be an acceptable generalization of differential forms. Thanks to these efforts, the best candidate for the construction of a noncommutative differential calculus is on the basis of the space of derivations as generalizations of vector fields. This formalism, which had been initiated and developed by Kozul and Dubois-Violette, has already provided the foundations of a noncommutative symplectic geometry for the study of quantum theories. [41, 45, 46, 82]

In a different story, the Connes–Kreimer Hopf algebraic renormalization is the direct result of the existence of the Birkhoff factorization on a class of Lie groups. The original source of this particular factorization is the multiplicativity of perturbative renormalization which is encoded by the theory of Rota–Baxter algebras. The determination of a class of Hopf subalgebras via Dyson–Schiwnger equations together with the renormalization of these equations under dimensional regularization had been applied to build a class of Dubois–Violette's differential graded algebras which encode the geometric information of these equations in the context of noncommutative differential forms. The basic idea in this approach is to associate a noncommutative algebra to each equation DSE and then build a theory of noncommutative (symplectic) geometry to encode Feynman diagrams which contribute to the solution of DSE and their renormalization process. One interesting output of this machinery is the description of the Connes–Kreimer non-perturbative renormalization group in the context of integrable systems [146]. Our main task in this part is to develop this approach and explain the construction of a noncommutative differential calculus theory for the topological Hopf algebra $\mathcal{S}^{\Phi}_{\text{graphon}}$ of Feynman graphons which is originated from the BPHZ renormalization (Theorem 2.4.1) and theory of Rota–Baxter algebras ([68]). Our study brings a new concept of integrable systems at the level of large Feynman diagrams. For this purpose, we need first to associate a (noncommutative) algebra to $\mathcal{S}^{\Phi}_{\rm graphon}$ and then build a theory of noncommutative differential forms on this algebra.

The BPHZ procedure contains two general steps namely, dimensional regularization and minimal subtraction defined by the Rota–Baxter map $R_{\rm ms}$ such that it acts on the regularization algebra. We want to show that the application of each step to Feynman graphons can lead us to the structure of a theory of noncommutative differential calculus. These calculi enable us to study the evolution of large Feynman diagrams under regularization and renormalization steps.

Theorem 4.2.1. *The minimal subtraction map $R_{\rm ms}$ in the BPHZ renormalization of Feynman graphons (Theorem 2.3.7 and Theorem 2.4.1) determines a noncommutative symplectic geometry model for the Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$.*

Proof. Consider $A_{\rm dr} := \mathcal{A}_+ \oplus \mathcal{A}_-$ as the algebra of Laurent series with finite pole parts which encodes dimensional regularization (i.e. regularization scheme) and $R_{\rm ms}$ as the linear map on $A_{\rm dr}$ which projects a series onto its corresponding pole parts. The pair $(A_{\rm dr}, R_{\rm ms})$ satisfies the conditions of a Rota–Baxter algebra which enables us to define a new family of convolution products on the space $L(\mathcal{S}^{\Phi}_{\rm graphon}, A_{\rm dr})$ of linear maps by the following procedure.

It is possible to lift $R_{\rm ms}$ onto $L(\mathcal{S}^{\Phi}_{\rm graphon}, A_{\rm dr})$,

$$\mathcal{R}(\phi) := R_{\rm ms} \circ \phi \tag{4.24}$$

to achieve the Rota–Baxter algebra $(L(\mathcal{S}^{\Phi}_{\rm graphon}, A_{\rm dr}), \mathcal{R})$. Set $\hat{\mathcal{R}} := Id - \mathcal{R}$ and for each $\lambda \in \mathbb{R}$ define a new class of Nijenhuis maps $\mathcal{R}_\lambda := \mathcal{R} - \lambda\hat{\mathcal{R}}$. Now define a new family of products on $L(\mathcal{S}^{\Phi}_{\rm graphon}, A_{\rm dr})$ of the form

$$\phi_1 \circ_\lambda \phi_2 := \mathcal{R}_\lambda(\phi_1) *_{\rm gr} \phi_2 + \phi_1 *_{\rm gr} \mathcal{R}_\lambda(\phi_2) - \mathcal{R}_\lambda(\phi_1 *_{\rm gr} \phi_2) \tag{4.25}$$

such that $*_{\rm gr}$ is the convolution product with respect to the coproduct $\Delta_{\rm graphon}$ on Feynman graphons (2.45) where we have

$$\psi_1 *_{\rm gr} \psi_2([W_\Gamma]) = \sum \psi_1([W_{\Gamma'}])\psi_2([W_{\Gamma''}]), \ \ \Delta_{\rm graphon}([W_\Gamma]) = \sum [W_{\Gamma'}] \otimes [W_{\Gamma''}]. \tag{4.26}$$

The non-cocommutativity of the renormalization Hopf algebra of Feynman graphons shows that the convolution product $*_{\rm gr}$ and new products $\circ_\lambda$ are noncommutative. The Nijenhuis property of $\mathcal{R}_\lambda$ shows that

$$\mathcal{R}_\lambda(\phi_1 \circ_\lambda \phi_2) = \mathcal{R}_\lambda(\phi_1) *_{\rm gr} \mathcal{R}_\lambda(\phi_2) \tag{4.27}$$

which supports the associativity of these new products. Set

$$C_\lambda^{\text{graphon}} := (L(\mathcal{S}^{\Phi}_{\text{graphon}}, A_{\text{dr}}), \circ_\lambda) \tag{4.28}$$

as the unital associative noncommutative algebra generated by the minimal subtraction map. In addition, for each λ, the commutator with respect to $\circ_\lambda$ determines a Lie bracket $[.,.]_\lambda$ on the space $L(\mathcal{S}^{\Phi}_{\text{graphon}}, A_{\text{dr}})$ given by

$$[\phi_1, \phi_2]_\lambda = [\mathcal{R}_\lambda(\phi_1), \phi_2] + [\phi_1, \mathcal{R}_\lambda(\phi_2)] - \mathcal{R}_\lambda[\phi_1, \phi_2]. \tag{4.29}$$

We are going to build a noncommutative differential calculus on $C_\lambda^{\text{graphon}}$ with respect to the Lie bracket $[.,.]_\lambda$.

Set $\text{Der}^\lambda_{\text{graphon}}$ as the space of all derivations on $C_\lambda^{\text{graphon}}$. It has all linear maps such as $\theta : C_\lambda^{\text{graphon}} \longrightarrow C_\lambda^{\text{graphon}}$ which enjoys the Leibniz rule. The Lie bracket $[.,.]_\lambda$ determines naturally the Poisson bracket $\{.,.\}_\lambda$ on $C_\lambda^{\text{graphon}}$ where for each $\phi \in C_\lambda^{\text{graphon}}$, its corresponding Hamiltonian derivation is defined by

$$\text{ham}(\phi) : \psi \longmapsto \{\phi, \psi\}_\lambda. \tag{4.30}$$

Set $\text{Ham}^\lambda_{\text{graphon}}$ as the $Z(C_\lambda^{\text{graphon}})$-module generated by all Hamiltonian derivations on $C_\lambda^{\text{graphon}}$.

Now define

$$\Omega^\bullet_{\lambda,\text{graphon}}(C_\lambda^{\text{graphon}}) := (\bigoplus_{n\geq 0} \Omega^n_{\lambda,\text{graphon}}(C_\lambda^{\text{graphon}}), d_\lambda) \tag{4.31}$$

as the differential graded algebra on $C_\lambda^{\text{graphon}}$ such that for each $n \geq 1$,

- $\Omega^n_{\lambda,\text{graphon}}(C_\lambda^{\text{graphon}})$ is the space of all $Z(C_\lambda^{\text{graphon}})$-multilinear antisymmetric mappings from $\text{Ham}^\lambda_{\text{graphon}} \times ...^n \times \text{Ham}^\lambda_{\text{graphon}}$ into $C_\lambda^{\text{graphon}}$. The zero component of this differential graded algebra is the initial algebra $C_\lambda^{\text{graphon}}$.

- For each $\omega \in \Omega^n_{\lambda,\text{graphon}}(C_\lambda^{\text{graphon}})$ and $\theta_i \in \text{Ham}^\lambda_{\text{graphon}}$, the anti-derivative degree one differential operator d_λ is defined by

$$d_\lambda \omega(\theta_0, ..., \theta_n) := \sum_{k=0}^{n} (-1)^k \theta_k \omega(\theta_0, ..., \hat{\theta_k}, ..., \theta_n) +$$

$$\sum_{0\leq r<s\leq n} (-1)^{r+s} \omega([\theta_r, \theta_s]_\lambda, \theta_0, ..., \hat{\theta_r}, ..., \hat{\theta_s}, ..., \theta_n) \tag{4.32}$$

such that we have $d_\lambda^2 = 0$.

Thanks to this differential graded (Lie) algebraic machinery, we can determine a class of symplectic structures generated by the Lie bracket $[.,.]_\lambda$. Define

$$\omega_\lambda : \text{Ham}^\lambda_{\text{graphon}} \times \text{Ham}^\lambda_{\text{graphon}} \longrightarrow C_\lambda^{\text{graphon}}$$

$$\omega_\lambda(\theta,\theta') := \sum_{i,j} u_i \circ_\lambda v_j \circ_\lambda [f_i, h_j]_\lambda \tag{4.33}$$

such that $\{f_1, ..., f_m, h_1, ..., h_n\} \subset C_\lambda^{\mathrm{graphon}}$, $\{u_1, ..., u_m, v_1, ..., v_n\} \subset Z(C_\lambda^{\mathrm{graphon}})$,

$$\theta = \sum_i u_i \circ_\lambda \mathrm{ham}(f_i), \quad \theta' = \sum_j v_j \circ_\lambda \mathrm{ham}(h_j). \tag{4.34}$$

ω_λ is a $Z(C_\lambda^{\mathrm{graphon}})$-bilinear anti-symmetric non-degenerate closed 2-form in $\Omega^2_{\lambda,\mathrm{graphon}}(C_\lambda^{\mathrm{graphon}})$.

For a given $f \in C_\lambda^{\mathrm{graphon}}$ with the corresponding symplectic vector field θ_f^λ, we have

$$\{f,g\}_\lambda := i_{\theta_f^\lambda}(d_\lambda g) \tag{4.35}$$

such that

$$i_\theta(\omega_0 d_\lambda \omega_1 ... d_\lambda \omega_n) = \sum_{j=1}^{n} (-1)^{j-1} \omega_0 d_\lambda \omega_1 ... \theta(\omega_j) ... d_\lambda \omega_n \tag{4.36}$$

is the super-derivation of degree -1. We can check that

$$\{f,g\}_\lambda = i_{\theta_f^\lambda} i_{\theta_g^\lambda} \omega_\lambda. \tag{4.37}$$

□

Theorem 4.2.2. *The dimensional regularization in the BPHZ renormalization of Feynman graphons (Theorem 2.3.7 and Theorem 2.4.1) determines a noncommutative symplectic geometry model for the Hopf algebra* $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$.

Proof. There exists a universal setting for the construction of a Nijenhuis algebra with respect to the commutative unital algebra A_{dr}. We present the product of formal series by $m(f,g) = [fg]$ and consider the graded tensor module $T(A_{\mathrm{dr}}) := \bigoplus_{n\geq 0} A_{\mathrm{dr}}^{\otimes n}$ generated by expressions such as $f_1 \otimes f_2 \otimes ... \otimes f_n$. From now we name each series in A_{dr} as a letter and each sequence $U := f_1 f_2 ... f_n$ of letters as a word with the length n. The empty word e which has the length zero is the unit object in $T(A_{\mathrm{dr}})$. By induction we can define the following shuffle type product on $T(A_{\mathrm{dr}})$

$$fU \circledcirc gV := f(U \circledcirc gV) + g(fU \circledcirc V) - e[fg](U \circledcirc V) \tag{4.38}$$

which is unital and associative. Thanks to (4.38), we can define the following quasi-shuffle type product on $\overline{T}(A_{\mathrm{dr}}) := \bigoplus_{n\geq 1} A_{\mathrm{dr}}^{\otimes n}$,

$$fU \ominus gV := [fg](U \circledcirc V) \tag{4.39}$$

which is also unital and associative. Now consider the linear map B_e^+ on $\overline{T}(A_{\mathrm{dr}})$ which sends each word U of length n to the new word eU of length

$n+1$. Thanks to investigations discussed in [55], the triple $(\overline{T}(A_{\mathrm{dr}}), \ominus, B_e^+)$ is a Nijenhuis algebra which enjoys the universal property in a category of Nijenhuis algebras generated by the initial algebra A_{dr}. Lift the linear map B_e^+ onto $L(\mathcal{S}^{\Phi}_{\mathrm{graphon}}, \overline{T}(A_{\mathrm{dr}}))$ to define the new Nijenhuis map

$$\mathcal{N}_{\mathrm{graphon}}(\psi) := B_e^+ \circ \psi. \tag{4.40}$$

The resulting Nijenhuis algebra is the key tool for us to build a new product $\circ_u$ on $L(\mathcal{S}^{\Phi}_{\mathrm{graphon}}, \overline{T}(A_{\mathrm{dr}}))$ defined by

$$\psi_1 \circ_u \psi_2 := \mathcal{N}_{\mathrm{graphon}}(\psi_1) *_{\ominus} \psi_2 + \psi_1 *_{\ominus} \mathcal{N}_{\mathrm{graphon}}(\psi_2) - \mathcal{N}_{\mathrm{graphon}}(\psi_1 *_{\ominus} \psi_2) \tag{4.41}$$

such that $*_{\ominus}$ is the convolution product with respect to the coproduct $\Delta_{\mathrm{graphon}}$ on Feynman graphons (2.45) and the product $\ominus$. We have

$$\psi_1 *_{\ominus} \psi_2([W_\Gamma]) = \sum \psi_1([W_{\Gamma'}]) \ominus \psi_2([W_{\Gamma''}]), \ \ \Delta_{\mathrm{graphon}}([W_\Gamma]) = \sum [W_{\Gamma'}] \otimes [W_{\Gamma''}]. \tag{4.42}$$

The non-cocommutativity of the renormalization Hopf algebra of Feynman graphons shows that the convolution product $*_{\ominus}$ and the new product $\circ_u$ are noncommutative. The Nijenhuis property shows that

$$\mathcal{N}_{\mathrm{graphon}}(\psi_1 \circ_u \psi_2) = \mathcal{N}_{\mathrm{graphon}}(\psi_1) *_{\ominus} \mathcal{N}_{\mathrm{graphon}}(\psi_2) \tag{4.43}$$

which supports the associativity of this new product. Set

$$C_u^{\mathrm{graphon}} := (L(\mathcal{S}^{\Phi}_{\mathrm{graphon}}, \overline{T}(A_{\mathrm{dr}})), \circ_u) \tag{4.44}$$

as the unital associative noncommutative algebra generated by dimensional regularization. In addition, the commutator with respect to the product $\circ_u$ determines a Lie bracket $[.,.]_u$ on the space $L(\mathcal{S}^{\Phi}_{\mathrm{graphon}}, \overline{T}(A_{\mathrm{dr}}))$ given by

$$[\psi_1, \psi_2]_u := [\mathcal{N}_{\mathrm{graphon}}(\psi_1), \psi_2] + [\psi_1, \mathcal{N}_{\mathrm{graphon}}(\psi_2)] - \mathcal{N}_{\mathrm{graphon}}[\psi_1, \psi_2]. \tag{4.45}$$

We are going to build a noncommutative differential calculus on C_u^{graphon} with respect to the Lie bracket $[.,.]_u$.

Set $\mathrm{Der}^u_{\mathrm{graphon}}$ as the space of all derivations on C_u^{graphon}. It has all linear maps such as $\theta : C_u^{\mathrm{graphon}} \longrightarrow C_u^{\mathrm{graphon}}$ which enjoys the Leibniz rule. The Lie bracket $[.,.]_u$ naturally determines the Poisson bracket $\{.,.\}_u$ on C_u^{graphon} where for each $\phi \in C_u^{\mathrm{graphon}}$, its corresponding Hamiltonian derivation is defined by

$$\mathrm{ham}(\phi) : \psi \longmapsto \{\phi, \psi\}_u. \tag{4.46}$$

Set $\mathrm{Ham}^u_{\mathrm{graphon}}$ as the $Z(C_u^{\mathrm{graphon}})$-module generated by all Hamiltonian derivations on C_u^{graphon}.

Now define

$$\Omega^{\bullet}_{u,\mathrm{graphon}}(C_u^{\mathrm{graphon}}) := (\bigoplus_{n \geq 0} \Omega^n_{u,\mathrm{graphon}}(C_u^{\mathrm{graphon}}), d_u) \tag{4.47}$$

as the differential graded algebra on C_u^{graphon} such that for each $n \geq 1$,

- $\Omega^n_{u,\text{graphon}}(C_u^{\text{graphon}})$ is the space of all $Z(C_u^{\text{graphon}})$-multilinear antisymmetric mappings from $\text{Ham}^u_{\text{graphon}} \times ...^n \times \text{Ham}^u_{\text{graphon}}$ into C_u^{graphon}. The zero component of this differential graded algebra is the initial algebra C_u^{graphon}.

- For each $\omega \in \Omega^n_{u,\text{graphon}}(C_u^{\text{graphon}})$ and $\theta_i \in \text{Ham}^u_{\text{graphon}}$, the anti-derivative degree one differential operator d_u is defined by

$$d_u\omega(\theta_0, ..., \theta_n) := \sum_{k=0}^{n}(-1)^k \theta_k \omega(\theta_0, ..., \hat{\theta_k}, ..., \theta_n)+$$

$$\sum_{0\leq r<s\leq n} (-1)^{r+s}\omega([\theta_r, \theta_s]_u, \theta_0, ..., \hat{\theta_r}, ..., \hat{\theta_s}, ..., \theta_n) \tag{4.48}$$

such that we have $d_u^2 = 0$.

Thanks to this differential graded (Lie) algebraic machinery, we can determine a class of symplectic structures generated by the Lie bracket $[.,.]_u$. Define

$$\omega_u : \text{Ham}^u_{\text{graphon}} \times \text{Ham}^u_{\text{graphon}} \longrightarrow C_u^{\text{graphon}}$$

$$\omega_u(\theta, \theta') := \sum_{i,j} u_i \circ_u v_j \circ_u [f_i, h_j]_u \tag{4.49}$$

such that $\{f_1, ..., f_m, h_1, ..., h_n\} \subset C_u^{\text{graphon}}$, $\{u_1, ..., u_m, v_1, ..., v_n\} \subset Z(C_u^{\text{graphon}})$,

$$\theta = \sum_i u_i \circ_u \text{ham}(f_i), \quad \theta' = \sum_j v_j \circ_u \text{ham}(h_j). \tag{4.50}$$

ω_u is a $Z(C_u^{\text{graphon}})$-bilinear anti-symmetric non-degenerate closed 2-form in $\Omega^2_{u,\text{graphon}}(C_u^{\text{graphon}})$.

For a given $f \in C_u^{\text{graphon}}$ with the corresponding symplectic vector field θ^u_f, we have

$$\{f, g\}_u := i_{\theta^u_f}(d_u g) \tag{4.51}$$

such that

$$i_\theta(\omega_0 d_u\omega_1 ... d_u\omega_n) = \sum_{j=1}^{n}(-1)^{j-1}\omega_0 d_u\omega_1 ... \theta(\omega_j) ... d_u\omega_n \tag{4.52}$$

is the super-derivation of degree -1. We can check that

$$\{f, g\}_u = i_{\theta^u_f} i_{\theta^u_g} \omega_u. \tag{4.53}$$

□

The structure of the modified version of the Connes–Kreimer renormalization group for Feynman graphons has been explained in Lemma 2.4.2 where we apply the filtration parameter on Feynman graphons (Theorem 2.3.9). Thanks to the built noncommutative differential geometry on $\mathcal{S}^{\Phi}_{\text{graphon}}$, we can provide a new geometric interpretation for the behavior of the Connes–Kreimer renormalization group when it acts on large Feynman diagrams.

Lemma 4.2.3. *Let $\{F_t\}_t$ be the renormalization group on Feynman graphons (defined by Lemma 2.4.2). For each t and any large Feynman diagram X, $F_t(X)$ is the convergent limit of the sequence $\{F_t(X_n)\}_{n\geq 1}$ with respect to the cut-distance topology.*

Proof. Consider the loop $\gamma_\mu \in \text{Loop}(\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C}), \mu)$ which encodes the Feynman rules characters in the renormalization Hopf algebra of Feynman graphons with respect to a given physical theory Φ. For a given Dyson–Schwinger equation DSE with the unique solution $X = \sum_{n\geq 0} X_n$, we have

$$\gamma_\mu(z)([W_X]) := U^z_\mu(X) \tag{4.54}$$

such that $U^z_\mu(X)$ is a Laurent series as the regularized large Feynman integral with respect to X. The one-parameter group $\{\theta_t\}_{t\in\mathbb{C}}$ sends the unlabeled graphon class $[W_X]$ to the filtration rank of the equation DSE (Theorem 2.3.9). The resulting renormalization group $\{F_t\}_t$ (i.e. Lemma 2.4.2) is a subgroup of $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$ which means that for each t, F_t is a linear homomorphism. On the other hand, thanks to Theorem 2.3.6, we know that the large Feynman diagram X is the convergent limit of the sequence of its partial sums with respect to the cut-distance topology. Therefore we have

$$F_t(X) = F_t(\lim_{m\to\infty} Y_m) = F_t(\lim_{m\to\infty} \sum_{n=1}^{m} X_n) =$$

$$\lim_{m\to\infty} \sum_{n=1}^{m} F_t(X_n) = \lim_{m\to\infty} \sum_{n=1}^{m} \lim_{z\to 0} \gamma_-(z)(X_n)\theta_{tz}(\gamma_-^{-1}(z)(X_n))$$

$$= \lim_{m\to\infty} \lim_{z\to 0} \sum_{n=1}^{m} \gamma_-(z)(X_n)\theta_{tz}(\gamma_-^{-1}(z)(X_n)) \tag{4.55}$$

such that according to Proposition 1.47 in [37], for each t, $F_t(X_n)$ is a polynomial in t. □

Corollary 4.2.4. *The Connes–Kreimer renormalization group on Feynman graphons determines an infinite dimensional integrable system.*

Proof. We work on the unital associative noncommutative algebra $C^{\text{graphon}}_0 := (L(\mathcal{S}^{\Phi}_{\text{graphon}}, A_{\text{dr}}), \circ_0)$ generated by the minimal subtraction map for $\lambda = 0$. Thanks to Theorem 4.2.1, consider the differential graded algebra

$$\Omega^{\bullet}_{0,\text{graphon}}(C^{\text{graphon}}_0) := (\bigoplus_{n\geq 0} \Omega^{n}_{0,\text{graphon}}(C^{\text{graphon}}_0), d_0) \tag{4.56}$$

with respect to the Lie bracket $[.,.]_0$. Each character F_t of the renormalization group $\{F_t\}_t$ given by Lemma 4.2.3 is an object in the algebra C_0^{graphon}. Therefore the motion integral equation with respect to the character F_{t_0} is given by the equation

$$\{f, F_{t_0}\}_0 = 0 \tag{4.57}$$

such that $f \in C_0^{\text{graphon}}$. Thanks to the existence of a noncommutative symplectic form ω_0 on C_0^{graphon} with respect to the Lie bracket $[.,.]_0$ (Theorem 4.2.1), the motion integral can be determined by the equation

$$\{f, F_{t_0}\}_0 = i_{\theta^0_{F_{t_0}}} i_{\theta^0_f} \omega_0 = w_0(\theta^0_{F_{t_0}}, \theta^0_f) = [f, F_{t_0}] = 0. \tag{4.58}$$

On the one hand, from the definition of the deformed Lie bracket $[.,.]_0$ and the idempotent Rota–Baxter property of $(A_{\text{dr}}, R_{\text{ms}})$, we have

$$\{F_t, F_s\}_0 = [R_{\text{ms}}(F_t), F_s] + [F_t, R_{\text{ms}}(F_s)] - R_{\text{ms}}([F_t, F_s]). \tag{4.59}$$

On the other hand, for each t, $F_t([W_\Gamma])$ is a polynomial in t which means that $R_{\text{ms}}(F_t([W_\Gamma])) = 0$ and in addition, for each s, t, $F_t * F_s = F_{t+s}$. Thanks to these facts, we can observe that for each s, t,

$$\{F_t, F_s\}_0 = 0. \tag{4.60}$$

□

Chapter 5

A theory of functional analysis for large Feynman diagrams

- ***The Haar integration on $\mathcal{S}^{\Phi,g}$ and its application***
- ***The Gâteaux differential calculus on $\mathcal{S}^{\Phi,g}$ and its application***

This chapter aims to provide the foundations of a functional analysis machinery for the study of large Feynman diagrams which contribute to solutions of Dyson–Schwinger equations. We build an integration theory and a differentiation theory for the functionals on the space $\mathcal{S}^{\Phi,g}$ (when the bare coupling constant g is strong) which can be embedded into the topological Hopf algebra $H^{\rm cut}_{\rm FG}(\Phi)$ of (large) Feynman diagrams equipped with the cut-distance topology. Actually, $H^{\rm cut}_{\rm FG}(\Phi)$ consists of all Feynman diagrams and their corresponding finite or infinite formal expansions where solutions of all non-perturbative Dyson–Schwinger equations belong to the boundary region. As we have shown in the previous parts, this enriched Hopf algebra of Feynman diagrams can be encoded via the Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$ of Feynman graphons. Therefore at the first step, we study measure theory on the set $\mathcal{S}^{\Phi,g}$ where we equip it with a new topological group structure which leads us to a new Haar measure integration theory for functionals on large Feynman diagrams. Then we deal with some applications of the resulting measure space where a new generalization of the classical Johnson–Lapidus Dyson series for large Feynman diagrams will be obtained. In addition, we work on the construction of a new Fourier transformation machinery on the Banach algebra $L^1(\mathcal{S}^{\Phi,g}, \mu_{\rm Haar})$ which enables us to describe the evolution of large Feynman diagrams on the basis of their corresponding partial sums under a functional setting. At the second step, we concern the Gâteaux differentiability of real valued functionals on $H^{\rm cut}_{\rm FG}(\Phi)$ where we obtain Taylor expansion representations for these functionals under some conditions.

The promising differential calculus and integration theory enable us to describe the dynamics of topological regions of Feynman diagrams on the basis of the behavior of functionals with respect to the built Haar integration theory and Gâteaux differentiation theory on Feynman graphons.

5.1 *The Haar integration on $\mathcal{S}^{\Phi,g}$ and its application*

For a given physical theory Φ with strong coupling constant $g \geq 1$, set $V(\Phi)$ as the set of all vertices which appear in Feynman diagrams and their corresponding formal expansions as interactions among elementary particles. This infinite countable set allows us to count interactions independent of their physical types. Set $K_{V(\Phi)}$ as the complete graph with $V(\Phi)$ as the set of vertices and all possible edges among these vertices except self-loops. The collection $\{0,1\}^{K_{V(\Phi)}}$ as the family of all functions from $K_{V(\Phi)}$ to $\{0,1\}$ allows us to characterize Feynman diagrams which contribute to physical theory Φ. Therefore, each (large) Feynman diagram Γ can be determined by its corresponding characteristic function χ_Γ which sends vertices $v \in K_{V(\Phi)}$ to 1 if $v \in \Gamma$ and sends other vertices to 0. For each edge $e \in K_{V(\Phi)}$, if V_e be the set of vertices in $V(\Phi)$ which are attached to the edge e, then the infinite Cartesian product $\times_{e \in K_{V(\Phi)}} \mathcal{P}(K_{V_e})$ has enough vertices and edges to contain all Feynman graphs in $H_{\mathrm{FG}}^{\mathrm{cut}}(\Phi)$ as subsets.

Lemma 5.1.1. *$\mathcal{S}^{\Phi,g}$ can be equipped with an abelian compact Hausdorff topological group structure.*

Proof. Thanks to characteristic functions, we can associate a function $f_X \in \{0,1\}^{K_{V(\Phi)}}$ to each large Feynman graph X for the identification of vertices and edges which contribute to the corresponding Dyson–Schwinger equation DSE. It means that we can embed the family $\mathcal{S}^{\Phi,g}$ into $\{0,1\}^{K_{V(\Phi)}}$ which is useful to define new addition and multiplication operators on (large) Feynman diagrams in terms of the pointwise addition and multiplication of their corresponding characteristic functions. Therefore we can lift the set $\mathcal{S}^{\Phi,g}$ onto a vector space generated by infinite graphs X which allows us to have a commutative $\mathbb{Z}_2$-algebra.

In addition, define a new binary operation on $\mathcal{S}^{\Phi,g}$ by the symmetric difference operator

$$(\Gamma_1, \Gamma_2) \longmapsto \Gamma_1 \triangle \Gamma_2. \tag{5.1}$$

By adding the empty graph $\mathbb{I}$ to $\mathcal{S}^{\Phi,g}$ as the zero element, $(\mathcal{S}^{\Phi,g}, \triangle)$ is an abelian group which can be equipped with a compatible topology to obtain a compact topological group. For this purpose, suppose α be a bijection between $K_{V(\Phi)}$ and the set of natural numbers $\mathbb{N}$. For the fixed coupling constant $g \geq 1$ and each $\epsilon > 0$, define a new map $d_{g,\alpha,\epsilon} : \mathcal{S}^{\Phi,g} \times \mathcal{S}^{\Phi,g} \longrightarrow [0,\infty)$ given by

$$d_{g,\alpha,\epsilon}(\Gamma_1, \Gamma_2) := \sum_{e \in \Gamma_1 \triangle \Gamma_2} (g+\epsilon)^{-\alpha(e)} \tag{5.2}$$

such that the sum is taken over all vertices e which belongs to only one of the large Feynman graphs Γ_1 or Γ_2. $d_{g,\alpha,\epsilon}$ is a translation invariant metric such that d_{g,α,ϵ_1} and d_{g,α,ϵ_2} have the equivalent topology. As the result, $\mathcal{S}^{\Phi,g}$ together with the symmetric difference operator and the topology generated by the metric $d_{g,\alpha,\epsilon}$ is a compact Hausdorff abelian topological group. □

Thanks to the translation-invariant metric $d_{g,\alpha,\epsilon}$ defined by Lemma 5.1.1, for each large Feynman diagram X define

$$\parallel X \parallel_{g,\alpha,\epsilon} := d_{g,\alpha,\epsilon}(\mathbb{I}, X). \tag{5.3}$$

such that $\mathbb{I}$ is the empty graph. In this setting, a sequence $\{\Gamma_n\}_{n \geq 1}$ of large Feynman diagrams in $\mathcal{S}^{\Phi,g}$ is convergent to a unique large Feynman diagram Γ, if each indicator sequence $\{\mathbf{1}_{e \in \Gamma_n}\}_{n \geq 1}$ converges to the indicator $\mathbf{1}_{e \in \Gamma}$ for any $e \in K_{V(\Phi)}$.

Theorem 5.1.2. *The topological group $\mathcal{S}^{\Phi,g}$ can be equipped with the Haar measure μ_{Haar}.*

Proof. Lemma 5.1.1 supports the existence of the unique Haar measure μ_{Haar} on $\mathcal{S}^{\Phi,g}$ originated from the compact topological structure. We build this measure which has a Bernoulli probability nature.

Consider the product σ-algebra $\sum_{\mathrm{prod}}$ on $\mathcal{S}^{\Phi,g}$ generated by cylinder sets

$$S_{\Gamma_0} := \times_{\Gamma \neq \Gamma_0}\{\mathbb{I}, \{\Gamma\}\} \times \{\Gamma_0\} \tag{5.4}$$

for each large Feynman diagram Γ_0 in $\mathcal{S}^{\Phi,g}$. Each Γ can be seen as an infinite countable subset of $K_{V(\Phi)}$ which contributes to the unique solution of an equation DSE.

Each function $P \in \{0,1\}^{K_{V(\Phi)}}$ identifies a new function $\tilde{P} : \mathcal{S}^{\Phi,g} \longrightarrow [0,1]$ which enables us to define the measure $\mu_{\tilde{P}}$ on the σ-algebra $\sum_{\mathrm{prod}}$. For finite intersections of cylinder sets $S_{\Gamma_1}, ..., S_{\Gamma_n}$, we have

$$\mu_{\tilde{P}}(S_{\Gamma_1} \cap S_{\Gamma_2} \cap ... \cap S_{\Gamma_n}) = \prod_{i=1}^{n} \tilde{P}(\Gamma_i) \tag{5.5}$$

for large Feynman diagrams $\Gamma_1, ..., \Gamma_n$ which contribute to $\mathcal{S}^{\Phi,g}$. In fact, $\mu_{\tilde{P}}$ is a probability measure on $\mathcal{S}^{\Phi,g}$ and we can present it as the following way

$$\mu_{\tilde{P}} := \prod_{X \in \mathcal{S}^{\Phi,g}} \mu_{\tilde{P},X}. \tag{5.6}$$

Now we need to show that the measure $\mu_{\tilde{P}}$ is the Haar measure. In other words, we claim that the Haar measure is equal with $\mu_{\tilde{P}}$ where $P(X) = 1/2$ for each large Feynman diagram $X \in \mathcal{S}^{\Phi,g}$.

For subsets Z_1, Z_2 of $K_{V(\Phi)}$, define

$$I(Z_1, Z_2) := \{\mathrm{DSE} \in \mathcal{S}^{\Phi,g} : Z_1 \subset X_{\mathrm{DSE}}, \quad Z_2 \subset K_{V(\Phi)} \backslash X_{\mathrm{DSE}}\} \tag{5.7}$$

and then consider the σ-algebra $\sum_{\mathcal{I}}$ generated by all sets $I(Z_1, Z_2)$. The σ-algebra $\sum_{\mathcal{I}}$ is the same as the σ-algebra generated by all sets $I(Z_1, Z_2)$ for disjoint sets Z_1, Z_2. In addition, we have $S_\Gamma = I(\mathbb{I}, \{\Gamma\})$ which leads us to show that $\sum_{\mathrm{prod}} = \sum_{\mathcal{I}}$.

Thanks to some standard analysis methods [138], we can determine the unique translation-invariant probability measure μ_{Haar} on the compact topological group $\mathcal{S}^{\Phi,g}$. For a given large Feynman diagram X and a subset Z of $K_{V(\Phi)}$, define

$$Z + X := \{\gamma \sqcup X : \quad \gamma \in Z\}. \tag{5.8}$$

Then we can show that

$$I(Z_1, Z_2) = I(\mathbb{I}, Z_1 \sqcup Z_2) + Z_2. \tag{5.9}$$

Thanks to this fact, for given large Feynman diagrams Γ_1, Γ_2, set $\Gamma = \Gamma_1 \sqcup \Gamma_2$. Then we have

$$\mu_{\mathrm{Haar}}(I(\Gamma_1, \Gamma_2)) = \mu_{\mathrm{Haar}}(I(\Gamma_1, \Gamma_2) + \Gamma_2) = \mu_{\mathrm{Haar}}(I(\mathbb{I}, \Gamma)) \tag{5.10}$$

which informs the translation-invariance. On the other hand, let the subset Z of $K_{V(\Phi)}$ contains a finite number of large Feynman diagrams $\Gamma_1, ..., \Gamma_n$. We can show that as the set

$$\mathcal{S}^{\Phi,g} = \coprod_{Z_0 \subset Z} I(Z_0, Z \backslash Z_0) \tag{5.11}$$

which leads us to obtain

$$\mu_{\text{Haar}}(I(Z_1, Z_2)) = 2^{-n} = \mu_{1/2}(I(Z_1, Z_2)). \tag{5.12}$$

In general, if (Ω, A) ba σ-algebra generated by a subset $C \subset A$ which is closed under finite intersections, then two probability measures on A are equal if and only if they agree on C. If C has an algebraic structure which is equipped by a probability measure μ, then we can extend μ to a unique measure on A [138]. Thanks to this fact, μ_{Haar} agrees with $\mu_{1/2}$ on $\sigma_{\mathcal{I}}$.

We can also check that $\sigma_{\mathcal{I}}$ is equal with the Borel σ-algebra generated by all open sets in $\mathcal{S}^{\Phi,g}$ with respect to the metric $d_{g,\alpha,\epsilon}$. For a given bijection $\alpha : K_{V(\Phi)} \longrightarrow \mathbb{N}$, set

$$E_n(\alpha) := \{e \in K_{V(\Phi)} : \quad \alpha(e) \leq n\}. \tag{5.13}$$

For each large Feynman diagram X in $\mathcal{S}^{\Phi,g}$, set $B(X, r, \| \, . \, \|_{g,\alpha,\epsilon})$ as the open ball in $(\mathcal{S}^{\Phi,g}, d_{g,\alpha,\epsilon})$ with center X and radius r. In addition, for given disjoint finite subsets $N_1, N_2 \subset \mathbb{N}$ such that $N_1 \cup N_2 = \{1, 2, ..., n\}$, define

$$I(N_1, N_2) := I(\alpha^{-1}(N_1), \alpha^{-1}(N_2)). \tag{5.14}$$

Then we have

$$I(N_1, N_2) = B(\alpha^{-1}(N_2), 2^{-n}, \| \, . \, \|_{g,\alpha,2}) \bigcup B(K_{V(\Phi)} \backslash \alpha^{-1}(N_1), 2^{-n}, \| \, . \, \|_{g,\alpha,2}) \tag{5.15}$$

such that $B(\alpha^{-1}(N_i), 2^{-n}, \| \, . \, \|_{g,\alpha,2})$ is the open ball in $(\mathcal{S}^{\Phi,g}, d_{g,\alpha,g+\epsilon=2})$ with center $\alpha^{-1}(N_i)$ and radius 2^{-n}.

Now a large Feynman diagram $X \in \mathcal{S}^{\Phi,g}$ can be described as the convergent limit of the sequence $\{\Gamma_n\}_{n \geq 1}$ such that $\Gamma_n := X \cap E_n(\alpha)$.

Furthermore, we know that $\mathcal{S}^{\Phi,g}$ is a compact Hausdorff topological group (Lemma 5.1.1). Therefore $K \subset \mathcal{S}^{\Phi,g}$ is compact iff $\mathcal{S}^{\Phi,g} \backslash K$ is open. It shows that the σ-algebra generated by all compact sets is the same as the Borel σ-algebra generated by all open sets.

As the result, μ_{Haar} on σ_{prod} determines uniquely the Haar measure. Therefore $\mu_{\text{Haar}} = \mu_{1/2}$. □

Theorem 5.1.3. *For a given bijection α, the Haar measure of any ball of radius $0 \leq r \leq 1$ in the normed vector space $(\mathcal{S}^{\Phi,g}, \| \, . \, \|_{g,\alpha,g+\epsilon=2})$ is r.*

Proof. The proof is a direct result of Theorem 5.1.2 and the proof of Theorem A in [81]. □

The resulting measure space enables us to initiate an integration theory on the family of (large) Feynman diagrams which can be interpreted in the context of the Riemann–Lebesgue integration theory on the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Theorem 5.1.4. *The integration theory on the measure space* $(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ *can be described by the Riemann–Lebesgue integration theory on real numbers with respect to the Borel* σ*-algebra generated by all open sets.*

Proof. Thanks to the structure of the topological group $\mathcal{S}^{\Phi,g}$ (Lemma 5.1.1) where the norm $\| \, . \, \|_{g,\alpha,2}$ (5.3) and the Haar measure μ_{Haar} (Theorem 5.1.2) are defined on large Feynman diagrams, we can adapt the proofs of Lemma 3.22 and Proposition 3.23 in [148] for large Feynman diagrams to obtain the following results.

(i) We can show that the norm $\| \, . \, \|_{g,\alpha,2}$ (as a real valued function on $\mathcal{S}^{\Phi,g}$) is the Haar measure-preserving map.

(ii) We can show that for any Lebesgue integrable real valued function f on $[0, 1]$,

$$\mathbb{E}_{\mu_{\text{Haar}}}[f(\| \, . \, \|_{g,\alpha,2})] = \int_0^1 f(x)dx. \tag{5.16}$$

(iii) We can show that for any integrable real valued function h on $\mathcal{S}^{\Phi,g}$,

$$\mathbb{E}_{\mu_{\text{Haar}}}[h] = \int_0^1 h((\| \, . \, \|_{g,\alpha,2})^{-1}(x))dx. \tag{5.17}$$

Therefore the Haar integration theory on $(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ can be described by transferring the Riemann–Lebesgue integration theory from the unit interval to large Feynman diagrams. □

The rest of this section provides some applications of this integration theory for functionals on $(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$.

Consider a single quantum particle which moves in a given potential such that its behavior can be studied by a class of functionals on $C[0, t]$ given by

$$Z(y) := \exp\{\int_{(0,t)} \theta(s, y(s))ds\} \tag{5.18}$$

where the complex valued function θ on $[0, t] \times \mathbb{R}^n$ is the given potential. This formulation is on the basis of the standard Lebesgue–Stieltjes measure but under some conditions it is possible to formulate these functionals with respect to other complex Borel measures. It has been shown that for each complex number with positive real part λ, the operators $K_\lambda(Z_n)$ exist for each n such that $Z_n(y) := (\int_{(0,t)} \theta(s, y(s))d\eta)^n$ and $K_\lambda(Z) = \sum_{n\geq 0} a_n K_\lambda(Z_n)$. The central motivation of this formulation was to deal with Feynman's operational calculus in QED and other quantum theories. [79]

A modification of the Johnson–Lapidus Dyson series for a measure space of Feynman diagrams which contribute to the topological Hopf algebra $H_{\rm FG}^{\rm cut}(\Phi)$ has been obtained in [148]. Thanks to the Haar integration theory on the topological group of large Feynman diagrams, we want to formulate the Johnson–Lapidus Dyson series on $\mathcal{S}^{\Phi,g}$ which leads us to understand the evolution of Dyson–Schwinger equations with respect to other large Feynman diagrams.

Theorem 5.1.5. *Let θ be a complex valued function on $\mathcal{S}^{\Phi,g}\times\mathbb{R}^2$ and $v(z) = \sum_{n\geq 0} a_n z^n$ with the radius of convergence strictly grater than $||\theta||_{\infty;\mu_{\rm Borel}}$. For a functional Z on the measure space $L^1(\mathcal{S}^{\Phi,g},\mu_{\rm Haar})$ of all complex valued $\mu_{\rm Haar}$-integrable functions on $\mathcal{S}^{\Phi,g}$ given by*

$$Z(F) := v(\int_{\mathcal{S}^{\Phi,g}} \theta(X,F(X))d\mu_{\rm Borel}) \tag{5.19}$$

, there exists a family of operators $\{K_\lambda(Z_n)\}_{n\in\mathbb{N}}$ such that parameters λ are complex numbers with positive real parts and $Z_n(F) := (\int_{\mathcal{S}^{\Phi,g}} \theta(X,F(X))d\mu_{\rm Borel})^n$. In addition, we have

$$K_\lambda(Z) = \sum_{n\geq 0} a_n K_\lambda(Z_n). \tag{5.20}$$

Proof. Theorem 5.1.2 and Theorem 5.1.4 enable us to understand the Haar integration theory on $\mathcal{S}^{\Phi,g}$ in the language of the Riemann–Lebesgue integration theory for real valued functions on the closed interval. In addition, we have discussed the equivalency between the product σ-algebra $\sum_{\rm prod}$ on cylinders determined by large Feynman diagrams and the Borel σ-algebra of open balls with respect to the norm $\|\ .\ \|_{g,\alpha,2}$. It enables us to determine uniquely the Borel measure $\mu_{\rm Borel}$ on $\mathcal{S}^{\Phi,g}$ corresponding to the Haar measure $\mu_{\rm Haar}$ (i.e. Theorem 5.1.2). Thanks to these facts, it does make sense to extend the classical Johnson-Lapidus Dyson series to the level of the Haar measure $\mu_{\rm Haar}$ on $\mathcal{S}^{\Phi,g}$.

We have shown the existence of a compact Hausdorff topological group structure on $\mathcal{S}^{\Phi,g}$. Thanks to standard methods in Analysis ([138]), it is easy to show that the topological space $C_c(\mathcal{S}^{\Phi,g})$ consisting of continuous functions on $\mathcal{S}^{\Phi,g}$ with compact support is dense in $L^1(\mathcal{S}^{\Phi,g},\mu_{\rm Haar})$. Apply (5.17) to transfer the Haar measure integral

$$\mathbb{E}_{\mu_{\rm Haar}}[h] = \int_{\mathcal{S}^{\Phi,g}} h(X)d\mu_{\rm Haar} \tag{5.21}$$

of each $h \in C_c(\mathcal{S}^{\Phi,g})$ to its corresponding Riemann–Lebesgue integral.

It remains only to lift the proof of the classical Johnson-Lapidus generalized Dyson series given in [79] onto $L^1(\mathcal{S}^{\Phi,g},\mu_{\rm Haar})$. □

Corollary 5.1.6. *(i) The Johnson–Lapidus Dyson series can describe the behavior of a combinatorial Dyson–Schwinger equation in a given potential.*

(ii) The functionals $K_\lambda(Z)$ (determined by Theorem 5.1.5) enable us to describe the evolution of each large Feynman diagram X in terms of a sequence of Dyson–Schwinger equations in $\mathcal{S}^{\Phi,g}$.

Proof. (i) Suppose $X_{\mathrm{DSE}}(g) = \sum_{n\geq 0} g^n X_n$ is a large Feynman diagram as the unique solution of an equation DSE in the normed vector space $(\mathcal{S}^{\Phi,g}, \| \,.\, \|_{g,\alpha,2})$. Thanks to the Hahn–Banach Theorem ([138]), there exists a continuous linear map $\psi_{\mathrm{DSE}} : \mathcal{S}^{\Phi,g} \longrightarrow \mathbb{R}$ such that

$$\psi_{\mathrm{DSE}}(X_{\mathrm{DSE}}) = \| X_{\mathrm{DSE}} \|_{g,\alpha,2}, \quad \| \psi_{\mathrm{DSE}} \| \leq 1 \tag{5.22}$$

where the operator norm $\| \psi_{\mathrm{DSE}} \|$ is defined

$$\| \psi_{\mathrm{DSE}} \| := \inf\{c \geq 0 : \ |\psi_{\mathrm{DSE}}(X)| \leq c \| X \|_{g,\alpha,2}, \ \forall X \in \mathcal{S}^{\Phi,g}\}. \tag{5.23}$$

Now apply Theorem 5.1.5 for $\psi_{\mathrm{DSE}} \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$.

(ii) If we apply the multi-scale renormalization machinery (Theorem 3.2.2) $R^{\mathrm{multi}}_{(\tau g,\Lambda_\tau),(\lambda\tau g,\lambda\Lambda_\tau)}\mathrm{DSE}$, then we can build the sequence $\{\mathrm{DSE}(\frac{n}{n+1}g)\}_{n\geq 1}$ of Dyson–Schwinger equations where we have

$$X_{\mathrm{DSE}}(\frac{n}{n+1}g) \subset X_{\mathrm{DSE}}(\frac{n+1}{n+2}g). \tag{5.24}$$

For each n, set χ^n_{DSE} as the characteristic function with respect to the large Feynman diagram $X_{\mathrm{DSE}}(\frac{n}{n+1}g)$ on $\mathcal{S}^{\Phi,g}$ such that $\chi^n_{\mathrm{DSE}} \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$. Now apply Theorem 5.1.5 to the sequence $\{\chi^n_{\mathrm{DSE}}\}_{n\geq 1}$ to obtain a description for the evolution of $X_{\mathrm{DSE}}(g)$ in terms of large sub-graphs. A free evolution from $X_{\mathrm{DSE}}(0) = \mathbb{I}$ (empty graph) to $X_{\mathrm{DSE}}(\frac{1}{2}g)$, interactions of particles in $X_{\mathrm{DSE}}(\frac{1}{2}g)$ with the potential θ, free evolution from $X_{\mathrm{DSE}}(\frac{1}{2}g)$ to $X_{\mathrm{DSE}}(\frac{2}{3}g)$, and so on up to n^{th} integration with θ at the level $X_{\mathrm{DSE}}(\frac{n}{n+1}g)$ followed by a free evolution from $X_{\mathrm{DSE}}(\frac{n}{n+1}g)$ to $X_{\mathrm{DSE}}(g)$ when n tends to infinity. □

Lemma 5.1.7. *Thanks to the symmetric difference as a binary operation on large Feynman diagrams, there exists a complex commutative Banach algebra structure on $L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$.*

Proof. We have seen that the binary operation $\triangle$ determines an abelian compact Hausdorff topological group structure on $\mathcal{S}^{\Phi,g}$ such that the empty graph $\mathbb{I}$ is the zero element of this group. Now define the following convolution product on $L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$

$$F_1 *_\triangle F_2(\Gamma_1)$$

$$= \int_{\mathcal{S}^{\Phi,g}} F_1(\Gamma_2)F_2(\Gamma_2^{-1}\triangle\Gamma_1)d\mu_{\mathrm{Haar}}(\Gamma_2), \quad F_1, F_2 \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}}) \tag{5.25}$$

such that Γ_2^{-1} is the inverse of the graph with respect to the group structure $\triangle$. The compatibility between the product (5.25) and the L^1 norm

$$\| F \| := \int_{\mathcal{S}^{\Phi,g}} | F(X) | \, d\mu_{\mathrm{Haar}}(X) \tag{5.26}$$

provides our promising Banach algebra. The commutativity of the group $(\mathcal{S}^{\Phi,g}, \triangle)$ guarantees the commutativity of this Banach algebra. In addition, we add the infinitesimal delta function δ as the multiplicative unit for this Banach algebra. We have

$$\int_{\mathcal{S}^{\Phi,g}} F(X)\delta(X)d\mu_{\text{Haar}}(X) = F(\mathbb{I}) \tag{5.27}$$

for each $F \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ and each large Feynman diagram $X \in \mathcal{S}^{\Phi,g}$ such that $\mathbb{I}$ is the empty graph. □

Proposition 5.1.8. *(i) Each functional $F \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ has a non-empty spectrum.*

(ii) The space $\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}))$ of all characters of the complex Banach algebra $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ is a compact Hausdorff topological space.

Proof. We need only to adapt the standard procedures in Functional Analysis ([138]) to achieve the results.

(i) We show that

$$\text{sp}(F) := \{\lambda \in \mathbb{C} : F - \lambda\delta \ \ \text{not invertible}\} \tag{5.28}$$

is non-empty. If $F = 0$, then thanks to the definition of the infinitesimal delta function, we have the result. If F be a non-zero functional, suppose its spectrum is empty which means that the new function

$$R : \mathbb{C} \longrightarrow L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}), \quad \lambda \longmapsto (F - \lambda\delta)^{-1} \tag{5.29}$$

is well-defined, holomorphic, non-constant and bounded. For any bounded linear functional Υ on $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$, define a new function $\tilde{\Upsilon}$ on $\mathbb{R}^2$ given by

$$\tilde{\Upsilon}(x, y) := \Upsilon(R(xe^{iy})). \tag{5.30}$$

We can show that $\tilde{\Upsilon}$ is continuously differentiable with respect to variables x and y. Now by differentiation under the integral sign from the holomorphic bounded function $K(x) := \int_0^{2\pi} \tilde{\Upsilon}(x, y)dy$, we have $K'(x) = 0$. Therefore K is a constant function which is a contradiction with the initial assumption.

(ii) The ideal generated by kernel of any character provides a natural correspondence between the set of maximal ideals of the Banach algebra $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ and the set of characters on the space $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$.

Each character $\psi \in \Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}))$ is actually an algebra homomorphism from $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ to $\mathbb{C}$ such that $\psi(\delta) = 1$. We can show that ψ is continuous of norm 1, otherwise there exists a function $F \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ such that $||F|| < 1$ and $\psi(F) = 1$. Apply the convolution product to define $G := \sum_{n \geq 1} G^n$. From the equation $G = F + FG$ we have

$$\psi(G) = \psi(F) + \psi(F)\psi(G) = 1 + \psi(G) \tag{5.31}$$

which shows a contradiction. So the norm of ψ is less than or equal to 1 and $\psi(\delta) = 1$ which implies that $||\psi|| = 1$. Thanks to this fact, we can see that $\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}}))$ is a closed subset of the unit ball of the dual space $L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})^*$ which is a compact Hausdorff space with respect to the weak-$\star$ topology. As the consequence, $\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}}))$ is a compact Hausdorff topological space. $\square$

Thanks to the Gelfand transform, define

$$L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}}) \longrightarrow C_0(\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}}))) \tag{5.32}$$

$$F \longmapsto \widetilde{F}, \quad \widetilde{F}(\psi) := \psi(F).$$

It is a norm decreasing algebraic homomorphism such that its image separates μ_{Haar}-integrable functions on $\mathcal{S}^{\Phi,g}$. It can be seen that

$$||\widetilde{F}||_\infty = \max\{|\lambda| : \quad \lambda \in \mathrm{sp}(F)\}. \tag{5.33}$$

Thanks to the Pontryagin duality Theorem [138], we can obtain a correspondence between elements of the topological space $\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}}))$ and elements of the Pontryagin dual. In this situation, the canonical isomorphism

$$\mathrm{ev}_{L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})}(X)(\rho) = \rho(X) \in S^1 \subset \mathbb{C} \tag{5.34}$$

can be applied to define a modification of the Fourier transformation on $L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$.

Definition 5.1.9. For a given Quantum Field Theory Φ with the strong coupling constant $g \geq 1$ and the corresponding collection $\mathcal{S}^{\Phi,g}$ of all large Feynman diagrams generated by Dyson–Schwinger equations, the Fourier transformation $\mathcal{F}$ on the complex commutative unital Banach algebra $L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$ is well-defined. For $F \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$, we have

$$\widehat{F}(\rho) = \int_{\mathcal{S}^{\Phi,g}} F(X)\overline{\rho(X)} d\mu_{\mathrm{Haar}}(X). \tag{5.35}$$

for any character ρ.

For functionals $G, H \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$, we have

$$\mathcal{F}\{G *_\triangle H\} = \mathcal{F}\{G\}\mathcal{F}\{H\}. \tag{5.36}$$

The original motivation to formulate the Gelfand transform (5.32) is providing a way to separate functionals in $L^1(\mathcal{S}^{\Phi,g}, \mu_{\mathrm{Haar}})$. Thanks to this idea, the Fourier transformation (5.35) encodes the mathematical procedure for the decomposition of the functional F in terms of large Feynman diagrams which contribute to Dyson–Schwinger equations in $\mathcal{S}^{\Phi,g}$. In particular, if we restrict our discussion to a fixed Dyson–Schwinger equation DSE with the unique solution X_{DSE}, then our generalized Fourier transformation describes the evolution of the large Feynman diagram X_{DSE} with respect to μ-integrable functions originated from large subdiagrams (or partial sums) which converges to X_{DSE}.

As the final note, we have built and developed a theory of Haar integration theory on large Feynman diagrams in the language of the classical Riemann–Lebesgue integral and it is useful if we think about an analogous version of the classical Newton–Leibniz differentiation theory for the level of large Feynman diagrams and metrics $d_{g,\alpha,\epsilon}$.

5.2 The Gâteaux differential calculus on $\mathcal{S}^{\Phi,g}$ and its application

In the second section of the previous chapter we have explained the construction of a noncommutative differential geometry model on the Hopf algebra $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ which is derived from the BPHZ renormalization process of Feynman graphons. In [148] we have applied the analysis of linear spaces to discuss the construction of a theory of differentiation on the space of Feynman diagrams in the context of unlabeled graphon classes and Gâteaux differentiability under the cut-distance topology where we worked on admissible directions to define well-defined differentiations. It has led us to obtain Taylor series type representations for continuous functionals on Feynman graphons on the basis of homomorphism densities. The homomorphism densities can be considered as functionals $W_{\Gamma} \longmapsto t(G, W_{\Gamma})$ on Feynman graphons such that G is an arbitrary finite graph. If G is a simple graph (such as rooted trees), then we can show that the corresponding homomorphism density is continuous with respect to the cut-distance topology and it is also L^1-integrable. Thanks to the disjoint union operator, we can build an algebraic structure on the linear span of homomorphism densities with respect to finite simple graphs which is dense in the space $C(\mathcal{S}^{\Phi}_{\mathrm{graphon}})$ of all continuous functions on $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ with respect to the topology of uniform convergence. In addition, we can compute the Gâteaux derivatives of homomorphism densities where their Fréchet differentiability can be achieved with respect to simple graphs (i.e. decorated non-planar rooted trees) under some conditions where we might need to remove the symmetric condition of Feynman graphons and work on Feynman bigraphons. Thanks to these observations, homomorphism densities play central roles for functional analysis of graphons.

In this section, we concern the question of how to endow with a differential calculus on large Feynman diagrams independent of any renormalization program. We consider the real or complex vector space $\mathcal{S}^{\Phi,g}$ generated by all Dyson–Schwinger equations in the physical theory Φ and equip this space with the cut-distance topology determined by

$$d(\mathrm{DSE}_1, \mathrm{DSE}_2) := d_{\mathrm{cut}}([f^{X_{\mathrm{DSE}_1}}], [f^{X_{\mathrm{DSE}_2}}]). \tag{5.37}$$

It enables us to define the cut-norm for each large Feynman diagram

$$\| X_{\mathrm{DSE}} \|_{\mathrm{cut}} = \sup_{A,B \subset [0,1]} | \int_{A \times B} f^{X_{\mathrm{DSE}}}(x,y)dxdy | \tag{5.38}$$

such that the supremum is taken over Lebesgue measurable subsets A, B of the closed interval. The resulting space can be interpreted as a closed topological subspace of the compact topological space of all unlabeled graphons. It means that we can consider $\mathcal{S}^{\Phi,g}$ as a Banach space and build a Gâteaux differential calculus on (large) Feynman diagrams. Then we apply the functional analysis of graphons ([39]) to obtain a new Gâteaux differential calculus machinery for the study of $C(\mathcal{S}^{\Phi,g})$.

Total derivative and directional derivatives are the most common differentiation machineries in finite dimensions such that their analogous versions in infinite dimensions are Fréchet derivative and Gâteaux derivative.

For a given function $F : X \longrightarrow Y$ between two Banach spaces (or normed vector spaces), the Gâteaux derivative at $x_0 \in X$ is by definition a bounded linear operator $T_{x_0} : X \longrightarrow Y \in \mathcal{B}(X,Y)$ such that for every $u \in X$,

$$\lim_{t \longrightarrow 0} \frac{F(x_0 + tu) - F(x_0)}{t} = T_{x_0} u. \tag{5.39}$$

If for some fixed u the limit

$$\delta_u F(x) := \frac{d}{dt}|_{t=0} F(x + tu) = \lim_{t \to 0} \frac{F(x + tu) - F(x)}{t} \tag{5.40}$$

exists, then we call F has a directional derivative at x in the direction u. Therefore F is Gâteaux derivative at x_0 if and only if all the directional derivatives $\delta_u F(x)$ exist and form a bounded linear operator

$$DF(x) : u \longmapsto \delta_u F(x). \tag{5.41}$$

T_{x_0} is called the Fréchet derivative of F at x_0, if the limit (in the sense of the Gâteaux derivative) exists uniformly in u on the unit ball in X. If we set $y = tu$ then t tends to zero is equivalent to y tends to zero. Now F is Fréchet differentiable at x_0 if for all y,

$$F(x_0 + y) = F(x_0) + T_{x_0}(y) + o(\| y \|) \tag{5.42}$$

which means that

$$\lim_{\|h\| \to 0} \frac{\| F(x+h) - F(x) - Th \|}{\| h \|} = 0 \tag{5.43}$$

holds. As we can see, the limit in the Fréchet derivative only depends on the norm of y where the operator T defines the natural linear approximation of F in a neighborhood of the point x_0. In this setting, we call $T_{x_0} = DF(x_0)$ as the derivative of F at x_0. In addition, we can show that being Fréchet differentiable at a point implies being Gâteaux differentiable at a point such that in this case the Gâteaux derivative is equal to the Fréchet derivative.

If F is Gâteaux differentiable on X, then we have the mean value formula

$$\parallel F(y) - F(x) \parallel \leq \parallel x - y \parallel \sup_{0\leq\theta\leq 1} \parallel DF(\theta x + (1-\theta)y) \parallel . \tag{5.44}$$

This enables us to show that if F is Gâteaux differentiable on an open neighborhood U of x and $DF(x)$ is continuous, then F is Fréchet differentiable at x. [60, 138]

We plan to study (smooth) real valued continuous functions on the Banach space $\mathcal{S}^{\Phi,g}$ in terms of their Taylor series representation under the higher orders Gâteaux differentiations. We show that the solution space of the natural generalization of the equation $\frac{d^n}{dx^n}F(x) \equiv 0$ to large Feynman diagrams namely, Gâteaux type differential equations with the general form

$$d^{N+1}F(X; Z_1, ..., Z_{N+1}) = 0 \tag{5.45}$$

for all large Feynman diagrams $X, Z_1, ...Z_{N+1} \in \mathcal{S}^{\Phi,g}$, can be described by homomorphism densities. Solutions of this class of equations enjoy the property

$$F(Z) = F(\mathbb{I}) + dF(\mathbb{I}; Z) + \frac{d^2F(\mathbb{I}; Z, Z)}{2} + ... + \frac{d^nF(\mathbb{I}; Z, ..., Z)}{n!}. \tag{5.46}$$

Definition 5.2.1. For a given function $F : \mathcal{S}^{\Phi,g} \longrightarrow \mathbb{R}$ and each large Feynman diagram X, the Gâteaux derivative exists at X in the direction $Y \in \mathcal{S}^{\Phi,g}$, if the limit

$$dF(X; Y) = \lim_{t\longrightarrow 0} \frac{F(X + tY) - F(Y)}{t} \tag{5.47}$$

exists. The higher orders of the Gâteaux differentiability can be defined by induction where for any $n \geq 2$, F is called n-time Gâteaux differentiable at X in directions $Z_1, ..., Z_n$ if at the first, the higher mixed Gâteaux derivatives

$$d^{n-1}F(X + \lambda Z_n; Z_1, ..., Z_{n-1}) \tag{5.48}$$

exist for each real number λ and at the second, the limit

$$d^nF(X; Z_1, ..., Z_n) =$$

$$\lim_{\lambda\longrightarrow 0} \frac{d^{n-1}F(X + \lambda Z_n; Z_1, ..., Z_{n-1}) - d^{n-1}F(X; Z_1, ..., Z_{n-1})}{\lambda} \tag{5.49}$$

exists.

It is easy to check that the Gâteaux derivatives $d^n F(X; Z_1, ..., Z_n)$ are multilinear maps in Z_i and in addition, for any permutation $\tau \in S_n$, we have

$$d^n F(X; Z_1, ..., Z_n) = d^n F(X; Z_{\tau(1)}, ..., Z_{\tau(n)}). \tag{5.50}$$

The description of large Feynman diagrams via Feynman graphons in $\mathcal{S}^{\Phi}_{\text{graphon}}$ allows us to think about the concept of density of a subgraph in the solution of a given Dyson–Schiwnger equation. From the Quantum Field Theory viewpoint, each subgraph may contain a class of subdivergencies derived from some particular virtual particles and their interactions. In this situation, one important question is to estimate the appearance of these subdivergencies in an infinite expansion of finite Feynman diagrams. The homomorphism density, as a class function on $\mathcal{S}^{\Phi}_{\text{graphon}}$ or $\mathcal{S}^{\Phi,g}$, is a useful tool to formulate this story.

Proposition 5.2.2. *The homomorphism density is a well-defined operator on large Feynman diagrams.*

Proof. Proposition 4.6 in [149] shows that the unique solution X_{DSE} of a given Dyson–Schwinger equation DSE can be interpreted as the convergent limit of the sequence $\{Y_m\}_{m\geq 1}$ of its partial sums with respect to the cut-distance topology. For each unlabeled Feynman graphon class $[W]$, we plan to characterize the homomorphism density $t(X_{\text{DSE}}, W)$ as the "limit" of the sequence $\{t(Y_m, W)\}_{m\geq 1}$ of homomorphism densities corresponding to finite expansions Y_m of finite graphs (which do not have self-loops but have loops). For each m, if Y_m has k_m vertices, we have

$$t(Y_m, W) = \int_{[0,1]^{k_m}} \prod_{(i,j)\in E(Y_m)} W(x_i, x_j) dx_1 ... dx_{k_m}. \tag{5.51}$$

For each $m \geq 1$, $Y_m := X_1 + ... + X_m$, by induction we can show that

$$t(Y_{m+1}, W) = t(Y_m, W) t(X_{m+1}, W). \tag{5.52}$$

In addition, the condition $d_{\text{cut}}(W', W) = 0$ implies $t(Y_m, W') = t(Y_m, W)$ which leads us to $t(X_{\text{DSE}}, W') = t(X_{\text{DSE}}, W)$. Therefore we can define a poset on homomorphism densities where a family of injections $\{f_{ij} : t(Y_i, -) \to t(Y_j, -)\}_{i\leq j}$ on the space of Feynman graphons can be formulated. This gives us an inverse system where its inverse limit can be identified as a subset of the direct product of the $t(Y_i, -)$s. This inverse limit can be considered as the homomorphism density with respect to the large Feynman diagram X_{DSE}. We have

$$t(X_{\text{DSE}}, -) = \lim_{\leftarrow m} t(Y_m, -)$$

$$= \{\overrightarrow{W} \in \prod_{m=1}^{\infty} t(Y_m, -) : \quad f_{ij}(W_j) = W_i, \ \forall i \leq j\} \subseteq \prod_{m=1}^{\infty} t(Y_m, -). \tag{5.53}$$

□

For $1 \leq n \leq \infty$, set $[n] := \{i \in \mathbb{N} : \quad i \leq n\}$. For a given Feynman graphon W, define a random graph $G(n, W)$ with vertex set $[n]$ (chosen points $\{x_1, ..., x_n\}$ at random from the closed unit interval) by letting ij be an edge in $G(n, W)$ with the probability $W(x_i, x_j)$. It is possible to build $G(n, W)$ for all n by first constructing $G(\infty, W)$ as an exchangeable random graph namely, its distribution is invariant under permutations of the vertices and every exchangeable random graph is a mixture of such graphs. Then take the subgraph defined by the first n vertices. In general, for a Feynman diagram Γ (as a labeled graph), the homomorphism density $t(\Gamma, W)$ equals the probability that Γ is a subgraph of $G(\infty, W)$ or of $G(n, W)$ for any $n \geq |\Gamma|$. In other words, the family $\{t(\Gamma, W)\}_\Gamma$ and the distribution of $G(\infty, W)$ determine each other. It is important to note that for given Feynman graphons W, W', $G(\infty, W)$ and $G(\infty, W')$ have the same distribution if and only if those two graphons are weakly equivalent.

Lemma 5.2.3. *Homomorphism densities provide a class of random graphs with respect to large Feynman diagrams.*

Proof. We know that $X_{\mathrm{DSE}} = \lim_{m\to\infty} Y_m$ with respect to the cut-distance topology such that $|Y_m| \to \infty$. For each $[k]$ there exists a random graph $Y_m[k]$ on the vertex set $[k]$ such that $\{Y_m[k]\}_{m\geq 1}$ converges to $X_{\mathrm{DSE}}[k]$ with respect to the metric

$$d_{\mathrm{dens}}(\Gamma, \Gamma') = \sum_i 2^{-i} |t(W_i, \Gamma) - t(W_i, \Gamma')| \tag{5.54}$$

which is equivalent to the cut-distance ([17, 77]). Therefore there exists a random infinite graph $\overline{X_{\mathrm{DSE}}}$ on $[\infty]$ such that $X_{\mathrm{DSE}}[k] \equiv \overline{X_{\mathrm{DSE}}}|_{[k]}$ with respect to the metric d_{dens} which means that

$$\lim_{m\to\infty} Y_m[k] = \overline{X_{\mathrm{DSE}}}|_{[k]}. \tag{5.55}$$

□

Corollary 5.2.4. *The distribution of the random graphs $Y_m[k]$ with respect to partial sums of X_{DSE} converges when m tends to infinity.*

Proof. Thanks to Lemma 5.2.3, for each $k \leq |Y_m|$, there exists $Y_m[k]$ as the random induced subgraph of Y_m with k vertices determined by selecting k separate vertices $v_1, ..., v_k$ of Y_m at uniformly random procedures. Now thanks to graphon representations of Feynman diagrams, it is enough to apply the definition of convergent sequences in the theory of graphons via random graphs ([77]) to the sequence $\{Y_m\}_{m\geq 1}$ which converges to X_{DSE}. □

Lemma 5.2.5. *Dyson–Schwinger equations which generate isomorphic Hopf subalgebras have the same homomorphism density.*

Proof. Suppose Hopf subalgebras H_{DSE_1}, H_{DSE_2} corresponding to the equations DSE_1 and DSE_2 are isomorphic which means that the unique solutions X_{DSE_1} and X_{DSE_1} are isomorphic infinite graphs. We can lift the weakly equivalent relation on graphons onto the level of large Feynman diagrams. We say that two large Feynamn diagrams $X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}$ are weakly equivalent if their corresponding labeled graphons have the same unlabeled measurable function almost everywhere. In other words, X_{DSE_1} and X_{DSE_2} are weakly equivalent if

$$d_{\mathrm{cut}}(f^{X_{\mathrm{DSE}_1}}, f^{X_{\mathrm{DSE}_2}}) = 0. \tag{5.56}$$

Thanks to Borgs–Chayes–Lovasz Theorem ([107]) and Proposition 5.2.2, we can show that two weakly equivalent large Feynamn diagrams $X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}$ have the same homomorphism density for all (finite) simple graphs. In other words,

$$t(H, f^{X_{\mathrm{DSE}_1}}) = t(H, f^{X_{\mathrm{DSE}_2}}). \tag{5.57}$$

□

In the rest of this part we try to show that homomorphism densities of the type $t(X_{\mathrm{DSE}}, -)$ can play the role of a basis for the space of smooth real valued functions on $\mathcal{S}^{\Phi,g}$.

Define a new graduation parameter on the collection $\mathcal{H}(\Phi)$ of all Feynman diagrams of a physical theory Φ in terms of the number of edges. For each n, set $\mathcal{H}_n(\Phi)$ as the isomorphism classes of Feynman diagrams with n internal and external edges, no isolated vertices, no self-loops but possible multi-edges. Set $\mathcal{H}_{\leq n}(\Phi) = \bigcup_{j\leq n} \mathcal{H}_j(\Phi)$. The homomorphism density for all Feynman graphs $\Gamma \in \mathcal{H}_j(\Phi)$ is well-defined.

Theorem 5.2.6. *For each $n \geq 1$, define*

$$F_n : \mathcal{S}^{\Phi}_{\mathrm{graphon}} \longrightarrow \mathbb{R}, \quad F_n(W_\Gamma) := \sum_{\gamma \in \mathcal{H}_{\leq n}(\Phi)} a_\gamma t(\gamma, W_\Gamma) \tag{5.58}$$

for some constants a_γ. Then
(i) F_n is continuous in the L^1-topology.
(ii) It is possible to lift F_n onto the space of large Feynman diagrams and define a new real valued map $\tilde{F}$ on $\mathcal{S}^{\Phi,g}$ which is continuous in the L^1-topology.

Proof. (i) Define a new real valued multilinear functional τ on $(\mathcal{S}^{\Phi}_{\mathrm{graphon}})^n$ given by

$$\tau((W_e)_{e\in E(\gamma)}) := \int_{[0,1]^{|\gamma|}} \prod_{e\in E(\gamma)} W_e(x_i, x_j) \prod_{i\in V(\gamma)} dx_i. \tag{5.59}$$

We can show that

$$|\tau((W_e)_{e\in E(\gamma)}) - \tau((W'_e)_{e\in E(\gamma)})| \leq \sum_{e\in E(\gamma)} \parallel W_e - W'_e \parallel_1 \tag{5.60}$$

which leads us to

$$|t(\gamma, W) - t(\gamma, W')| \leq |E(\gamma)| \parallel W_e - W'_e \parallel_1 . \tag{5.61}$$

(ii) Thanks to Proposition 5.2.2, we plan to lift the above process onto the level of large Feynman diagram X_{DSE} with the partial sums Y_m, $m \geq 1$. It is enough to extend the relation (5.61) to $Y_{m+1} = Y_m + X_{m+1}$. We have

$$\begin{aligned} &\big|t(Y_m \sqcup X_{m+1}, W) - t(Y_m \sqcup X_{m+1}, W')\big| = \\ &\big|t(Y_m \sqcup X_{m+1}, W) - t(Y_m \sqcup X_{m+1}, W') \pm t(Y_m, W')t(X_{m+1}, W)\big| = \\ &\big|t(X_{m+1}, W)\big(t(Y_m, W) - t(Y_m, W')\big) + t(Y_m, W')\big(t(X_{m+1}, W) - t(X_{m+1}, W')\big)\big| \\ &\leq |t(X_{m+1}, W)||(t(Y_m, W) - t(Y_m, W'))| + |t(Y_m, W')||t(X_{m+1}, W) - t(X_{m+1}, W')| \\ &\leq |t(X_{m+1}, W)|\big(|E(Y_m)| \parallel W_e - W'_e \parallel_1\big) + |t(Y_m, W')|\big(|E(X_{m+1})| \parallel W_e - W'_e \parallel_1\big) \end{aligned} \tag{5.62}$$

such that $\parallel W_e \parallel_\infty, \parallel W'_e \parallel_\infty \leq 1$.

Lift the multilinear operator τ onto the multilinear operator $\tilde{\tau}$ defined as a bounded operator on the Banach space $\mathcal{S}^{\Phi,g}$. Now define the new map $\tilde{F}$ on $\mathcal{S}^{\Phi,g}$ given by

$$\tilde{F}(X_{\mathrm{DSE}}) := \prod_{m=1}^{\infty} F_m(W_{Y_m}) \tag{5.63}$$

such that each term $F_m(W_{Y_m})$ is a L^1-continuous function. Therefore $\tilde{F}$, as the product of continuous functions, is also continuous with respect to the L^1- topology. □

The space $\mathcal{W}_{[0,1]}$ of all (bi-)graphons can be embedded into the vector space $\mathcal{W}$ of bounded (symmetric) measurable functions $f : [0,1]^2 \longrightarrow \mathbb{R}$ which is equipped by a semi-norm. Under weakly equivalent relation $\sim$, we can build a complete metric structure on the quotient space $\mathcal{W}_{[0,1]}/\sim$. The topological space $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$ of all unlabeled graphons which contribute to the representations of Feynman diagrams and Dyson–Schwinger equations sits inside $\mathcal{W}_{[0,1]}/\sim$. Each $[W] \in \mathcal{S}^{\Phi}_{\mathrm{graphon}}$ is a class of bounded (symmetric) measurable functions from $[0,1]^2 \longrightarrow [0,1]$ up to relabeling which is generated by rooted tree representations of (large) Feynman diagrams. Rooted trees are simple graphs where the adjacency matrices determine their corresponding graphon classes. Orientations on decorated non-planar rooted trees, which encode positions of nested loops in the main Feynman diagram, inform us that we might need only the upper part or the lower part of the adjacency matrix for the reconstruction of a Feynman diagram from its graphon representation. It means that we do not need the symmetric property of graphons and we can work only on the bounded measurable functions $f : [0,1]^2 \longrightarrow [0,1]$ up to relabeling. This class of objects, which are known as bi-graphons in Combinatorics, enables us to have Fréchet differentiability of homomorphism densities.

Lemma 5.2.7. *(i) The Homomorphism densities on $\mathcal{S}^{\Phi}_{\text{graphon}}$ are Fréchet differentiable.*

(ii) The Homomorphism densities on $\mathcal{S}^{\Phi,g}$ are Fréchet differentiable.

Proof. (i) We compute the Fréchet derivatives of the homomorphism densities on $\mathcal{S}^{\Phi}_{\text{graphon}}$ for ladder trees l_1, l_2, l_3 and the rooted tree $\bigvee$ where vertices $2, 3$ are adjacent to the root 1.

For the tree with only one vertex, $t(l_1, -) \equiv 1$ which is obviously Fréchet differentiable.

For the oriented decorated ladder tree l_2 with two vertices $1, 2$ and one edge e_{12} from 1 to 2, the Gâteaux derivative can be computed by

$$d(t(H, W_\Gamma); W_{\Gamma'}) =$$

$$\int_{[0,1]^k} \sum_{(i_1,j_1)\in E(H)} W_{\Gamma'}(x_{i_1}, x_{j_1}) \prod_{(i,j)\in E(H)\backslash(i_1,j_1)} W_\Gamma(x_i, x_j) dx_1 ... dx_k \quad (5.64)$$

which leads us to compute the unique Fréchet derivative by the linear map

$$W_{\Gamma'} \longmapsto d(t(l_2, W_\Gamma); W_{\Gamma'}) = \int_{[0,1]^2} W_{\Gamma'}(x_1, x_2) dx_1 dx_2. \quad (5.65)$$

We have

$$\lim_{W_{\Gamma'}\to 0} \frac{\left| t(H, W_\Gamma + W_{\Gamma'}) - t(H, W_\Gamma) - \int_{[0,1]^2} W_{\Gamma'}(x_1, x_2) dx_1 dx_2 \right|}{\| W_{\Gamma'} \|_{\text{cut}}} \quad (5.66)$$

$$= \lim_{W_{\Gamma'}\to 0} \frac{0}{\| W_{\Gamma'} \|_{\text{cut}}} = 0$$

which approves the Fréchet differentiability in terms of the formula (5.43).

For the oriented decorated ladder tree l_3 with three vertices $1, 2, 3$ and two edges e_{12}, e_{23} which connect the vertices 1 to 2 and 2 to 3, the unique candidate for the Fréchet derivative of $t(l_3, -)$ should be

$$W_{\Gamma'} \longmapsto d(t(l_3, W_\Gamma); W_{\Gamma'}) =$$

$$\int_{[0,1]^3} W_{\Gamma'}(x_1, x_2) W_\Gamma(x_2, x_3) + W_\Gamma(x_1, x_2) W_{\Gamma'}(x_2, x_3) dx_1 dx_2 dx_3 \quad (5.67)$$

$$= 2 \int_{[0,1]^3} W_\Gamma(x_1, x_2) W_{\Gamma'}(x_2, x_3) dx_1 dx_2 dx_3.$$

$t(l_3, -)$ is Fréchet differentiable if the following limit exists and equals to zero,

$$\lim_{W_{\Gamma'}\to 0} \frac{\left| t(l_3, W_\Gamma + W_{\Gamma'}) - t(l_3, W_\Gamma) - d(t(l_3, W_\Gamma); g) \right|}{\| W_{\Gamma'} \|_{\text{cut}}}$$

$$= \lim_{W_{\Gamma'} \to 0} \frac{\big| \int_{[0,1]^3} W_{\Gamma'}(x_1,x_2) W_{\Gamma'}(x_2,x_3) dx_1 dx_2 dx_3 \big|}{\parallel W_{\Gamma'} \parallel_{\rm cut}}. \tag{5.68}$$

If this limit is not zero, then there are some below boundaries $c > 0$ which means that we can define a sequence $\{W_n\}_{n \geq 1}$ of graphons which converges to zero with respect to the cut-norm but the limit (5.68) does not zero when n tends to infinity. In other words, we might have

$$0 < c \leq \frac{1}{n} = \big| \int_{[0,1]^2, x_2 \in [0,1/n]} 1 dx_1 dx_2 dx_3 \big| =$$
$$\big| \int_{[0,1]^2, x_2 \in [0,1/n]} W_n(x_1,x_2) W_n(x_2,x_3) dx_1 dx_2 dx_3 \big| \leq \tag{5.69}$$
$$\big| \int_{[0,1]^3} W_n(x_1,x_2) W_n(x_2,x_3) dx_1 dx_2 dx_3 \big|.$$

This situation supports the existence of sequences of graphons such as $W_n = \mathbf{1}(\min(x_1,x_2) < 1/n)$ which satisfy the above inequality. On the other hand, we can build the topological renormalization Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$ of Feynman graphons by measurable bounded functions from $[0,1]^2$ to $[0,1]$ in terms of working only on the upper parts or lower parts of the adjacency matrices of oriented decorated non-planar rooted trees. In this non-symmetric setting, the sequences such as $\{W_n\}_{n \geq 1}$ of graphons does not belong to $\mathcal{S}^{\Phi}_{\rm graphon}$. As the consequence, the only lower boundary for Feynman graphons is zero itself which means that the limit (5.68) is zero.

By a similar discussion, we can show the existence of Fréchet derivative for other oriented rooted trees H which contains the tree $\bigvee$ where vertices $2,3$ are adjacent to the root 1. In this situation, we need to deal with

$$\lim_{n \to \infty} \frac{\big| t(H, W_\Gamma + W_{\Gamma'}) - t(H, W_\Gamma) - d(t(H,W_\Gamma); W_{\Gamma'}) \big|}{\parallel W_{\Gamma'} \parallel_{\rm cut}} \tag{5.70}$$

where if this limit is not zero then we get some lower boundaries $c > 0$ such that

$$c^{|E(H)|-2} \int_{[0,1]^3} W_{\Gamma'}(x_1,x_2) W_{\Gamma'}(x_2,x_3) dx_1 dx_2 dx_3 \leq$$
$$\int_{[0,1]^{|V(H)|}} W_{\Gamma'}(x_1,x_2) W_{\Gamma'}(x_2,x_3) \prod_{(ij) \in E(H) \backslash \{(1,2),(2,3)\}} W_\Gamma(x_i,x_j) dx_1 ... dx_{|V(H)|}$$
$$\leq t(H, W_\Gamma + W_{\Gamma'}) - t(H, W_\Gamma) - d(t(H,W_\Gamma); W_{\Gamma'}). \tag{5.71}$$

This situation allows us to determine sequences of graphons which do not belong to $\mathcal{S}^{\Phi}_{\rm graphon}$. Therefore the limit (5.70) should be zero.

(ii) Objects in the Banach space $\mathcal{S}^{\Phi,g}$ are large Feynman diagrams namely, infinite formal expansions of Feynman diagrams which have nested or overlapping loops. Thanks to Theorem 5.2.2, homomorphism densities on large

Feynman diagrams can be computed in terms of homomorphism densities of finite partial sums. For large Feynman diagrams X, Z, we have

$$t(X,Z) = \lim_{\leftarrow m} t(Y_m, Z) \tag{5.72}$$

such that for each $m \geq 1$,

$$t(Y_m, Z) = \prod_{i=1}^{m} t(X_i, W_Z) \tag{5.73}$$

such that $X(g) = \sum_{n\geq 0} g^n X_n$ and W_Z is a Feynman graphon which lives in $\mathcal{S}^{\Phi}_{\text{graphon}}$ where it can not have a non-zero lower boundary. Furthermore, thanks to the formula (5.42), we know that the Fréchet differentiability depends only on the norm W_Z where by applying (i), each $t(X_m, W_Z)$ is Fréchet differentiable. Thanks to the product rule, $t(Y_m, Z)$ as the product of Fréchet differentiable functions is also Fréchet differentiable for each m. Since $t(X, Z)$ can be identified by a subset of the direct product $\prod_{m=1}^{\infty} t(Y_m, Z)$, $t(X, Z)$ will be also Fréchet differentiable. □

Lemma 5.2.8. *For a given C^n $(n > 0)$ class function $G : \mathcal{S}^{\Phi,g} \longrightarrow \mathbb{R}$, $d^n G(\mathbb{I}; Z_1, ..., Z_n)$ is a symmetric $S_{[0,1]}$-invariant multilinear functional.*

Proof. We can extend the functions

$$\mathcal{G}_{X,Z}(\lambda_1, ..., \lambda_m) := G(X + \lambda_1 Z_1 + ... + \lambda_m Z_m) \tag{5.74}$$

to C^n functions on $\mathbb{R}^m$ where the equality of mixed partial derivatives show that for any permutation $\sigma \in S_n$, we have

$$d^n G(X; Z_1, ..., Z_n) = d^n G(X; Z_{\sigma(1)}, ..., Z_{\sigma(n)}). \tag{5.75}$$

In addition, we can extend $d^n G(\mathbb{I}; Z_1, ..., Z_n)$ multilinearly to each

$$(\text{span}_{\mathbb{R}}(\Gamma_1, ..., \Gamma_m))^n. \tag{5.76}$$

□

Corollary 5.2.9. *Let $\tilde{F} : \mathcal{S}^{\Phi,g} \longrightarrow \mathbb{R}$ be a L^1-continuous functional (determined by Theorem 5.2.6) such that for some $N \geq 1$, $\tilde{F}$ is $N+1$ times Gâteaux differentiable. Then for each large Feynman diagram X and also $Z_1, ..., Z_{N+1}$ in the Banach space $\mathcal{S}^{\Phi,g}$,*

$$d^{N+1}\tilde{F}(X; Z_1, ..., Z_{N+1}) = 0$$

if and only if there exists a unique family $\{a_\gamma\}_\gamma$ of real constants such that $\tilde{F}(X) = \sum_{\gamma \in \mathcal{H}_{\leq N}(\Phi)} a_\gamma t(\gamma, W_X)$.

Proof. Thanks to Definition 5.2.1, Proposition 5.2.2, Theorem 5.2.6, Lemma 5.2.7, Lemma 5.2.8, we can apply the main result in [39]. □

Corollary 5.2.10. *Let $G : \mathcal{S}^{\Phi,g} \longrightarrow \mathbb{R}$ be a continuous functional with respect to the cut-distance topology and smooth with respect to the Gâteaux derivation. For each large Feynman diagram X define the following sequence of Taylor polynomials*

$$P_n(X) := \sum_{m=0}^{n} \frac{1}{m!} d^m G(0; X, ..., X), \quad \forall n \geq o \tag{5.77}$$

which converges to $G(X)$ when n tends to infinity. In addition, let the Taylor expansion

$$\sum_{m=0}^{\infty} \sum_{\gamma \in \mathcal{H}_m(\Phi)} a_\gamma t(\gamma, W_X) \tag{5.78}$$

is absolutely convergent to $P(G)(X)$ such that

$$\sum_{\gamma \in \mathcal{H}_m(\Phi)} a_\gamma t(\gamma, W_X) = \frac{1}{m!} d^m G(0; X, ..., X). \tag{5.79}$$

Then $G(X) = P(G)(X)$.

Proof. A sufficient condition under which the Taylor series of a smooth function on the space of unlabeled graphons is convergent has been obtained in [39]. Now it is enough to adapt that procedure for to the level of Feynman graphons (which is already addressed in [148]) and then lift it onto the level of $\mathcal{S}^{\Phi,g}$ in terms of Definition 5.2.1, Proposition 5.2.2, Theorem 5.2.6, Lemma 5.2.7, Lemma 5.2.8 and Corollary 5.2.9. □

Chapter 6

The intrinsic foundations of QFT under a non-perturbative setting

- *Information flow via Feynman graphons*
- *Quantum logic via non-perturbative propositional calculus*

Quantum concepts such as entanglement and superposition have been applied as the fundamental tools for a theory of quantum computation which performs operations on information on the basis of quantum bits. The sate of a quantum bit lives in a superposition of two orthonormal states $|\psi >:= \alpha|0> +\beta|1>$ such that α, β are complex numbers. The measurement of one quantum bit collapses the wave function of the other quantum bit. Generally speaking, the theory of quantum entanglement deals with three fundamental subjects which could be studied under deterministic and indeterministic settings. The first challenge is to explain how we can detect optimally entanglement under theoretical models and experimental tests. The second challenge is to build theoretical models and experimental tests which reverse an inevitable process of degradation of entanglement. The third challenge is to design computational algorithms which enable us to characterize, control and quantify entanglement. The main objective in dealing with these challenges is to find a way to estimate optimally the amount of quantum entanglement of compound system in an unknown state if only incomplete data in the form of average values of some operators detecting entanglement are accessible. In this direction, a notion of minimization of entanglement has been formulated under a chosen measure of entanglement with constrains in the form of incomplete set of data from experiment. In addition, theory of positive maps has been also developed to provide strong tools for the detection of entanglement. [28, 75, 76, 135, 139]

Entanglement in Quantum Field Theories have also been considered recently where the measurements of the amount of entanglement in a quantum system with infinity degrees of freedom were modeled under some settings such as entropy, kinematic entanglement, particle mixing and oscillations, theory of neutrino oscillations and entangled space-time points. Entropy is on the basis of partitioning an extended quantum system into two complementary subsystems and calculating the entanglement entropy defined as the von Neumann entropy of the reduced density matrix of one subsystem. This treatment does not provide information about the entanglement between two non-complementary parts of a larger system because of the existence of a mixed state. Negativity is one interesting tool to deal with this issue in Quantum Field Theory. Multi-mode entanglement of single-particle states has been concerned via particle mixing and flavor oscillations. It is shown that in Quantum Field Theory these phenomena exhibit a fine structure of quantum correlations as multi-mode multi-particle entanglement appears. Quantum information theory is capable to provide appropriate tools to quantify the content of multi-particle flavor entanglement in QFT systems where the multi-particle flavor-species entanglement associated with flavor oscillations of the QFT neutrino system has been studied in terms of the particle-antiparticle species as further quantum modes. Neutrino oscillations are due to neutrino mixing and neutrino mass differences. Theory of entanglement in neutrino oscillations is another progress in this direction where mode entanglement can be expressed in terms of flavor transition probabilities. Charged-current weak-interaction processes together with their associated charged leptons enable us to identify flavor neutrinos. Neutrino oscillations and CP violation concern neutrino mixing such that neutrino masses as corrections to Standard Model play their essential roles in the procedure. [4, 13, 14, 15, 28, 69, 80]

In the first part of this chapter our original effort is to build a new mathematical model for the description of information flow among virtual particles and elementary particles in interacting gauge field theories. This mathematical model enables us to analyze quantum entanglement via fundamental tools in Category Theory and Theoretical Computer Science. We apply Dyson–Schwinger equations (their combinatorial versions) as the building blocks of information flow among distant elementary particles in a system with infinite number of freedom. The cut-distance topology is applied to construct topological regions around elementary particles which encode passing information. The whole machinery will be encapsulated via a new lattice model of intermediate algorithms which contribute to transferring information among entangled particles. [151]

Passing from Classical Physics to Quantum Physics changes the logical foundations of our mathematical frameworks. If we have a rigorous formulation for the logic of quantum systems with infinite degrees of freedom, then it definitely helps us to develop our knowledge about entanglement machinery in QFTs. The logical foundations of Quantum Mechanics were firstly built in the context of propositional calculus ([10]) and then thanks to applications of Category Theory, some advanced topos models for the logical descriptions of physical phenomena have been designed and developed. Thanks to this modern perspective, Classical Physics has been reconstructed on the category of sets while Quantum Physics has been reconstructed on the category of presheaves on a particular base category. This categorical machinery has been developed to QFT models where nowadays we have some topos models for gauge field theories. [1, 8, 47, 48, 49, 50, 51, 104, 108, 111, 123]

The original aim of the propositional calculus in logic is to evaluate propositions with the general form " *the physical quantity such as A of a given system S has a value in the subset Δ of real numbers.*" In this context, the main task is to find what truth-values such propositions have in a given state of the system and how the truth-value changes with the state in time. In Classical Physics, there is a space of states such as points in a topological space equipped with some additional structures (such as Poisson brackets, Symplectic forms, ...) such that in any given state, each physical quantity has its value and each proposition of the form $A \in \Delta$, which is represented by some Borel subsets of the state space, has a truth-value true or false. The Borel subsets of the state space form a Boolean σ-algebra which means that the logic of classical physical systems is definitely a Boolean logic. This description enables us to label Classical Physics as a realist theory. Quantum Physics does not have this explicit realistic nature and according to the Kochen–Specker Theorem, there is no state space of a quantum system analogous to the classical state space. As the assumptions of this Theorem, the physical quantities are represented as real-valued functions on the hypothetical state space of a quantum system and then it is shown that such a space does not exist and it is impossible to assign values to all physical quantities at once and therefore it is also impossible to assign true or false values to all propositions. Birkhoff and von Neumann built the foundations of an instrumentalist approach to quantum logic where upon measurement of the physical quantity A, we could find the result belong in Δ with a determined probability. In this approach, pure states are represented by unit vectors in one particular Hilbert space and propositions with the general form $A \in \Delta$ are represented by projection operators on this Hilbert space. These projections form a non-distributive lattice. Set $\hat{E}[A \in \Delta]$ as the projection which represents the proposition $A \in \Delta$. The probability of $A \in \Delta$ being true in a given state $|\psi >$ is determined by

$$P(A \in \Delta|\psi >) :=< \psi|\hat{E}[A \in \Delta]|\psi >\in [0, 1]. \tag{6.1}$$

Non-distribuitivity, dependence on measurement tools and the use of real numbers as continuum are the most fundamental and conceptual issues of this instrumentalist approach and its generalizations. Thanks to topos theory, a new contextual form of quantum logic has been built where it is possible to reconstruct the foundations of physical theories in the context of search for a suitable representation in a topos of a certain formal language. [1, 31, 32, 104, 111]

In the second part of this chapter our original task is to build a new mathematical model for the description of logical propositions of non-perturbative aspects in gauge field theories with strong couplings. We explain the fundamental structure of a new topos of presheaves which is capable to encode topological regions of elementary particles and the strength of coupling constants. This topos has enough physical information to evaluate logical propositions about infinite formal expansions of Feynman diagrams which contribute to quantum motions. [152]

6.1 *Information flow via Feynman graphons*

The mathematical formulation of Standard Model in the context of Noncommutative Geometry can motivate to bring a new approach for the description of quantum entanglement in gauge field theories. In Standard Model we have six quarks, six leptons and gauge bosons which are responsible to carry fundamental forces. Gauge bosons describe exchanging information between elementary particles and their interactions in strong, weak and electromagnetic forces. For example, the exchanging virtual photons (as the gauge boson in quantum electrodynamics) makes transferring information as the force between two electrons which is repulsive. Gluons are involved gauge bosons in strong interactions among hadrons (i.e. six quarks) which live in the nucleus of an atom. Electrons and neutrinos do not feel strong nuclear force. Every charge particle feels the electromagnetic force. $W^{\pm}, Z$ are involved gauge bosons in weak interactions where everything is effected by weak nuclear force. Graviton is the theoretical candidate for gauge bosons of gravity which effects everything. The modified versions of Standard Model aim to describe the contribution of gravity. Heavier gauge bosons are bosons of the fundamental force with the shortest range of effect. Photons are massless which means that the electromagnetic force has infinite range. $W^{\pm}, Z$ bosons are extremely heavy and they have very short range. For example, a neutron can decay into a proton and the gauge boson W^{-} where at the very short time, this boson quickly decays into an electron and an antielectron neutrino. A proton can decay into a neutron and the gauge boson W^{+} where at the very short time this boson quickly decays into a positron and a neutrino. Protons and neutrons are built by quarks. $W^{\pm}$ bosons can contribute to exchanging a type of quark to another type where as the result a proton converts to a neutron and vice versa. Since $W^{\pm}$ are heavy, they need to borrow energy to perform this exchange and then they should pay back the energy by converting to pairs (positron, neutrino) or (electron, antielectron neutrino) very quickly. Quarks enjoy the Pauli exclusion principle which means that quarks should be in different quantum states. This distinction is encoded by colors. Gluons govern any possible interactions among quarks which convert or exchange the colors of quarks by absorbtion or emission of gluons. Gluons can also produce other gluons and they glue quarks together. Force between two quarks is independent of distance between them and therefore we need infinite amount of energy to separate quarks. This fact, known as quark confinement, tells us that we can not isolate a quark. Thanks to gluons, strong force also governs the existence of protons and neutrons together inside the nucleus but the force at this level is not independent of distance. Theoretically, the amount of energy can be converted to a pair of quark and anti-quark where some interactions could happen to exchange colours. [37, 94, 129, 136, 137, 163]

It is possible to encapsulate all possible interactions in terms of Green's functions where their self-similar nature enable us to study interactions in the context of fixed point equations of Green's functions namely, Dyson–Schwinger equations. The strength of the fundamental forces dictate the appearance of perturbative, asymptotic freedom or non-perturbative behaviors to these equations. It is mentioned that gauge bosons provide information exchange and here we plan to mathematically describe the existence of information flow among elementary particles at strong levels of the coupling constants in interacting gauge field theories via towers of Dyson–Schwinger equations, cut-distance topological regions of Feynman diagrams which contribute to solutions of these equations and the vacuum energy. The vacuum energy guarantees the existence of virtual particles in the vacuum state which will be used in our setting.

Remark 6.1.1. The vacuum state in free field theory can be described in terms of a tensor product of the Fock space vacuum states for each independent field mode where there is no entanglement between the field modes at different momenta. The full vacuum state in interacting Quantum Field Theory can be described in terms of a superposition of Fock basis states where the modes of different momenta are entangled.

The existence of divergencies originated from non-perturbative aspects is a strong evidence which inform us the indeterministic nature of Quantum Field Theories with strong coupling constants. Our mathematical model shows a deep dependence of the quantum entanglement on the indeterminateness of elementary particles in gauge field theories with strong enough couplings.

Definition 6.1.2. For each n, consider γ_n^p as a (1PI) primitive Feynman diagram which contribute to present some interactions of an elementary particle p with other (virtual) particles in the physical theory. For each Feynman diagram Γ, $B^+_{\gamma_n^p}(\Gamma)$ builds a new disjoint union of graphs as the result of all possibilities for the insertion of Γ into γ_n^p in terms of the types of vertices in γ_n^p and types of external edges in Γ. Each family $\{B^+_{\gamma_n^p}\}_{n\geq 0}$ of this class of Hochschild one cocycles can determine a particular Dyson–Schwinger equation DSE_p which encodes a collection of possible interactions between p and other (virtual) particles in the physical theory.

We plan to describe quantum entanglement in the language of lattice theory. As reminding, a lattice is a partially ordered set such that each pair of its elements has a unique join $\vee$ which is the least upper bound and a unique meet $\wedge$ which is the greatest lower bound. A lattice is called bounded if there exist the greatest element and the least element. A lattice is called distributive, if the operations meet and join obey the distributive conditions.

Theorem 6.1.3. *The information flow between the particle p and all unobserved intermediate states can be described in terms of a lattice of topological Hopf subalgebras.*

Proof. Intermediate states address virtual particles. Dyson–Schwinger equations are the best tools for us to build Hopf subalgebras of the Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ of Feynman diagrams. Thanks to the cut-distance topology defined on Feynman graphons (Theorem 2.3.7), we can naturally equip each Hopf subalgebra H_{DSE} with this topology such that the distance between Feynman diagrams Γ_1, Γ_2 is given by

$$d(\Gamma_1, \Gamma_2) := d_{\mathrm{cut}}([f^{\Gamma_1}], [f^{\Gamma_2}]). \tag{6.2}$$

$[f^{\Gamma_i}]$ is the unique unlabeled graphon class with respect to the graph Γ_i and

$$d_{\text{cut}}([f^{\Gamma_1}],[f^{\Gamma_2}]) = \inf_{\phi,\psi}\sup_{A,B}|\int_{A\times B} f^{\Gamma_1}(\phi(x),\phi(y)) - f^{\Gamma_2}(\psi(x),\psi(y))dxdy| \tag{6.3}$$

where the infimum is taken over all different relabelings of ϕ,ψ for the labeled graphons f^{ϕ}, f^{ψ}, respectively. The supremum is taken over all Lebesgue measurable subsets A, B of the closed interval.

In addition, the coproduct of H_{DSE} is a linear bounded operator on the cut-normed space of Feynman diagrams which leads us to consider each $H_{\text{DSE}}^{\text{cut}}$ as a topological Hopf subalgebra.

Thanks to Definition 6.1.2, choose an equation DSE_p which contains some interactions related to the particle p. For each $j \geq 1$, build a new collection $\{\Gamma_n^{(j)}\}_{n\geq 1}$ of primitive graphs

$$\Gamma_n^{(j)} := \Gamma_1^{(j-1)} + ... + \Gamma_n^{(j-1)} \tag{6.4}$$

such that $\Gamma_n^{(0)} = \gamma_n^p$ for each $n \geq 1$.

Thanks to the Milnor–Moore Theorem ([126]), the Connes–Kreimer renormalization Hopf algebra $H_{\text{FG}}(\Phi)$ is isomorphic to the graded dual of the universal enveloping algebra of the Lie algebra of primitive elements in $H_{\text{FG}}(\Phi)^*$ where for each Feynman diagram Γ, we can consider its corresponding infinitesimal character Z_Γ in the dual space. Since the renormalization coproduct is a linear map, we can check easily that the sum of a finite number of primitive graphs is primitive. Therefore for each $j \geq 1$ and $n \geq 1$, the operator $B^+_{\Gamma_n^{(j)}}$ is the corresponding Hochschild one cocyle such that for each Feynman diagram Γ, this operator concerns all possible situations for the insertion of Γ into the disjoint unions of Feynman diagrams $\Gamma_1^{(j-1)}, ..., \Gamma_n^{(j-1)}$. In addition, by induction, we can show that for each j, the resulting graph $B^+_{\Gamma_n^{(j)}}(\Gamma)$ covers $B^+_{\Gamma_n^{(j-1)}}(\Gamma)$ as a subgraph.

For each $j \geq 1$, we can consider the Dyson–Schwinger equation

$$\text{DSE}_p^{(j)} :=< \{B^+_{\Gamma_n^{(j)}}\}_{n\geq 1} >, \tag{6.5}$$

with the corresponding Hopf subalgebra $H_{\text{DSE}_p^{(j)}}$. There exists a natural injective Hopf algebra homomorphism from $H_{\text{DSE}_p^{(j)}}$ to $H_{\text{DSE}_p^{(j+1)}}$ which leads us to build the following increasing chain of Hopf subalgebras

$$H_{\text{DSE}_p} \leq H_{\text{DSE}_p^{(1)}} \leq H_{\text{DSE}_p^{(2)}} \leq ... \leq H_{\text{DSE}_p^{(j)}} \leq .... \tag{6.6}$$

If $X_{\text{DSE}_p^{(j)}} = \sum_{n_{(j)}\geq 0} g^{n_{(j)}} X^j_{n_{(j)}}$ is the unique solution of the equation $\text{DSE}_p^{(j)}$, then for each $n_{(j)} \geq 1$, set

$$H(X_1^j, ..., X^j_{n_{(j)}}) \tag{6.7}$$

as the graded Hopf subalgebra of $H_{\mathrm{DSE}_p^{(j)}}$ free commutative generated algebraically by graphs $X_1^j, ..., X_{n_{(j)}}^j$.

Our plan is to equip these Hopf subalgebras with the cut-distance topology and then consider the corresponding completed versions of these Hopf subalgebras. We have discussed that solutions of Dyson–Schwinger equations are actually graph limits of sequences of finite Feynman diagrams with respect to the cut-distance topology. The Hopf subalgebra H_{DSE} generated by a given Dyson–Schwinger equation DSE is graded with respect to the number of internal edges or the number of independent loops. We have

$$H_{\mathrm{DSE}} = \bigoplus_{n \geq 0} H_{\mathrm{DSE},(n)} \tag{6.8}$$

such that $H_{\mathrm{DSE},(n)}$ is the homogeneous component of degree n and

$$H_{\mathrm{DSE},(p)} H_{\mathrm{DSE},(q)} \subset H_{\mathrm{DSE},(p+q)},$$

$$\Delta(H_{\mathrm{DSE},(n)}) \subset \bigoplus_{p+q=n} H_{\mathrm{DSE},(p)} \otimes H_{\mathrm{DSE},(q)}, \quad S(H_{\mathrm{DSE},(p)}) \subset H_{\mathrm{DSE},(p)}. \tag{6.9}$$

Define a new parameter val for Feynman diagrams in H_{DSE} given by

$$\mathrm{val}(\Gamma) := \mathrm{Max}\{n \in \mathbb{N} : \quad \Gamma \in \bigoplus_{k \geq n} H_{\mathrm{DSE},(k)}\} \tag{6.10}$$

which determines the n-adic metric on H_{DSE} defined by the formula

$$d(\Gamma_1, \Gamma_2) := 2^{-\mathrm{val}(\Gamma_1 - \Gamma_2)}. \tag{6.11}$$

Thanks to Proposition 4.6 in [149], the n-adic metric enables us to build a sequence $\{R_n(\mathrm{DSE})\}_{n \geq 1}$ of random graphs which is convergent to the unique solution X_{DSE} with respect to the cut-distance topology. Now if we apply the graphon representations of Feynman diagrams, then we can embed H_{DSE} into the renormalization topological Hopf algebra of Feynman graphons $\mathcal{S}^{\Phi}_{\mathrm{graphon}}$. Equip H_{DSE} with the cut norm and add graph-limits to it to obtain a topological Hopf algebra. It is important to note that the coproduct $\Delta_{H_{\mathrm{DSE}}}$ is a linear bounded map on the normed space H_{DSE} which makes it a continuous operator. Thanks to the graduation parameter, the antipode is also a continuous operator in this setting. Denote $H^{\mathrm{cut}}_{\mathrm{DSE}}$ as the resulting topological Hopf algebra.

Thanks to the explained machinery, set $H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(j)}}$ as the resulting topological Hopf subalgebra corresponding to each equation $\mathrm{DSE}_p^{(j)}$. The family $\mathcal{C}_p$ of these topological Hopf subalgebras can be equipped by the following binary relation

$$V \preccurlyeq W \Longleftrightarrow \tag{6.12}$$

there exists a finite sequence of topological Hopf subalgebras $V_1, ..., V_r \in \mathcal{C}_p$ together with injective morphisms $V \to V_1 \to V_2 \to ... \to V_r \to W$ which connect V to W. As the result, $(\mathcal{C}_p, \preccurlyeq)$ is a lattice of topological Hopf subalgebras with the greatest lower bound $H(X^{(0)}_{1_{(0)}})^{\text{cut}}$. For each subset $\{V_1, V_2\}$ of $(\mathcal{C}_p, \preccurlyeq)$, if $V_1 \preccurlyeq V_2$ then define

$$V_1 \wedge V_2 := V_1, \quad V_1 \vee V_2 := V_2. \tag{6.13}$$

The lattice $(\mathcal{C}_p, \preccurlyeq)$ enables us to relate a class of Dyson–Schwinger equations derived from the basic equation DSE_p to each other with respect to morphisms among their corresponding topological Hopf subalgebras. This means that the lattice $(\mathcal{C}_p, \preccurlyeq)$ mathematically describes the transferring of information from one equation to another which includes information flow between p and its intermediate states. □

Definition 6.1.4. Set R^p as the smallest collection of all Feynman diagrams in Φ which contribute to the equations DSE_p and $\text{DSE}_p^{(j)}$ for all $j \geq 1$. Then equip it with the cut-distance topology and add garph limits to obtain a complete topological region $\overline{R^p}$.

Thanks to Theorem 6.1.3, the lattice $(\mathcal{C}_p, \preccurlyeq)$ shows us the information flow processes among all (virtual) particles which contribute to the topological region $\overline{R^p}$.

Theorem 6.1.5. *There exists a lattice of topological Hopf subalgebras which describes the information flow in an entangled system of elementary particles in an interacting Quantum Field Theory.*

Proof. At the first step, we are going to show the existence of a class of Dyson–Schwinger equations for the description of information flow between two space-time far distant particles which belong to an entangled system in interacting Quantum Field Theory.

Thanks to Theorem 6.1.3, we already have described the entanglement process in a topological region around an elementary particle on the basis of the cut-distance topology. Here we need to show the possibility of information flow between two entangled particles p, q while p does not belong to $\overline{R^q}$, q does not belong to $\overline{R^p}$ and $\overline{R^p} \cap \overline{R^q} = \emptyset$.

We have identified topological subspaces $\overline{R^p}$ and $\overline{R^q}$ of the topological Hopf algebra $H^{\text{cut}}_{\text{FG}}(\Phi)$ in terms of their contribution to the entanglement of intermediate states (Definition 6.1.4). Now thanks to the metric (6.2), define the distance between this class of regions by

$$d(\overline{R^p}, \overline{R^q}) := \inf\{d(X, Y) : \quad X \in \overline{R^p},\ Y \in \overline{R^q}\}. \tag{6.14}$$

We want to show the existence of topological regions such as $R^{c_{pq}}$ in $H^{\text{cut}}_{\text{FG}}(\Phi)$ with the following conditions

$$\overline{R^p} \cap \overline{R^{c_{pq}}} \neq \emptyset, \quad \overline{R^q} \cap \overline{R^{c_{pq}}} \neq \emptyset. \tag{6.15}$$

For $d(\overline{R^p}, \overline{R^q}) > 0$, there exists $j_1, j_2 \geq 0$ such that the corresponding equations $\mathrm{DSE}_p^{(j_1)}$ and $\mathrm{DSE}_q^{(j_2)}$ have the following conditions

$$X_{\mathrm{DSE}_p^{(j_1)}} = \lim_{n\to\infty} \sum_{k=0}^{n} X_k^{(j_1)}, \quad X_{\mathrm{DSE}_q^{(j_2)}} = \lim_{n\to\infty} \sum_{k=0}^{n} X_k^{(j_2)} \tag{6.16}$$

$$d(\overline{R^p}, \overline{R^q}) = d_{\mathrm{cut}}(X_{\mathrm{DSE}_p^{(j_1)}}, X_{\mathrm{DSE}_q^{(j_2)}}) > 0. \tag{6.17}$$

For each $\epsilon > 0$, we can determine Hochschild one cocycles $B^+_{\gamma_n^\epsilon,p}$, $n \geq 1$ which fulfills the following conditions:

- Each γ_n^ϵ is a finite primitive (1PI) Feynman diagram such that

$$\forall n \geq 1, \quad \gamma_n^\epsilon \notin R^p, \quad \gamma_n^\epsilon \notin R^q, \quad \gamma_n^\epsilon \in H_{\mathrm{FG}}(\Phi). \tag{6.18}$$

- The equation $\mathrm{DSE}_p^{(\epsilon)}$ as the Dyson–Schwinger equation originated from the family $\{B^+_{\gamma_n^\epsilon,p}\}_{n\geq 1}$ with the unique solution $X_\epsilon^p = \sum_{n\geq 0} X_n^{(\epsilon)p}$ has the following property that there exists $N_\epsilon \in \mathbb{N}$ such that for each $n > N_\epsilon$, we have $d(X_n^{(j_1)}, X_n^{(\epsilon)p}) \leq \epsilon$.

Thanks to the triangle inequality of the cut-distance metric, we can obtain

$$d(X_{\mathrm{DSE}_p^{(j_1)}}, X_\epsilon^p) \leq \epsilon \tag{6.19}$$

In addition, for each $\epsilon > 0$, we can determine Hochschild one cocycles $B^+_{\eta_n^\epsilon,q}$, $n \geq 1$ which fulfills the following conditions:

- Each η_n^ϵ is a finite primitive (1PI) Feynman diagram such that

$$\forall n \geq 1, \quad \eta_n^\epsilon \notin R^p, \quad \eta_n^\epsilon \notin R^q, \quad \eta_n^\epsilon \in H_{\mathrm{FG}}(\Phi). \tag{6.20}$$

- The equation $\mathrm{DSE}_q^{(\epsilon)}$ as the Dyson–Schwinger equation originated from the family $\{B^+_{\eta_n^\epsilon,q}\}_{n\geq 1}$ with the unique solution $X_\epsilon^q = \sum_{n\geq 0} X_n^{(\epsilon)q}$ has the following property that there exists $N'_\epsilon \in \mathbb{N}$ such that for each $n > N'_\epsilon$, we have $d(X_n^{(j_2)}, X_n^{(\epsilon)q}) \leq \epsilon$.

Thanks to the triangle inequality of the cut-distance metric, we can obtain

$$d(X_{\mathrm{DSE}_q^{(j_2)}}, X_\epsilon^q) \leq \epsilon. \tag{6.21}$$

The vacuum in an interacting physical theory can be described as a homogeneous system of virtual particles where its states are invariant by all transformations of the invariance group. Some particles in the vacuum have negative energies where without violating the conservation laws they can annihilate ([136]). Thanks to this fact, we can determine $R^{c_{pq}}$ as the smallest topological subset of $H_{\mathrm{FG}}^{\mathrm{cut}}(\Phi)$ consisting of Feynman graphs which contribute to equations of the types $\mathrm{DSE}_p^{(\epsilon)}$ and $\mathrm{DSE}_q^{(\epsilon)}$. This region contains virtual particles (created by the vacuum energy) which has separate contributive parts with topological regions R^p and R^q. The relations (6.19) and (6.21) guarantee that

$$R^p \cap R^{c_{pq}} \neq \emptyset, \quad R^q \cap R^{c_{pq}} \neq \emptyset. \tag{6.22}$$

Now if we apply Theorem 6.1.3, then graphs which belong to the region $\overline{R^{c_{pq}}}$ (as the completion of $R^{c_{pq}}$ with respect to the cut-distance topology) make informational bridges between entangled particles p, q and their corresponding intermediate states (virtual particles) which live in $\overline{R^p} \cup \overline{R^q} \cup \overline{R^{c_{pq}}}$.

At the second step, we want to formulate the above machinery in the language of lattice theory.

Suppose DSE_p and DSE_q are the basic Dyson–Schwinger equations corresponding to entangled particles p and q. Thanks to the built lattice by Theorem 6.1.3, let $(\mathcal{C}_p, \preccurlyeq)$ be the lattice of topological Hopf subalgebras $H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(j)}}$ generated by equations of the type $\mathrm{DSE}_p^{(j)}$ which live in the topological region $\overline{R^p}$ and let $(\mathcal{C}_q, \preccurlyeq)$ be the lattice of topological Hopf subalgebras $H^{\mathrm{cut}}_{\mathrm{DSE}_q^{(l)}}$ generated by equations of the type $\mathrm{DSE}_q^{(l)}$ which live in the topological region $\overline{R^q}$.

Thanks to the previous part of the proof, we can show the existence of $j_1, j_2 \geq 0$ such that $\mathrm{DSE}_p^{(j_1)}$ and $\mathrm{DSE}_p^{(j_2)}$ contribute in the description of the distance between two topological regions $\overline{R^p}$ and $\overline{R^q}$ (i.e. metric (6.14)). Set

$$j_1^* := \mathrm{Min}\{j_1 : \mathrm{DSE}_p^{(j_1)}\}, \quad j_2^* := \mathrm{Min}\{j_2 : \mathrm{DSE}_q^{(j_2)}\}. \tag{6.23}$$

Consider the sub-lattice $(\mathcal{C}_p^{j_1^*}, \preccurlyeq)$ which contains only the first j_1^* columns of the original lattice $(\mathcal{C}_p, \preccurlyeq)$ and the sub-lattice $(\mathcal{C}_q^{j_2^*}, \preccurlyeq)$ which contains only the first j_2^* columns of the original lattice $(\mathcal{C}_q, \preccurlyeq)$. These two sub-lattices have the greatest lower bound and the smallest upper bound.

Thanks to the structure of the topological region $R^{c_{pq}}$, we can build a new lattice $(\mathcal{C}_{c_{pq}}^{j_1^* j_2^*}, \preccurlyeq)$ which is the result of the disjoint union of the sub-lattices $(\mathcal{C}_p^{j_1^*}, \preccurlyeq)$ and $(\mathcal{C}_q^{j_2^*}, \preccurlyeq)$ which are connected to each other by topological Hopf algebra homomorphisms associated to Dyson–Schwinger equations of the types $\mathrm{DSE}_p^{(\epsilon)}$ and $\mathrm{DSE}_q^{(\epsilon)}$. We have

$$H^{\mathrm{cut}}_{\mathrm{DSE}_p} \leq H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(1)}} \leq \ldots \leq H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(j_1^*)}} \tag{6.24}$$

$$H^{\mathrm{cut}}_{\mathrm{DSE}_q} \leq H^{\mathrm{cut}}_{\mathrm{DSE}_q^{(1)}} \leq \ldots \leq H^{\mathrm{cut}}_{\mathrm{DSE}_q^{(j_2^*)}} \tag{6.25}$$

which belong to the new lattice in terms of one of the following topological Hopf algebra homomorphisms

$$H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(j_1^*)}} \longrightarrow H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(\epsilon)}} \longrightarrow H^{\mathrm{cut}}_{\mathrm{DSE}_c^{(k)}} \longrightarrow H^{\mathrm{cut}}_{\mathrm{DSE}_q^{(\epsilon)}} \longrightarrow H^{\mathrm{cut}}_{\mathrm{DSE}_q^{(j_2^*)}} \tag{6.26}$$

or

$$H^{\mathrm{cut}}_{\mathrm{DSE}_q^{(j_2^*)}} \longrightarrow H^{\mathrm{cut}}_{\mathrm{DSE}_q^{(\epsilon)}} \longrightarrow H^{\mathrm{cut}}_{\mathrm{DSE}_c^{(k)}} \longrightarrow H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(\epsilon)}} \longrightarrow H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(j_1^*)}} \tag{6.27}$$

such that $H^{\text{cut}}_{\text{DSE}_c^{(k)}}$ is the topological Hopf subalgebra associated to the Dyson–Schwinger equation $\text{DSE}_c^{(k)}$ which lives in the region $R^{c_{pq}}$ and derived from the virtual particle c. $\square$

The Milnor–Moore theorem provides a correspondence between pro-unipotent Lie groups and graded commutative Hopf algebras ([27, 121]). This correspondence enables us to translate the determination of Hopf subalgebraic structures in the renormalization Hopf algebra to a problem in Lie groups. One interesting application of Galois theory is to find a fundamental interrelationship between subgroups of the group of all automorphisms and intermediate algorithmic structures which live in the middle of programs and computable functions [172, 173]. Dyson–Schwinger equations can be applied for the determination of substructures in the renormalization Hopf algebra $H_{\text{FG}}(\Phi)$ and the determination of Lie subgroups of the complex Lie group $\mathbb{G}_\Phi(\mathbb{C})$ of characters. Therefore these non-perturbative equations play the middle bridge between Computation Theory and Quantum Field Theory [52, 147].

Theorem 6.1.3 and Theorem 6.1.5 have explained the entanglement of elementary particles in the context of the existence of substructures raised via Dyson–Schwinger equtations. This mathematical setting addresses a deep connection between the concept of information flow in Quantum Field Theory and the existence of subobjects inside an object originated from the Galois Fundamental Theorem. The immediate consequence of this investigation is to recognize a new approach to quantum entanglement in the language of intermediate algorithms in Theoretical Computer Science. We deal with this interesting challenge on the basis of the representation theory of Lie groups.

Theorem 6.1.6. *The intermediate algorithms corresponding to Lie subgroups of the complex Lie group $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$ can encode the information flow in an entangled system in an interacting Quantum Field Theory Φ in terms of a lattice of Lie subgroups.*

Proof. At the first step, we plan to show that the information flow between the particle p and all unobserved intermediate states can be encoded via a lattice of Lie subgroups of $\mathbb{G}_\Phi(\mathbb{C})$. Thanks to Theorem 6.1.3, the entanglement of the particle p and its related virtual particles is encapsulated by the lattice $(\mathcal{C}_p, \preceq)$ of topological Hopf subalgebras. Each pair of Hopf subalgebras $H_{\text{DSE}_p^{(k)}} \preceq H_{\text{DSE}_p^{(l)}}$ in this lattice determines the injective Hopf algebra homomorphism

$$i_{kl} : H_{\text{DSE}_p^{(k)}} \longrightarrow H_{\text{DSE}_p^{(l)}}. \tag{6.28}$$

The passing from Hopf subalgebras to Lie subgroups can be formulated by applying the contravariant functor Spec which sends a commutative algebra to a topological space. For each object $H_{\mathrm{DSE}_p^{(k)}} \in (\mathcal{C}_p, \preceq)$, $\mathrm{Spec}(H_{\mathrm{DSE}_p^{(k)}})$ is the set of all prime ideals of the commutative algebra $H_{\mathrm{DSE}_p^{(k)}}$ equipped with the Zariski topology and the structure sheaf. The homomorphism i_{kl} can be lifted onto the surjective homomorphism

$$\widetilde{i}_{kl} : \mathrm{Spec}(H_{\mathrm{DSE}_p^{(l)}}) \longrightarrow \mathrm{Spec}(H_{\mathrm{DSE}_p^{(k)}}) \tag{6.29}$$

of affine group schemes. For a fixed Hopf subalgebra $H_{\mathrm{DSE}_p^{(k)}}$, $\mathrm{Spec}(H_{\mathrm{DSE}_p^{(k)}})$ is a representable covariant functor which sends a topological space to a group. Set $G_{p^{(k)}} := \mathrm{Spec}(H_{\mathrm{DSE}_p^{(k)}})(\mathbb{C})$ as the complex Lie subgroup corresponding to the Hopf subalgebra $H_{\mathrm{DSE}_p^{(k)}}$ such that its group structure is given by the convolution product generated by the coproduct $\Delta_{H_{\mathrm{DSE}_p^{(k)}}}$. Thanks to this setting, we can build a new lattice $(\mathcal{G}_p, \succeq)$ of complex Lie groups such that

$$G \succeq K \Longleftrightarrow \tag{6.30}$$

There exists a finite sequence of complex Lie subgroups $G_1, ..., G_r \in \mathcal{G}_p$ together with surjective group homomorphisms $G \to G_1 \to G_2 \to ... \to G_r \to K$ which connect G to K. In addition, for each $n \geq 1$, define $G(X_1^j, ..., X_n^j)$ as the finite dimensional complex Lie subgroup corresponding to the free commutative graded Hopf subalgebra $H(X_1^j, ..., X_n^j)$ of $H_{\mathrm{DSE}_p^{(j)}}$. The lattice $(\mathcal{G}_p, \succeq)$ of Lie groups encodes the information flow between p and its related virtual particles.

At the second step, we plan to show that there exists a lattice of Lie subgroups which describes the information flow in an entangled system of elementary particles in a given interacting gauge field theory. For this purpose, we need to build a lattice of Lie subgroups for the description of the quantum entanglement process between space-time far distant elementary particles which belong to an entangled system in the physical theory Φ.

Theorem 6.1.5 determines a lattice $(\mathcal{C}_{c_{pq}}^{j_1^* j_2^*}, \preceq)$ of Hopf subalgebras which describes the entanglement process. Thanks to the first part of the Proof, it is possible to lift the increasing chains (6.24), (6.25) onto the following decreasing chains of Lie subgroups

$$G_{p^{(j_1^*)}} \geq G_{p^{(j_1^*-1)}} \geq ... \geq G_{p^{(1)}} \geq G_{\mathrm{DSE}_p} \tag{6.31}$$

$$G_{q^{(j_2^*)}} \geq G_{q^{(j_2^*-1)}} \geq ... \geq G_{q^{(1)}} \geq G_{\mathrm{DSE}_q} \tag{6.32}$$

with the corresponding group homomorphisms

$$G_{q^{(j_2^*)}} \longrightarrow G_{q^{(\epsilon)}} \longrightarrow G_{c^{(k)}} \longrightarrow G_{p^{(\epsilon)}} \longrightarrow G_{p^{(j_1^*)}} \tag{6.33}$$

or

$$G_{p^{(j_1^*)}} \longrightarrow G_{p^{(\epsilon)}} \longrightarrow G_{c^{(k)}} \longrightarrow G_{q^{(\epsilon)}} \longrightarrow G_{q^{(j_2^*)}} \tag{6.34}$$

such that $G_{c^{(k)}}$ is the complex Lie subgroup corresponding to the Hopf subalgebra $H_{\mathrm{DSE}_c^{(k)}}$ generated by the equation $\mathrm{DSE}_c^{(k)}$ which lives in the topological region $\overline{R^{c_{pq}}}$. The existence of the virtual particle c, which contributes to interactions of the particles p, q, is the consequence of the energy of the vacuum in interacting QFT. Now we can define a new lattice

$$(\mathcal{G}_{c_{pq}}^{j_1^* j_2^*}, \succeq) \tag{6.35}$$

of Lie subgroups and Lie group epimorphisms. This lattice encodes the entanglement processes between p and q.

As we have shown in the previous parts of this work, the renormalization Hopf algebra of Feynman graphons $\mathcal{S}^{\Phi}_{\text{graphon}}$ is capable to recover the renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ and Hopf subalgebras generated by all Dyson–Schwinger equations. Therefore for each Dyson–Schwinger equation DSE with the corresponding Hopf subalgebra H_{DSE}, we can embed the associated complex Lie subgroup $G_{\mathrm{DSE}}(\mathbb{C})$ into the complex Lie group $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$. □

The study of Dyson–Schwinger equations had been developed to a categorical setting where we associated a category of geometric objects to each system of these equations. Then we have embedded this class of categories into the universal Connes–Marcolli category $\mathcal{E}^{\mathrm{CM}}$ of flat equi-singular vector bundles. Thanks to this machinery, some new geometric and combinatorial tools for the computation of non-perturbative parameters have already been obtained [140, 142, 144, 147]. Thanks to Theorem 6.1.6, now it is possible to describe quantum entanglement in the context of the representation theory of Lie groups where we can address a new application of Tannakian formalism to Quantum Field Theory.

Theorem 6.1.7. *There exists a lattice of Tannakian subcategories which describes quantum entanglement in interacting Quantum Field Theory.*

Proof. At the first step, we show the existence of a lattice $(\mathrm{Cat}_p, \succeq)$ of Tannakian subcategories which encodes the quantum entanglement between an elementary particle p and all unobserved intermediate states (as virtual particles).

Thanks to Theorem 6.1.3, the entanglement of the particle p and its related virtual particles is encapsulated by the lattice $(\mathcal{C}_p, \preceq)$ of topological Hopf subalgebras.

Each pair of objects $H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(k)}} \preccurlyeq H^{\mathrm{cut}}_{\mathrm{DSE}_p^{(l)}}$ determines the natural injective Hopf algebra homomorphism

$$i_{kl} : H_{\mathrm{DSE}_p^{(k)}} \longrightarrow H_{\mathrm{DSE}_p^{(l)}} \tag{6.36}$$

which can be lifted onto the surjective group homomorphism

$$\widetilde{i}_{kl} : G_{p^{(l)}} \longrightarrow G_{p^{(k)}}. \tag{6.37}$$

For each object $G_{p^{(l)}}$ of the lattice $(\mathcal{G}_p, \succeq)$, let

$$G^*_{p^{(l)}} := G_{p^{(l)}} \rtimes \mathbb{G}_m \tag{6.38}$$

such that $\mathbb{G}_m$ is the multiplicative group which acts on the original group. Define $\mathrm{Rep}_{G^*_{p^{(l)}}}$ as the category of finite dimensional representations of the complex Lie group $G^*_{p^{(l)}}$ which is a neutral Tannakian category. Thanks to the representation theory of affine group schemes ([121]), the surjective morphism $\widetilde{i}_{kl}$ allows us to send each representation $\sigma : G_{p^{(k)}} \longrightarrow GL_V$ to a representation

$$\sigma \circ \widetilde{i}_{kl} : G_{p^{(l)}} \longrightarrow GL_V \tag{6.39}$$

which leads us to achieve an exact fully faithful functor

$$\mathrm{Rep}_{G^*_{p^{(k)}}} \longrightarrow \mathrm{Rep}_{G^*_{p^{(l)}}}. \tag{6.40}$$

This information is enough to build a new lattice $(\mathrm{Cat}_p, \succeq)$ of subcategories such that

$$\mathrm{Rep}_{H^*} \succeq \mathrm{Rep}_{K^*} \Longleftrightarrow \tag{6.41}$$

there exists a finite sequence of subcategories $\mathrm{Rep}_{H^*_1}, ..., \mathrm{Rep}_{H^*_t} \in \mathrm{Cat}_p$ together with exact fully faithful functors

$$\mathrm{Rep}_{K^*} \to \mathrm{Rep}_{H^*_1} \to ... \to \mathrm{Rep}_{H^*_t} \to \mathrm{Rep}_{H^*} \tag{6.42}$$

derived from the epimorphisms $\widetilde{i}_{kl}$.

At the second step, we show the existence of a lattice of Tannakian subcategories for the description of the entanglement between space-time far distant elementary particles which belong to an entangled system.

Thanks to Theorem 6.1.5, the information flow between p and other distant particle q is described by the lattice $(\mathcal{C}^{j^*_1 j^*_2}_{c_{pq}}, \preceq)$ of topological Hopf subalgebras which inherits a lattice $(\mathcal{G}^{j^*_1 j^*_2}_{c_{pq}}, \succeq)$ of Lie subgroups (Theorem 6.1.6). The decreasing chains (6.31) and (6.32) can be lifted onto the categorical setting to achieve the following chains of categories and exact fully faithful functors

$$\mathrm{Rep}_{G^*_{\mathrm{DSE_p}}} \geq \mathrm{Rep}_{G^*_{p^{(1)}}} \geq ... \geq \mathrm{Rep}_{G^*_{p^{(j^*_1-1)}}} \geq \mathrm{Rep}_{G^*_{p^{(j^*_1)}}} \tag{6.43}$$

$$\mathrm{Rep}_{G^*_{\mathrm{DSE_q}}} \geq \mathrm{Rep}_{G^*_{q^{(1)}}} \geq ... \geq \mathrm{Rep}_{G^*_{q^{(j^*_2-1)}}} \geq \mathrm{Rep}_{G^*_{q^{(j^*_2)}}}. \tag{6.44}$$

We can connect these two chains to each other in terms of one of the following sequences of exact fully faithful functors

$$\mathrm{Rep}_{G^*_{q^{(j^*_2)}}} \longrightarrow \mathrm{Rep}_{G^*_{q^{(\epsilon)}}} \longrightarrow \mathrm{Rep}_{G^*_{c^{(k)}}} \longrightarrow \mathrm{Rep}_{G^*_{p^{(\epsilon)}}} \longrightarrow \mathrm{Rep}_{G^*_{p^{(j^*_1)}}} \tag{6.45}$$

or

$$\mathrm{Rep}_{G^*_{p^{(j_1^*)}}} \longrightarrow \mathrm{Rep}_{G^*_{p^{(\epsilon)}}} \longrightarrow \mathrm{Rep}_{G^*_{c^{(k)}}} \longrightarrow \mathrm{Rep}_{G^*_{q^{(\epsilon)}}} \longrightarrow \mathrm{Rep}_{G^*_{q^{(j_2^*)}}}. \qquad (6.46)$$

This information is enough to define $(\mathrm{Cat}^{j_1^* j_2^*}_{c_{pq}}, \succeq)$ as a lattice of Tannakian subcategories which encodes the entanglement process between p and q. □

Theorem 6.1.8. *Flat equi-singular vector bundles provide a new geometric description of the information flow in interaction Quantum Field Theories on the basis of the Riemann–Hilbert correspondence.*

Proof. Flat equi-singular vector bundles have been applied for the construction of the Connes–Marcolli category $\mathcal{E}^{\mathrm{CM}}$ which is a neutral Tannakian category. It is isomorphic to the category $\mathrm{Rep}_{\mathbb{U}^*}$ of finite dimensional representations of the universal affine group scheme $\mathbb{U}^*$. [37]

This universal category is rich enough to recover all categories $\mathrm{Rep}_{G^*_{p^{(j)}}}$ as subcategories which enable us to define a surjective functor

$$\pi_j^* : \mathrm{Rep}_{\mathbb{U}^*} \longrightarrow \mathrm{Rep}_{G^*_{p^{(j)}}} \qquad (6.47)$$

of categories. Now if we apply the chain (6.45) or (6.46), then we can determine one of the functors

$$\rho^{pq}_{j_1^* j_2^*} : \mathrm{Rep}_{G^*_{p^{(j_1^*)}}} \longrightarrow \mathrm{Rep}_{G^*_{q^{(j_2^*)}}} \qquad (6.48)$$

or

$$\rho^{pq}_{j_2^* j_1^*} : \mathrm{Rep}_{G^*_{q^{(j_2^*)}}} \longrightarrow \mathrm{Rep}_{G^*_{p^{(j_1^*)}}}. \qquad (6.49)$$

These functors allow us to formulate one of the following equations

$$\pi_{j_2^*} = \rho^{pq}_{j_1^* j_2^*} \circ \pi_{j_1^*}, \quad \text{or} \quad \pi_{j_1^*} = \rho^{pq}_{j_2^* j_1^*} \circ \pi_{j_2^*} \qquad (6.50)$$

at the level of functors. They are the key tools for us to determine some flat equi-singular vector bundles which contribute to the information flow in entangled systems. □

Corollary 6.1.9. *There exists a category of mixed Tate motives which interprets the information flow in an entangled system of particles in an interacting Quantum Field Theory.*

Proof. The category $\mathcal{E}^{\mathrm{CM}}$ is equivalent to the motivic category

$$\mathcal{TM}_{\mathrm{mix}}(\mathrm{Spec}\ \mathcal{O}[1/N]) \qquad (6.51)$$

of mixed Tate motives (i.e. Proposition 1.110, Corollary 1.111 in [37]). Thanks to Theorem 6.1.8, neutral Tannakian subcategories $\mathrm{Rep}_{G^*_{p^{(j_1^*)}}}$ and

$\mathrm{Rep}_{G^*_{q^{(j_2^*)}}}$ can be embedded into this motivic category. Therefore the information flow is equivalent to determining subcategories of the category $\mathcal{TM}_{\mathrm{mix}}(\mathrm{Spec}\ \mathcal{O}[1/N])$ which contain those mixed Tate motives identified by motivic Galois groups $G^*_{p^{(j_1^*)}}$ and $G^*_{q^{(j_2^*)}}$. We denote the resulting motivic subcategories with $\mathrm{Mot}(G^*_{p^{(j_1^*)}})$ and $\mathrm{Mot}(G^*_{q^{(j_2^*)}})$, respectively. On the other hand, thanks to Theorem 6.1.6, we have already determined a new class of Dyson–Schwinger equations $\mathrm{DSE}_c^{(k)}$ which describes the information flow in the topological region $\overline{R^{c_{pq}}}$. Now by applying Theorem 6.1.7, we have the category $\mathrm{Rep}_{G^*_{c^{(k)}}}$ of representations with respect to this class of Dyson–Schwinger equations which encodes the information flow. This category can be also embedded into $\mathcal{E}^{\mathrm{CM}}$ which leads us to characterize another class of mixed Tate motives identified with the motivic Galois group $G^*_{c^{(k)}}$. We denote the resulting motivic subcategory with $\mathrm{Mot}(G^*_{c^{(k)}})$. The disjoint unions of objects of these motivic subcategories make a new subcategory

$$\mathrm{Mot}_{pq} := \mathrm{Mot}(G^*_{p^{(j_1^*)}}) \bigsqcup \mathrm{Mot}(G^*_{c^{(k)}}) \bigsqcup \mathrm{Mot}(G^*_{q^{(j_2^*)}}) \tag{6.52}$$

of $\mathcal{TM}_{\mathrm{mix}}(\mathrm{Spec}\ \mathcal{O}[1/N])$. Mot_{pq} is a category of mixed Tate motives which contribute to the entanglement processes between space-time far distant particles p, q via virtual particles of the vacuum energy. □

The explained mathematical machinery for the description of quantum entanglement is on the basis of the strength of the bare or effective coupling constants of the physical theories where we deal with Dyson–Schwinger equations. We can show that our machinery still works after changing the scale of the momenta (i.e. running coupling constant).

Theory of renormalization group is the key tool in Quantum Field Theory to study the changes of the dynamics of a quantum system with infinite degrees of freedom when the scales of some physical parameters such as momentum, energy and mass have been changed. It allows us to concern the possibility of exchanging information from scale to scale in the appearance of uncertainty principle. We apply the Connes–Marcolli universal affine group scheme to define a suitable renormalization group which encodes the information flow under the re-scaling of the momentum parameter.

Corollary 6.1.10. *The information flow between an elementary particle p and all unobserved intermediate states exists independent of changing the scale of the momenta of particles.*

Proof. The universal category $\mathcal{E}^{\mathrm{CM}}$ is isomorphic to the category $\mathrm{Rep}_{\mathbb{U}^*}$ with respect to the universal affine group scheme $\mathbb{U}^*$. The Connes–Marcolli universal shuffle type Hopf algebra of renormalization $H_{\mathbb{U}}$ is the result of the graded dual of the universal enveloping algebra of the free graded Lie algebra $L_{\mathbb{U}}$ which is generated by elements e_{-n} of degree $-n$ for each $n > 0$. The

Milnor–Moore Theorem determines its corresponding affine group scheme $\mathbb{U}$. The sum $e := \sum_n e_{-n}$ is an element of the Lie algebra $L_{\mathbb{U}}$ where thanks to the pro-unipotent structure of $\mathbb{U}$, we can lift it onto the morphism $\mathrm{rg} : \mathbb{G}_a \to \mathbb{U}$. [37]

Now apply Theorem 1.106 in [37] to determine a graded representation

$$\tau_p : \mathbb{U}^*(\mathbb{C}) \to G^*_{\mathrm{DSE}_p} \tag{6.53}$$

such that the resulting map $\tau_p \circ \mathrm{rg}$ provides the renormalization group with respect to the equation DSE_p. By this method, we can build a renormalization group with respect to each Dyson–Schwinger equation $\mathrm{DSE}_p^{(j)}$ for each $j \geq 1$. These renormalization groups make the possibility to study the behavior of the information flow in the topological regions such as R^p under changing the scales of the momentum parameter. □

6.2 *Quantum logic via non-perturbative propositional calculus*

The fundamental purpose in this section is to address a new category model for the study of logical propositions about large Feynman diagrams by applying Feynman graphons. We explain the construction of a topos model which concerns the strength of the (bare) coupling constants in the structure of the base category. It enables us to study the logical evaluation of non-perturbative parameters originated from Dyson–Schwinger equations. We will determine a new class of computable Heyting algebras which concerns logical propositions about topological regions of Feynman diagrams.

Generally speaking, a quantum system is described by its von Neumann algebra $\mathcal{B}(H)$ of observables which contains all bounded operators on an infinite dimensional separable Hilbert space H. Each physical quantity A has a corresponding self-adjoint operator $\hat{A}$ in $\mathcal{B}(H)$ and vice versa. Set $\mathcal{V}(H)$ as the poset of all unital abelian von Neumann subalgebras of $\mathcal{B}(H)$ which can be seen also as the context category. For objects $V_1 \subset V_2$ in the context category, the subalgebra V_1 has less number of self-adjoint operators and less number of projections than the subalgebra V_2. The restriction process from the subalgebra V_2 to the subalgebra V_1 or the lifting process from V_1 onto V_2 are fundamental issues in propositional calculus of quantum systems. These translation issues have been studied under coarse-graining process. On the one hand, it enables us to map self-adjoint operators and projections from V_2 to V_1. For a proposition $A \in \Delta$ about a given physical quantity A, suppose its corresponding projection $\hat{P}^{\Delta}_A$ belongs to V_2 but not belong to V_1. It means that this proposition can not be evaluated from the perspective of V_1. The daseinisation process has been designed to adapt the projection $\hat{P}^{\Delta}_A$ and the proposition $A \in \Delta$ to V_1 by making them coarser. On the other hand, every self-adjoint operator and every projection in V_1

belong also to V_2 but the embedding of the smaller subalgebra into the larger one requires some extra structures and objects which live in V_2. To concern this issue has led people to build a topos of contravariant functors from the context category $\mathcal{V}(H)$ to the category **Set**. This categorical setting has been developed very fast for the reconstruction of physical theories in the context of higher order logic. [1, 10, 48, 49, 50, 51, 104, 111, 123, 153]

The original motivation for the construction of a new topos model is to provide a new analogous of this propositional calculus for the study of situations beyond perturbation theory in Quantum Field Theories with strong couplings in the context of logical conceptions. Our topos model shows the importance of the strength of the (bare) couplings in the construction of the category of context (i.e. base category). The context category of our new topos is actually more complicated than $\mathcal{V}(H)$ because the inclusion $H_{\mathrm{DSE}_1} \subset H_{\mathrm{DSE}_2}$ between two Hopf subalgebras in this base category does not mean in general that the equation DSE_1 should have less physical information than the equation DSE_2. We can remind the calculus of ordinals in Set Theory, where we deal with different types of infinities, and see that sometimes a subset of a set and the set can have the same cardinal. Therefore coarse-graining process is not noticed in the foundations of our topos model and we need to concern other parameters to deal with propositions at the level of large Feynman diagrams.

Let us give a short overview about the concept of topos . The fundamental motivation for the study of topos came from the concept of abstraction in Mathematics. In fact, Category Theory, as a modern discipline, comes to the game whenever we plan to study a general theory of structures. Categories enable us to concern mathematical structures in terms of interrelationships among objects (which are formally known as morphisms) while under a set theoretic perspective, we choose to deal with properties of mathematical structures on the basis of elements and membership relations. Many basic concepts such as spaces and elements in Set Theory can be replaced by objects and arrows in the categorical setting, respectively. It is reasonable to think about Category Theory as a generalization of Set Theory where we are capable to study a mathematical structure in terms of its relations with other structures. This approach leads us to a universal fundamental language in dealing with mathematical structures where we have general powerful tools such as functors between categories and natural transformations between functors instead of the equality relation between elements of sets in Set Theory. Actually, the language of Category Theory provides a new understanding of the notion of "element" of an object in a mathematical structure which is more general than its set theoretic version. Each arrow is indeed a generalized element of its own codomain which means that each object X can be described in terms of consisting of different collections of arrows $Y \longrightarrow X$. This interpretation is known as the varying of elements of X over the stages Y which corresponds to the notion of absolute element x of a set

X in Set Theory. This story is encapsulated in terms of a map $x : \{*\} \longrightarrow X$ where we enable to address the terminal object underlying the categorical setting. Questions about the existence of a class of categories which could be regarded as a categorical-theoretic replacement and generalization of the category of sets and functions have led people to build elementary topos and Grothendieck topos such that the second class is known as a replacement for the notion of "space". The concept of topos has all tools of the set-theoretical world which are necessary for the construction of mathematical structures and their models under a categorical configuration. It provides a generalized version of the notions of space and logic where we enable to interpret it as a categorical-theoretic generalization of the structure of a universe of sets and functions that disappears certain logical and geometric restrictions of the base mathematical structure. Generally speaking, for a given mathematical theory, we can have a treatment to evaluate and study the theory under different stages with respect to objects of a fixed base category. So there is a chance to consider possible relations among toposes in the context of a special family of functors which are called geometric morphisms. A fundamental example of a topos tells us more about the value of this modern categorical methodology in dealing with mathematical structures. Consider the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ of contravariant functors from the base category $\mathcal{C}$ to the standard category $\mathbf{Set}$ of sets and functions. Elements of that mathematical theory corresponded to the base category $\mathcal{C}$, which have been already modeled as objects in the mentioned topos, become representable in terms of set-valued functors over the base category $\mathcal{C}$. Roughly speaking, a topos is a Cartesian closed category with equalisers and subobject classifier. In other words, it has terminal object, equalisers, pullbacks, all other limits, exponential objects and subobject classifier. The category $\mathbf{Set}$ of all sets and functions is an example of a topos which is the basis for the construction of more complicated toposes such as the Grothendieck topos of sheaves over a given base category. [78, 105, 106, 108, 133, 154]

It is time to explain the construction of a new topos which encodes non-perturbative aspects of gauge field theories.

Definition 6.2.1. Topological Hopf algebras $H^{\mathrm{cut}}_{\mathrm{DSE}(\lambda \mathrm{g})}$ generated by solutions of Dyson–Schwinger equations in a given gauge field theory Φ with the strong bare coupling constant g and their closed Hopf subalgebras can be organized in the context of a poset structure. For each pair (H_1, H_2) of objects, we can define arrows pointing from H_1 to H_2 (i.e. $H_1 \leq H_2$) if and only if there exists a homomorphism $H_1 \longrightarrow H_2$ of Hopf algebras which is continuous with respect to the cut-distance topology. This poset can be seen also as a category denoted by $\mathcal{C}^{\mathrm{non},g}_{\Phi}$.

The existence of the graduation parameters on the topological Hopf algebra $H^{\mathrm{cut}}_{\mathrm{FG}}$ allow us to describe it in terms of infinite systems of Dyson–Schwinger equations which means that this Hopf algebra can belong to the category $\mathcal{C}_\Phi^{\mathrm{non},g}$. Therefore $\mathcal{C}_\Phi^{\mathrm{non},g}$ can be considered as a small category. In addition, the category $\mathcal{C}_\Phi^{\mathrm{non},g}$ is capable for the description of topological neighborhoods around a single Feynman diagram which contributes to some Dyson–Schwinger equations. Thanks to the metric defined by cut-norm, these topological regions are Hausdorff which means that we can separate (large) Feynman diagrams from each other.

Lemma 6.2.2. *There exists a topos structure on the small category* $\mathcal{C}_\Phi^{\mathrm{non,g}}$.

Proof. The natural choice is the topos of presheaves on the category $\mathcal{C}_\Phi^{\mathrm{non,g}}$ (as the category of context). We denote this category by $\mathbf{T}_\Phi^{\mathrm{non},g}$.

An object in $\mathbf{T}_\Phi^{\mathrm{non},g}$ is a contravariant functor from the category $\mathcal{C}_\Phi^{\mathrm{non},g}$ to the standard category **Set** of sets and functions.

A morphism between a pair (F_1, F_2) of objects is a natural transformation such as $\eta : F_1 \longrightarrow F_2$ which is actually a family of morphisms $\{\eta_H : F_1(H) \longrightarrow F_2(H)\}_{H \in \mathrm{Obj}(\mathcal{C}_\Phi^{\mathrm{non},g})}$ which respect to the contravariant property. It means that for each morphism $f : H_1 \to H_2$ in $\mathcal{C}_\Phi^{\mathrm{non},g}$, we have $\eta_{H_1} \circ F_1(f) = F_2(f) \circ \eta_{H_2}$.

A sieve on an object $H \in \mathcal{C}_\Phi^{\mathrm{non},g}$ is defined as a collection S of morphisms $f : H \longrightarrow H'$ in $\mathcal{C}_\Phi^{\mathrm{non},g}$ with the property that if f belongs to S and if $g : H' \longrightarrow H''$ is any other morphism in $\mathcal{C}_\Phi^{\mathrm{non},g}$, then $g \circ f : H \longrightarrow H''$ also belongs to S.

The terminal object $1 : \mathcal{C}_\Phi^{\mathrm{non},g} \longrightarrow \mathbf{Set}$ can be defined by $1(H) := \{*\}$ at all stages H in $\mathcal{C}_\Phi^{\mathrm{non},g}$, if $f : H \longrightarrow H'$ is a morphism in $\mathcal{C}_\Phi^{\mathrm{non},g}$ then $1(f) : \{*\} \longrightarrow \{*\}$. It is a terminal object because for any other presheaf F we can define a unique natural transformation $\eta : F \longrightarrow 1$ such that its components $\eta_H : F(H) \longrightarrow 1(H) = \{*\}$ are the constant maps $\Gamma \longmapsto *$ for all $\Gamma \in F(H)$.

The subobject classifier Ω^{non} is a presheaf $\Omega^{\mathrm{non}} : \mathcal{C}_\Phi^{\mathrm{non},g} \longrightarrow \mathbf{Set}$ such that for any object $H \in \mathcal{C}_\Phi^{\mathrm{non},g}$, $\Omega^{\mathrm{non}}(H)$ is identified by the set of all sieves on H. If $f : H' \longrightarrow H''$ is a morphism in $\mathcal{C}_\Phi^{\mathrm{non},g}$, then $\Omega^{\mathrm{non}}(f) : \Omega^{\mathrm{non}}(H'') \longrightarrow \Omega^{\mathrm{non}}(H')$ is given by

$$\Omega^{\mathrm{non}}(f)(S) := \{h : H'' \longrightarrow H''', \quad h \circ f \in S\} \tag{6.54}$$

for all sieves S which lives in $\Omega^{\mathrm{non}}(H)$. □

Heyting algebras are practical models of the intuitionistic logic where we do not have the law of excluded middle. It means that the proposition $\phi \vee \neg\phi$ is not intuitionistically valid. [72, 104]

Definition 6.2.3. A Heyting algebra A is a bounded distributive lattice with the largest element 1 and the smallest element 0 which obeys this condition that for each couple (a, b) of its elements there exists a greatest element $x \in A$ such that $a \wedge x \leq b$. This particular element is called the relative pseudo-complement of a with respect to b. A is called a complete Heyting algebra, if it is a complete lattice.

Theorem 6.2.4. *The topos $\boldsymbol{T}_{\Phi}^{\mathrm{non},g}$ encodes the evaluation of propositions about topological regions of Feynman diagrams.*

Proof. Truth objects corresponding to cut-distance topological regions of Feynman diagrams can be determined by the Heyting algebraic structure defined naturally on the subobject classifier of the topos $\mathbf{T}_{\Phi}^{\mathrm{non},g}$.

For a given topological Hopf algebra $H_{\mathrm{DSE}}^{\mathrm{cut}}$, consider the space $\Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ which contains all sieves on $H_{\mathrm{DSE}}^{\mathrm{cut}}$. Now for arbitrary collections S_1, S_2 of sieves on $H_{\mathrm{DSE}}^{\mathrm{cut}}$ which live in $\Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})$, the partial order relation on $\Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ can be determined naturally by the relation

$$S_1 \leq S_2 \Longleftrightarrow S_1 \subset S_2 \tag{6.55}$$

which leads us to make the following elementary logical statements

$$S_1 \wedge S_2 := S_1 \cap S_2, \qquad S_1 \vee S_2 := S_1 \cup S_2,$$

$$S_1 \Longrightarrow S_2 :=$$

$$\{f : H_{\mathrm{DSE},1}^{\mathrm{cut}} \longrightarrow H_{\mathrm{DSE},2}^{\mathrm{cut}} \text{ s.t. } \forall g : H_{\mathrm{DSE},2}^{\mathrm{cut}} \longrightarrow H_{\mathrm{DSE},3}^{\mathrm{cut}},\ g \circ f \in S_1 \Longrightarrow g \circ f \in S_2\}. \tag{6.56}$$

The negation of an element S is defined by the proposition $\neg S := S \Longrightarrow 0$ which means that

$$\neg S := \{f : H_{\mathrm{DSE},1}^{\mathrm{cut}} \longrightarrow H_{\mathrm{DSE},2}^{\mathrm{cut}} \text{ s.t. } \forall g : H_{\mathrm{DSE},2}^{\mathrm{cut}} \longrightarrow H_{\mathrm{DSE},3}^{\mathrm{cut}},\quad g \circ f \notin S\}. \tag{6.57}$$

Thanks to the defined partial order (6.55), for $S_1, S_2 \in \Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})$, there exists a proposition $S_1 \Rightarrow S_2$ of $\Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ with the property that for all $S \in \Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})$,

$$S \leq (S_1 \Rightarrow S_2) \Leftrightarrow S \wedge S_1 \leq S_2. \tag{6.58}$$

In addition, the unit element in $\Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ is the principal sieve on $H_{\mathrm{DSE}}^{\mathrm{adic}}$ and the null element is the empty sieve $\emptyset$.

The presheaf Ω^{non} as the subobject classifier shows that subobjects of any object F in the topos $\mathbf{T}_{\Phi}^{\mathrm{non},g}$ are in an one to one correspondence with morphisms such as $\chi : F \longrightarrow \Omega^{\mathrm{non}}$. In other words, for a subobject K of F, its associated characteristic morphism χ^K is determined by its components $\chi^K_{H_{\mathrm{DSE}}^{\mathrm{cut}}} : F(H_{\mathrm{DSE}}^{\mathrm{cut}}) \longrightarrow \Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})$ where

$$\chi^K_{H_{\mathrm{DSE}}^{\mathrm{cut}}}(A) := \{f : H_{\mathrm{DSE}}^{\mathrm{cut}} \longrightarrow H_{\mathrm{DSE},1}^{\mathrm{cut}} : \quad F(f)(A) \in K(H_{\mathrm{DSE},1}^{\mathrm{cut}})\}, \tag{6.59}$$

for all $A \in F(H_{\mathrm{DSE}}^{\mathrm{cut}})$, is actually a sieve on $H_{\mathrm{DSE}}^{\mathrm{cut}}$. Furthermore, each morphism $\chi : F \longrightarrow \Omega^{\mathrm{non}}$, which is a natural transformation between presheaves, defines a subobject K^{χ} of F which is given by

$$K^{\chi}(H_{\mathrm{DSE}}^{\mathrm{cut}}) := \chi^{-1}_{H_{\mathrm{DSE}}^{\mathrm{cut}}}\{1_{\Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}})}\}. \tag{6.60}$$

As the conclusion, for each equation DSE, $(\Omega^{\mathrm{non}}(H_{\mathrm{DSE}}^{\mathrm{cut}}), \leq, \wedge, \vee, \rightarrow)$ is our promising Heyting algebra. □

A Heyting algebra is called finitely free, if it is generated by the equivalence classes of formulas of finite number of propositional variables under provable equivalence in the intuitionistic logic.

A subset A of natural numbers is called computable if there exists an algorithm to decide whether a natural number belongs to A or not. In other words, A is computable if its corresponding characteristic function is computable. An algebraic structure is called computable if its domain can be identified with a computable set of natural numbers where the (finitely many) operations and relations on the structure are computable. If the structure is infinite, people usually identify the cardinal of its domain with the symbol ω. The computable dimension of a computable structure is the number of classically isomorphic computable copies of the structure up to the computable isomorphism. [153]

Definition 6.2.5. A Heyting algebra $(H, \leq, \wedge, \vee, \rightarrow)$ is called computable, if H and all its corresponding operations are computable.

For a given Heyting algebra with one generator, there exist infinitely nonequivalent intuitionistic formulas of one propositional variable. The connection between free Heyting algebras and the intuitionistic logic leads us to the concept of "computable dimension" for Heyting algebras in particular the ones which encode the logic of the topos $\mathbf{T}_{\Phi}^{\text{non},g}$.

Theorem 6.2.6. *There exists a computable Heyting algebra which encodes truth objects associated to topological regions of Feynman diagrams which contribute to the unique solution of the Dyson–Schwinger equation DSE in a given gauge field theory Φ.*

Proof. We work on the topos $\mathbf{T}_{\Phi}^{\text{non},g}$. Thanks to Theorem 6.2.4, we can associate the Heyting algebra $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$ to each combinatorial Dyson–Schwinger equation DSE. Therefore the subobject classifier Ω^{non} has the internal structure of our interesting Heyting algebra (as the algebraic structure appropriate for the intuitionistic logic). We want to lift this logical type of algebra onto a enriched version $\hat{\Omega}^{\text{non}}$ which is computable at the level of dimension.

Consider the propositional intuitionistic logic over the given language $(\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}}), \wedge, \vee, \rightarrow, \bot, \top)$ such that $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$ can be seen as the collection of propositional formulas in infinitely many variables modulo equivalence under the intuitionistic logic where $\wedge, \vee, \rightarrow$ are the logical connectives, $\bot$ is false and $\top$ is truth. The codes for formulas such as $\phi \wedge \psi$, $\phi \vee \psi$ or $\phi \longrightarrow \psi$ are always greater than the codes for ϕ and ψ.

The intuitionistic propositional logic is decidable ([32, 157]) which means that we need a finite constructive process to apply uniformly to every propositional formula to understand either it produces an intuitionistic proof of the formula or it shows no such proof can exist. Therefore we have a computable copy of the free Heyting algebra on ω generators. Now we can consider elements of $\Omega^{\mathrm{non}}(H^{\mathrm{cut}}_{\mathrm{DSE}})$ as the equivalence classes $[\phi]$ under provable equivalence in the intuitionistic logic which leads us to the following computational operations

$$[\phi] \leq [\psi] \Longleftrightarrow \phi \rightarrow \psi \ \text{ is provable under the intuitionistic logic,}$$

$$[\phi] \wedge [\psi] = [\phi \wedge \psi], \qquad [\phi] \vee [\psi] = [\phi \vee \psi]. \tag{6.61}$$

The plan is to build $\hat{\Omega}^{\mathrm{non}}$ as a computable copy which is not computability isomorphic to Ω^{non}. Let $\alpha_s(n)$ be a label at stage s determined by the domain of $\hat{\Omega}^{\mathrm{non}}$ in the construction process. It is a propositional formula in the intuitionistic logic such that
- $\alpha(n) = \lim_s \alpha_s(n)$,
- For $n \neq m$, the propositional formulas $\alpha(n)$ and $\alpha(m)$ are not intuitionistically equivalent,
- For each intuitionistic propositional formula ϕ there exists such n such that $\alpha(n)$ is intuitionistically equivalent to ϕ,
- Morphisms with the general form $\phi_e : \hat{\Omega}^{\mathrm{non}} \longrightarrow \Omega^{\mathrm{non}}$ can be applied to deal with the diagonalization against all possible computable isomorphisms ([157]).

Once we define the join, meet or relative pseudo-complement of elements, these relationships never change in future stages. Therefore, the function α, which indicates an isomorphism map between $\hat{\Omega}^{\mathrm{non}}$ and Ω^{non}, makes $\hat{\Omega}^{\mathrm{non}}$ computable. $\square$

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