Cartan subalgebras of $\mathfrak{gl}_\infty$

Karl-Hermann Neeb * and Ivan Penkov **

Abstract. Let V be a vector space over a field $\mathbb{K}$ of characteristic zero and V_* be a space of linear functionals on V which separate the points of V. We consider $V\otimes V_*$ as a Lie algebra of finite rank operators on V, and set $\mathfrak{gl}(V,V_*):=V\otimes V_*$. We define a Cartan subalgebra of $\mathfrak{gl}(V,V_*)$ as the centralizer of a maximal subalgebra every element of which is semisimple, and then give the following description of all Cartan subalgebras of $\mathfrak{gl}(V,V_*)$ under the assumption that $\mathbb{K}$ is algebraically closed. A subalgebra of $\mathfrak{gl}(V,V_*)$ is a Cartan subalgebra if and only if it equals $\oplus_j(V_j\otimes(V_j)_*)\oplus(V^0\otimes V_*^0)$ for some one-dimensional subspaces $V_j\subseteq V$ and $(V_j)_*\subseteq V_*$ with $(V_i)_*(V_j)=\delta_{ij}\mathbb{K}$ and such that the spaces $V_*^0=\cap_j(V_j)^\perp\subseteq V_*$ and $V^0=\cap_j((V_j)_*)^\perp\subseteq V$ satisfy $V_*^0(V^0)=\{0\}$. We then discuss explicit constructions of subspaces V_j and $(V_j)_*$ as above. Our second main result claims that a Cartan subalgebra of $\mathfrak{gl}(V,V_*)$ can be described alternatively as a locally nilpotent self-normalizing subalgebra whose adjoint representation is locally finite, or as a subalgebra $\mathfrak{h}$ which coincides with the maximal locally nilpotent $\mathfrak{h}$-submodule of $\mathfrak{gl}(V,V_*)$, and such that the adjoint representation of $\mathfrak{h}$ is locally finite.

AMS Subject Classification 2000: Primary 17B65, Secondary 17B20.

Introduction

It is an interesting question which class of subalgebras of an infinite-dimensional Lie algebra, over a field $\mathbb{K}$ of characteristic zero, play a role similar to Cartan subalgebras of a finite-dimensional Lie algebra. Despite the fact that infinite-dimensional Lie algebras have been studied extensively in the last 30 years, there is no definitive answer to this question. The best understood cases are those of Kac–Moody algebras and extended affine Lie algebras (see [BP95], [PK83], [AABGP97] and the references therein), whose specific is that their Cartan subalgebras are finite-dimensional. The simplest example of an infinite-dimensional Lie algebra whose Cartan subalgebras are no longer finite-dimensional, is the Lie algebra $\mathfrak{gl}_\infty$ of infinite matrices with finitely many non-zero entries in $\mathbb{K}$, and in the literature there is no systematic investigation of all Cartan subalgebras of $\mathfrak{gl}_\infty$. The purpose of the present paper is to fill in this gap for $\mathfrak{gl}_\infty$ and to the larger class of Lie algebras $\mathfrak{gl}(V,V_*)$ defined below.

The following three definitions of a Cartan subalgebra $\mathfrak{h}$ of a finite-dimensional Lie algebra $\mathfrak{g}$ are equivalent:

(C1) $\mathfrak{h}$ is a locally nilpotent self-normalizing subalgebra;

(C2) $\mathfrak{h}$ coincides with the maximal locally nilpotent $\mathfrak{h}$-submodule of $\mathfrak{g}$, i.e., $\mathfrak{h}=\mathfrak{g}^0(\mathfrak{h})$, where

$$\mathfrak{g}^0(\mathfrak{h})=\{x\in\mathfrak{g}\colon(\exists n\in\mathbb{N})\ \ (\operatorname{ad}\mathfrak{h})^n(x)=\{0\}\};$$

(C3) $\mathfrak{h}$ is a locally nilpotent subalgebra which coincides with the set of all elements $x\in\mathfrak{g}$ for which $\operatorname{ad}x$ commutes with the abelian subalgebra $(\operatorname{ad}\mathfrak{h})_s$ consisting of all semisimple parts $(\operatorname{ad}h)_s$ in the Jordan decomposition $\operatorname{ad}h=(\operatorname{ad}h)_s+(\operatorname{ad}h)_n$ of $\operatorname{ad}h$ for elements $h\in\mathfrak{h}$ ($(\operatorname{ad}h)_n$ stands for the nilpotent part).

* Work supported in part by the University of California at Riverside and by a grant of the DFG.

** Work supported in part by MSRI, by an NSF grant, and by the Max Planck Institut für Mathematik in Bonn.

Since $\mathfrak{g}$ is assumed finite-dimensional, "locally nilpotent" in the conditions (C1)-(C3) is of course equivalent to "nilpotent", but we have stated (C1)-(C3) in a form suitable also for the more general class of locally finite Lie algebras we consider in this paper. The fact that (C1) is equivalent to (C2) is well-known ([Bou90, Ch. VII]). The equivalence of (C2) and (C3) follows from the equalities

$$\mathfrak{g}^0(\mathfrak{h}) = \bigcap_{h\in\mathfrak{h}} \mathfrak{g}^0(\operatorname{ad} h) = \bigcap_{h\in\mathfrak{h}} \ker(\operatorname{ad} h)_s = \mathfrak{z}_{\mathfrak{g}}((\operatorname{ad}\mathfrak{h})_s).$$

If $\mathfrak{g}$ is reductive over an algebraically closed field, every Cartan subalgebra $\mathfrak{h}$ is maximal toral, i.e., for every $0 \neq h \in \mathfrak{h}$, $\operatorname{ad} h$ is diagonalizable, and $\mathfrak{h}$ is maximal with this property. This is the key to one of the most important properties of a Cartan subalgebra $\mathfrak{h}$ of a reductive Lie algebra $\mathfrak{g}$: that (after a possible field extension) $\mathfrak{h}$ yields a root decomposition of $\mathfrak{g}$.

Let V be a fixed (arbitrary) vector space over $\mathbb{K}$ and V^* be its dual space. In what follows we set $\mathfrak{g} = \mathfrak{gl}(V, V_*) := V \otimes V_*$, considered as a Lie algebra of finite rank operators on V, where $V_* \subseteq V^*$ is a subspace separating the points of V. If J is a set, we write $\mathbb{K}^{(J)}$ for the free vector space over J and $(e_j)_{j\in J}$ for the canonical basis elements. The standard pairing $\mathbb{K}^{(J)} \times \mathbb{K}^{(J)} \to \mathbb{K}$ induces an injection $\mathbb{K}^{(J)} \hookrightarrow (\mathbb{K}^{(J)})^* \cong \mathbb{K}^J$, and for $V = V_* = \mathbb{K}^{(J)}$, $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{K}) := \mathfrak{gl}(V, V_*)$ is the Lie algebra of $J \times J$-matrices with finitely many non-zero entries. We set also $\mathfrak{gl}_\infty := \mathfrak{gl}(\mathbb{N}, \mathbb{K})$.

All Cartan subalgebras yielding a root decomposition of $\mathfrak{gl}_\infty$, i.e., the so called splitting Cartan subalgebras, are well understood, see [NS01], [St01] and [PS03]. It is also known that there are maximal toral subalgebras of $\mathfrak{gl}_\infty$ which do not yield a root decomposition, and therefore also no generalized root decomposition ([PS03]). In particular, even if $\mathbb{K}$ is algebraically closed, none of the conditions (C1)-(C3) implies the existence of a generalized root decomposition related to $\mathfrak{h}$. In this paper we put the condition (C3) in the spotlight, as it relates $\mathfrak{h}$ in a most transparent way with the abelian subalgebra $\mathfrak{h}_s$, consisting of the semisimple parts of all $h \in \mathfrak{h}$, and in this way carries the most resemblance with the finite-dimensional case. More precisely, we define a Cartan subalgebra of $\mathfrak{gl}(V, V_*)$ as a subalgebra satisfying (C3).

Our main result is a description, in terms of linear algebra, of all Cartan subalgebras of $\mathfrak{g} = \mathfrak{gl}(V, V_*)$ for which $\mathfrak{h}_s$ is toral. (The latter condition is automatic if $\mathbb{K}$ algebraically closed.) In particular we prove that all Cartan subalgebras are abelian. As a corollary we obtain that there are at most three types of Cartan subalgebras for which $\mathfrak{h}_s$ is toral: the ones for which the inclusion $\mathfrak{h}_s \subseteq \mathfrak{h}$ is proper, the toral ones, i.e. those for which $\mathfrak{h} = \mathfrak{h}_s$, and finally, the splitting ones for which $\mathfrak{h} = \mathfrak{h}_s$ and $\mathfrak{g}$ has a root decomposition with respect to $\mathfrak{h}$. We consider examples of pairs V, V_* for which not all types of Cartan subalgebras occur, and we show that all three types do occur for $\mathfrak{gl}(J, \mathbb{K})$.

As each Cartan subalgebra of $\mathfrak{gl}(V, V_*)$ is abelian, its adjoint module is trivial and in particular locally finite. It is not clear if this latter property holds for any subalgebra of $\mathfrak{gl}(V, V_*)$ (in particular of $\mathfrak{gl}_\infty$) which satisfies (C1) or (C2). However, our second main result claims that if one strengthens (C1) and (C2) by the very natural additional requirement that the adjoint module of $\mathfrak{h}$ be locally finite, then the so obtained new conditions (C1') and (C2') are equivalent to (C3) for subalgebras of $\mathfrak{g} = \mathfrak{gl}(V, V_*)$.

Acknowledgments. We thank Helmut Strade for sharing an inspiring idea about describing the toral Cartan subalgebras of $\mathfrak{gl}_\infty$. We thank also Ivan Dimitrov and Gregg Zuckerman for the permission to present in the Appendix a proposition they recently proved.

I. Preliminaries and notation

All vector spaces (and Lie algebras) are defined over $\mathbb{K}$ and $\overline{\mathbb{K}}$ stands for the algebraic closure of $\mathbb{K}$. The superscript $*$ denotes dual space. The vector spaces V and V_* are fixed as above, and the sign $\perp$ always refers to the pairing $V \times V_* \to \mathbb{K}$. If $\mathfrak{k}$ is a Lie algebra, $U(\mathfrak{k})$ stands for the enveloping algebra of $\mathfrak{k}$. In this paper $\mathbb{N} := \{1, 2, 3, ...\}$.

We call a Lie algebra $\mathfrak{k}$ *locally finite* (resp., *locally nilpotent*) if every finite subset of $\mathfrak{k}$ is contained in a finite-dimensional (resp., nilpotent) subalgebra. We call an $\mathfrak{k}$-module M *locally finite* if each element $m \in M$ is contained in a finite-dimensional submodule, and we call M *locally nilpotent* if for any $m \in M$ there exists an $i \in \mathbb{N}$ with $\mathfrak{k}^i \cdot m = \{0\}$. Furthermore, we say that an $\mathfrak{k}$-module is a *generalized weight* $\mathfrak{k}$*-module* if $M = \bigoplus_{\lambda \in \mathfrak{k}^*} M^\lambda(\mathfrak{k})$, where

$$M^\lambda(\mathfrak{k}) := \{m \in M : (\exists i \in \mathbb{N})(\forall x \in \mathfrak{k})\ (x - \lambda(x)\mathbf{1})^i \cdot m = 0\}.$$

We define M to be a *weight module*, if, in addition,

$$M^\lambda(\mathfrak{k}) = M_\lambda(\mathfrak{k}) := \{m \in M : (\forall x \in \mathfrak{k})\ x \cdot m = \lambda(x)m\}$$

for each $\lambda \in \mathfrak{k}^*$. The *support* in $\mathfrak{k}^*$ of a module M is the set

$$\operatorname{supp} M := \{\alpha \in \mathfrak{k}^* : M^\alpha(\mathfrak{k}) \neq \{0\}\}.$$

If $\mathfrak{k}$ is any Lie algebra and $\mathfrak{h} \subseteq \mathfrak{k}$ is a locally nilpotent Lie subalgebra, we say that $\mathfrak{k}$ admits a *generalized* $\mathfrak{h}$*-root decomposition* (resp., an $\mathfrak{h}$*-root decomposition*) if $\mathfrak{k}$ is a generalized $\mathfrak{h}$-weight module (resp., an $\mathfrak{h}$-weight module) with respect to the adjoint action, and, in addition, $\mathfrak{h}$ coincides with the maximal locally nilpotent $\mathfrak{h}$-submodule of $\mathfrak{k}$, i.e., $\mathfrak{h} = \mathfrak{k}^0(\mathfrak{h})$.

In this paper we denote by $\mathfrak{g}$ a fixed Lie algebra of the form $\mathfrak{gl}(V, V_*)$, where $V_* \subseteq V^*$ is a subspace separating the points of V. Typical examples of this situation are as follows.

(a) $V_* = V^*$. Then $\mathfrak{gl}(V, V_*)$ is the Lie algebra of finite rank operators on V.
(b) $V = \mathbb{K}^{(J)} = V_*$ for a set J. Then $\mathfrak{gl}(V, V_*) \cong \mathfrak{gl}(J, \mathbb{K})$.
(c) V is a locally convex real or complex vector space and V_* is the space of continuous linear functionals. As a consequence of the Hahn-Banach Extension Theorem, V_* separates the points of V. Here $\mathfrak{gl}(V, V_*)$ is the Lie algebra of continuous finite rank operators on V.

In general, the structure of the Lie algebra $\mathfrak{gl}(V, V_*)$ depends essentially on the choice of the subspace V_*. As the following proposition shows, this is not the case when V and V_* are of countable dimension.

Proposition I.1. *If V and V_* are of countable dimension, then $\mathfrak{gl}(V, V_*) \cong \mathfrak{gl}_\infty$.*

Proof. We have to find a basis $(f_n)_{n\in\mathbb{N}}$ of V for which the dual basis $(f_n^*)_{n\in\mathbb{N}} \subseteq V^*$ spans V_*.

Fix a basis $(e_n)_{n\in\mathbb{N}}$ of V and a basis $(\varphi_n)_{n\in\mathbb{N}}$ of V_*. We first change the enumeration of the basis $(\varphi_n)_{n\in\mathbb{N}}$ by a permution $\sigma : \mathbb{N} \to \mathbb{N}$ according to the following rule. Put $V_n := \operatorname{span}\{e_1, \ldots, e_n\}$ for $n \in \mathbb{N}$, and let σ_1 be the minimal number j with $\varphi_j(e_1) \neq 0$. Inductively we proceed as follows. If $\sigma_1, \ldots, \sigma_k$ are chosen such that the restrictions of $\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_k}$ to V_k are linearly independent, then we choose σ_{k+1} as the minimal element in $\mathbb{N} \setminus \{\sigma_1, \ldots, \sigma_k\}$ for which the restriction of $\varphi_{\sigma_{k+1}}$ to V_{k+1} is linearly independent from the restrictions of $\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_k}$. As the sequence $(\varphi_n)_{n\in\mathbb{N}}$ separates the points of V, the above procedure never stops and defines an injection $\sigma : \mathbb{N} \to \mathbb{N}$. To see that σ is surjective, hence a permutation, we argue by contradiction. Assume that σ is not surjective and pick the minimal element $m \in \mathbb{N} \setminus \sigma(\mathbb{N})$. Suppose that $\{1, \ldots, m-1\} \subseteq \{\sigma_1, \ldots, \sigma_k\}$. Then there exist $\lambda_1, \ldots, \lambda_k$ such that the linear functional

$$\varphi'_m := \varphi_m - \sum_{j=1}^{k} \lambda_j \varphi_{\sigma_j}$$

vanishes on $e_1, \ldots, e_k$. From the linear independence of the sequence $(\varphi_n)_{n\in\mathbb{N}}$ we infer that φ'_m is non-zero, so that there exists a minimal $N \in \mathbb{N}$ with $\varphi_m(e_N) \neq 0$. Then the restrictions of $\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_{N-1}}, \varphi'_m$ to V_N are linearly independent, hence the restrictions of $\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_{N-1}}, \varphi_m$ to V_N are linearly independent. Thus $\sigma_N = m$, in contradiction with the choice of m. This proves that σ is a permutation, and hence that the functionals φ_{σ_k} form a basis of V_*.

Let $(V_*)_n := \operatorname{span}\{\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_n}\}$. Then $(V_*)_n|_{V_n} = V_n^*$, and we can choose inductively a basis $(f_n)_{n\in\mathbb{N}}$ for which $V_n = \operatorname{span}\{f_1, \ldots, f_n\}$ for $n \in \mathbb{N}$, and $\varphi_{\sigma_j}(f_i) = \delta_{ij}$ for $i \geq j$. In the next step we alterate the basis $(\varphi_{\sigma_n})_{n\in\mathbb{N}}$ of V_* to a basis $(\nu_n)_{n\in\mathbb{N}}$ with $(V_*)_n = \operatorname{span}\{\nu_1, \ldots, \nu_n\}$ and $\nu_j(f_i) = \delta_{ij}$ for all $i, j \in \mathbb{N}$, i.e., $(\nu_n)_{n\in\mathbb{N}}$ is the dual basis to $(f_n)_{n\in\mathbb{N}}$. This proves that the pair V, V_* is equivalent to the pair $\mathbb{K}^{(\mathbb{N})}, \mathbb{K}^{(\mathbb{N})}$ with the standard pairing. ∎

The next proposition shows that the requirement $\dim V = \dim V_*$ does not always lead to a pairing equivalent to the standard pairing $\mathbb{K}^{(J)} \times \mathbb{K}^{(J)} \to \mathbb{K}$ for some J.

Proposition I.2. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and V be an infinite-dimensional Hilbert space and V_* be the space of continuous linear functionals on V. Then there is no (vector space) basis of V such that the dual basis belongs to V_*.*

Proof. We argue by contradiction and assume that $(f_j)_{j\in J}$ is a (vector space) basis of V for which the dual basis belongs to V_*. Let $(e_s)_{s\in S}$ be an orthonormal Hilbert basis of V. Each e_s is a finite linear combination $e_s = \sum_{j\in J} a_{js} f_j$. Let $S_0 \subseteq S$ be an infinite countable subset. Then the set $J_0 := \{j \in J : (\exists s \in S_0) a_{js} \neq 0\}$ is also countable. Furthermore,

$$H := \bigcap_{j\in J\setminus J_0} \ker f_j^*$$

is a closed subspace of V with $(f_j)_{j\in J_0}$ as a (vector space) basis. On the other hand, Baire's Category Theorem implies that H is not the union of an ascending chain of finite-dimensional subspaces, hence not of countable dimension. Contradiction. ∎

Any element $x \in \mathfrak{g} = \mathfrak{gl}(V, V_*)$ is a finite rank operator on V, hence has a Jordan decomposition $x = x_s + x_n$ into a semisimple part x_s and a nilpotent part x_n. As $\mathfrak{g}$ is locally finite, the operator $\operatorname{ad} x$ is locally finite for any $x \in \mathfrak{g}$, and has a Jordan decomposition $\operatorname{ad} x = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$. As $\operatorname{ad} x_s$ is semisimple and $\operatorname{ad} x_n$ is nilpotent, both Jordan decompositions are compatible, i.e., $(\operatorname{ad} x)_s = \operatorname{ad} x_s$, $(\operatorname{ad} x)_n = \operatorname{ad} x_n$.

We call a subalgebra $\mathfrak{t}$ of a Lie algebra $\mathfrak{k}$ *toral* if for every element $x \in \mathfrak{t}$ the operator $\operatorname{ad} x : \mathfrak{k} \to \mathfrak{k}$ is diagonalizable. In particular, every non-zero element of a toral subalgebra of $\mathfrak{gl}(V, V_*)$ is semisimple (and if $\mathbb{K}$ is algebraically closed, a subalgebra is toral if and only if all its nonzero elements are diagonalizable).

Lemma I.3. *Every toral subalgebra of a Lie algebra is abelian.*

Proof. Let $x, y \in \mathfrak{t}$. Since $\operatorname{ad} x|_{\mathfrak{t}}$ is diagonalizable, we can write y as $y = \sum_\lambda y_\lambda$ with $[x, y_\lambda] = \lambda y_\lambda$ for $\lambda \in \mathbb{K}$. Then, for any λ, $(\operatorname{ad} y_\lambda)^2(x) = 0$, and as $\operatorname{ad} y_\lambda$ is also diagonalizable, $[y_\lambda, x] = (\operatorname{ad} y_\lambda)(x) = 0$. Therefore $[y, x] = \sum_\lambda [y_\lambda, x] = 0$. ∎

For any subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$, we denote by $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, by $\mathfrak{z}(\mathfrak{a})$ - the center of $\mathfrak{a}$, and by $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$ - the normalizer of $\mathfrak{a}$ in $\mathfrak{g}$.

Lemma I.4. *Let $\mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{gl}(V, V_*)$ be a locally nilpotent subalgebra, and $\mathfrak{h}_s = \{h_s : h \in \mathfrak{h}\}$ be the set of semisimple Jordan components of elements of $\mathfrak{h}$. Then the following assertions hold:*

(1) *$\mathfrak{h}_s$ is an abelian Lie algebra;*
(2) *$\mathfrak{h} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s)$;*
(3) *$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s)$;*
(4) *$\mathfrak{h} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s)$ is a self-normalizing subalgebra of $\mathfrak{g}$.*

Proof. (2) For each finite-dimensional nilpotent Lie algebra $\mathfrak{k}$ the set $(\operatorname{ad} \mathfrak{k})_s$ commutes with $\operatorname{ad} \mathfrak{k}$, which implies $[\mathfrak{h}_s, \mathfrak{h}] \subseteq \mathfrak{z}(\mathfrak{h})$. Hence $(\operatorname{ad} \mathfrak{h}_s)^2(\mathfrak{h}) = \{0\}$, and the semisimplicity of the elements of $\mathfrak{h}_s$ leads to $[\mathfrak{h}_s, \mathfrak{h}] = \{0\}$.
(1) For $x, y \in h$ assertion (2) implies that y_s commutes with x, and hence also with x_s. Therefore $x_s + y_s$ is semisimple and (2) shows that it commutes with $x_n + y_n$. From the finite-dimensional case we derive that $x_n + y_n$ is nilpotent, thus $x + y = (x_s + y_s) + (x_n + y_n)$ is the additive Jordan decomposition of $x + y$. This implies that $\mathfrak{h}_s$ is a subspace, hence, in view of (2), an abelian Lie algebra.
(3) In view of (2), for each $x \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ and each $h \in \mathfrak{h}$ we have the relation $[h, x] \in \mathfrak{h}$. Since the semisimple semisimple part $(\operatorname{ad} h)_s = \operatorname{ad} h_s$ of $\operatorname{ad} h$ can be obtained by applying a polynomial without constant term to $\operatorname{ad} h$, we also obtain $(\operatorname{ad} h_s)(x) \in \mathfrak{h}$, so (2) leads to $(\operatorname{ad} h_s)^2(x) = 0$. As $\operatorname{ad} h_s$ is semisimple, we obtain $[h_s, x] = 0$. This proves (3).
(4) follows directly from (3). ∎

II. Maximal toral subalgebras

Lemma II.1. *An element $x \in \mathfrak{g}$ is* ad*-diagonalizable if and only if x is diagonalizable as an operator on V.*

Proof. Decompose V as a direct sum $U \oplus W$, where $x \cdot W = \{0\}$ and U is a finite-dimensional x-invariant subspace. Then $V_* = W^\perp \oplus U^\perp$, $W^\perp \cong U^*$, and $x \in \mathfrak{gl}(U, W^\perp) \cong U \otimes U^*$.

If x is diagonalizable as an operator on V, then $\mathrm{ad}\, x$ is diagonalizable with eigenvalues $\lambda_i - \lambda_j$, where λ_i are the eigenvalues of x.

Assume now that $\mathrm{ad}\, x$ is diagonalizable and observe that this implies that $\mathrm{ad}\, x\,|_{\mathfrak{gl}(U,W^\perp)}$ is diagonlizable. This implies that x is semisimple. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of x in $\overline{\mathbb{K}}$. Then $\lambda_i - \lambda_j$ are the eigenvalues of $\mathrm{ad}\, x$, and $\lambda_i - \lambda_j \in \mathbb{K}$. We may therefore write $\lambda_i = \lambda + \mu_i$ with $\mu_i \in \mathbb{K}$. As the set of all λ_i is invariant under the Galois group $\mathrm{Aut}_{\mathbb{K}}(\overline{\mathbb{K}})$, the affine space generated by all λ_i contains a fixed point, i.e., an element of $\mathbb{K}$. On the other hand, this affine space is contained in $\lambda + \mathbb{K}$, which gives $\lambda \in \mathbb{K}$. Therefore $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, i.e., x is diagonalizable on U, and therefore on V. ■

In this section we consider a fixed toral subalgebra $\mathfrak{t} \subseteq \mathfrak{g} = \mathfrak{gl}(V, V_*)$. We write V' (resp., V'_*, $\mathfrak{g}'$) for the maximal locally finite $\mathfrak{t}$-submodule of V (resp., V_*, $\mathfrak{g}$). Since each element $x \in \mathfrak{t}$ is diagonalizable (Lemma II.1), the action of $\mathfrak{t}$ on the locally finite modules V' and V'_* is simultaneously diagonalizable, i.e., V' and V'_* are weight $\mathfrak{t}$-modules. Let

$$V' = \bigoplus_{\alpha \in \operatorname{supp} V} V^\alpha \quad \text{and} \quad V'_* = \bigoplus_{\beta \in \operatorname{supp} V_*} V_*^\beta$$

be the corresponding weight decompositions.

Lemma II.2.

(1) $\mathfrak{t} \cdot V \subseteq V'$ *and* $\mathfrak{t} \cdot V_* \subseteq V'_*$.

(2) $\mathfrak{g}' = V' \otimes V'_*$.

(3) $\mathfrak{g}' = V' \otimes V'_*$ *is an associative subalgebra of $\mathfrak{g} = V \otimes V_*$ and a weight $\mathfrak{t}$-module with respect to the adjoint action. w*

Proof. (1) For $x \in \mathfrak{t}$, $x \cdot V$ is a finite-dimensional subspace of V which is y-invariant for every $y \in \mathfrak{t}$ as $\mathfrak{t}$ is abelian. Hence $x \cdot V \subset V'$. Similarly $x \cdot V_* \subset V'_*$.

(2) is a direct consequence of Proposition A in the Appendix.

(3) $V' \otimes V'_*$ is obviously an associative subalgebra of $V \otimes V_*$. Furthermore, $V' \otimes V'_*$ is the tensor product of the weight $\mathfrak{t}$-modules V' and V'_*, and is thus itself a weight $\mathfrak{t}$-module. ■

In view of Lemma II.2(3), the weight decompositions of V' and V'_* yield the root decomposition

$$\mathfrak{g}' = \bigoplus_{\alpha,\beta} V^\alpha \otimes V_*^\beta = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) \oplus \Big(\bigoplus_{\gamma \neq 0} \mathfrak{g}^\gamma\Big), \quad \text{where} \quad \mathfrak{g}^\gamma = \bigoplus_{\alpha+\beta=\gamma} V^\alpha \otimes V_*^\beta.$$

Furthermore, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) \subseteq \mathfrak{g}'$ implies

$$\mathfrak{t} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{g}^0 = \bigoplus_{\alpha \in (\operatorname{supp} V) \cap (-\operatorname{supp} V_*)} V^\alpha \otimes V_*^{-\alpha}.$$

In the sequel we are mainly interested in the centralizers of maximal toral subalgebras. We start by a description of maximal toral subalgebras in terms of their action on V and V_*.

Proposition II.3. *The subalgebra* $\mathfrak{t}$ *is maximal toral if and only if the following conditions are satisfied:*
(M1) $(\operatorname{supp} V)\setminus\{0\} = -(\operatorname{supp} V_*)\setminus\{0\}$ *and* $\dim V^\alpha = \dim V_*^{-\alpha} = 1$ *for any* $\alpha \in \operatorname{supp} V\setminus\{0\}$*;*
(M2) $V_*^0(V^0) = \{0\}$ *and* $(V^0 \otimes V_*^0)\cap \mathfrak{t} = \{0\}$*;*
(M3) $\mathfrak{t} = \bigoplus_{0\neq\alpha\in\operatorname{supp} V} V^\alpha \otimes V_*^{-\alpha}$*;*
(M4) $V_*^0 = \bigcap_{0\neq\alpha\in\operatorname{supp} V}(V^\alpha)^\perp$ *and* $V^0 = \bigcap_{0\neq\beta\in\operatorname{supp} V_*}(V_*^\beta)^\perp$*.*
If these conditions are satisfied, then

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}\oplus(V^0\otimes V_*^0).$$

Proof. Assume first that $\mathfrak{t}$ is maximal toral.
(M1) From $\mathfrak{t}\subseteq\bigoplus_{\alpha\in(\operatorname{supp} V)\cap(-\operatorname{supp} V_*)} V^\alpha\otimes V_*^{-\alpha}$, $V_*^\beta(V^\alpha)=\{0\}$ for $\beta\neq-\alpha$, and $\mathfrak{t}\cdot V^\alpha\neq\{0\}$ for $\alpha\neq 0$, it follows that $V_*^{-\alpha}(V^\alpha)\neq\{0\}$. Pick $v\in V^\alpha$ and $\varphi\in V_*^{-\alpha}$ with $\varphi(v)=1$. Then $v\otimes\varphi\in\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$ is diagonalizable, hence contained in $\mathfrak{t}$ by maximality. Therefore

$$\{0\} = [v\otimes\varphi, V^\alpha\otimes V_*^{-\alpha}].$$

This implies in particular

$$\{0\} = [v\otimes\varphi, (\ker\varphi\cap V^\alpha)\otimes V_*^{-\alpha}] = (\ker\varphi\cap V^\alpha)\otimes\varphi.$$

Thus $\ker\varphi\cap V^\alpha = \{0\}$, which yields $V^\alpha = \mathbb{K}v$ for $0\neq\alpha\in(\operatorname{supp} V)\cap(-\operatorname{supp} V_*)$. We likewise see that $V_*^{-\alpha}=\mathbb{K}\varphi$. In particular, $V^\alpha\otimes V_*^{-\alpha}\subseteq\mathfrak{t}$. As $\mathfrak{t}\cdot V' = \bigoplus_{0\neq\alpha\in\operatorname{supp} V} V^\alpha$ and $\mathfrak{t}\cdot V_*' = \bigoplus_{0\neq\beta\in\operatorname{supp} V_*} V_*^\beta$, we further see that $\operatorname{supp} V_*\setminus\{0\} = -\operatorname{supp} V\setminus\{0\}$.
(M2) Suppose that there exists $\varphi\in V_*^0$ and $v\in V^0$ with $\varphi(v)=1$. As above, we see that $v\otimes\varphi\in\mathfrak{t}$, contradicting $(v\otimes\varphi)\cdot v = v$ and $v\in V^0$. Therefore $V_*^0(V^0)=\{0\}$, which in turn implies that each element in $V^0\otimes V_*^0$ is nilpotent. Hence $\mathfrak{t}\cap(V^0\otimes V_*^0)=\{0\}$.
(M3) Since $\mathfrak{t}$ contains all the spaces $V^\alpha\otimes V_*^{-\alpha}$ for $\alpha\neq0$ and is contained in $\bigoplus_{\alpha\in\operatorname{supp} V} V^\alpha\otimes V_*^{-\alpha}$, we obtain

$$\mathfrak{t} = (\mathfrak{t}\cap(V^0\otimes V_*^0))\oplus(\bigoplus_{0\neq\alpha\in\operatorname{supp} V} V^\alpha\otimes V_*^{-\alpha}).$$

Now (M3) follows from (M2).
(M4) follows from the equality $V^0 = \{v\in V : \mathfrak{t}\cdot v = \{0\}\}$ as, in view of (M3), the space $\{v\in V:\mathfrak{t}\cdot v=\{0\}\}$ coincides with the common annihilator of the spaces $V_*^{-\alpha}$, $\alpha\neq0$. A similar argument applies to V_*^0.

Conversely, assume that (M1)-(M4) are satisfied. Then $\dim V^\alpha\otimes V_*^{-\alpha}=1$ for $0\neq\alpha\in\operatorname{supp} V$, and $V_*^\beta(V^\alpha)\neq\{0\}$ for $\beta\neq-\alpha$ imply that $\mathfrak{t}$ is abelian and that each element of $\mathfrak{t}$ is diagonalizable. Therefore $\mathfrak{t}$ is a toral subalgebra of $\mathfrak{g} = V\otimes V_*$ (Lemma II.1). The centralizer of $\mathfrak{t}$ in $\mathfrak{g}$ is contained in $\mathfrak{g}'$ and coincides with $\mathfrak{t}\oplus(V^0\otimes V_*^0)$. Now (M2) implies that each element in $V^0\otimes V_*^0$ is nilpotent, so $\mathfrak{t}$ is maximal toral.

Finally,

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{g}^0(\mathfrak{t}) = \bigoplus_{\alpha+\beta=0} V^\alpha\otimes V_*^\beta = \mathfrak{t}\oplus(V^0\otimes V_*^0). \qquad \blacksquare$$

Corollary II.4. *If* $\mathfrak{t}$ *is a maximal toral subalgebra, then* $(\operatorname{supp} V)\setminus\{0\}\subset\mathfrak{t}^*$ *is a linearly independent set.*

Proof. The statement follows from the equality $\mathfrak{t} = \bigoplus_{0\neq\alpha\in\operatorname{supp} V} V^\alpha\otimes V_*^{-\alpha}$ and from the fact that α vanishes on $\bigoplus_{\delta\neq\alpha\in\operatorname{supp} V} V^\delta\otimes V_*^{-\delta}$. ■

The next proposition shows that for a maximal toral subalgebra $\mathfrak{t}$ the spaces V^α for $\alpha\neq0$ determine the space V^0 (resp., V_*^β for $\beta\neq0$ determine V_*^0).

Proposition II.5. *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a maximal toral subalgebra. Then*

$$V^0 = \bigcap_{0\neq\alpha\in\operatorname{supp} V} \Big(\bigcap_{\alpha\neq\delta\in\operatorname{supp} V} (V^\delta)^\perp\Big)^\perp \quad \text{and} \quad V_*^0 = \bigcap_{0\neq\beta\in\operatorname{supp} V_*} \Big(\bigcap_{\beta\neq\eta\in\operatorname{supp} V_*} (V_*^\eta)^\perp\Big)^\perp.$$

Proof. By Proposition II.3, $V_*^0 = \bigcap_{0\neq\alpha\in\operatorname{supp} V} (V^\alpha)^\perp$. Fix $0 \neq \alpha \in \operatorname{supp} V$ and pick $f_\alpha \in V^\alpha$ and $f_\alpha^* \in V_*^{-\alpha}$ with $f_\alpha^*(f_\alpha) = 1$. Consider an element $\varphi \in \bigcap_{\alpha\neq\delta\in\operatorname{supp} V} (V^\delta)^\perp$. Then $\varphi - \varphi(f_\alpha) f_\alpha^* \in V_*^0$ leads to $\varphi \in V_*^0 + V_*^{-\alpha}$, and therefore to

$$V_*^0 + V_*^{-\alpha} = \bigcap_{\alpha\neq\delta\in\operatorname{supp} V} (V^\delta)^\perp.$$

As $V_*^0(V^0) = \{0\}$, we have

$$V^0 = \bigcap_{0\neq\alpha\in\operatorname{supp} V} (V_*^{-\alpha})^\perp = \bigcap_{0\neq\alpha\in\operatorname{supp} V} (V_*^0 + V_*^{-\alpha})^\perp = \bigcap_{0\neq\alpha\in\operatorname{supp} V} \bigcap_{\alpha\neq\delta\in\operatorname{supp} V} (V^\delta)^\perp.$$

The second equality is established in a similar way. ■

III. The structure of Cartan subalgebras

Definition III.1. (a) We define a *Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{gl}(V, V_*)$ as a locally nilpotent subalgebra $\mathfrak{h}$ with $\mathfrak{h} = \mathfrak{z}_\mathfrak{g}(\mathfrak{h}_s)$.
(b) A *toral Cartan subalgebra* of $\mathfrak{g}$ is a toral subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ with $\mathfrak{z}_\mathfrak{g}(\mathfrak{t}) = \mathfrak{t}$.
(c) A *generalized splitting Cartan subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ for which $\mathfrak{g}$ has a generalized root decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha)$, where $\Delta := (\operatorname{supp} \mathfrak{g}) \setminus \{0\}$. The Cartan subalgebra $\mathfrak{h}$ is *splitting* if, in addition, $\mathfrak{g}$ is a weight $\mathfrak{h}$-module. ■

As all toral subalgebras are abelian by Lemma I.3, toral Cartan subalgebras are in particular Cartan subalgebras. For the same reason, toral Cartan subalgebras are maximal abelian, hence in particular maximal toral subalgebras of $\mathfrak{g}$, and are therefore covered by Proposition II.3. Moreover, if $\mathfrak{h}$ is a generalized splitting Cartan subalgebra, then the generalized root spaces are common eigenspaces of $(\operatorname{ad} \mathfrak{h})_s = \operatorname{ad} \mathfrak{h}_s$ corresponding to non-zero eigenfunctionals. This immediately implies that $\mathfrak{z}_\mathfrak{g}(\mathfrak{h}_s) = \mathfrak{g}^0(\mathfrak{h}) = \mathfrak{g}_0(\mathfrak{h}_s) = \mathfrak{h}$. Therefore $\mathfrak{h}$ is a Cartan subalgebra in the sense of Definition III.1(a).

Lemma III.2. *For a maximal toral subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ the following are equivalent:*
(1) $\mathfrak{t}$ *is a toral Cartan subalgebra.*
(2) $V^0 = \{0\}$ *or* $V_*^0 = \{0\}$.

Proof. This follows from the equality $\mathfrak{z}_\mathfrak{g}(\mathfrak{t}) = \mathfrak{t} \oplus (V^0 \otimes V_*^0)$ (Proposition II.3). ■

Proposition III.3. *For any maximal toral subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, $\mathfrak{h} := \mathfrak{z}_\mathfrak{g}(\mathfrak{t})$ is an abelian self-normalizing subalgebra of $\mathfrak{g}$ with $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h}) = \mathfrak{z}_\mathfrak{g}(\mathfrak{h}_s)$, and in particular a Cartan subalgebra.*

Proof. By Proposition II.3,

$$\mathfrak{h} = \mathfrak{t} \oplus (V^0 \otimes V_*^0).$$

Furthermore, the equality $V_*^0(V^0) = \{0\}$ implies that $V^0 \otimes V_*^0$ is an abelian Lie algebra such that $(V^0 \otimes V_*^0)^2 = \{0\}$, and thus $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ with $\mathfrak{h}_s = \mathfrak{t}$ and $\mathfrak{h}_n = V^0 \otimes V_*^0$. Hence $\mathfrak{h} = \mathfrak{z}_\mathfrak{g}(\mathfrak{h}_s)$, and Lemma I.4 implies that $\mathfrak{h}$ is self-normalizing. Finally, $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathfrak{h}) \subseteq \mathfrak{g}^0(\mathfrak{t}) = \mathfrak{z}_\mathfrak{g}(\mathfrak{t}) = \mathfrak{h}$ shows that $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$. ■

The following theorem is our first main result. It implies that if $\mathbb{K}$ is algebraically closed, all Cartan subalgebras of $\mathfrak{g}$ are centralizers of maximal toral subalgebras, and hence are as in Proposition III.3.

Theorem III.4. (Structure Theorem for Cartan Subalgebras) *Let $\mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{gl}(V, V_*)$ be a Cartan subalgebra for which the abelian subalgebra $\mathfrak{h}_s \subseteq \mathfrak{g}$ is toral. (The latter is automatic when $\mathbb{K} = \overline{\mathbb{K}}$.) Then*

(1) $\mathfrak{h}_s$ *is a maximal toral subalgebra of* $\mathfrak{g}$ *with* $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s)$*;*

(2) $\mathfrak{h} = \mathfrak{h}_s \oplus (V^0 \otimes V_*^0) = (\bigoplus_{0 \neq \alpha \in \operatorname{supp} V} V^\alpha \otimes V_*^{-\alpha}) \oplus (V^0 \otimes V_*^0)$*, where* $V' = \bigoplus_{\alpha \in \operatorname{supp} V} V^\alpha$ *and* $V_* = \bigoplus_{\beta \in \operatorname{supp} V_*} V_*^\beta$ *are the* $\mathfrak{h}_s$*-module weight decompositions of* V' *and* V_*'*;*

(3) $\mathfrak{h}$ *is abelian;*

(4) *if* $\mathfrak{h}$ *is a generalized splitting Cartan subalgebra, then* $\mathfrak{h}$ *is splitting.*

Proof. (1) Let $\mathfrak{t} \supseteq \mathfrak{h}_s$ be a toral subalgebra. Then $\mathfrak{t}$ is abelian (Lemma I.3) and therefore $\mathfrak{t} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s) = \mathfrak{h}$, i.e., $\mathfrak{t} \subseteq \mathfrak{h}_s$.

(2) follows from (1) and Proposition II.3.

(3) is a direct consequence of (1), (2), and the equality $V_*^0(V^0) = \{0\}$ (Proposition II.3(M2)).

(4) If $\mathfrak{g}$ has a generalized $\mathfrak{h}$-root decomposition, then $\mathfrak{g}$ is a locally finite $\mathfrak{h}_s$-module. Therefore

$$\mathfrak{g} = \bigoplus_{\alpha,\beta} V^\alpha \otimes V_*^\beta = V' \otimes V_*'$$

by Lemma II.2, and $V' = V$ and $V_*' = V_*$. Furthermore, $V_*^0(V^0) = \{0\}$ and $V_*^0(V^\alpha) = \{0\}$ for $\alpha \neq 0$, i.e., $V_*^0(V) = \{0\}$. Hence $V_*^0 = \{0\}$ (and similarly $V^0 = \{0\}$). This implies that $\mathfrak{h} = \mathfrak{h}_s$, i.e., that $\mathfrak{h}$ is splitting. ■

In [PS03] a statement similar to Theorem III.4(4) is established. Namely the main result of [PS03] claims that, if $\mathbb{K} = \overline{\mathbb{K}}$ any subalgebra $\mathfrak{h}$ which yields a generalized root decomposition of $\mathfrak{g}$ is a splitting Cartan subalgebra.

Corollary III.5. *Any Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$ *is abelian.*

Proof. If $\mathbb{K} = \overline{\mathbb{K}}$ the statement is proved Theorem III.4. Let

$$\overline{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}} = \mathfrak{gl}(V \otimes_{\mathbb{K}} \overline{\mathbb{K}}, V_* \otimes_{\mathbb{K}} \overline{\mathbb{K}}).$$

Then $\overline{\mathfrak{h}} := \mathfrak{h} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ is a locally nilpotent subalgebra of $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{h}}_s = \mathfrak{h}_s \otimes_{\mathbb{K}} \overline{\mathbb{K}}$, which directly implies that

$$\mathfrak{z}_{\overline{\mathfrak{g}}}(\overline{\mathfrak{h}}_s) = \mathfrak{z}_{\overline{\mathfrak{g}}}(\mathfrak{h}_s) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s) \otimes_{\mathbb{K}} \overline{\mathbb{K}} = \overline{\mathfrak{h}}.$$

Therefore $\overline{\mathfrak{h}}$ is a Cartan subalgebra of $\overline{\mathfrak{g}}$. Hence $\overline{\mathfrak{h}}$ is abelian by Theorem III.4, and consequently $\mathfrak{h}$ is abelian. ■

If $\mathbb{K}$ is algebraically closed, Theorem III.4 enables us to give a description of all Cartan subalgebras of $\mathfrak{gl}(V, V_*)$ in terms of pure linear algebra.

We define a *dual system of one dimensional subspaces* to be a family $(V_j)_{j \in J}$ of one-dimensional subspaces of V, together with a family of one-dimensional subspaces $((V_j)_*)_{j \in J}$ of V_* such that $(V_i)_*(V_j) = \delta_{ij}\mathbb{K}$.

Lemma III.6. *Let* $(V_j)_{j \in J}$, $((V_j)_*)_{j \in J}$ *be a dual system of one-dimensional subspaces. Then it is maximal if and only if the spaces* $V_*^0 := \cap_j (V_j)^\perp \subseteq V_*$ *and* $V^0 := \cap_j ((V_j)_*)^\perp \subseteq V$ *satisfy* $V_*^0(V^0) = \{0\}$.

Proof. If $V_*^0(V^0) \neq \{0\}$, there is an element $e \in V^0$ and an element $e_* \in V_*^0$ with $e_*(e) = 1$, therefore the dual system $(V_j)_{j \in J}$, $((V_j)_*)_{j \in J}$ can be extended by the pair of one-dimensional spaces $\mathbb{K}e$, $\mathbb{K}e_*$. Thus the maximality of the family $(V_j)_{j \in J}$ implies $V_*^0(V^0) = \{0\}$. Conversely, it is clear that $V_*^0(V^0) \neq \{0\}$ if the dual system is not maximal. ■

The existence of maximal dual families of one-dimensional subspaces follows easily from Zorn's Lemma. Proposition II.3, Theorem III.4, and Lemma III.6 imply immediately the following proposition.

Proposition III.7. *Let $(V_j)_{j\in J}$, $((V_j)_*)_{j\in J}$ be a dual system of one-dimensional subspaces satisfying the equivalent conditions of* Lemma III.6. *Then $\mathfrak{t} := \bigoplus_{j\in J} V_j \otimes (V_j)_* \subseteq \mathfrak{g}$ is a maximal toral subalgebra and $\mathfrak{h} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$ is a Cartan subalgebra of $\mathfrak{gl}(V, V_*)$. If $\mathbb{K} = \overline{\mathbb{K}}$, every Cartan subalgebra of $\mathfrak{g}$ is obtained by this construction.* ∎

Theorem III.4 and Proposition III.7 imply that, there are the following (mutually exclusive) alternatives for a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with toral $\mathfrak{h}_s$:

(I) $\mathfrak{h} \neq \mathfrak{h}_s$;

(II) $\mathfrak{h} = \mathfrak{h}_s$ is toral but not splitting;

(III) $\mathfrak{h} = \mathfrak{h}_s$ is splitting.

Clearly not all cases will always occur, as for instance case (III) implies that the dimensions of V and V_* coincide, while for an infinite-dimensional V they a priori need not coincide. Moreover, Proposition I.2 shows that, when $\dim V$ is uncountable, equality of the dimensions of V and V_* is not sufficient for the occurence of case (III). The next three propositions describe precisely which cases among (I)-(III) occur in the following situations: when $V = V_* = \mathbb{K}^{(J)}$ for an infinite set J, when V is a Hilbert space (here $\mathbb{K} = \mathbb{R}, \mathbb{C}$) and V_* is the space of continuous linear functionals on V, and when V is an arbitrary infinite-dimensional vector space and $V_* = V^*$.

Proposition III.8. *Let J be an infinite set. Then in $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{K})$ all three types* (I)–(III) *of Cartan subalgebras, and, moreover, all dimensions of the spaces $V^0 \subseteq V$ and $V_*^0 \subseteq V_*$ do occur.*

Proof. Here $V = \mathbb{K}^{(J)} = V_*$. Write J as a disjoint union $J = J_0 \sqcup J_1$ and assume that $\eta\colon J_0 \to J_1$ is a surjection such that the inverse image of every element in J_1 is infinite. Fix a decomposition into two disjoint subsets $J_1 = J_1^+ \sqcup J_1^-$ and put $J_0^\pm := \eta^{-1}(J_1^\pm)$. If $(e_j)_{j\in J}$ denotes the canonical basis of V, set

$$V_j := \begin{cases} \mathbb{K}(e_j + e_{\eta(j)}), & j \in J_0^+ \\ \mathbb{K}e_j, & j \in J_0^- \end{cases}$$

and

$$(V_j)_* := \begin{cases} \mathbb{K}e_j, & j \in J_0^+ \\ \mathbb{K}(e_j + e_{\eta(j)}), & j \in J_0^- . \end{cases}$$

The families $(V_j)_{j\in J_0}$, and $((V_*)_j)_{j\in J_0}$, satisfy $(V_i)_*(V_j) = \delta_{ij}\mathbb{K}$ for $i, j \in J_0$ and thus form a dual system of one-dimensional subspaces. Furthermore, if $\alpha \in V_*$ vanishes on all V_j, then $\alpha(e_j) = -\alpha(e_{\eta(j)})$ holds for each $j \in J_0^+$. For $i := \eta(j)$ we then have $\alpha(e_i) = \alpha(e_j)$ for infinitely many indices j with $\eta(j) = i$. This implies that $\alpha(e_i) = 0$ for $i \in J_1^+$, and likewise $\alpha(e_j) = 0$ for $j \in J_0^+$. We also have $\alpha(e_j) = 0$ for $j \in J_0^-$, and therefore

$$V_*^0 = \cap_j (V_j)^\perp = \operatorname{span}\{e_j : j \in J_1^-\} \cong \mathbb{K}^{J_1^-}.$$

In a similar way we obtain

$$V^0 = \cap_j ((V_j)_*)^\perp = \operatorname{span}\{e_j : j \in J_1^+\} \cong \mathbb{K}^{J_1^+}.$$

In particular, $V_*^0(V^0) = 0$, i.e. the dual system $(V_j)_{j\in J_0}$, $((V_*)_j)_{j\in J_0}$ is maximal. Consequently, for any infinite countable set J the spaces V^0 and V_*^0 can have arbitrary prescribed dimensions less or equal $|J|$. ∎

Proposition III.9. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and V be an infinite-dimensional Hilbert space and V_* be the space of continuous linear functionals on V. Then any Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{gl}(V, V_*)$ with toral $\mathfrak{h}_s$ has type* (I) *or* (II), *and both cases are possible.*

Proof. Proposition I.2 implies that case (III) does not occur. To construct a Cartan subalgebra of type (II), fix an orthonormal Hilbert basis $(e_j)_{j\in J}$ of V. Set $V_j := \mathbb{K}e_j$, and $(V_j)_* = \mathbb{K}e_j^*$, where $v^*(x) := \langle x, v\rangle$ is the linear functional corresponding to $v \in V$. Then $(V_j)_{j\in J}$, $((V_j)_*)_{j\in J}$

is a dual system of one-dimensional subspaces with $V^0 = \{0\} = V_*^0$, and it defines a Cartan subalgebra of $\mathfrak{gl}(V, V_*)$ of type (II).

A Cartan subalgebra of type (I) can be constructed as follows. Here we assume that $J = \mathbb{N}$ and set $f_n := e_1^* + e_{n+1}^* - 2e_{n+2}^*$ for $n \in \mathbb{N}$. Let $V_n := \mathbb{K}e_{n+1}$ and $(V_n)_* := \mathbb{K}f_n$. Then $V_*^0 = \mathbb{K}e_1^*$ and

$$V^0 = \bigcap_{n\in\mathbb{N}} \ker f_n = \mathbb{K}\sum_{n\geq 2} 2^{-n} e_n \subseteq (V_*^0)^\perp.$$

Hence the maximal dual system of one dimensional spaces $(V_n)_{n\in\mathbb{N}}$, $((V_n)_*)_{n\in\mathbb{N}}$ defines a Cartan subalgebra of $\mathfrak{gl}(V, V_*)$ of type (I). ■

Proposition III.10. *Let V be an infinite-dimensional vector space and $V_* = V^*$. Then any Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{gl}(V, V_*)$ with toral $\mathfrak{h}_s$ has type* (II).

Proof. The occurence of type (III) implies equality of the dimensions of V and V_*, which is not the case when $V_* = V^*$. It remains to show that a Cartan subalgebra cannot have type (I). Assume to the contrary that, for some J, $(V_j)_{j\in J}$, $((V_j)_*)_{j\in J}$ form a maximal dual system of one-dimensional spaces for which $V^0 \neq \{0\}$, $V_*^0 \neq \{0\}$. Fix $0 \neq e \in V^0$. Then, since $V^* = V_*$, there is a linear functional $e_* \in V_*^0$ with $e_*(e) = 0$. Consequently $V_*^0(V^0) \neq \{0\}$, which is a contradiction. ■

We complete this section by addressing the problem of conjugacy for maximal toral subalgebras and thus also for Cartan subalgebras.

Proposition III.11. *All splitting Cartan subalgebras of $\mathfrak{g} = \mathfrak{gl}(V, V_*)$ are conjugate under the group $\mathrm{GL}(V, V_*) := \{g \in \mathrm{GL}(V) : g^* \cdot V_* = V_*\}$. In general $\mathrm{GL}(V, V_*)$-conjugate maximal toral subalgebras have equal respective dimensions of the subspaces V^0 and V_*^0, but equality of those dimensions is not sufficient for $\mathrm{GL}(V, V_*)$-cojugacy.*

Proof. If $\mathfrak{h}$ is a splitting Cartan subalgebra, Proposition II.3 implies that there is a basis $(v_\alpha)_{\alpha\in A}$ of V and a dual basis $(v_\alpha^*)_{\alpha\in A}$ in V_* such that

$$\mathfrak{h} = \bigoplus_{\alpha\in A} (\mathbb{K}v_\alpha \otimes \mathbb{K}v_\alpha^*).$$

In other words, $\mathfrak{h}$ is the set of all elements of $\mathfrak{g}$ which are represented by diagonal matrices with respect to the basis $(v_\alpha)_{\alpha\in A}$. This implies immediately the conjugacy of all splitting Cartan subalgebras of $V \otimes V_*$ under $\mathrm{GL}(V, V_*)$ (cf. [NS01] for the case $\mathfrak{gl}(J, \mathbb{K})$).

It is clear that, if two maximal toral subalgebras are $\mathrm{GL}(V, V_*)$-conjugate, their respective dimensions of the spaces V^0 and V_*^0 coincide. The following example shows that this is not sufficient for $\mathrm{GL}(V, V_*)$-conjugacy.

Set $V = V_* := \mathbb{K}^{(\mathbb{N})}$. Fix an injective map $\eta: \mathbb{N} \to \mathbb{N}$ with $\eta(n) > n$ for each $n \in \mathbb{N}$, and let $S: V \to V, e_n \mapsto e_{\eta(n)}$ be the corresponding shift operator (where $(e_n)_{n\in\mathbb{N}}$ is the standard basis of $\mathbb{K}^{(\mathbb{N})}$). Then the endomorphism $A := \mathbf{1} - S \in \mathrm{End}(V)$ is injective because S has no eigenvectors in V, and is obviously not surjective as $e_n \notin A(V)$ for any n. Furthermore, the matrix $A^\top := \mathbf{1} - S^\top$, considered as an operator on V_*, is locally unipotent with inverse given by $\sum_{n\in\mathbb{N}_0} (S^\top)^n$, and consequently $A^*|_{V_*}$ is an automorphism of V_*.

As A is injective, the one-dimensional subspaces $V_n := A(\mathbb{K}e_n)$ satisfy the conditions of Proposition III.7 with $(V_n)_* := A^*|_{V^*}(\mathbb{K}e_n)$ and

$$V^0 = \bigcap_{n\in\mathbb{N}} (V_n^*)^\perp = V_*^\perp = \{0\} \quad \text{and} \quad V_*^0 = \bigcap V_n^\perp = \ker A^*|_{V_*} = \{0\}.$$

Since A is not surjective, $V' = \mathrm{span}\{V_n : n \in \mathbb{N}\} \neq V$. Therefore the toral subalgebra $\mathfrak{h}$ is not splitting and is not conjugate to any splitting Cartan subalgebra. ■

IV. Alternative characterizations of Cartan subalgebras

In this and in the section we consider $\mathfrak{g} = \mathfrak{gl}(V, V_*)$ for an arbitrary subspace $V_* \subseteq V^*$ separating the points of V.

The following theorem is our second main result.

Theorem IV.1. *A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra if and only if it satisfies one of the following two equivalent conditions:*

(C1') *$\mathfrak{h}$ is a locally nilpotent self-normalizing subalgebra whose adjoint module is locally finite;*

(C2') *$\mathfrak{h}$ coincides with the maximal locally nilpotent $\mathfrak{h}$-submodule of $\mathfrak{g}$ and the adjoint module of $\mathfrak{h}$ is locally finite.* ∎

First we observe that conditions (C1'), (C2') and (C3) are satisfied for a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ if and only if they are satisfied for the subalgebra $\overline{\mathfrak{h}} \subseteq \overline{\mathfrak{g}}$, where $\overline{\mathfrak{k}} = \mathfrak{k} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$. Therefore, without restricting generality, we will assume throughout this section that $\mathbb{K}$ is algebraically closed.

The fact that (C1') and (C2') are satisfied for a Cartan subalgebra follows immediately from Proposition III.3 and Theorem III.4. The following Lemma IV.2 and Proposition IV.4 imply that any subalgebra $\mathfrak{h}$ satisfying (C1') or (C2') is a Cartan subalgebra. Note that, since $\mathbb{K} = \overline{\mathbb{K}}$, the subalgebra $\mathfrak{h}_s \subseteq \mathfrak{h}$ of any subalgebra $\mathfrak{h}$ satisfying (C1') or (C2') is toral.

Lemma IV.2. *The conditions (C1') and (C2') are equivalent.*

Proof. Assume that (C1') is satisfied. Consider the maximal locally nilpotent $\mathfrak{h}$-submodule $\mathfrak{g}^0(\mathfrak{h}) \subseteq \mathfrak{g}$. Since the adjoint representation of $\mathfrak{h}$ is locally finite, we have $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathfrak{h})$. Indeed, otherwise for some $h \in \mathfrak{h}$ the finite-dimensional submodule $U(\mathfrak{h}) \cdot h$ would have an $\mathfrak{h}$-eigenvector of non-zero eigenvalue, which would contradict the local nilpotence of $\mathfrak{h}$. To prove that $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$, assume to the contrary that $h \in \mathfrak{g}^0(\mathfrak{h})$ is such that $\mathfrak{h}^n \cdot h \in \mathfrak{h}$ for a minimal $n > 0$. Then $\mathfrak{h}^{n-1} \cdot h \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, contradicting the minimality of n. This shows that $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$, i.e., that (C1') implies (C2').

Conversely, let (C2') be satisfied. Since $\mathfrak{h} \cdot \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{h}$ and $\mathfrak{h}$ is a locally nilpotent $\mathfrak{h}$-module, $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ is also a locally nilpotent $\mathfrak{h}$-module. Therefore $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}^0(\mathfrak{h})$. Since $\mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$, this gives $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{h}$, i.e., $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. ∎

Lemma IV.3. *Condition (C2)' implies $V_*^0(V^0) = \{0\}$, where $V^0 := V^0(\mathfrak{h}_s)$, $V_*^0 := V_*^0(\mathfrak{h}_s)$.*

Proof. By Proposition A in the Appendix, $\mathfrak{g}' = V' \otimes V_*'$, where the superscript $'$ indicates maximal locally finite $\mathfrak{h}_s$-submodule. Furthermore, the assumption that the adjoint action of $\mathfrak{h}$ on $\mathfrak{h}$ is locally finite implies $\mathfrak{h} \subseteq \mathfrak{g}'$. Hence the generalized weight $\mathfrak{h}_s$-module decomposition of V' and V_*' and the equality $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$ yield

$$\mathfrak{h} = \bigoplus_{\alpha \in \operatorname{supp} V} (V^\alpha \otimes V^{-\alpha}) = \mathfrak{h}_s \oplus (V^0 \otimes V_*^0).$$

The local nilpotence of $\mathfrak{h}$ implies now $V_*^0(V^0) = \{0\}$. ∎

Proposition IV.4. *Condition (C2') implies that $\mathfrak{h}$ is a Cartan subalgebra.*

Proof. The equality $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$ implies that $\mathfrak{h}$ is locally nilpotent. Therefore Lemma I.4 yields the inclusion $\mathfrak{h} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s)$. It remains to establish the opposite inclusion $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s) \subseteq \mathfrak{h}$.

For $h \in \mathfrak{h}_s$ put $U := h \cdot V$ and $W := \{v \in V : h \cdot v = 0\}$. Then U is a finite-dimensional space and $V = U \oplus W$, $V^* = W^\perp \oplus V^\perp$, and $W^\perp \simeq U^*$. As $h \in \mathfrak{z}(\mathfrak{h})$, $\mathfrak{h}$ preserves the four spaces $U, W, U^\perp$, and $W^\perp$. Therefore the projection

$$p_U \colon \mathfrak{g} = V \otimes V_* \to U \otimes W^\perp \cong \mathfrak{gl}(U, W^\perp)$$

with kernel $V \otimes U^\perp + W \otimes V_*$ is $\mathfrak{h}$-equivariant. This gives

$$p_U(\mathfrak{h}) = p_U(\mathfrak{g}^0(\mathfrak{h})) = \mathfrak{gl}(U, W^\perp)^0(\mathfrak{h}) = \mathfrak{h} \cap \mathfrak{gl}(U, W^\perp) =: \mathfrak{h}_U.$$

The centralizer of h in $\mathfrak{g} = V \otimes V_*$, and therefore the subalgebra $\mathfrak{h}$, is contained in $U \otimes W^\perp + W \otimes U^\perp$, thus

$$\mathfrak{h} = \mathfrak{h}_U + (\mathfrak{h} \cap \ker p_U) = \mathfrak{h}_U + \mathfrak{h} \cap \mathfrak{gl}(W, U^\perp).$$

As $\mathfrak{gl}(W, U^\perp)$ commutes with $\mathfrak{gl}(U, W^\perp)$, we conclude that

$$\mathfrak{gl}(U, W^\perp)^0(\mathfrak{h}_U) = \mathfrak{gl}(U, W^\perp)^0(\mathfrak{h}) = \mathfrak{h}_U,$$

and hence that $\mathfrak{h}_U$ is a Cartan subalgebra of $\mathfrak{gl}(U, W^\perp) \cong \mathfrak{gl}(U)$. As $\mathfrak{gl}(U)$ is reductive, we have $\mathfrak{h}_U \subseteq \mathfrak{h}_s$ and $\mathfrak{h}_n \cdot U = \{0\}$.

By considering $\mathfrak{h}_U \subseteq \mathfrak{gl}(U, W^\perp)$, we now see that the weight spaces of $\mathfrak{h}_s$ in V and V_* with non-zero weights are one-dimensional. From the preceding argument we further conclude that $\mathfrak{h}_n \circ \mathfrak{h}_s = \{0\}$ in $\operatorname{End} V$, which, in view of $[\mathfrak{h}_s, \mathfrak{h}_n] = \{0\}$, implies $\mathfrak{h}_n \cdot V \subseteq V^0$. By Proposition A in the Appendix,

$$\mathfrak{g}' = (V \otimes V_*)' = V' \otimes V_*' = \bigoplus_{\alpha,\beta} V^\alpha \otimes V_*^\beta.$$

In particular $\mathfrak{h} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s) \subseteq \mathfrak{g}'$, and therefore

$$\mathfrak{h} \subseteq \bigoplus_\alpha V^\alpha \otimes V_*^{-\alpha}.$$

For $0 \neq \alpha$ we have seen above that $\dim V^\alpha = \dim V_*^{-\alpha} = 1$, so that

$$\bigoplus_{\alpha \neq 0} V^\alpha \otimes V_*^{-\alpha} \subseteq \mathfrak{h}_s,$$

because $\mathfrak{h}_s \cdot V^\alpha \neq \{0\}$ implies $V_*^{-\alpha}(V^\alpha) \neq \{0\}$. If $A \in \mathfrak{h}_s \cap \big(V^0 \otimes V_*^0\big)$, then A annihilates all weight spaces V^α with $\alpha \neq 0$, and it also annihilates V^0. Therefore $A \cdot V \subseteq V'$ leads to $A^2 = \{0\}$,k and hence to $A = 0$ as $\mathfrak{h}_s$ consists of semisimple elements (Lemma II.1). This proves

$$\mathfrak{h}_s = \bigoplus_{\alpha \neq 0} V^\alpha \otimes V_*^{-\alpha},$$

which in turn yields

$$V^0 = \bigcap_{0 \neq \alpha} (V_*^{-\alpha})^\perp \quad \text{and} \quad V_*^0 = \bigcap_{0 \neq \alpha} (V^\alpha)^\perp.$$

Since $V_*^0(V^0) = \{0\}$ by Lemma IV.3, Proposition II.3 implies that $\mathfrak{h}_s$ is maximal toral and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s) = \mathfrak{h}_s \otimes (V^0 \otimes V_*^0)$ is abelian. Therefore

$$V^0 \otimes V_*^0 \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h},$$

in particular $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_s) \subseteq \mathfrak{h}$. ■

It is an interesting open problem whether conditions (C1), (C2) and (C3) are equivalent for $\mathfrak{gl}(V, V_*)$. Our results reduce the problem to the question of existence for subalgebras of $\mathfrak{gl}(V, V_*)$ satisfying (C1) or (C2) and such that their adjoint representation is not locally finite.

V. The structure of $\mathfrak{g}'$

In this section $\mathfrak{t}$ is a fixed maximal toral subalgebra of $\mathfrak{g} = \mathfrak{gl}(V, V_*)$, $\mathfrak{h} := \mathfrak{z}_\mathfrak{g}(\mathfrak{t}) = \mathfrak{t} \oplus (V^0 \otimes V_*^0)$, and the superscript ' indicates maximal locally finite $\mathfrak{t}$-submodule.

Lemma V.1. *The subalgebra $\mathfrak{h} \subseteq \mathfrak{g}'$ is a maximal toral Cartan subalgebra of $\mathfrak{g}'$ with $\mathfrak{z}(\mathfrak{g}') = V^0 \otimes V_*^0$. Furthermore, the decomposition $\mathfrak{g}' = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha)$ is a root decomposition of $\mathfrak{g}'$ with respect to $\mathfrak{h}$.*

Proof. The fact that the decomposition $\mathfrak{g}' = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha)$ is an $\mathfrak{h}$-root decomposition is clear from the very definition of this decomposition and from the equality $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$ (Proposition III.3).

The equality

$$[V^0 \otimes V_*^0, V^\alpha \otimes V_*^\beta] = \{0\}$$

for all α, β (Proposition II.3) implies that $V^0 \otimes V_*^0 \subseteq \mathfrak{z}(\mathfrak{g}')$. Conversely, the inclusion $\mathfrak{z}(\mathfrak{g}') \subseteq \mathfrak{z}_\mathfrak{g}(\mathfrak{t}) = \mathfrak{h}$ implies easily that $\mathfrak{z}(\mathfrak{g}')$ is not larger than $V^0 \otimes V_*^0$. Furthermore, the equality $\mathfrak{h} = \mathfrak{t} + \mathfrak{z}(\mathfrak{g}')$ shows that the adjoint action on $\mathfrak{g}'$ of every element $x \in \mathfrak{h}$ is diagonalizable, i.e., that $\mathfrak{h}$ is a toral Cartan subalgebra of $\mathfrak{g}'$. The maximality of $\mathfrak{g}$ is an immediate corollary of the equallity $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$. ■

As a consequence of Lemma V.1, the following theorem ([St99, Th. I.4]) applies to the pair $(\mathfrak{g}', \mathfrak{h})$. If $\mathfrak{k}$ is a locally finite Lie algebra which admits a root decomposition with respect to some subalgebra $\mathfrak{h}_\mathfrak{k}$, we call a root α *integrable* if the subalgebra of $\mathfrak{k}$ generated by the root spaces $\mathfrak{k}_{\pm\alpha}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{K})$.

Theorem V.2. (Levi decomposition of locally finite split Lie algebras) *Let $\mathfrak{k}$ be a locally finite Lie algebra with root decomposition $\mathfrak{k} = \mathfrak{h}_\mathfrak{k} \oplus (\bigoplus_{\alpha\in\Delta(\mathfrak{k})} \mathfrak{k}_\alpha)$ with respect to a toral Cartan subalgebra $\mathfrak{h}_\mathfrak{k}$. Denote the set of integrable root by $\Delta_i(\mathfrak{k}) \subseteq \Delta(\mathfrak{k})$.*
(1) The subspace $\mathfrak{s} = \operatorname{span} \check{\Delta}_i(\mathfrak{k}) + \bigoplus_{\alpha\in\Delta_i(\mathfrak{k})} \mathfrak{k}_\alpha$ is a semisimple subalgebra of $\mathfrak{k}$.
(2) Let $\Delta_n(\mathfrak{k}) := \Delta(\mathfrak{k}) \setminus \Delta_i(\mathfrak{k})$. The subspace $\mathfrak{r} := \mathfrak{z}_{\mathfrak{h}_\mathfrak{k}}(\mathfrak{s}) + \bigoplus_{\alpha\in\Delta_n(\mathfrak{k})} \mathfrak{k}_\alpha$ is the unique maximal locally solvable ideal of $\mathfrak{k}$, and $\mathfrak{u} := \mathfrak{z}(\mathfrak{k}) + \bigoplus_{\alpha\in\Delta_n(\mathfrak{k})} \mathfrak{k}_\alpha$ is the unique maximal locally nilpotent ideal of $\mathfrak{k}$.
(3) If $\mathfrak{a}$ is a vector space complement to the subspace $\mathfrak{z}(\mathfrak{k}) + \operatorname{span} \check{\Delta}_i(\mathfrak{k})$ in $\mathfrak{h}_\mathfrak{k}$, we have $\mathfrak{k} \cong \mathfrak{u} \rtimes (\mathfrak{s} \rtimes \mathfrak{a})$, where the Lie algebra $\mathfrak{l} := \mathfrak{s} \rtimes \mathfrak{a}$ is almost reductive, i.e., has a semisimple commutator algebra. ■

Lemma V.3. *The subspaces $V' \otimes V_*^0$ and $V^0 \otimes V_*'$ are abelian ideals of $\mathfrak{g}'$ with*

$$[V' \otimes V_*^0, V^0 \otimes V_*'] = V^0 \otimes V_*^0 = \mathfrak{z}(\mathfrak{g}').$$

Proof. The statement follows from the equalities

$$[V' \otimes V_*', V^0 \otimes V_*'] = (V^0 \otimes V_*') \cdot (V' \otimes V_*') = V^0 \otimes V_*',$$

$$[V' \otimes V_*', V' \otimes V_*^0] = (V' \otimes V_*') \cdot (V' \otimes V_*^0) = V' \otimes V_*^0$$

and

$$[V^0 \otimes V_*', V^0 \otimes V_*'] = \{0\} = [V' \otimes V_*^0, V' \otimes V_*^0],$$

which in turn follow from the equality $V_*^0(V^0) = \{0\}$. ■

Proposition V.4. *If $0 \neq \alpha, \delta \in \mathrm{supp}\, V$, then the functional $\alpha - \delta \in \mathfrak{h}^*$ is an integrable root of $\mathfrak{g}'$ with*

$$\mathfrak{g}'_{\alpha-\delta} = V^\alpha \otimes V_*^{-\delta}.$$

If $V_*^0 \neq \{0\}$, then a functional $0 \neq \alpha \in \mathrm{supp}\, V$ is a non-integrable root with

$$\mathfrak{g}'_\alpha = V^\alpha \otimes V'_*,$$

and if $V^0 \neq \{0\}$, a functional $0 \neq -\alpha \in \mathrm{supp}\, V_*$ is a non-integrable root with

$$\mathfrak{g}'_{-\alpha} = V^0 \otimes V_*^{-\alpha}.$$

The space

$$\mathfrak{u} := V' \otimes V_*^0 + V^0 \otimes V_* = \bigoplus_{0 \neq \alpha \in \mathrm{supp}\, V} (V^0 \otimes V_*^{-\alpha} + V^\alpha \otimes V_*^0)$$

is the maximal locally nilpotent ideal of $\mathfrak{g}'$.

Proof. For $0 \neq \alpha, \delta \in \mathrm{supp}\, V$ we have a root $\alpha - \delta \in \Delta$ with $\mathfrak{g}'_{\alpha-\delta} \supseteq V^\alpha \otimes V_*^{-\delta}$. Let $0 \neq f_\alpha \in V^\alpha$, $\alpha \neq 0$. Define $f_\alpha^* \in V_*^{-\alpha}$ by $f_\alpha^*(f_\alpha) = 1$. Then

$$h_{\alpha,\delta} := [f_\alpha \otimes f_\delta^*, f_\delta \otimes f_\alpha^*] = f_\alpha \otimes f_\alpha^* - f_\delta \otimes f_\delta^*$$

satisfies $\alpha(h_{\alpha,\delta}) = 1$ and $\delta(h_{\alpha,\delta}) = -1$. Therefore the roots $\alpha - \delta$ are integrable, which implies in particular that $\mathfrak{g}'_{\alpha-\delta}$ is one-dimensional, so that $\mathfrak{g}'_{\alpha-\delta} = V^\alpha \otimes V^\delta$.

Furthermore, Lemma V.3 implies that the root spaces $\mathfrak{g}'_\alpha = V^\alpha \otimes V_*^0 + V^0 \otimes V_*^{-\alpha}$, for $0 \neq \alpha \in \mathrm{supp}\, V$, are contained in the maximal locally nilpotent ideal of $\mathfrak{g}'$.

The remaining assertions are direct consequences of Theorem V.2. ∎

The following theorem is a direct corollary of Theorem V.2 via the information provided by Lemma V.3 and Proposition V.4.

Theorem V.5. (Structure Theorem for $\mathfrak{g}'$) *The Lie algebra $\mathfrak{g}'$ is isomorphic to the semidirect product $\mathfrak{u} \rtimes \mathfrak{l}$, where*

$$\mathfrak{u} := V^0 \otimes V'_* + V' \otimes V_*^0$$

is the Lie algebra with bracket

$$[(v \otimes \varphi), (y \otimes \psi)] = \varphi(y) \cdot v \otimes v^*,$$

and $\mathfrak{l} \cong W' \otimes W_* = \mathfrak{gl}(W, W_*)$ for $W := \mathfrak{t} \cdot V$ and $W_* := \mathfrak{t} \cdot V_*$. ∎

Appendix. A useful general proposition

The following proposition was communicated to us by I. Dimitrov and G. Zuckerman and is a generalization version of a lemma which we had proved for the specific purposes of the present paper.

Let U and W be vector spaces. To any element $x \in U \otimes W$ we assign a subspace $U_x \subset U$ in the following way. Write x as $\sum_j u_j \otimes w_j$ with linearly independent $w_j \in W$, and set $U_x := \mathrm{span}\{u_i\}$. In a similar way we assign to x a subspace $W_x \subset W$. To check that U_x (and similarly W_x) does not depend on the presentation of x as $\sum_j u_j \otimes w_j$, it suffices to identify U_x with the image of the linear operator $\psi(x) \in \mathrm{Hom}(W^*, U)$, where ψ is the canonical inclusion

$$U \otimes W \hookrightarrow \mathrm{Hom}(W^*, U), \quad \psi(u \otimes w)(\alpha) := \alpha(w)u.$$

This is a straightforward checking which we omit.

Clearly, $\dim U_x < \infty$ as $\psi(x)$ is an operator of finite rank. Note also that for any subspace $Y \subset U$ we have

$$Y \otimes W = \{x \in U \otimes W : U_x \subset Y\},$$

and similarly for any $z \in W$,

$$U \otimes Z = \{x \in U \otimes W : W_x \subset Z\}.$$

Let now $\mathfrak{k}$ be a Lie algebra. For any $\mathfrak{k}$-module Q, we denote by Q' the maximal locally finite $\mathfrak{k}$-submodule of Q.

Proposition A. *For any $\mathfrak{k}$-modules M and N, we have*

$$M' \otimes N' = (M \otimes N)'.$$

Proof. The inclusion $M' \otimes N' \subset (M \otimes N)'$ is obvious.

Fix $0 \neq x \in (M \otimes N)'$ and a basis $x_1, \ldots, x_n$ of $U(\mathfrak{k}) \cdot x$ with $x_1 = x$. Set $Y := M_{x_1} + \ldots + M_{x_n}$ and $Z := N_{x_1} + \ldots + N_{x_n}$. Since $x \in M_x \otimes N_x \subset Y \otimes Z$, it suffices to prove that $Y \subset M'$ and $Z \subset N'$. We will show that $Z \subset N'$ (the argument for Y is completely similar), which will follows from $U(\mathfrak{k}) \cdot Z \subset Z$. For this it is enough to verify that $k \cdot N_{x_i} \subset Z$ for any i and any $k \in \mathfrak{k}$.

Fix k and i and write x_i as $\sum_j m_j \otimes n_j$ with linearly independent $m_j \in M$. Then $k \cdot x_i = \sum_j k \cdot m_j \otimes n_j + \sum_j m_j \otimes k \cdot n_j$. Since $k \cdot x_i \in Y \otimes Z$ and $\sum_j k \cdot m_j \otimes n_j \in M \otimes Z$, we have $\sum_j m_j \otimes k \cdot n_j \in M \otimes Z$. Therefore $N_{\sum_j m_j \otimes k \cdot n_j} \subset Z$, i.e., $k \cdot n_j \in Z$. As the n_j generate N_{x_i}, this implies $k \cdot N_{x_i} \in Z$. ∎

References

[AABGP97] B. Allison, S. Azam, S. Berman, Y. Gao, and A. Pianzola, "Extended affine Lie algebras and their root systems,", Memoirs Amer. Math. Soc. **603**, 1997.

[BP95] Y. Billig and A. Pianzola, *On Cartan subalgebras*, Journ. Algebra **171** (1995), 397–412.

[Bou90] N. Bourbaki, "Groupes et algèbres de Lie, Chapitres VII-VIII," Masson, Paris, 1990.

[DP99] I. Dimitrov and I. Penkov, *Weight modules of direct limit Lie algebras*, Internat. Math. Res. Notices **5** (1999), 223–249.

[NS01] K.-H. Neeb and N. Stumme, *On the classification of locally finite split simple Lie algebras*, Journ. reine u. angew. Math. **533** (2001), 25–53.

[PS03] I. Penkov, and H. Strade, *Locally finite Lie algebras with root decomposition*, Arch. der Math. (2003), to appear.

[PK83] D. Peterson, and V. Kac, *Infinite flag varieties and conjugacy theorems*, Proc. Nat. Acad. Sci. USA **80** (1983), 1778-1782.

[St99] N. Stumme, *The structure of locally finite split Lie algebras*, Journ. Algebra **220** (1999), 664–693.

[St01] N. Stumme, *Automorphisms and conjugacy of compact real forms of the classical infinite-dimensional matrix Lie algebras*, Forum Math. **13** (2001), 817–851.

K.-H. N.:
Fachbereich Mathematik AG5
Schlossgartenstr. 7
D-64289 Darmstadt, Germany
e-mail: neeb@mathematik.tu-darmstadt.de

I.P.:
Department of Mathematics,
University of Califronia at Riverside
Riverside, CA 92521, USA
e-mail: penkov@math.ucr.edu