Parameterized Plane Curves, Minkowski Caustics, Minkowski Vertices and Conservative Line Fields

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0. Introduction

Some time ago I made the following observation ([T1]) generalizing the classical 4-vertex theorem. Consider a smooth closed strictly convex parameterized curve $\gamma(t)$ in the oriented affine plane. The acceleration vectors $\gamma''(t)$ (where prime denotes d/dt) generate a smooth line field $l(t)$ along the curve. Assume that these lines rotate in the same sense along γ; analytically this means that $[\gamma''(t), \gamma'''(t)] \neq 0$ for all t (where $[\ ,\ ]$ denotes the determinant of two vectors).

Theorem 0.1. *For a generic curve $\gamma(t)$ the envelope of the one-parameter family of lines $l(t)$ has at least 4 cusp singularities.*

If $\gamma(t)$ is a strictly convex curve in the arc-length parameterization then the lines $l(t)$ are perpendicular to γ and their envelope is the caustic of the curve. The singularities of the caustic correspond to the vertices of the cuvre, i.e., to its curvature extrema. Thus Theorem 0.1 is a generalization of the 4-vertex theorem which asserts that a smooth closed convex plane curve has at least 4 vertices.

The trick used in [T1] to prove the theorem does not explain its relation to concepts of differential geometry, in particular, whether Theorem 0.1 can be interpreted as a 4-vertex theorem. The purpose of this paper is to provide such an explanation. I will show that Theorem 0.1 is a 4-vertex theorem in Minkowski geometry in the plane associated with the parameterized curve $\gamma(t)$.

An appropriate point of view is that of contact geometry which makes it possible to naturally extend many a familiar results from the Euclidean setting to the more general Minkowski and Finsler ones. For an approach to the 4-vertex theorem and related results as theorems of symplectic and contact topology see, e.g., [A 1, A 4].

1. Finsler metric from the contact geometrical viewpoint

Finsler geometry describes the propagation of light in an inhomogeneous anisotropic medium. This means that the velocity of light depends on the point and the direction. There are two equivalent descriptions of this process corresponding to the Lagrangian and the Hamiltonian approaches in classical mechanics.

On the one hand, one may study the rays of light, that is, the shortest paths between points. The optical properties of a medium are described by a strictly convex smooth hypersurface, called the *indicatrix*, in the tangent space at each point. The indicatrix consists of the velocity vectors of the propagation of light at a point in all directions. It plays the role of the unit sphere in Riemannian geometry.

The distance $d(x, y)$ between points x and y is the least time it takes light to travel from x to y. If the indicatrices are not centrally symmetric this distance may be not symmetric: $d(x, y) \neq d(y, x)$. However it still satisfies the triangle inequality:

$$d(x, y) + d(y, z) \geq d(x, z).$$

Minkowski geometry is a particular case of Finsler geometry in affine space in which the indicatrices of all points are identified by parallel translations. The rays of light in Minkowski geometry are straight lines.

On the other hand, one may study the wave fronts. The wave front of a point is the hypersurface that consists of points which light can reach from the given point in a fixed time. A wave front is characterized by its contact elements (hyperplanes in the tangent spaces at the points of the front tangent to it) cooriented by the direction of the time evolution of the front. This evolution is described by a vector field in the space of all cooriented contact elements.

We recall in this section (without proofs) the relevant facts from symplectic and contact geometry – see [A 2, A 3].

Let M^n be a smooth manifold and $\pi : T^*M \to M$ its cotangent bundle. When the need be one introduces local coordinates in T^*M

$$(q, p) = (q_1, ..., q_n, p_1, ..., p_n),$$

where q are position coordinates in M and p are the corresponding momenta coordinates in the fibers of the cotangent bundle. Denote by λ_0 the Liouville differential 1-form on T^*M. The value of λ_0 on a tangent vector v to T^*M at point (x, θ), where $x \in M, \theta \in T_x^*M$, is, by definition, $\theta(d\pi(v))$. In coordinates, $\lambda_0 = pdq$ $(= \sum p_i dq_i)$. The 2-form $d\lambda_0$ is the canonical symplectic form in T^*M.

The space of cooriented contact elements is the spherization ST^*M of the cotangent bundle. Consider the principle $\mathbf{R}_+^*$ - bundle

$$p : T^*M - M \to ST^*M$$

($T^*M - M$ is the complement to the zero section); its fiber over a cooriented contact element consists of the linear functionals vanishing on this contact element and positive on its positive side. The codimension 1 distribution $Ker\ \lambda_0$ on $T^*M - M$ projects to the canonical contact structure in ST^*M.

Finsler metric on M is determined by a (Hamiltonian) function H on T^*M. This function satisfies the following assumptions:

1). H is a nonnegative function, homogeneous of degree 1 in momenta, i.e., $H(q, tp) = tH(q, p)$ for all $t > 0$;

2). The level hypersurface $S = H^{-1}(1)$ is fiberwise star-shaped, i.e., each intersection $S_x = S \cap T_x^*M$, $x \in M$, transversely intersects every ray from the origin in the linear space T_x^*M.

3). The level hypersurface $S = H^{-1}(1)$ is fiberwise quadratically convex, i.e., each intersection $S_x = S \cap T_x^*M$, $x \in M$, is quadratically convex in the linear space T_x^*M.

The surface S_x is sometimes called the *figuratrix*. It may be thought of as the set of "unit covectors" in T_x^*M.

Denote by ξ the Hamiltonian vector field of H, that is the field such that $i_\xi d\lambda_0 = dH$. In local coordinates,

$$\xi = H_p\, \partial/\partial q - H_q\, \partial/\partial p.$$

The field ξ is tangent to the hypersurface S. Let ϕ_t be the time-t map of the flow ξ. Denote by λ the restriction of the Liouville form to S.

Theorem 1.1. *Let the Hamilton function satisfy the above conditions 1)–2). Then:*
a). The form λ is a contact form, that is, $\lambda \wedge (d\lambda)^{n-1} \neq 0$ everywhere on S.
b). The field ξ is the characteristic vector field of the form λ, that is, $i_\xi d\lambda = 0$, $\lambda(\xi) = 1$, and the flow ϕ_t preserves the form λ for all t.

The hypersurface S being fiberwise star-shaped, it is identified with ST^*M, and the contact form λ determines the canonical contact structure in ST^*M. Conversely, a contact form λ for the canonical contact structure in ST^*M is a section ϕ of the bundle $p : T^*M - M \to ST^*M$ such that $\phi^*\lambda_0 = \lambda$. The image of this section is a fiberwise star-shaped hypersurface $S \subset T^*M$, and one can reconstruct the homogeneous Hamilton function H by $S = H^{-1}(1)$.

The one-parameter group ϕ_t describes the time evolution of cooriented contact elements of M mentioned at the beginning of the section. This flow will be referred to as the *geodesic flow* in the space of cooriented contact elements.

Example. Let M be a Riemannian manifold and $H(q,p) = |p|$. Then ξ is the usual geodesic flow: each coorented contact element moves with the unit speed in its positive normal direction.

We assume that the figuratrices S_x are quadratically convex. The indicatrix I_x at point $x \in M$ consists of the velocity vectors of the foot points of the contact elements in S_x under the flow ξ. That is,

$$I_x = \{d\pi(\xi(x,\theta))\}, \quad \theta \in S_x \subset T_x^*M.$$

Definition. Let X be a smooth strictly convex star-shaped hypersurface in vector space V. For every $x \in X$ there exists a unique functional $y \in V^*$ such that $y(x) = 1$ and $\operatorname{Ker} y = T_xX$. The set of such functionals for all $x \in X$ is called the *dual hypersurface* and is denoted by X^*.

Note that X^* is strictly convex and star-shaped too; note also that $(X^*)^* = X$.

Theorem 1.2. *The indicatrix I_x and the figuratrix S_x are dual to each other for every $x \in M$.*

To the field of indicatrices a (Lagrangian) function L on the tangent bundle TM corresponds: this function is homogeneous of degree 1 in tangent vectors, and $L^{-1}(1) \cap T_xM = I_x$ for all $x \in M$. This function gives the length of a tangent vector in Finsler geometry. Trajectories of light in Finsler geometry are the extremals of the functional $\int L(q,\dot{q})\, dt$.

Theorem 1.3. *These extremals are the projections to M of the trajectories of the vector field ξ.*

Thus the Hamiltonian vector field ξ of the Hamiltonian function H describes the propagation of light in an inhomogeneous anisotropic medium. In the case of Minkowski geometry H depends on the momenta variables only. The trajectories of light in Minkowski geometry are straight lines, and the indicatrix is identified with the time-1 front of the origin. The cooriented contact elements of this front are the time-1 images in the geodesic flow of all contact elements at the origin.

Let $N \subset \mathbf{R}^n$ be a cooriented hypersurface in Minkowski space. The geodesic flow trajectories of the foot points of the cooriented contact elements of N will be called (Minkowski) *normals* of N. Note that the normals may change if the coorientation of N is reversed. The reader interested in differential geometry of Finsler manifolds is referred to [Ru], and to [Bu, Gu] for the case of Minkowski geometry.

2. Minkowski geometry associated with a parameterized curve

Return to the situation of Introduction: $\gamma(t)$ is a smooth closed strictly convex parameterized plane curve satisfying the condition $[\gamma''(t), \gamma'''(t)] \neq 0$ for all t. The lines $l(t)$ generated by the acceleration vectors $\gamma''(t)$ constitute a smooth transverse line field along $\gamma(t)$. The condition $[\gamma''(t), \gamma'''(t)] \neq 0$ ensures that infinitesimally close lines from the family $l(t)$ intersect, therefore their envelope is bounded.

Give γ the inward coorientation. Then γ determines a curve $\tilde{\gamma}$ in the space of cooriented contact elements of the plane. The curve $\tilde{\gamma}$ is Legendrian, that is, tangent to the contact structure in the space of cooriented contact elements.

Theorem 2.1. *There exists a unique, up to a multiplicative constant, Minkowski metric in the plane such that the lines $l(t)$ are the Minkowski normals of the cooriented curve γ.*

Proof. Identify the tangent planes at different points with $\mathbf{R}^2$ by parallel translations. Consider the curve $S(t) = \gamma'(t) \subset \mathbf{R}^2$. Since γ is strictly convex, S is star-shaped. Moreover, $S'(t) = \gamma''(t)$, therefore $[S'(t), S''(t)] \neq 0$ for all t. Thus the curve S is strictly convex.

Assume that the curve $\gamma(t)$ is oriented counterclockwise. Identify the tangent and cotangent planes by the bilinear form $[\ ,\ ]$: a vector v is considered as the covector $[v,\ ]$. Then one may consider S as a curve in the dual plane $(\mathbf{R}^2)^*$.

Let H be the homogeneous of degree 1 function in $(\mathbf{R}^2)^*$ whose level curve $H^{-1}(1)$ is S. Consider H as a function on $T^*\mathbf{R}^2$ depending on the momenta only, and let ξ be its Hamiltonian vector field. We claim that the cooriented contact element of the curve γ at point $\gamma(t)$ is translated by the field ξ in the direction of the line $l(t)$. The desired Minkowski metric is determined then by the Hamiltonian function H, as explained in the previous section.

To start with, the trajectories of ξ project to straight lines; in local coordinates, $\xi = H_p\ \partial/\partial q$. Since $H(\gamma'(t)) = 1$, one has $dH(\gamma''(t)) = 0$ (here γ' and γ'' are considered as vectors in $(\mathbf{R}^2)^*$). The differential $dH \in ((\mathbf{R}^2)^*)^* = (\mathbf{R}^2)$, and, as a vector in $(\mathbf{R}^2)$, this is $H_p\ \partial/\partial q$. In view of the chosen identification of $(\mathbf{R}^2)^*$ with $(\mathbf{R}^2)$, the equality

$dH(\gamma'') = 0$ reads $[H_p\ \partial/\partial q, \gamma''] = 0$ in $(\mathbf{R}^2)$. Thus $H_p\ \partial/\partial q$ is collinear with γ'' at every point of the curve.

Conversely, the same argument shows that if the trajectories of the Hamiltonian field ξ project to the lines $l(t)$ then H is constant on the curve $S(t) = \gamma'(t)$. This, along with the homogenuity, determines H, and therefore the Minkowski metric, up to a multiplicative constant.

Remark. Suppose a smooth strictly convex closed nonparameterized plane curve γ is given. Then a choice of a homogeneous function H in $(\mathbf{R}^2)^*$, such that $S = H^{-1}(1)$ is star-shaped, determines a parameterization $\gamma(t)$ with the property that the trajectories of the Hamiltonian vector field ξ project to the lines generated by the vectors $\gamma''(t)$ along γ. Considering S as a curve in $\mathbf{R}^2$, this parameterization is defined by the requirement: $\gamma'(t) \in S$ for all t.

To describe the indicatrix of the Minkowski geometry constructed in the above theorem one needs the following lemma. As before, we identify the tangent and the cotangent planes by the bilinear form $[\ ,\ ]$.

Lemma 2.2. *Let $S(t)$ be a parameterized strictly convex star-shaped curve in $(\mathbf{R}^2)^*$. Then the dual curve is $S^*(t) = S'(t)/[S(t), S'(t)]$.*

Proof. By definition, the dual curve consists of the vectors $S^*(t) \in \mathbf{R}^2$ such that $< S(t), S^*(t) >= 1$ and $< S'(t), S^*(t) >= 0$. Clearly, the curve $S^*(t) = S'(t)/[S(t), S'(t)]$ satisfies both equalities, and Lemma follows.

This lemma, applied to the curve $S(t) = \gamma'(t)$, along with Theorem 1.2, implies the formula for the indicatrix:

$$I(t) = \gamma''(t)/[\gamma'(t), \gamma''(t)].$$

This formula gives the plane projection of the velocity vector of the cooriented contact element of the curve γ at point $\gamma(t)$ in the geodesic flow. Notice that the original parameterization $\gamma(t)$ is not, in general, by arc-length in the constructed Minkowski geometry.

Next, we give some explicit formulas in Euclidean terms. Let $\alpha(t)$ be the angle made by the tangent vector $\gamma(t)$ with a fixed direction. Set $f(\alpha(t)) = \log|\gamma'(t)|$; then

$$\gamma'(t) = e^{f(\alpha(t))}\ (\cos\alpha(t),\ \sin\alpha(t)).$$

The plane projection of the vector of the geodesic flow at point $\gamma(t)$ is given by the formula

$$e^{-f(\alpha(t))}\ (f'(\alpha(t))\cos\alpha(t) - \sin\alpha(t),\ f'(\alpha(t))\sin\alpha(t) + \cos\alpha(t)),$$

where prime means $d/d\alpha$. The function $f(\alpha)$ determines the Minkowski metric. The convexity condition for the indicatrix reads: $1 + (f')^2 > f''$. If $\gamma(t)$ is arc-length parametrized then $f(\alpha) = 1$ identically, and the Minkowski metric is the Euclidean one.

3. Osculating indicatrices and Minkowski caustic

Consider the curve $J \subset \mathbf{R}^2$ centrally symmetric to the indicatrix I with respect to the origin, and coorient it inwards. Since I is the time-1 front of the origin, the time-1 map

ϕ_1 of the geodesic flow takes the foot points of all the cooriented contact elements of J to the origin. If the curve J is a source of light in our anisotropic Minkowski plane then light from all points of J focuses at the origin in unit time.

Let γ be a nonparametrized closed strictly convex curve in Minkowski plane, cooriented inwards. For every point $x \in \gamma$ there exists a unique curve $J(x)$, homothetic to J (that is, obtained from J by a dilation with a positive coefficient and a parallel translation) which is second order tangent to γ at x.

Definitions. Call $J(x)$ the *osculating indicatrix* of γ at x. The coefficient $r(x)$ of the dilation that takes J to $J(x)$ is called the (Minkowski) *curvature radius* of γ at x. The center of $J(x)$, i.e., the image of the origin under the homothety that takes J to $J(x)$, is called the (Minkowski) *center of curvature* of γ at x. A point $x \in \gamma$ is called (Minkowski) *vertex* if the osculating indicatrix is third order tangent to γ at x. Call the envelope Γ of the Minkowski normals to γ its (Minkowski) *caustic*.

Remark. The curvature radius at $x \in \gamma$ is the focusing time for light, propagating from a small piece of γ around x in the direction of the coorientation. This time is positive if the coorientation vectors point to the convex side of the curve, and negative otherwise.

If the metric is Euclidean all these notions coincide with the usual ones, e.g., the osculating indicatrix is the osculating circle, etc. We list below a number of properties of osculating circles and Euclidean caustics subject to a generalization in the Minkowski setting.

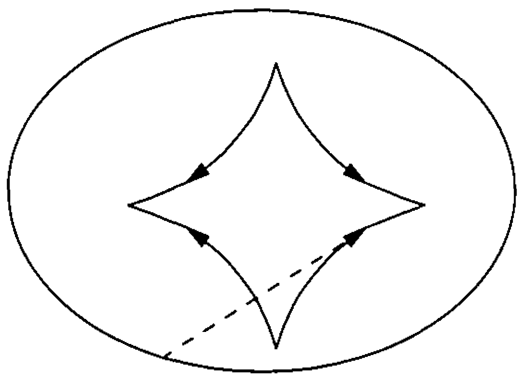

Figure 1

1). The caustic of a curve is the locus of its centers of curvature.

2). A vertex of a curve corresponds to a singularity of its caustic.

3). A vertex is an extremum of the curvature radius.

4). The caustic of a generic curve is a piecewise smooth curve with an even number of cusps and without inflection points.

5). If a caustic is bounded then the alternating sum of the lengths of its smooth pieces equals zero.

6). A curve γ is described by the free end of a stretched string developing from its caustic Γ.

7). (Kneser's theorem). The osculating circles of an arc of a curve, free from vertices, are pairwise disjoint and lie one inside the other.

In the case of Minkowski geometry these properties still make sense (using the above definitions) except for 5) and 6) which require an explanation because Minkowski length of a curve depends on its orientation.

Give the normals of γ the inward orientation; then every smooth piece of Γ gets an orientation too. The length of a smooth oriented piece of Γ is understood to be its length in Minkowski geometry. In this way property 5) makes sense – see Figure 1.

To explain property 6) consider a smooth arc of the caustic, oriented as above, and let A and B be two its points such that A precedes B on the arc. Consider the tangent segments to Γ at A and B which are normals to γ, oriented "from γ". Let r and R be their respective Minkowski lengths and L be the Minkowski length of the arc AB of the caustic. Property 6) asserts that $R - r = L$ – see Figure 2.

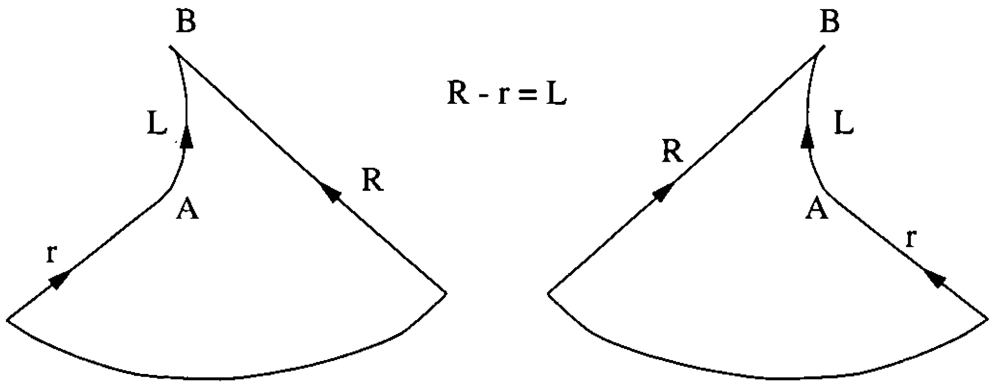

Figure 2

The next theorem is, possibly, not new (at least some of its statements) – cf. [Gu].

Theorem 3.1. *The properties 1)–7) hold true in the Minkowski setting.*

Proof. As before, H denotes the Hamiltonian function associated with the Minkowski metric, $S = H^{-1}(1) \subset T^*\mathbf{R}^2$ and $\pi : S \to \mathbf{R}^2$ is the projection. Let $\tilde{\gamma}$ be the lift of the cooriented curve γ to S (considered as the space of cooriented contact elements of the plane), and $Z \subset S$ be the cylinder that consists of the trajectories of the Hamiltonian vector field ξ through $\tilde{\gamma}$. Denote by $\tilde{\Gamma} \subset Z$ the curve consisting of points at which the rank of the projection $\pi|_Z$ is less than 2. Thus $\tilde{\Gamma}$ is the set of points at which the fibers of π are tangent to Z. Since the trajectories of ξ project diffeomorphically to the plane the rank of $\pi|_Z$ equals 1 along $\tilde{\Gamma}$. The curve $\tilde{\Gamma}$ projects to the caustic Γ.

To prove property 1), consider the osculating indicatrix $J(x)$ at $x \in \gamma$, cooriented inwards. Then $\tilde{J}(x) \subset S$ is tangent to $\tilde{\gamma}$ at point $\tilde{x}$, the cooriented contact element of γ at x. Let $r(x)$ be the curvature radius of γ at x. Then $\phi_{r(x)}(\tilde{J}(x))$ is a fiber of π. Therefore a fiber of π is tangent to the curve $\phi_{r(x)}(\tilde{\gamma}) \subset Z$, and hence $\phi_{r(x)}(\tilde{x}) \in \tilde{\Gamma}$. It remains to note that $\pi(\phi_{r(x)}(\tilde{x}))$ is the center of curvature of γ at x.

Likewise, if $x \in \gamma$ is a vertex then $\tilde{\Gamma}$ is tangent to the curve $\phi_{r(x)}(\tilde{J}(x))$ at point $\phi_{r(x)}(\bar{x})$. Therefore $\tilde{\Gamma}$ is tangent to a fiber of π, so Γ has a singularity at the respective center of curvature. Property 2) follows. It follows also that the singularities of the caustic are the singularities of the projection $\pi : \tilde{\Gamma} \to \Gamma$; the curve $\tilde{\Gamma}$ is smooth.

Next, note that an orientation of γ gives Γ a coorientation. Give Γ an orientation; then the pair (orienting vector, coorienting vector) is either a positive or a negative frame along each smooth piece of Γ. The positive and negative pieces alternate, so the number of cusps is even.

Consider the space of oriented lines in the plane (topologically, the cylinder); the tangent lines to the caustic constitute a curve σ in this space. The family of Minkowski normals to γ being smooth, the curve σ is smooth as well. An inflection of Γ would correspond to a singularity of σ. Thus Γ is inflection free, and property 4 follows. Note that an inflection of Γ corresponds to the tangency between $\tilde{\Gamma}$ and a trajectory of the field ξ. Therefore $\tilde{\Gamma}$ is transverse to ξ.

Vertices correspond to the stationary osculating circles, therefore they are extrema of the curvature radius. Conversely, consider a critical value of the curvature radius at $x \in \gamma$, and assume that the caustic is smooth at the corresponding curvature center. Then the direction of Γ is parallel to the tangent line to γ at x. However the tangent line to Γ is the Minkowski normal to γ at x which is transverse to γ. Property 3 follows.

One may use the Minkowski length of the tangent segment to Γ from γ, that is, the curvature radius r, as a local parameter on a smooth oriented piece of the caustic. The velocity vector $\partial\Gamma/\partial r$ at a point of Γ is the projection under $d\pi$ of the vector ξ at the corresponding point of $\tilde{\Gamma}$. Therefore the vector $\partial\Gamma/\partial r$ belongs to the indicatrix, and the parameterization $\Gamma(r)$ is by arc-length. Property 6 follows. Property 5 is obtained from 6 by summation over smooth pieces of the caustic.

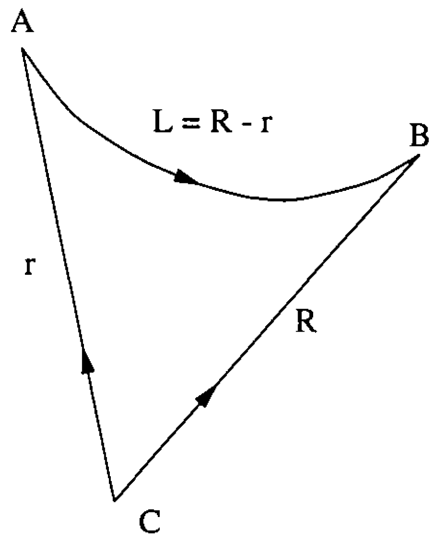

Figure 3

Equivalently, the argument from the preceding paragraph means that the Minkowski length of a smooth arc δ of the caustic, oriented as above, equals the integral of the contact form λ over the lifted arc $\tilde{\delta} \subset \tilde{\Gamma}$. Likewise, r and R are the respective integrals of λ over the trajectory segments of the field ξ. Since $i_\xi d\lambda = 0$, the integral of $d\lambda$ over the quadrilaterals in Z, bounded by the trajectories of ξ and the curves $\tilde{\delta}$ and $\tilde{\gamma}$, vanishes. Applying the Stokes theorem and taking into account that $\lambda = 0$ on $\tilde{\gamma}$, the equality $L - R + r = 0$ follows.

To prove property 7), the Kneser theorem, assume that two osculating indicatrices intersect at some point C. Let A and B be the respective centers of curvature such that A precedes B on the oriented smooth piece of the caustic, and let r and R be the corresponding curvature radii. Then the length of the oriented segments CA and CB equal r and R, respectively. By property 6) the Minkowski length of the arc AB equals $R - r$, and this violates the triangle inequality – see Figure 3.

Remark. The definitions given at the beginning of this section extend to complete Finsler metrics without conjugate points. Properties 1) – 7) hold in this case as well, and the proof goes through without change.

Returning to the situation of Introduction one sees that Theorem 0.1 is the 4-vertex theorem in the Minkowski geometry associated with a parametrized curve (as explained in Section 2). In particular, the envelope of the lines $l(t)$ is the Minkowski caustic. The explicit formulas are as follows. The caustic is

$$\Gamma(t) = \gamma(t) + \frac{[\gamma'(t), \gamma''(t)]}{[\gamma''(t), \gamma'''(t)]}\, \gamma''(t),$$

and the radius of curvature is

$$r(t) = \frac{[\gamma'(t), \gamma''(t)]^2}{[\gamma''(t), \gamma'''(t)]}.$$

Examples.

1). Let γ be a nonparametrized smooth closed strictly convex plane curve and O be its interior point. Take O as the origin in $\mathbf{R}^2$. There exists a parameterization $\gamma(t)$ such that $[\gamma(t), \gamma'(t)] = 1$ for all t. Then $\gamma''(t)$ is collinear with $\gamma(t)$, and the caustic in the corresponding Minkowski geometry degenerates to the point O. All points of γ are Minkowski vertices, and all osculating indicatrices coincide with the curve itself.

2). Let a parameterization $\gamma(t)$ satisfy $[\gamma'(t), \gamma''(t)] = 1$ for all t (an affine parameter). The indicatrix in the corresponding Minkowski geometry is given by the formula $I(t) = \gamma''(t)$. The lines $l(t)$, generated by the vectors $\gamma''(t)$, are called affine normals of the curve. The line $l(t)$ is tangent to the curve that consists of midpoints of the segments, bounded by the intersections of γ with the lines, parallel to the tangent line to γ at point $\gamma(t)$ – see Figure 4. The envelope of the affine normals is called the affine caustic.

Differentiating the equality $[\gamma'(t), \gamma''(t)] = 1$ one finds: $\gamma'''(t) = -k(t)\ \gamma'(t)$ where the function $k(t)$ is called the affine curvature. The affine curvature is reciprocal to the curvature radius in the corresponding Minkowski geometry. Critical points of the affine curvature are called affine vertices (or sextactic points). A smooth closed convex curve has at least 6 affine vertices (see [Bl 1]); thus a generic affine caustic has at least 6 cusps.

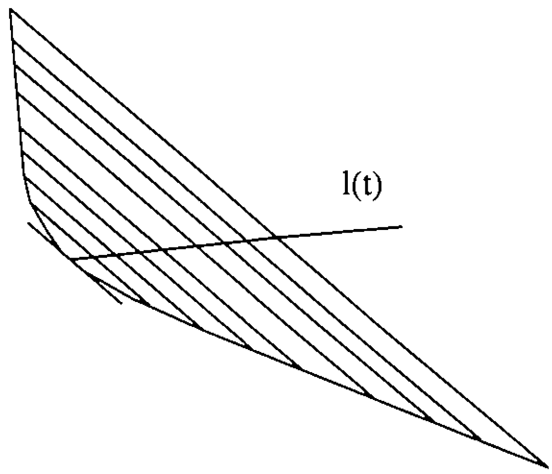

Figure 4

Affine vertices are points of 5-th order contact of the curve with a conic; at an ordinary point the order of contact is one less.

To conclude this section, note that the Minkowski metric gives rise to a symplectic form ω in the space C of oriented lines in the plane. Indeed, C is identified with the space of trajectories of the geodesic flow ξ. Let λ be the contact form in the space of cooriented contact elements associated with the Hamilton function H (see Theorem 1.1). Then the 2-form $d\lambda$ descends to C; this is the symplectic form in question.

The family of Minkowski normals to γ is a curve $\sigma \subset C$. Let $\sigma_0 \subset C$ be the curve that consists of oriented lines through a fixed point x in the plane.

Lemma 3.2. *The ω-area of the region in C between the curves σ and σ_0 equals zero.*

Proof. Denote by $\tilde{\gamma}_0$ the set of cooriented contact elements with the foot point at x. Then $\tilde{\gamma}_0$ is a Legendrian curve. The projections of $\tilde{\gamma}$ and $\tilde{\gamma}_0$ along the trajectories of ξ are the curves σ and σ_0. The area under consideration is the integral of the form $d\lambda$ over a film spanned by $\tilde{\gamma}$ and $\tilde{\gamma}_0$. By the Stokes theorem, this area equals

$$\int_{\tilde{\gamma}} \lambda - \int_{\tilde{\gamma}_0} \lambda = 0$$

since both curves are Legendrian.

In particular, the curves σ and σ_0 intersect at least twice. Therefore at least two Minkowski normals to γ pass through an arbitrary point x in the plane. If the Minkowski metric is associated with a parameterized curve $\gamma(t)$ then the corresponding values of t are the critical points of the function $[\gamma(t) - x, \gamma'(t)]$.

Remark. In the Euclidean case a convex closed curve has at least 2 double normals (chords, perpendicular to the curve at both ends). This is still true in the Minkowski setting, provided the indicatrix is centrally symmetric, but does not seem to hold in general.

4. Minkowski vertices and Chebyshev systems

In this section we give proofs of the 4-vertex theorem in the Minkowski setting, different from the one in [T1]. The arguments used are more-or-less standard in the Euclidean case. We emphasize that the 4-vertex theorem in Minkowski geometry is not new: an equivalent statement can be found, e.g., in [Bl 2].

Let J have the same meaning as in the previous section and $J(t)$ be some parameterization of this curve, $0 \le t \le T$. Let $\gamma(t)$ be a strictly convex closed smooth curve, parameterized so that the tangent vector $\gamma'(t)$ has the same direction as $J'(t)$ for all t. Denote by $r(t)$ the Minkowski curvature radius at point $\gamma(t)$ and by $k(t) = 1/r(t)$ the Minkowski curvature. Fix a linear coordinate system in the plane, and let $(\gamma_1(t), \gamma_2(t))$ be the coordinates of the point $\gamma(t)$.

Lemma 4.1. *The function $k'(t)$ is L_2–orthogonal to the functions $\{1, \gamma_1(t), \gamma_2(t)\}$ on the circle $\mathbf{R}/T\mathbf{Z}$.*

Proof. Clearly, $\int_0^T k'(t)dt = 0$. A curve, homothetic to $J(t)$ with the coefficient $r(t)$, is second order tangent to $\gamma(t)$. Therefore $\gamma'(t) = r(t)J'(t)$. One has:

$$\int_0^T k'(t)\gamma(t)dt = -\int_0^T k(t)\gamma'(t)dt = -\int_0^T J'(t)dt = 0.$$

Thus $k'(t)$ is orthogonal to $\gamma_1(t)$ and $\gamma_2(t)$.

If the Minkowski metric is associated with a parameterized curve $\gamma(t)$, as in Section 2, then the above argument boils down to the easily verified identity:

$$\frac{[\gamma''(t), \gamma'''(t)]}{[\gamma'(t), \gamma''(t)]^2}\,\gamma'(t) = -\Big(\frac{\gamma''(t)}{[\gamma'(t), \gamma''(t)]}\Big)'.$$

Definition. A $(2n+1)$-dimensional space of functions on the circle is called a *Chebyshev system* if every function from this space has at most $2n$ zeroes, multiplicities counted.

The functions $\{1, \gamma_1(t), \gamma_2(t)\}$ constitute a Chebyshev system: zeroes of a fuction $a + b\ \gamma_1(t) + c\ \gamma_2(t)$ are the intersections of the line $a + bx + cy$ with the curve γ, and γ is strictly convex. Since Minkowski vertices of γ are critical points of its Minkowski curvature, the 4-vertex theorem follows from the next result from [A4, G-M-O].

Theorem 4.2. *A function f, orthogonal to a $(2n+1)$-dimensional Chebyshev system on the circle, has at least $2n+2$ distict zeroes.*

Sketch of Proof. Assume f has $2n$ simple roots $x_1, ..., x_{2n}$. There exists a function g in the Chebyshev system with zeroes at $x_1, ..., x_{2n}$. By definition of Chebyshev systems, this function has no other zeroes. Thus the constant sign intervals of f and g coincide, and $\int fg \neq 0$, a contradiction. (The argument adopts to the general case of fewer and, possibly, multiple roots).

Next, we construct the support function of the curve $\gamma(t)$. Let O be the origin in $\mathbf{R}^2$. The tangent lines to γ at point $\gamma(t)$ and to J at point $J(t)$ are parallel; let $p(t)$ be the coefficient of the homothety with the center at O that takes the latter line to the former.

Definition. The periodic function $p(t)$ is called the *support function* of the curve $\gamma(t)$.

A curve is uniquely determined by its support function. In the Euclidean case the support function is the signed distance from the origin to the oriented tangent lines to γ.

Let $S(t)$ be the parametrized figuratrix, considered as a curve in $\mathbf{R}^2$, and let $(S_1(t), S_2(t))$ be its linear coordinates. Consider the collection of curves obtained from $J(t)$ by parallel translations and dilations with positive coefficients.

Lemma 4.3. *The support functions of these curves are the functions* $\{a+b\, S_1(t)+c\, S_2(t)\}$ *where* a, b, c *are constants and* $a > 0$.

Proof. Clearly, the support function of $J(t)$ is 1. By Lemma 2.2., $[J(t), S(t)] = 1$ and $[J'(t), S(t)] = 0$. Thus the linear functional $[\ \ , S(t)]$ equals 1 on the tangent line to the curve J at point $J(t)$. It follows that the support function $p(t)$ of a curve $\gamma(t)$ equals $[\gamma(t), S(t)]$. Applying the dilation with coefficient a and the parallel translation through vector v to the curve $J(t)$ one obtains the support function

$$p(t) = [aJ(t) + v, S(t)] = a + [v, S(t)].$$

The result follows.

The curve $S(t)$ being strictly convex, the vectors $S'(t)$ and $S''(t)$ are everywhere linearly independent. Thus

$$S'''(t) = u(t)\ S''(t) + v(t)\ S'(t)$$

for some T-periodic functions $u(t), v(t)$. Consider the linear differential operator on the circle $\mathbf{R}/T\mathbf{Z}$:

$$L = (d/dt)^3 - u(t)\ (d/dt)^2 - v(t)\ d/dt.$$

The kernel of L consists of the functions $a+b\, S_1(t)+c\, S_2(t)$. It follows from strict convexity of S that these functions constitute a Chebyshev system.

Example. If the parameterization $S(t)$ is an affine one then the operator L equals $(d/dt)^3 + k(t)\ (d/dt)$ where $k(t)$ is the affine curvature.

Definition. A linear differential operator of odd degree is called *disconjugate* on the circle $\mathbf{R}/T\mathbf{Z}$ if every function in its kernel is T-periodic and this kernel is a Chebyshev system.

The operator L is disconjugate. In the Euclidean case $S(t)$ is the unit circle, and $L = (d/dt)^3 + d/dt$. Disconjugate operators enjoy the following property proved in [A4, G-M-O].

Theorem 4.4. *Let* L *be a disconjugate differential operator on the circle of degree* $2n+1$. *For every smooth function* f *on the circle the function* $L(f)$ *has at least* $2n+2$ *distinct zeroes.*

Vertices of a curve $\gamma(t)$ present themselves as follows in terms of the support function $p(t)$.

Lemma 4.5. *A point $\gamma(t_0)$ is a Minkowski vertex if and only if $L(p)(t_0) = 0$.*

Proof. Let $p_0(t)$ be the support function of the osculating indicatrix at point $\gamma(t_0)$. Then $(j^2p)(t_0) = (j^2p_0)(t_0)$. If $\gamma(t_0)$ is a vertex then the 3-jets are equal: $(j^3p)(t_0) = (j^3p_0)(t_0)$. Since $L(p_0) = 0$, one has: $L(p)(t_0) = 0$.

Conversely, if $L(p)(t_0) = 0$ then

$$p'''(t_0) = u(t_0)\; p''(t_0) + v(t_0)\; p'(t_0),$$

and the function p_0 satisfies the same equation. Since the 2-jets of p and p_0 at t_0 coincide, it follows that $p'''(t_0) = p_0'''(t_0)$ as well. Therefore the osculating indicatrix is third order tangent to γ at point $\gamma(t_0)$.

Thus Theorem 4.4 again implies the Minkowski 4-vertex theorem.

Remark. In the Euclidean case the following result holds ([Bl 2]): if a closed convex curve intersects a circle at $2n$ points then it has at least $2n$ vertices. Does a similar result hold in the Minkowski setting?

5. Conservative transverse line fields

In this section we discuss the following problem: given a smooth strictly convex closed plane curve γ and a smooth transverse line field l along it, when a parameterization $\gamma(t)$ exists such that the line $l(t)$ at point $\gamma(t)$ is generated by the acceleration vector $\gamma''(t)$ for all t?

Definition. A transverse line field along a closed plane curve, generated by the acceleration vectors for some parameterization of the curve, is called *conservative*.

Clearly, not every line field is conservative: consider, for example, a field of lines that everywhere make an acute angle with the curve. Theorem 0.1 provides a necessary condition: the envelope of the lines from a conservative line field has at least 4 cusps. Lemma 3.2 gives another one: there exist at least 2 tangent lines to this envelope through every point in the plane.

We start with the following situation. Let M^3 be a contact manifold and $\tilde{\gamma} \subset M$ be a closed smooth Legendrian curve. Recall that the characteristic line field η of a contact form λ is the field $Ker\ d\lambda$. Assume that the contact distribution along $\tilde{\gamma}$ is coorientable; then it can be determined by a contact form. Let η be a line field along $\tilde{\gamma}$, transverse to the contact distribution.

Question: When does a contact form exists in a vicinity of $\tilde{\gamma}$ for which η is the characteristic field?

Be this the case we call the field η *characteristic.*

Let λ be some contact form near $\tilde{\gamma}$ and v be a vector field along $\tilde{\gamma}$ that generates the line field η. Consider the 1-form $(i_v\; d\lambda)/\lambda(v)$ and set:

$$\beta(\tilde{\gamma}, \eta) = \int_{\tilde{\gamma}} \frac{i_v\; d\lambda}{\lambda(v)}.$$

Theorem 5.1. *The number $\beta(\tilde{\gamma}, \eta)$ does not depend on the choice of the contact form λ nor the vector field v. This number vanishes if and only if the field η is characteristic.*

Proof. Clearly, $(i_v\ d\lambda)/\lambda(v)$ does not change if v is multiplied by a nonvanishing function. Let $\lambda_1 = f\lambda$ with $f \neq 0$ be another contact form. Then $d\lambda_1 = df \wedge \lambda + f d\lambda$. One has:

$$\int_{\tilde{\gamma}} \frac{i_v\ d\lambda_1}{\lambda_1(v)} = \int_{\tilde{\gamma}} \frac{f\ i_v\ d\lambda + df(v)\ \lambda - \lambda(v)\ df}{f\ \lambda(v)} = \int_{\tilde{\gamma}} \frac{i_v\ d\lambda}{\lambda(v)} + \int_{\tilde{\gamma}} \frac{df(v)}{f\ \lambda(v)}\lambda - \int_{\tilde{\gamma}} \frac{df}{f}.$$

The second integral on the right hand side vanishes because $\tilde{\gamma}$ is a Legendrian curve, tangent to the kernel of $df(v)\lambda/f\lambda(v)$, and so does the third because df/f is an exact 1-form. Thus $\beta(\tilde{\gamma}, \eta)$ does not depend on the choices involved.

If η is characteristic for a contact form λ then $i_v\ d\lambda = 0$, so $\beta(\tilde{\gamma}, \eta) = 0$. Conversely, let $\beta(\tilde{\gamma}, \eta) = 0$. A neighbourhood of $\tilde{\gamma}$ in M is contactomorphic to a neighbourhood of the zero section in the space of 1-jets J^1S^1 (see [Ar 3]). That is, there exist coordinates (x, y, z), $x \in S^1$, $y, z \in \mathbf{R}^1$ in which the contact structure is given by the 1-form $\lambda_0 = dz - ydx$, and $\tilde{\gamma}$ is the curve $y = z = 0$. Since η is transverse to the contact structure one may assume it to be generated by the vector field

$$v = a(x)\ \partial/\partial x + b(x)\ \partial/\partial y + \partial/\partial z,$$

where $a(x)$ and $b(x)$ are functions on the circle. Then

$$\beta(\tilde{\gamma}, \eta) = \int_{\tilde{\gamma}} \frac{i_v\ d\lambda_0}{\lambda_0(v)} = -\int b(x)\ dx.$$

If $\beta(\tilde{\gamma}, \eta)$ vanishes then there exists a function $g(x)$ such that $b(x) = g'(x)$.

Next, a direct computation shows that the characteristic line field of the contact form $e^{f(x,y,z)}\ \lambda_0$ is generated by the vector field

$$f_y\ \partial/\partial x\ - (f_x + yf_z)\ \partial/\partial y + (1 + yf_y)\ \partial/\partial z,$$

which equals, along $\tilde{\gamma}$,

$$u = f_y\ \partial/\partial x - f_x\ \partial/\partial y + \partial/\partial z.$$

Therefore, setting $f(x, y, z) = a(x)y - g(x)$, one has: $v = u$, and the field η is characteristic.

Thus the characteristic line fields constitute a codimension 1 subspace in the (infinite dimensional) space of line fields along $\tilde{\gamma}$, transverse to the contact structure.

Return to the situation at the beginning of the section. Let γ be a smooth strictly convex closed curve, cooriented inwards, and l be a smooth transverse line field along γ. As before, $\tilde{\gamma}$ is the Legendrian curve in the space of cooriented contact elements $ST^*\mathbf{R}^2$, corresponding to γ. For every point $x \in \gamma$ consider the family of cooriented contact elements along the line $l(x)$, parallel to the contact element of γ at x. This gives a line field η along $\tilde{\gamma}$, a lift of the field l. The field η is transverse to the contact structure.

Choose a parameterization $\gamma(t)$, $0 \leq t \leq T$, and a vector field $u(t)$ along γ that generates the line field $l(t)$.

Lemma 5.2. *One has:*

$$\beta(\tilde{\gamma}, \eta) = \int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]} dt.$$

Proof. Let v be the lift of u to $ST^*\mathbf{R}^2$ that generates the field η. In Theorem 2.1 a Hamilton function H in $ST^*\mathbf{R}^2$ is constructed, associated with the parameterization $\gamma(t)$ (one does not need the assumption $[\gamma''(t), \gamma'''(t)] \neq 0$ here). The space $ST^*\mathbf{R}^2$ is identified with $\mathbf{R}^2 \times S$, where the star-shaped curve $S \subset (\mathbf{R}^2)^*$, the level curve of H, consists of the covectors $[\gamma'(t), \ ]$. The corresponding contact form λ is the restriction of the Liouville form pdq to $\mathbf{R}^2 \times S$. The curve $\tilde{\gamma}$ is given by the formula:

$$\tilde{\gamma}(t) = (\gamma(t), \ [\gamma'(t), \ \ ]).$$

It follows that $\lambda(v(t)) = [\gamma'(t), u(t)]$. Likewise,

$$(i_{v(t)} d\lambda)(\tilde{\gamma}'(t)) = (i_{v(t)} dp \wedge dq)(\tilde{\gamma}'(t)) = [\gamma''(t), u(t)].$$

Therefore

$$\int_{\tilde{\gamma}} \frac{i_v \, d\lambda}{\lambda(v)} = \int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]} dt.$$

Lemma is proved.

In particular, the value of the integral

$$\int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]} dt$$

does not depend on the parameterization $\gamma(t)$ nor on the choice of the vector field $u(t)$. Denote this integral by $\alpha(\gamma, l)$.

Lemma 5.3. *The line field l along γ is conservative if and only if the line field η along $\tilde{\gamma}$ is characteristic.*

Proof. If l is generated by the vectors $\gamma''(t)$ then η consists of the characteristic directions of the contact form in $ST^*\mathbf{R}^2$, associated with the parameterization $\gamma(t)$ in Theorem 2.1 (cf. the proof of the preceding lemma).

Conversely, a contact form λ along $\tilde{\gamma}$, whose characteristics are the lines η, is a field of covectors p along γ which vanish on the tangent lines to γ at the respective points. Define the parameterization $\gamma(t)$ by the condition: $[\gamma'(t), \ ] = p\ (\gamma(t))$ for all t. Then the contact form in $ST^*\mathbf{R}^2$, associated with this parameterization according to Theorem 2.1, coincides with λ along $\tilde{\gamma}$. Therefore the lines $l(t)$ are generated by the vectors $\gamma''(t)$.

Combining Theorem 5.1, Lemma 5.2 and 5.3, one arrives to the following result (discovered in [T 2] and proved therein by a direct computation).

Theorem 5.4. *A transverse line field l along a smooth strictly convex closed plane curve γ is conservative if and only if $\alpha(\gamma, l) = 0$.*

Thus conservative line fields constitute a codimension one subspace in the space of transverse line fields along a closed curve.

Example. L. Guieu and V. Ovsienko studied the following situation in [G-O]. Given a smooth convex closed plane curve consider the field of lines connecting each point of the curve with a focus of its osculating conic at this point (see Example 2 in Section 3). This line field is conservative, and its envelope, called the gravitational caustic in [G-O], has at least 6 cusps.

Consider a curve γ with a transverse line field l. A (partial) diffeomorphism of the plane F takes γ to a new curve $F(\gamma)$ with the transverse line field $dF(l)$. The field $dF(l)$ does not have to be conservative even if l is.

Example. Let γ be the unit circle, l consists of its normals, and F is given near γ in polar coordinates by the formula: $(\alpha, r) \to (\alpha + r, r)$. Then $F(\gamma) = \gamma$, and the lines $dF(l)$ make a constant acute angle with the circle.

However the following result holds (to answer a question by V. Arnold).

Theorem 5.5. *Every projective transformation of the plane takes the conservative line fields to the conservative ones.*

Proof. Consider $\mathbf{R}^2$ as the plane $\{z = 1\}$ in Euclidean 3-space, and let

$$\pi : (x, y, z) \to (x/z, y/z)$$

be the projection of the half-space $\mathbf{R}^3_+ = \{z > 0\}$ on $\mathbf{R}^2$. Consider a parameterized curve $\Gamma(t) \subset \mathbf{R}^3_+$, and let $\gamma(t) = \pi(\Gamma(t))$.

Claim: the field $(d\pi)(\Gamma''(t))$ is conservative along the curve $\gamma(t)$.

Indeed, a direct computation (which is left to the reader) shows that

$$(d\pi)(\Gamma''(t)) = \gamma''(t) + 2\,\frac{z'(t)}{z(t)}\,\gamma'(t).$$

Therefore

$$\alpha(\gamma,\ (d\pi)(\Gamma''(t))) = -\int 2\,\frac{z'(t)}{z(t)}\,dt = -2\int d\ \log\ z(t) = 0.$$

The claim follows from Theorem 5.4.

Let A be a linear transformation of space. Then $F = \pi A : \mathbf{R}^2 \to \mathbf{R}^2$ is a projective transformation, and all projective transformations are obtained this way. Consider a curve $\gamma(t) \subset \mathbf{R}^2$, and let $l(t)$ be generated by the acceleration vectors $\gamma''(t)$. Let $\Gamma(t) = A(\gamma(t))$; assume, without loss of generality, that $\Gamma(t) \subset \mathbf{R}^3_+$. One has: $\Gamma''(t) = A(\gamma''(t))$, and it follows from the above claim that the field $(d\pi)(\Gamma''(t))$ is conservative along the curve $\pi(\Gamma(t))$. Thus the line field $dF(l)$ is conservative along the curve $F(\gamma)$.

Remark. Theorem 5.5 shows that the notion of the conservative line fields along closed curves is a projective, and not an affine, one. The theory of this paper can be extended to spherical curves in the spirit of [A 5].

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