BIRATIONAL TRANSFORMATIONS OF WEIGHTED GRAPHS

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Dedicated to Masayoshi Miyanishi

Abstract. We introduce the notion of a standard weighted graph and show that every weighted graph has an essentially unique standard model. Moreover we classify birational transformations between such models. Our central result shows that these are composed of elementary transformations. The latter ones are defined similarly to the well known elementary transformations of ruled surfaces.

In a forthcoming paper, we apply these results in the geometric setup to obtain standard equivariant completions of affine surfaces with an action of certain algebraic groups. We show that these completions are unique up to equivariant elementary transformations.

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Acknowledgements: This research was started during a visit of the first two authors at the Fourier Institute, Grenoble, and continued during a visit of two of us at the Max Planck Institute of Mathematics in Bonn. We thank these institutions for their generous support and excellent working conditions. The research was also partially supported by NSA Grant no. MDA904-00-1-0016 and the DFG-Schwerpunkt Komplexe Geometrie.

1991 Mathematics Subject Classification: 05C99, 14R99, 14J50.

Key words: Weighted graph, standard model, minimal model, birational transformation, elementary transformation.

1. Introduction

Birational transformations of weighted graphs were studied by many authors, mainly due to their importance for understanding completions of algebraic surfaces, see e.g. [DG, Da_1, Da_2, FZ, Fu, Hi_2, Mu, Ne, Ra, Ru]. Danilov and Gizatullin [DG] were the first to introduce several special forms of linear graphs like semistandard, m-standard or quasistandard ones to deduce their interesting results on automorphism groups of affine algebraic surfaces that are completed by a chain of rational curves. More recently Daigle [Da_1, Da_2] studied standard models of weighted trees and showed that any such tree has a unique standard model in its birational equivalence class.

In this paper we generalize this theory to arbitrary weighted graphs. The Reduction Theorem 2.15 shows that any graph, possibly with cycles, loops and multiple edges, admits a standard model. Moreover, this standard model is essentially unique in its birational equivalence class, see Corollary 3.33.

A major part of the paper is devoted to the study of birational transformations of standard weighted graphs. Any such transformation preserves the branching points, see Lemma 2.4. Therefore it is sufficient to classify them for linear chains and for circular graphs. In both cases we show that one can decompose such a transformation into simpler ones called *moves, shifts and turns*, see Propositions 3.4, 3.7 and Theorem 3.18. In particular (see Theorem 3.1) any birational transformation between standard models is composed of elementary transformations, which are defined similar to those for ruled surfaces (see Definition 2.10).

It is worthwhile to compare our result with those in the paper of Danilov and Gizatullin [DG]. Indeed, Theorem 1 in [DG] implies in particular that any biregular map between two affine surfaces completed by standard linear chains[1] of rational curves, is a product of birational elementary transformations and some standard reconstructions of good completions of our surfaces. While proving this theorem, the control on the indeterminacy points of the underlying birational map was important. Since we deal with general graphs which include non-simply connected ones and do not necessarily satisfy the Hodge index theorem, we do not have an underlying birational map of algebraic surfaces. Therefore, our pivotal point is different. Given a birational transformation of two standard weighted graphs A and B, we look for dominant maps from a third graph $\Gamma \to A$ and $\Gamma \to B$. Actually we decompose our birational transformation into a sequence of elementary transformations dominated at every step by Γ so that one can apply our results to the corresponding situation of boundary divisors of algebraic surfaces. The role of indeterminacy points in [DG] is played by the vertices of Γ that are not contracted in both A and B.

To complete the picture, we survey in the Appendix some well known facts on the adjacency matrix and the discriminant of a weighted graph and their behaviour under birational transformations. In particular, we compute the spectra of standard weighted graphs.

In the subsequent paper [FKZ] we will apply our results in the geometric setting to obtain equivariant standard completions of affine surfaces equipped with an effective action of certain algebraic groups, cf. [DG, §6].

[1]I Notice that in [DG] a graph is called standard, if it is semistandard, m-standard or quasistandard, whereas our standard graphs are 0-standard in the terminology of [DG].

2. Weighted graphs

2.1. **Generalities.** A *(combinatorial) graph* consists of a nonempty set of vertices $\Gamma^{(0)}$ and a set of edges $\Gamma^{(1)}$ together with a boundary map ∂ which associates to every edge $e \in \Gamma^{(1)}$ the set $\partial(e)$ consisting of one or two vertices, called the *end points* of e. An edge e with just one end point is a simple loop. In this subsection we consider weighted graphs with arbitrary real weights of vertices, and we denote by $|\Gamma|$ the number of vertices of Γ. All our graphs are assumed to be finite.

The *degree* (or the *valency*) $\deg_\Gamma(v)$ of a vertex $v \in \Gamma^{(0)}$ is the number of edges adjacent to v, where we count the loops at v twice. The *branches* of a connected graph Γ at v are the connected components of the graph $\Gamma \ominus v$ obtained from Γ by deleting the vertex v and all its incident edges. In case $\deg(v) > 2$ we call v a *branching point*; if $\deg(v) \le 1$, v is called an *end vertex* or a *tip*, and a *linear vertex* if $\deg(v) = 2$.

A graph is said to be *linear* or a *chain* if it has two end vertices and all other vertices are linear. By a *circular* graph we mean a connected graph with only linear vertices. We let $B(\Gamma)$ denote the set of all branching points of Γ. A connected graph Γ with $B(\Gamma) = \emptyset$ is either linear or circular.

The connected components of $\Gamma \ominus B(\Gamma)$ will be called the *segments* of Γ. Clearly the segments of Γ are either linear or circular weighted graphs, as they do not include the branching points of Γ. Moreover for a connected graph Γ a circular segment can appear if and only if Γ is circular itself.

The *branching number* at v is $\nu_\Gamma(v) = \max\{0, \deg(v) - 2\}$, and the *total branching number* is

$$\nu(\Gamma) = \sum_{v \in B(\Gamma)} \nu_\Gamma(v)\,.$$

2.1. We denote by $[[w_0, \ldots, w_n]]$ a chain with linearly ordered vertices $v_0 < v_1 < \ldots < v_n$, where $w_i = v_i^2 \in \mathbb{R}$ is the weight of v_i:

$$\begin{array}{ccccccc} v_0 & & v_1 & & & & v_n \\ \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ \\ w_0 & & w_1 & & & & w_n \end{array}\;.$$

Similarly, we denote by $((w_0, \ldots, w_n))$ a circular graph with cyclically ordered vertices $v_0 < \ldots < v_n < v_0 \ldots$ with weights $w_i = v_i^2 \in \mathbb{R}$ of v_i.

2.2. Given two ordered linear chains L with vertices $v_1 < \ldots < v_k$ and M with vertices $u_1 < \ldots < u_l$ we denote by LM their *join* that is, the ordered linear chain with vertices $v_1 < \ldots < v_k < u_1 < \ldots < u_l$. We let L^{-1} be the chain L with the reversed ordering $v_k < \ldots < v_1$. For a sequence of ordered linear chains $L_1, \ldots, L_n$, we let $((L_1 \ldots L_n))$ be the circular cyclically ordered graph made of their join.

Definitions 2.3. For a weighted graph Γ, an *inner blowup* $\Gamma' \to \Gamma$ at an edge e with end vertices $v_0, v_1 \in \Gamma^{(0)}$ consists in introducing a new vertex $v \in (\Gamma')^{(0)}$ of weight -1 subdividing e in two edges e' and e'' with $\partial(e') = \{v_0, v\}$ and $\partial(e'') = \{v, v_1\}$, and diminishing by 1 the weights of v_0 and v_1 (in case where e is a loop i.e., $v_0 = v_1$, the weight of v_0 is diminishing by 2). An *outer blowup* $\Gamma' \to \Gamma$ at a vertex v_0 of Γ consists in introducing a new vertex v of weight -1 and a new edge e with end vertices v_0, v, and diminishing by 1 the weight of v_0. In both cases, the inverse procedure is called *blowdown* of v.

The graph Γ is *minimal* if it does not admit any blowdown. Clearly Γ is minimal if and only if every segment of Γ is.

A *birational transformation* of a graph Γ into another one Γ' is a sequence of blowing ups and downs. We write $\Gamma \sim \Gamma'$ or $\Gamma \dashrightarrow \Gamma'$ if such a transformation does exist, and $\Gamma \to \Gamma'$ if Γ is obtained from Γ' by a sequence of only blowups. In the latter case we say that Γ *dominates* Γ', and we call $\Gamma \to \Gamma'$ a *domination*. If $\Gamma \to \Gamma'$ is a domination and v is a vertex of Γ', we denote by $\hat{v}$ the corresponding vertex of Γ called the *proper transform* or the *preimage* of v. Similarly, for a subgraph A of Γ' with vertices $\{a_i\}$, $\hat{A}$ stands for a subgraph of Γ with vertices $\{\hat{a}_i\}$.

Any birational transformation $\gamma : \Gamma_1 \to \Gamma_2$ fits into a commutative diagram

$$\begin{tikzcd} & \Gamma \arrow[dl] \arrow[dr] & \\ \Gamma_1 \arrow[rr, dashed, "\gamma"] & & \Gamma_2\,, \end{tikzcd}$$

where $\Gamma \to \Gamma_i$, $i = 1, 2$, are dominations. Moreover we may suppose that this decomposition is *relatively minimal* that is, no (-1)-vertex of Γ is contracted in both directions, see [FZ, Appendix, Remark A.1(1)].

Clearly, the topological (homotopy) type of a graph is birationally invariant whereas $\nu(\Gamma)$ is not, in general.

We recall the following facts, see [FZ, Appendix to §4] and also [Da_1, Cor. 3.6], [Ru]. We provide a short argument in our more general setting.

Lemma 2.4. *Let Γ dominate a minimal weighted graph Γ_1.*

(a) If a branching point v of degree r in Γ is not contained in $\widehat{B(\Gamma_1)}$, then Γ has at least $r-2$ branches at v contractible inside Γ and contracted in Γ_1.
(b) If $\Gamma \to \Gamma_1$, $\Gamma \to \Gamma_2$ is a pair of dominations and Γ_1, Γ_2 are minimal then $\widehat{B(\Gamma_1)} = \widehat{B(\Gamma_2)}$.

Proof. To show (a) we note that a vertex $v \in B(\Gamma)$ of degree $r \geq 3$ can become at most linear only after contracting $r-2$ branches of Γ at v. Moreover, blowdowns in one of them do not affect the other ones, so each of them must be contractible.

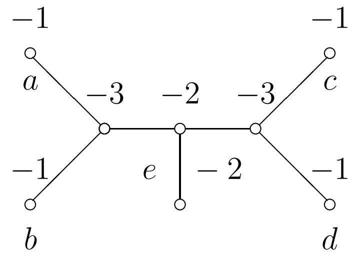
To show (b), suppose on the contrary that there exists a vertex $\hat{v}_1 \in \widehat{B(\Gamma_1)} \ominus \widehat{B(\Gamma_2)}$, and let $r \geq 3$ be the degree of $\hat{v}_1$ in Γ. According to (a) there are $r-2$ contractible branches of Γ at $\hat{v}_1$ that are blown down in Γ_2. Since $v_1 \in B(\Gamma_1)$, at least one of these branches is not blown down completely in Γ_1. This contradicts the minimality assumption. □

Corollary 2.5. *The number of branching points of a minimal graph Γ, their degrees and the total branching number $\nu(\Gamma)$ are birational invariants. In particular, a weighted graph Γ that can be transformed into one with fewer branching points is not minimal.*

Thus the only birationally non-rigid elements of a minimal graph can be its segments. A graph with no segments is birationally rigid that is, has a birationally unique minimal model. A segment of a graph can eventually be non-rigid even being minimal, see an example in 2.10 below. However the number of such segments and their types (linear or circular) remain stable under birational transformations.

Examples 2.6. The graph

$\Gamma :$



admits two different contractions $\Gamma \to A$ and $\Gamma \to B$ to the linear chains $A = B := [[0,0]]$, namely by contracting the subgraphs $\Gamma \ominus \{a,b\}$, $\Gamma \ominus \{c,d\}$, respectively. We note that the extremal linear branch T of Γ consisting just of the vertex e is contracted in both A and B, although it is not contractible and $e \notin \hat{A} \cup \hat{B}$. Thus for a pair of dominations $\Gamma \to A, \Gamma \to B$, an extremal linear branch T of Γ contracted in both A and B is not necessarily contractible.

In the special case, where Γ dominates two circular graphs, we have the following simple observation.

Lemma 2.7. *If $\Gamma \to A$, $\Gamma \to B$ is a relatively minimal pair of dominations of minimal circular graphs A, B then Γ is as well circular.*

Proof. Since blowups do not change the topological type of the graph, there is a unique circular subgraph $\Gamma' \subseteq \Gamma$ dominating both A and B. If $\Gamma' \neq \Gamma$ then there is a branching point $c \in B(\Gamma)$ on Γ' and a branch T at c which is a nonempty tree disjoint to Γ'. But then T is contractible and is contracted in both A and B, which contradicts our assumption of relative minimality. $\square$

2.2. **Admissible transformations.** Let us introduce the following notion.

Definition 2.8. A birational transformation of weighted graphs $\Gamma \dashrightarrow \Gamma'$ which consists in a sequence

$$\gamma : \quad \Gamma = \Gamma_0 \overset{\gamma_1}{\dashrightarrow} \Gamma_1 \overset{\gamma_2}{\dashrightarrow} \cdots \overset{\gamma_n}{\dashrightarrow} \Gamma_n = \Gamma' \ ,$$

where each γ_i is either a blowdown or a blowup, is called *admissible*[2] if the total branching number remains constant at every step. For instance, a blowup $\Gamma' \to \Gamma$ is admissible if it is inner or performed in an end vertex of Γ, and a blowdown is admissible if its inverse is so. Clearly, a composition of admissible transformations is admissible.

More restrictively, we call γ *inner* if the γ_i are either admissible blowdowns or inner blowups. Thus the inverse γ^{-1} is admissible but not necessarily inner, see also Definition 2.10 below.

The following proposition gives a precision of Theorem 3.2 in [Da_2], which says that any birational transformation of minimal graphs can be replaced by an admissible one.

Proposition 2.9. *If Γ dominates two minimal graphs Γ_1 and Γ_2 then there exists an admissible transformation of Γ_1 into Γ_2 such that every step is dominated by Γ. In other words, there is a birational transformation of Γ_1 into Γ_2 such that each step is dominated by Γ and the total branching number stays constant.*

[2] Or *strict* in the terminology of [Da_2].

Proof. We may assume that Γ minimally dominates Γ_1 and Γ_2 that is, none of the (-1)-vertices of degree ≤ 2 in Γ is contracted in both Γ_1 and Γ_2. Let b be a branching point in Γ not contained in $B(\Gamma_1)\hat{} = B(\Gamma_2)\hat{}$. By Lemma 2.4(a), for $i = 1, 2$ there is a branch C_i of Γ at b that is contractible inside Γ and is contracted in Γ_i. Since Γ dominates Γ_1 and Γ_2 relatively minimally we have $C_1 \neq C_2$, so these are disjoint. Letting Γ/C_i be the result of contracting C_i inside Γ we obtain the diagram

$$\begin{tikzcd}
 & & \Gamma \arrow[dl] \arrow[dr] & & \\
 & \Gamma/C_1 \arrow[dl] \arrow[dr] & & \Gamma/C_2 \arrow[dl] \arrow[dr] & \\
\Gamma_1 & & \Gamma' & & \Gamma_2\,,
\end{tikzcd}$$

where Γ' is the minimal graph obtained from $\Gamma/(C_1 \cup C_2)$ by blowing down successively all (-1)-vertices of degree ≤ 2. The total branching number of Γ/C_1 and Γ/C_2 is strictly smaller than that of Γ. Using induction on this number the result follows. $\square$

2.3. **Elementary transformations.** In this subsection we assume that our weighted graph Γ is nonempty, connected and has *only integral weights*. In our principal result (see Reduction Theorem 2.15 below) we use the following operation on weighted graphs, cf. e.g. [DG].

Definition 2.10. Given an at most linear vertex v of Γ with weight 0 one can perform the following transformations. If v is linear with neighbors v_1, v_2 then we blow up the edge connecting v and v_1 in Γ and blow down the proper transform of v:

$$(1)\qquad \cdots\underset{w_1-1}{\overset{v_1}{\circ}}\!\!\text{——}\!\!\underset{0}{\overset{v'}{\circ}}\!\!\text{——}\!\!\underset{w_2+1}{\overset{v_2}{\circ}}\cdots \quad\dashrightarrow\quad \cdots\underset{w_1-1}{\overset{v_1}{\circ}}\!\!\text{——}\!\!\underset{-1}{\overset{v'}{\circ}}\!\!\text{——}\!\!\underset{-1}{\overset{v}{\circ}}\!\!\text{——}\!\!\underset{w_2}{\overset{v_2}{\circ}}\cdots \quad\longrightarrow\quad \cdots\underset{w_1}{\overset{v_1}{\circ}}\!\!\text{——}\!\!\underset{0}{\overset{v}{\circ}}\!\!\text{——}\!\!\underset{w_2}{\overset{v_2}{\circ}}\cdots\,.$$

Similarly, if v is an end vertex of Γ connected to the vertex v_1 then one proceeds as follows:

$$(2)\qquad \cdots\underset{w_1-1}{\overset{v_1}{\circ}}\!\!\text{——}\!\!\underset{0}{\overset{v'}{\circ}} \quad\dashrightarrow\quad \cdots\underset{w_1-1}{\overset{v_1}{\circ}}\!\!\text{——}\!\!\underset{-1}{\overset{v'}{\circ}}\!\!\text{——}\!\!\underset{-1}{\overset{v}{\circ}} \quad\longrightarrow\quad \cdots\underset{w_1}{\overset{v_1}{\circ}}\!\!\text{——}\!\!\underset{0}{\overset{v}{\circ}}\,.$$

These operations (1) and (2) and their inverses will be called *elementary transformations* of Γ. If such an elementary transformation involves only an inner blowup then we call it *inner*. Thus (1) and (2) are inner whereas the inverse of (2) is not as it involves an outer blowup. Clearly, elementary transformations are admissible in the sense of Definition 2.8.

Examples 2.11. 1. If $[[w_0, \ldots, w_n]]$ is a weighted linear graph with $w_k = 0$ for some k, $0 \leq k \leq n$ (see 2.1) then

$$[[w_0, \ldots, w_{k-1} + 1,\ 0,\ w_{k+1} - 1, \ldots w_n]] \dashrightarrow [[w_0, \ldots, w_{k-1},\ 0,\ w_{k+1}, \ldots w_n]]\,,$$

as well as its inverse are elementary transformations. It is inner unless $k = n$, and the inverse elementary transformation is inner unless $k = 0$.

2. Iterating inner elementary transformations as in (1), for $a \in \mathbb{Z}$ we can transform a linear subgraph $L = [[w_1, 0, w_2]]$ of a segment of Γ into $[[w_1 - a, 0, w_2 + a]]$ leaving $\Gamma \ominus L$ unchanged (see [Da$_2$, L. 4.15]). In particular, $[[w_1, 0, 0]]$ can be transformed into

$[[0, 0, w_1]]$. Thus we can "move" pairs of vertices of weight 0 within segments of Γ by means of a sequence of inner elementary transformations.

To simplify notation we will write $[[0_m]]$, $[[(a)_m]]$ for the linear chain of length m with all weights equal to 0 or a, respectively. Using the preceding examples we easily deduce the following facts (cf. [Da$_2$, Lemmas 4.15-4.16]).

Lemma 2.12. *Below L stands for a linear subchain of a segment Σ of Γ. By a sequence of inner elementary transformations of Γ we can transform*

(a) $L = [[0_{2k}, w_1, \ldots, w_n]]$ *into* $L^* = [[w_1, \ldots, w_n, 0_{2k}]], \quad \forall k, n \geq 1;$

(b) $L = [[w_1, 0_{2k+1}, w_2]]$ *into* $[[w_1 - a, 0_{2k+1}, w_2 + a]], \quad \forall a \in \mathbb{Z}, \quad \forall k \geq 0;$

(c) $L = [[0_{2k+1}, w_0, \ldots, w_n]]$ *into* $[[0_{2k+1}, w_0 - a, w_1, \ldots, w_n]], \quad \forall a, k, n \geq 0\,.$

These elementary transformations leave $\Gamma \ominus L$ unchanged except possibly for the case (c), where the vertex connected to the leftmost vertex of L will change its weight. Allowing as well outer elementary transformations (c) holds for all $a \in \mathbb{Z}$.

Proof. (a) follows by applying Example 2.11(2) repeatedly. To prove (b), by Example 2.11(2) we can transform $[[w_1, 0_{2k+1}, w_2]]$ into $[[w_1, 0_{2k-1}, -a, 0, w_2 + a]]$ by a sequence of inner elementary transformations. Hence (b) follows by induction. Finally, a chain L as in (c) can be transformed into $[[0, w_0, 0_{2k}, w_1, \ldots, w_n]]$. Applying elementary transformations as in Definition 2.10(2) repeatedly we can transform the latter chain into $[[0, w_0 - a, 0_{2k}, w_1, \ldots, w_n]]$ and then into $[[0_{2k+1}, w_0 - a, w_1, \ldots, w_n]]$. Note that in the case $a \geq 0$ only inner transformations are required, see Example 2.11(1). $\square$

2.4. **Standard and semistandard graphs.**

Definition 2.13. A non-circular weighted graph Γ will be called *standard* if each of its nonempty segments L is one of the linear chains

$$[[0_{2k}, w_1, \ldots, w_n]] \quad \text{or} \quad [[0_{2k+1}]]\,, \tag{3}$$

where $k, n \geq 0$ and $w_i \leq -2\,\forall i$. Similarly, a circular graph will be called *standard* if it is one of the graphs

$$((0_{2k}, w_1, \ldots, w_n)), \qquad ((0_l, w)) \quad \text{or} \quad ((0_{2k}, -1, -1)), \tag{4}$$

where $k, l \geq 0$, $n > 0$, $w \leq 0$ and $w_1, \ldots, w_n \leq -2$. We note that for $w = 0$ the second graph in (4) becomes $((0_{l+1}))$.

A graph Γ will be called *semistandard* if each of its nonempty segments is either a standard circular graph or one of the linear chains

$$[[0_l, w_1, \ldots, w_n]] \quad \text{or} \quad [[0_l, w_1, \ldots, w_n, 0]] \tag{5}$$

where $l, n \geq 0$ and $w_i \leq -2$ for $1 \leq i \leq n$. Thus every standard graph is also semistandard.

For the standard linear chain $L = [[0_{2k}, w_1, \ldots, w_n]]$ in (3) let us call the chain $L^* = [[0_{2k}, w_n, \ldots, w_1]] = [[w_1, \ldots, w_n, 0_{2k}]]$ the *reversion* of L. By Lemma 2.12.a the reversion L^* can be obtained from L by a sequence of inner elementary transformations.

Remarks 2.14. 1. *Every semistandard linear chain can be transformed into a standard one by a sequence of elementary transformations.* For it is one of the chains in (5), and by Lemma 2.12 it can be transformed into a chain as in (3).

2. If $\Gamma = [[w]]$ ($w \neq -1$) has only one vertex then it is either standard or $w > 0$. In the latter case after w blowups we obtain the chain $[[0, -1, -2, \ldots, -2]]$ of length $w+1$, which transforms as before into a standard chain $[[0, 0, -2, \ldots, -2]]$ of length $w + 1$. This transformation is not composed of elementary ones, since it does not preserve the length.

3. Omitting an end vertex from a semistandard linear chain yields again a semistandard chain.

By a result of Daigle [Da_2, Thm. 4.23] every weighted tree can be transformed into a unique standard one[3], see also [DG] for related results. We give a simplified proof of this reduction result for a general weighted graph in the following more precise form.

Theorem 2.15. *For a minimal graph Γ the following hold.*

(a) If Γ has at least two vertices then Γ admits an inner transformation into a semistandard graph, i.e. it can be transformed into a semistandard one via a sequence of admissible blowdowns and inner blowups.

(b) Γ allows an admissible transformation into a standard graph.

Proof. (b) follows from (a) in view of Remarks 2.14.1 and 2. To show (a), after performing a suitable sequence of inner blowups $\tilde{\Gamma} \to \Gamma$ we can achieve that all weights of vertices inside the segments of $\tilde{\Gamma}$ become ≤ 0, and because of the minimality assumption, these segments remain non-contractible. The result is now a consequence of the following claim applied to each segment Σ of $\tilde{\Gamma}$.

Claim: A non-contractible segment Σ with all weights ≤ 0 can be transformed into one of the graphs in (3), (4) or (5) by means of a sequence of admissible blowdowns and inner elementary transformations.

Proof of the claim. We proceed by induction on the length of Σ. Our assertion trivially holds if Σ is of length 1. If Σ contains a subgraph $[[w_1, -1, w_2]]$ with $w_1, w_2 < 0$ then we can contract the (-1)-vertex to obtain a segment with a smaller number of vertices. Similarly, if $\Sigma = [[-1, w_1, \ldots]]$ with $w_1 < 0$ then again we can contract the (-1)-vertex and get a segment of smaller length.

Using Lemma 2.12 and contractions as above we can collect inner vertices of weight 0 in Σ. If Σ is non-circular then we can also assume that they are on the left. In this way we obtain one of the following nonempty graphs:

$$[[0_m, w_0, \ldots, w_n]], \qquad [[0_m, w_0, \ldots, w_{n+1}, 0]] \quad \text{or} \quad ((0_m, w_0, \ldots, w_{n+1}))\,, \tag{6}$$

where $m \geq 0$, the sequence (w_i) can be empty, $w_0, w_{n+1} \leq -1$ and $w_1, \ldots, w_n \leq -2$.

Let us consider the first graph in (6). If $w_0 \leq -2$ then it is semistandard. If $m = 2k+1$ is odd and $w_0 = -1$ then by Lemma 2.12.c it can be transformed into $[[0_{2k+1}, -2, w_1, \ldots, w_n]]$ by a sequence of inner elementary transformations. In the case $m = 0$ and $w_0 = -1$ we can contract the (-1)-vertex and apply the induction hypothesis. If $m = 2k > 0$ is even and $w_1 = -1$, then we can first contract the (-1)-vertex to obtain the chain $[[0_{2k-1}, 1, w_1 + 1, \ldots, w_n]]$ and then transform this by inner

[3] A *canonical* linear chain in the terminology of [Da_2] is different from our standard one, but in a recent version of [Da_2] these coincide.

elementary transformations into $[[0_{2k}, w_1+1, \ldots, w_n]]$, see again Lemma 2.12.c. Now the result follows by induction.

Similarly, combining the same operations and reversing the ordering, if necessary, provides a reduction to a semistandard graph in the other two cases in (6). The details are left to the reader. $\square$

2.5. **Zigzags and standard zigzags.** Let V be a normal affine surface and X be a completion of V by a divisor D with simple normal crossings (or by an SNC-divisor, for short) so that D is contained in the regular part X_{reg} of X. The dual graphs $\Gamma(D)$ of such D are restricted by the Hodge index theorem. We use the following terminology.

Definition 2.16. An SNC divisor $D \subseteq X_{\mathrm{reg}}$ with irreducible components $C_1, \ldots, C_n$ in a complete algebraic surface X will be called a *zigzag* if the following conditions are satisfied.

⋄ The curves C_i are rational $\forall i = 1, \ldots, n$.

⋄ The dual graph of D is a linear chain $L = [[w_0, w_1, \ldots, w_n]]$ such that the adjacency matrix[4] $I(L)$ has exactly one positive eigenvalue.

A zigzag will be called (semi)standard if its dual graph (also called a zigzag) has the corresponding property, see Definition 2.13.

We remind the reader that the number of positive eigenvalues is a birational invariant of a graph, see 4.1 in the Appendix below. Thus a chain birationally equivalent to a zigzag is again a zigzag. We also note that our terminology is different from the one introduced in [DG]. Indeed, in [DG] by a standard zigzag the authors mean an m-standard, quasistandard or semistandard zigzag, whereas our standard zigzags are 0-standard in the sense of [DG]. In the following lemma we describe the dual graphs of (semi)standard zigzags (cf. [Da_2, Prop. 7.8]).

Lemma 2.17. *The possible standard zigzags are the chains*

$$[[0]]\,, \quad [[0,0,0]] \quad \textit{and} \quad [[0,0,w_1,\ldots,w_n]], \quad \textit{where} \quad n\ge 0,\, w_j \le -2\ \forall j, \tag{7}$$

whereas for the semistandard ones we have additionally the possibilities

$$[[0,w_1,\ldots,w_n]]\,,\ [[0,w_1,0]], \qquad \textit{where } n\ge 0 \quad \textit{and} \quad w_j \le -2\ \forall j\,. \tag{8}$$

Proof. This is an immediate consequence of the following claim[5].

Claim: A zigzag Γ cannot contain two vertices v_i and v_j of weight ≥ 0 unless they are joined by an edge, or Γ has at most length 3.

Otherwise, using the labelling as in (2.1), we first perform suitable inner blowups in Γ to make the weights $v_i^2 = v_j^2 = 0$ and then, using Lemma 2.12, we move the 0-weight on the left to v_0 and the one on the right to v_n. By means of further elementary transformations we can assign to v_1 and v_{n-1} arbitrary weights e.g., ≥ 2. But then the adjacency matrix $I(\Gamma)$ has the symmetric submatrix $\begin{pmatrix} v_1^2 & v_1.v_{n-1} \\ v_1.v_{n-1} & v_{n-1}^2 \end{pmatrix}$, which is positive definite since $v_1.v_{n-1} \le 1$ and $v_1^2, v_{n-1}^2 \ge 2$. This contradicts the assumption that $I(\Gamma)$ has only one positive eigenvalue, proving the claim. $\square$

[4]See the Appendix below.

[5]Alternatively, one can derive the result from Lemma 4.6, cf. also Proposition 4.11.b.

3. Birational transformations of standard graphs

The central result of this section is the following structure theorem for birational transformations of standard graphs.

Theorem 3.1. *Any birational transformation of one standard graph into another one can be decomposed into a sequence of elementary transformations.*

More precisely, if $\Gamma \to A$, $\Gamma \to B$ is a pair of dominations of standard graphs then B can be obtained from A by a sequence of elementary transformations such that every step is dominated by some inner blowup of Γ.

In Subsections 3.1 and 3.3 we completely describe admissible transformations of linear and circular graphs, respectively. In Subsection 3.4 the proof of 3.1 is then reduced to these special cases.

3.1. **Admissible transformations of standard linear chains.**

3.2. In this subsection we consider a diagram

$$\begin{tikzcd} & \Gamma \arrow[dl] \arrow[dr] & \\ A \arrow[rr, dashed] & & B \end{tikzcd} \tag{9}$$

where Γ, A, B are linear chains. We let "$<$" be an ordering on Γ, and we consider the induced ordering of A and B, respectively. The main results are Propositions 3.4 and 3.7 below, where we describe completely all such birational transformations. One of the key observations in the proofs is provided by the following lemma.

Lemma 3.3. *Let Γ be a linear chain, and let $\Gamma \to A$, $\Gamma \to B$ be a pair of dominations, where the vertices $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_m$ of A and B, respectively, are ordered upon an ordering in Γ. Then the following hold.*

(a) If $n \geq 2$ and $a_1^2 = b_1^2 = 0$ then $m \geq 2$ and $\hat{a}_2 = \hat{b}_2$ in Γ.

(b) If $n, m \geq 3$ and for some k, l with $1 < k < n$, $1 < l < m$ we have $a_k^2 = b_l^2 = 0$ then $a_{k-1} < b_{l-1}$ if and only if $a_{k+1} < b_{l+1}$.

Proof. To show (a), let us note first that in the case $m = 1$ the subchain $\Gamma_{<\hat{a}_2}$ is properly contained in Γ, so by Zariski's Lemma 4.14 its intersection matrix is negative definite. As it contracts to $[[0]]$ in A this gives a contradiction. Hence $m \geq 2$.

To show the remaining assertion, let us suppose e.g. that $\hat{a}_2 < \hat{b}_2$. Then the subchain $\Gamma_{<\hat{b}_2}$ of Γ is blown down to $[[0]]$ in B. By Zariski's Lemma 4.14, every proper subgraph of $\Gamma_{<\hat{b}_2}$ is negative definite. However, by assumption $\Gamma_{<\hat{b}_2}$ properly contains $\Gamma_{<\hat{a}_2}$, and $\Gamma_{<\hat{a}_2}$ is not negative definite as it is contracted to $[[0]]$ in A. This contradiction proves that indeed $\hat{a}_2 = \hat{b}_2$.

To deduce (b), we consider the open intervals Γ_A between $\hat{a}_{k-1}$ and $\hat{a}_{k+1}$ and Γ_B between $\hat{b}_{l-1}$ and $\hat{b}_{l+1}$ in Γ. Clearly Γ_A and Γ_B are contracted to $[[0]]$ in A, B, respectively. If e.g. $\hat{a}_{k-1} < \hat{b}_{l-1} < \hat{b}_{l+1} \leq \hat{a}_{k+1}$ then the inclusion $\Gamma_B \subseteq \Gamma_A$ would be proper contradicting Zariski's Lemma 4.14. The proof in the other cases is similar. $\square$

The following results show that a non-trivial admissible birational transformation between two standard linear chains can exist exclusively for two chains of the same odd length with all zero weights.

Proposition 3.4. *Let Γ, A, B be linear chains, and let $\Gamma \to A$, $\Gamma \to B$ be a relatively minimal pair of dominations. Assume that A, B are standard and at least one of them is different from $[[0_{2k+1}]]$ for all $k \geq 0$. If the groups of zeros in A, B are both on the left then $\Gamma \to A$ and $\Gamma \to B$ are isomorphisms.*

Proof. We denote as before by $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_m$ the vertices of A, B, respectively, ordered upon an ordering in Γ. We proceed by induction on the length of A.

If $a_i^2 \leq -2$ $\forall i$ then also $\hat{a}_i^2 \leq -2$ $\forall i$, so by relative minimality there is no (-1)-vertex in Γ blown down in B. Hence $\Gamma \to B$ is an isomorphism. Since B is minimal, Γ is minimal too and so $\Gamma \to A$ as well is an isomorphism. The same conclusion holds in the case where $b_j^2 \leq -2$ $\forall j$.

Thus we may restrict to the case where $a_1^2 = 0 = b_1^2$. By our assumptions one of the chains A, B has length ≥ 2 and so, by Lemma 3.3(a), $n, m \geq 2$ and $\hat{a}_2 = \hat{b}_2$ in Γ. The graphs

$$A' = A \ominus \{a_1, a_2\}, \quad B' = B \ominus \{b_1, b_2\} \quad \text{and} \quad \Gamma' = \Gamma_{>\hat{a}_2}$$

are linear, A', B' are still standard and $\Gamma' \to A'$, $\Gamma' \to B'$ is a relatively minimal pair of dominations. Note that by our assumption, $A' \neq [[0_{2k-1}]]$ or $B' \neq [[0_{2k-1}]]$. Using induction we get that $\Gamma' \to A'$, $\Gamma' \to B'$ are isomorphisms. Now

$$A'' = A \ominus A', \quad B'' = B \ominus B'$$

are both dominated by $\Gamma_{\leq \hat{a}_2}$. Since $a_2^2 = b_2^2 = 0$, by Lemma 3.3 $\hat{a}_1 = \hat{b}_1$. Taking into account the equality $\hat{a}_2 = \hat{b}_2$ it follows that $\hat{a}_1$ and $\hat{a}_2$ are neighbors in Γ and so $\Gamma \to A$ and $\Gamma \to B$ are isomorphisms as required. □

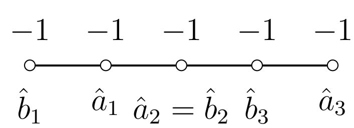
Next we describe the birational transformations of the standard graphs $[[0_{2k+1}]]$.

Definition 3.5. For $A = [[0_{2k+1}]]$ with vertices $a_1, a_2, \ldots, a_{2k+1}$, we let $\tau : A \dashrightarrow B$ be the birational transformation consisting of the outer blowup at a_1 and the inner blowups at the edges $[a_{2i}, a_{2i+1}]$, $i = 1, \ldots, k$ followed by the contraction of the vertices $\hat{a}_{2i+1}$, $i = 0, \ldots, k$. Thus τ fits the diagram (9) with $\Gamma = [[(-1)_{3k+2}]]$ and $B = [[0_{2k+1}]]$. We call τ the *left move* and τ^{-1} the *right move.*

A move τ admits a decomposition into a sequence of elementary transformations consisting in the above blowups and blowdowns, once at time.

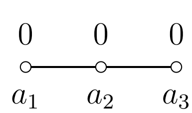
Example 3.6. The linear chain

$\Gamma = [[(-1)_5]]$:

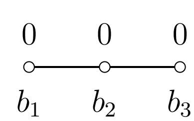


dominates two linear chains

$A = [[0_3]]$:



and $B = [[0_3]]$:



resulting in a left move $\tau : A \dashrightarrow B$.

Proposition 3.7. *Let $A = [[0_{2k+1}]]$, $B = [[0_{2l+1}]]$ and let Γ be a linear chain. If $\Gamma \to A$, $\Gamma \to B$ is a pair of dominations then $k = l$ and the resulting birational transformation $A \dashrightarrow B$ is equal to τ^s for some $s \in \mathbb{Z}$. In particular it is a composition of elementary transformations.*

Proof. We may assume that $k \le l$. With the notation as before, by Lemma 3.3(a) we have $\hat{a}_{2i} = \hat{b}_{2i}$ in Γ $\forall i = 1, \ldots, k$ and, similarly, $\hat{a}_{2k-2i} = \hat{b}_{2l-2i}$ $\forall i = 0, \ldots, k-1$. This is only possible if $k = l$. If $\hat{a}_1 = \hat{b}_1$ in Γ then applying Lemma 3.3(b) repeatedly we obtain that $\hat{a}_{2i+1} = \hat{b}_{2i+1}$ for $i = 1, \ldots, k$ and so $\hat{A} = \hat{B}$.

Now assume that $\hat{a}_1 < \hat{b}_1$. Using again Lemma 3.3(b) gives that $\hat{a}_{2i+1} < \hat{b}_{2i+1}$ for $i = 1, \ldots, k$. Hence to obtain Γ from B all the edges $[b_{2i}, b_{2i+1}]$ are blown up, and furthermore an outer blowup is performed at the vertex b_1. Consequently the linear chain $B' = \tau(B)$ is dominated as well by Γ. Moreover the distance of $\hat{a}_1$ to $\hat{B}'$ in Γ is strictly smaller than the distance to $\hat{B}$. Using induction on this number the assertion follows. $\square$

3.2. **Linear dominations of semistandard chains.** It follows from Propositions 3.4, 3.7 and Remark 2.14 that every birational transformation $A \dashrightarrow B$ of semistandard linear chains dominated by a linear chain Γ can be decomposed into a sequence of elementary transformations. Later on we need the stronger result which says that one can dominate A, B and the intermediate graphs even by a suitable *inner* blowup of Γ. To deduce this fact we need the following observation.

Lemma 3.8. *Let $\Gamma' \to \Gamma$ be a domination of weighted graphs and let $\gamma : \tilde{\Gamma} \dashrightarrow \Gamma$ be an admissible transformation. Then there is a commutative diagram*

$$\begin{tikzcd} \tilde{\Gamma}' \arrow[r, "\gamma'"] \arrow[d] & \Gamma' \arrow[d] \\ \tilde{\Gamma} \arrow[r, dashed, "\gamma"] & \Gamma , \end{tikzcd}$$

where the solid arrows are dominations and γ' is admissible. Moreover, if γ is inner then also γ' can be chosen to be inner.

Proof. Decomposing γ into a sequence of admissible blowups and blowdowns, it is enough to consider the following 3 cases: (i) γ is an admissible blowup in $\tilde{\Gamma}$ of a vertex v of Γ, (ii) γ is an inner blowup at an edge $e = [v_1, v_2]$ of Γ, or (iii) γ is an outer blowup at an end vertex v_0 of Γ. So in case (i) $\tilde{\Gamma}$ has 1 vertex more than Γ, and in cases (ii), (iii) it has 1 vertex less than Γ.

In case (i) Γ and then also Γ' dominates $\tilde{\Gamma}$, so we can choose $\tilde{\Gamma}' = \Gamma'$ and $\gamma' =$id. In case (ii), if the edge e is blown up in Γ' then we can again choose $\tilde{\Gamma}' = \Gamma'$ and $\gamma' =$id. Otherwise the proper transforms $\hat{v}_1$ and $\hat{v}_2$ of v_1, v_2 in Γ' are neighbors. Blowing up the edge $[\hat{v}_1, \hat{v}_2]$ we obtain a graph $\tilde{\Gamma}'$ with the desired properties.

Similarly, if in case (iii) $\Gamma' \to \Gamma$ factors through an outer blowup at v_0, we can choose $\tilde{\Gamma}' = \Gamma'$ and $\gamma' =$id. Otherwise the proper transform $\hat{v}_0$ of v_0 in Γ' remains an end vertex, and we can choose γ' to be the outer blowup at $\hat{v}_0$. $\square$

Remark 3.9. The pair of dominations $\tilde{\Gamma}' \to \Gamma'$, $\tilde{\Gamma}' \to \tilde{\Gamma}$ is not necessarily relatively minimal. Indeed, the same vertex can appear or disappear under $\Gamma' \to \Gamma$ and $\tilde{\Gamma} \dashrightarrow \Gamma$, cf. Remark 3.31 below.

Now we can deduce the following result.

Proposition 3.10. *If a linear chain Γ dominates two semistandard chains*[6] *A and B then B can be obtained from A by a sequence of elementary transformations such that every step is dominated by some inner blowup of Γ.*

Proof. As noted at the beginning of this subsection, A can be obtained from B by a sequence of elementary transformations and so, in particular, A and B have the same length. We denote the vertices of A and B as before by $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$, respectively.

We proceed by induction on n. If both chains A and B are standard then the result is a consequence of Propositions 3.4 and 3.7. In particular this settles the case $n = 1$. Let us assume for the rest of the proof that $n \geq 2$.

If all weights in A or in B are ≤ -2 then $\hat{A} = \hat{B}$ similarly as in the proof of Proposition 3.4. Hence we may assume in the sequel that both A and B have zero weights. Up to interchanging A and B or reversing both chains we have to consider the following two cases.

Case 1: $a_1^2 = b_1^2 = 0$, Case 2: $a_1^2 = b_n^2 = 0$, but $a_n^2 \neq 0$ and $b_1^2 \neq 0$.

In case (1), by Lemma 3.3.a we have $\hat{a}_2 = \hat{b}_2$ and so, as in the proof of Proposition 3.4, $\Gamma' = \Gamma_{>\hat{a}_2}$ dominates the semistandard chains

$$A' = A \ominus \{a_1, a_2\} \quad \text{and} \quad B' = B \ominus \{b_1, b_2\}\,,$$

while $\Gamma'' = \Gamma_{<\hat{a}_2}$ dominates the chains $A'' = \{a_1\}$ and $B'' = \{b_1\}$. Applying the induction hypothesis and the case $n = 1$, B', B'' can be obtained from A', A'', respectively, by a sequence of elementary transformations such that every step is dominated by some inner blowup of Γ', Γ'', respectively. After performing the same sequence of elementary transformations in A we may assume that $A' = B'$ and $A'' = B''$ and so we are done.

In case (2) we have

$$A = [[0, \alpha, \ldots]] \quad \text{and} \quad B = [[\ldots, \beta, 0]]\,.$$

⋄ If $\alpha = a_2^2 = 0$ then we can move this pair of zeros in A to the right by a sequence of inner elementary transformations and so reduce to the case already treated. Indeed, by Lemma 3.8 the resulting new chain A is again dominated by an inner blowup of Γ. Thus we may suppose that $\alpha = a_2^2 \leq -2$ and, symmetrically, $\beta = b_{n-1}^2 \leq -2$, so A and B have the only zero weights $a_1^2 = b_n^2 = 0$.

⋄ If $\hat{a}_1$ is not an end vertex then Γ dominates as well the chain $A_1 = [[0, \alpha + 1, \ldots]]$ obtained from A by an elementary transformation involving an outer blowup. Using decreasing induction on $-\alpha$ the result can thus be reduced either to the case treated above where $a_1^2 = a_2^2 = 0$, or to the case where $\hat{a}_1$ is the leftmost vertex of Γ. Clearly the same argument applies to b_n provided that $\hat{b}_n$ is not an end vertex of Γ.

⋄ So we are finally reduced to the case where both $\hat{a}_1$ and $\hat{b}_n$ are end vertices of Γ, $\hat{a}_1$ is the leftmost one and $\hat{b}_n$ is the rightmost one. We may suppose that the pair of dominations $\Gamma \to A$, $\Gamma \to B$ is relatively minimal so that every (-1)-vertex in Γ is

[6] See Definition 2.13.

contained in $\hat{A} \cup \hat{B}$. By virtue of (5) $\hat{a}_1$ and $\hat{b}_n$ are the only possible (-1)-vertices in Γ. If $\hat{a}_1^2 = 0$ then $\hat{a}_1$ cannot be contracted in B and its image in B is the only 0-vertex b_n. Therefore $\hat{a}_1 = \hat{b}_n$ is the only end vertex of Γ, so $A = B = [[0]]$, which contradicts our assumption that $n \geq 2$. Thus $\hat{a}_1^2 < 0$, and passing from A to Γ an inner blowup at the edge $[a_1, a_2]$ was performed. Hence $\hat{a}_2^2 < a_2^2 = \alpha$ and, likewise, $\hat{b}_n^2 < b_n^2 = \beta$, $a_1^* \neq \{\hat{a}_1\}$ and $b_n^* \neq \{\hat{b}_n\}$, where a^* stands for the total transform of a in Γ. Moreover $a_1^2 = 0$ forces that $\hat{a}_1$ must be contained in b_n^* and so $\hat{b}_{n-1} < \hat{a}_1$ in Γ. Similarly, $\hat{b}_n$ must be contained in a_1^* and so $\hat{b}_n < \hat{a}_2$ in Γ contradicting the fact that $\hat{a}_1$ and $\hat{b}_n$ are the end vertices of Γ. This concludes the proof. $\square$

3.3. **The circular case.** In this subsection we treat birational transformations of standard circular graphs. Unlike in the linear case, there are many birational transformations of such graphs as soon as they contain 0-vertices. Nevertheless, all these transformations are composed of simple ones, which we call *turns* and *shifts*, see Theorem 3.18 below. Let us introduce the following notions.

Definition 3.11. A circular graph A will be called *almost standard* if it is standard with all weights < 0 [7] or if it can be written in the form

$$A = ((0_{2k+1},\, \alpha_0, \ldots,\, \alpha_n)), \quad k \geq 0,$$

where the sequence $(\alpha_0, \ldots,\, \alpha_n)$ can be either empty, or equal to one of

$$(w) \quad \text{with} \quad w \leq 0, \quad (0, -1), \quad (-1, 0), \quad (0, -1, -1) \quad \text{or} \quad (-1, -1, 0)\,,$$

or satisfies the conditions

$$n \geq 1, \quad \alpha_0,\, \alpha_n \leq 0, \quad \alpha_0 + \alpha_n \leq -2 \quad \text{and} \quad \alpha_i \leq -2 \;\forall i = 1, \ldots, n-1\,.$$

Remarks 3.12. 1. We note that for any sequence $(\alpha_0, \ldots, \alpha_n)$ as in Definition 3.11 the standard form of the chain $[[\alpha_0, \ldots, \alpha_n]]$ contains at most two zeros. Hence its intersection form has at most 1 positive eigenvalue, see Proposition 4.11 in the Appendix.

2. Slightly more generally, the standard form of the chain $[[0_{2l}, \alpha_0, \ldots, \alpha_n]]$ contains at most $2l + 2$ zeros.

Clearly every standard circular graph is almost standard. To treat birational transformations of standard graphs it is convenient to consider these more generally for almost standard graphs. In a special case their classification is simple.

Lemma 3.13. *Let A, B be almost standard circular graphs and let $\Gamma \to A$ and $\Gamma \to B$ be dominations. If all weights of A or B are ≤ -1, then $\hat{A} = \hat{B}$.*

Proof. If all weights of A or B are ≤ -2 then with the same argument as in the proof of Proposition 3.4 it follows that $\hat{A} = \hat{B}$. This argument also works if A or B is one of the circular graph $((-1, -1))$, $((-1))$, whence the result. $\square$

In the case that A or B contain 0-vertices the situation is much more complicated. In analogy with moves of the linear chains $[[0_{2k+1}]]$ (see Definition 3.5), we introduce the following operation on such almost standard circular graphs.

[7] that is (see (4)) A is one of the graphs $((w))$, $w \leq -1$, $((-1, -1))$, $((\alpha_1, \ldots,\, \alpha_n))$, $\alpha_i \leq -2\, \forall i$.

Definition 3.14. Let $A = ((0_{2k+1}, \alpha_0, \ldots, \alpha_n))$ be an almost standard circular graph. By a *shift* we mean the birational transformation

$$\sigma : A = ((0_{2k+1}, \alpha_0, \ldots, \alpha_n)) \dashrightarrow A' = ((0_{2k+1},\, \alpha_0 - 1,\, \alpha_1,\, \ldots,\, \alpha_{n-1},\, \alpha_n + 1))$$

composed of a sequence of elementary transformations, which send the almost standard graph

$$A = ((0_{2k+1},\, \alpha_0, \ldots,\, \alpha_n)) = ((\alpha_n, 0_{2k+1},\, \alpha_0, \ldots,\, \alpha_{n-1}))$$

as in 3.11 with $n \geq 1$ into

$$((\alpha_n + 1, 0_{2k+1},\, \alpha_0 - 1,\, \alpha_1,\, \ldots,\, \alpha_{n-1})) = ((0_{2k+1},\, \alpha_0 - 1,\, \alpha_1,\, \ldots,\, \alpha_{n+1},\, \alpha_n + 1))\,,$$

see Lemma 2.12(b).

The resulting graph $A' = \sigma(A)$ is almost standard provided that $\alpha_n \leq -1$. Thus the shift of at least one of the graphs A or

$$A^* := ((0_{2k+1},\, \alpha_n,\, \alpha_{n-1},\, \ldots,\, \alpha_1,\, \alpha_0))$$

is again almost standard. Since A can be written in the form A^*, the inverse of a shift is again a shift.

Remarks 3.15. 1. Any almost standard circular graph $((0_{2k+1},\, \alpha_0, \ldots, \alpha_{n-1},\, \alpha_n))$ with $n \geq 1$ can be transformed into a standard one $((0_{2k+2},\, \alpha_1, \ldots, \alpha_{n-1},\, \alpha_n + \alpha_0))$ by a sequence of shifts.

2. A shift transforms a standard circular graph $A = ((0_{2k}, \alpha_1, \ldots, \alpha_n))$ with $k \geq 1$ and $\alpha_1, \ldots, \alpha_n \leq -2$ into $((0_{2k-1}, -1, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n + 1))$. Thus by a sequence of shifts we can transform A into the standard graph

$$((0_{2k-1}, \alpha_n, \alpha_1 \ldots, \alpha_{n-1}, 0)) = ((0_{2k}, \alpha_n, \alpha_1, \ldots, \alpha_{n-1})).$$

Hence, if for A as above and $B = ((0_{2k}, \beta_1 \ldots, \beta_n))$, the sequences $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are equal up to a cyclic permutation and reversion, then A and B are birationally equivalent via a sequence of shifts.

For instance, the standard circular graphs

$$((0, 0, -3, -5, -2)), \quad ((0, 0, -2, -3, -5)) \quad \text{and} \quad ((0, 0, -5, -2, -3))$$

obtained one from another by cyclic permutations of the nonzero weights, are birationally equivalent via a sequence of shifts.

3. Let $a_1, \ldots, a_{2k+1}, a_{2k+2}, \ldots a_{2k+n+2}$ be the vertices of A numbered according to the ordering of weights in $((0_{2k+1}, \alpha_0, \ldots, \alpha_n))$. Then the vertices a_i with i even or $i \geq 2k+2$ are not blown down by a shift σ as in Definition 3.14. More precisely, if A and A' are dominated by a graph Γ, where $A' = \sigma(A) = ((0_{2k+1}, \alpha_0 - 1, \ldots, \alpha_n + 1))$ has vertices $a'_1, \ldots, a'_{2k+1}, a'_{2k+2}, \ldots a'_{2k+n+2}$, then $\hat{a}_i = \hat{a}'_i$ in Γ for i even or $i \geq 2k+2$. Indeed, performing elementary transformations of a chain $[[w, 0, w']]$ at the 0-vertex in the middle, the two outer vertices will not be blown down.

4. Implicitly, the definition of shifts addresses as well cyclic renumbering. However, ignoring this procedure will not create serious ambiguity.

Definition 3.16. Let $A = ((0_{2k+1}, \alpha_0, \ldots, \alpha_n))$ be an almost standard circular graph. A *turn* $\tau : A \dashrightarrow A$ consists in a sequence of elementary transformations sending A first into

$$((0_{2k-1}, \alpha_0, 0, 0, \alpha_1, \ldots, \alpha_n)), \qquad \text{then into} \qquad ((0_{2k-1}, \alpha_0, \alpha_1, 0, 0, \alpha_2, \ldots, \alpha_n))$$

(see Lemma 2.12.a), until we arrive at

$$A' = ((0_{2k-1}, \alpha_0, \dots, \alpha_n, 0, 0)) = ((0_{2k+1}, \alpha_0, \dots, \alpha_n)) \cong A\,.$$

The inverse birational transformation will also be called a turn.

In analogy with Remark 3.15(2) we make the following observation.

Remark 3.17. Let $a_1, \dots, a_{2k+1}, a_{2k+2}, \dots a_{2k+n+2}$ be the vertices of A ordered correspondingly to the weights $((0_{2k+1}, \alpha_0, \dots, \alpha_n))$. Assume that Γ dominates both A and $A' = ((0_{2k-1}, \alpha_0, \dots, \alpha_n, 0, 0))$, where A' with vertices $a'_1, \dots, a'_{2k+n+2}$ is obtained from A by a turn as in Definition 3.16. Then in Γ we have $\hat{a}_i = \hat{a}'_i$ for $i \le 2k$ and $i = 2k+n+2$.

The following theorem gives a complete description of birational transformations between standard circular graphs.

Theorem 3.18. *Any birational transformation of standard circular graphs $A \dashrightarrow B$ is either an isomorphism or it can be written as a composition of turns and shifts.*

Before turning to the proof we mention the following corollary.

Corollary 3.19. *Any birational transformation of standard circular graphs $A \dashrightarrow B$ can be written as sequence of elementary transformations. More precisely, if $\Gamma \to A$ and $\Gamma \to B$ are dominations then A can be obtained from B by a sequence of elementary transformations such that every step is dominated by some inner blowup of Γ.*

Proof. The first part is an immediate consequence of Theorem 3.18. The second part follows from this in view of Lemmata 2.7 and 3.8. □

Theorem 3.18 is shown in 3.27 and is a consequence of Lemmata 3.13 above and 3.23, 3.26 below. The strategy of the proof is as follows. Consider a pair of dominations $\Gamma \to A$ and $\Gamma \to B$, where A and B are standard circular graphs. By Lemma 3.23 below, applying shifts we may achieve that $\hat{A} \cap \hat{B} \neq \emptyset$, and moreover, $\hat{a} = \hat{b}$ for some vertices a of A and b of B with $a^2 = b^2 = 0$. In a second step we reduce the statement to the linear case by restricting to $A \ominus \{a\}$, $B \ominus \{b\}$ and $\Gamma \ominus \{\hat{a} = \hat{b}\}$.

The following simple example shows that the case $\hat{A} \cap \hat{B} = \emptyset$ indeed occurs.

Example 3.20. Contracting in $\Gamma = ((-3, -1, -2, -2, -1))$ alternatively the subchains $[[-3, -1, -2]]$ or $[[-2, -2, -1]]$ yields the standard circular graphs $A = B = ((0, 0))$, and we have $\hat{A} \cap \hat{B} = \emptyset$.

To deal with the case $\hat{A} \cap \hat{B} = \emptyset$ we introduce a portion of notation.

3.21. We let $\Gamma \to A$, $\Gamma \to B$ be a relatively minimal pair of dominations of almost standard circular graphs A and B satisfying $\hat{A} \cap \hat{B} = \emptyset$. We also let $\hat{A}_1, \dots, \hat{A}_s$ and $\hat{B}_1, \dots, \hat{B}_t$ be the connected components of the graphs $\hat{A}$ and $\hat{B}$, respectively. Then A_i, B_i are connected subchains of A, B, respectively, so that

$$A = ((A_1, \dots, A_s)) \quad \text{and, similarly,} \quad B = ((B_1, \dots, B_t))\,.$$

The nonempty linear subchain, call it X_i, between $\hat{A}_i$ and $\hat{A}_{i+1}$ in Γ is contracted in A, so it contains a (-1)-vertex. By the relative minimality assumption this vertex must be in $\hat{B}$. Hence X_i includes some component $\hat{B}_j$. Similarly, between $\hat{B}_j$ and $\hat{B}_{j+1}$

there is a unique component $\hat{A}_i$. This implies that $s = t$, and with an appropriate enumeration of the components B_i we can write

$$\Gamma = ((E_1\,\hat{A}_1\,F_1\,\hat{B}_1\,E_2\,\hat{A}_2\,F_2\,\hat{B}_2\,\ldots\,E_s\,\hat{A}_s\,F_s\,\hat{B}_s))\,.$$

Using indices in $\mathbb{Z}/s\mathbb{Z}$ (so $E_{i+s} = E_i\ \forall i$) the chains

$$X_i = F_i\hat{B}_iE_{i+1} \qquad \text{and} \qquad Y_i = E_i\hat{A}_iF_i$$

are contractible, so they contain at least one (-1)-vertex lying in $\hat{B}_i$ and $\hat{A}_i$, respectively.

Lemma 3.22. *With the assumptions and notation as in 3.21 the following hold.*

(1) $\hat{A}_i$ and A_i have no vertex of weight ≥ 0 in their interior.
(2) $\hat{A}_i$ contains a (-1)-vertex but not the string $[[-1,-1]]$. Consequently, A_i has no chain $[[-1,-1]]$ in its interior.
(3) Each A_i has at least 2 vertices.
(4) It is not possible that $\hat{A}_i = [[\ldots,-1]]$ and $\hat{A}_{i+1} = [[-1,\ldots]]$ simultaneously.
(5) One of the end vertices of A_i has weight 0.

The same assertions hold for the components B_i.

Proof. (1) and (2) follow from the fact that $\hat{A}_i$ is part of a contractible chain and so every subchain of $\hat{A}_i$ has a negative definite intersection matrix.

To show (3), assume that A_i has only one vertex. As $\hat{A}_i$ contains a (-1)-vertex, necessarily $A_i = [[0]]$ in this case. But then to obtain Γ from A a blowup would occur only on one side of A_i, which leads to a contradiction since the chains $X_i = F_j\hat{B}_jE_{j+1}$ are all nonempty.

If (4) were violated then between A_i and A_{i+1} just one blowup would occur, so B_i would be a chain of length 1 contradicting (3).

To deduce (5), if for some i none of the end vertices of A_i were of weight 0 then by (1) the end vertices of $\hat{A}_i$ would be of weight ≤ -2. Hence a (-1)-vertex of $\hat{A}_i$ must lie in the interior of $\hat{A}_i$ and so it is a (-1)-vertex in A_i too. Thus $A = ((0_{2k+2},-1,-1))$ or $A = ((0_{2k+1},\alpha_0,\alpha_1,\ldots,\alpha_n))$ with $\alpha_0 = -1$ or $\alpha_n = -1$ in Definition 3.11. However in both cases any connected linear subchain of A satisfying (1) with a (-1)-vertex in its interior has an end 0-vertex. □

Lemma 3.23. *Let A and B be almost standard circular graphs, and let $\Gamma \to A$, $\Gamma \to B$ be a relatively minimal pair of dominations with $\hat{A} \cap \hat{B} = \emptyset$. Then A can be transformed by a finite sequence of shifts all dominated by Γ into a new almost standard circular graph A' satisfying $\hat{A}' \cap \hat{B} \neq \emptyset$.*

Proof. By Lemma 3.13 A is almost standard of the form $A = ((0_{2k+1},\alpha_0,\ldots,\alpha_n))$ as in Definition 3.11. For $n \geq 1$, since $\alpha_0 + \alpha_n \leq -1$ and A can be written in the form $((0_{2k+1},\alpha_n,\ldots,\alpha_0))$, we may assume that $\alpha_n \leq -1$. Using Lemma 3.22(1) and (3), at least k among the components A_i, say, $A_1,\ldots,A_k$, are equal to $[[0,0]]$. If $\alpha_0 \leq -1$ then by virtue of Lemma 3.22(5), up to reversion there is just one extra component $A_{k+1} = [[\alpha_0,\ldots,\alpha_n,0]]$. If $\alpha_0 = 0$ then either there is just one extra component $A_{k+1} = [[0,\alpha_1,\ldots,\alpha_n,0]]$, or $n \geq 2$ and there are 2 extra components

$$A_{k+1} = [[0,\alpha_1,\ldots,\alpha_l]] \quad \text{and} \quad A_{k+2} = [[\alpha_{l+1},\ldots,\alpha_n,0]], \quad \text{where} \quad 1 \leq l \leq n-1\,.$$

Actually the latter case cannot occur. Indeed by Lemma 3.22(4), if one of the chains $\hat{A}_i$ ends with a (-1)-vertex, say on the right, then all of them, in particular A_{k+1}, will have a (-1)-vertex on the right. But this is impossible since $\alpha_i \le -1$ $\forall i = 1, \dots, n$.

Thus for any $n \ge 0$,

$$A_i = [[0, 0]] \quad \forall i = 1, \dots, k \qquad \text{and} \qquad A_{k+1} = [[\alpha_0, \dots, \alpha_n, 0]] \,.$$

Let us first assume that $\hat{A}_i$ has a vertex of weight -1 on the right for at least one i in the range $1 \le i \le k+1$. Clearly Γ dominates the graph obtained from A by blowing up all edges between A_i and A_{i+1} (using cyclic indices as before). Therefore Γ dominates both A and the semistandard circular graph

$$A' := \sigma(A) = ((0_{2k+1}, \alpha_0 - 1, \alpha_1, \dots, \alpha_{n-1}, \alpha_n + 1))$$

obtained from A by a shift. Consider the blowdown Γ' of the (-1)-vertex on the right of $\hat{A}_i$. By construction A' and B are still dominated by Γ'. Applying induction on the number of vertices of Γ, the result follows.

For the rest of the proof we may, and we will, assume that $\hat{A}_i$ has no (-1)-vertex on the right, so $\hat{A}_i = [[-1, w_i]]$ for $i = 1, \dots, k$. If $\alpha_0 = 0$, then by symmetry the same argument as before works. Thus we may assume that $\alpha_0 < 0$. Using Lemma 3.22(4) $\hat{A}_{k+1}$ has no end vertex of weight -1. Since it contains a (-1)-vertex (see Lemma 3.22(2)) it follows that this vertex lies in the interior of $\hat{A}_{k+1}$ and so it is also a (-1)-vertex of A_{k+1}. As $\alpha_n \le -1$ and inspecting 3.11 this is only possible when $\alpha_n = -1$. Now we can reduce to the case already treated: Γ dominates both A and the semistandard graph $A' = ((0_{2k+1}, \alpha_0 - 1, \alpha_1, \dots, \alpha_{n-1}, 0)) = ((0_{2k+1}, \alpha'_0, \dots, \alpha'_n))$, where

$$(\alpha'_0, \dots, \alpha'_n) = (0, \alpha_0 - 1, \alpha_1, \dots, \alpha_{n-1}) \,.$$

Replacing A by A' and applying the previous case the result follows. $\square$

3.24. Because of Lemma 3.13 we may, and we will, assume in the rest of the proof of Theorem 3.18 that A and B are both almost standard with a vertex of weight 0. Let $\Gamma \to A$ and $\Gamma \to B$ be a pair of dominations with $\hat{A} \cap \hat{B} \neq \emptyset$. We fix an orientation of Γ and orient A and B accordingly. With these orientations let us write

$$A = ((0_{2k+1}, \alpha_0, \dots, \alpha_n)) \quad \text{and} \quad B = ((0_{2l+1}, \beta_0, \dots, \beta_m))$$

as in 3.11. Using the same ordering let us denote the vertices of A and B by $a_1, \dots, a_{2k+n+2}$ and $b_1, \dots, b_{2l+m+2}$, respectively.

With this notation we have the following result.

Lemma 3.25. *For any vertex a of an almost standard circular graph*

$$A = ((0_{2k+1}, \alpha_0, \dots, \alpha_n)) \quad \textit{with} \quad k, n \ge 0,$$

there exists a sequence of shifts and turns transforming A into an almost standard circular graph $A' = ((0_{2k+1}, \alpha'_0, \dots, \alpha'_n))$ so that at any step a is not blown down and one of the following conditions is satisfied.

(i) $\alpha'_n \le -1$ and the image of a in A' is the 0-vertex a'_1; or

(ii) $n = 0$ and the image of a in A' is the vertex a'_{2k+2} of weight α'_0.

Moreover, if A was standard then A' can be assumed to be standard.

Proof. If $n \geq 1$ then we may suppose that $\alpha_n \leq -1$, since otherwise we can write A in the form $((0_{2k+1}, 0, \alpha_0, \ldots, \alpha_{n-1}))$. Let $a = a_i$ with the notations as in 3.24. According to the value of i we consider the following cases (a)-(d).

(a) $i = 2k + 2 + j$, where $0 \leq j \leq n$. If $n = 0$ then $j = 0$ and $a = a_{2k+2}$ as needed in (ii). Otherwise according to Remark 3.15(1) and (3) there is a sequence of shifts which do not contract a_i and transform A into a standard circular graph

$$A_1 = ((0_{2k+1},\ \alpha_0 + \alpha_n,\ \alpha_1, \ldots, \alpha_{n-1}, 0)) = ((0_{2k+1}, 0, \alpha_0 + \alpha_n,\ \alpha_1, \ldots, \alpha_{n-1}))\,.$$

If $i = 2k + n + 2$ then the image of a in A_1 occupies the 1-st position, as required in (i). Otherwise we continue as in Remark 3.15(2) and transform A_1 by a sequence of shifts into a standard graph

$$A' = ((0_{2k+2}, \alpha_j, \ldots, \alpha_{n-1},\ \alpha_0 + \alpha_n,\ \alpha_1, \ldots, \alpha_{j-1}))$$

moving the vertex a to the 1-st position.

(b) Suppose now that $1 \leq i = 2l + 1 \leq 2k + 1$. By a sequence of turns moving the $2l$ zeros on the left of a_i to the right, we can transform A into

$$A_1 = ((0 = a^2,\ 0_{2k},\ \alpha_0,\ \alpha_1, \ldots, \alpha_n))$$

so that a is not blown down and is placed onto the 1-st position, as needed.

(c) Suppose that $n \geq 1$ and $1 \leq i = 2l \leq 2k + 1$. In view of Remark 3.15(3), a is not blown down under the shifts which transform A into a standard circular graph

$$A_1 = ((0_{2k+1},\ \alpha_0 + \alpha_n,\ \alpha_1, \ldots, \alpha_{n-1}, 0)) = ((0_{2k+2},\ \alpha_0 + \alpha_n,\ \alpha_1, \ldots, \alpha_{n-1}))\,.$$

Now the vertex $a = a_i = a_{2l}$ has been moved to the position $2l + 1$. This provides a reduction to the previous case, so we can further move a to the 1-st position.

(d) Finally, if $n = 0$ and $1 \leq i = 2l \leq 2k+1$ then a can be moved to the last position by a sequence of $k - l + 1$ turns, see Remark 3.17.

The reasoning in all 4 cases shows that if we start with a standard circular graph A then also A' will be standard, finishing the proof of the lemma. □

We are now ready to complete the proof of Theorem 3.18 by the following lemma.

Lemma 3.26. *With the notations and assumptions as in 3.24 suppose that one of the graphs A, B is standard. If $\hat{a}_i = \hat{b}_j$ for some i, j then A can be obtained from B by a sequence of turns and shifts such that after an appropriate inner blowup of Γ, every step is dominated by Γ.*

Proof. By Lemma 3.25 it suffices to distinguish the following cases.

Case 1: $n = m = 0$ and $i = 2k + 2$, $j = 2l + 2$;

Case 2: $n = 0$, $i = 2k + 2$ and $j = 1$;

Case 3: $i = j = 1$.

Moreover by *loc.cit.* in the case $n \geq 1$ we may assume that $\alpha_n \leq -1$.

We note that under our assumptions the birational map $A \dashrightarrow B$ restricts to one between the chains $A' := A \ominus \{a_i\}$ and $B' := B \ominus \{b_j\}$ dominated by the chain $\Gamma' := \Gamma \ominus \{\hat{a}_i = \hat{b}_j\}$.

• In case 1 the chains $A' = [[0_{2k+1}]]$ and $B' = [[0_{2l+1}]]$ are birationally equivalent, so by Proposition 3.7 $k = l$ and $A' \dashrightarrow B'$ is equal to τ^s, where τ is a move. This amounts to an s-iterated shift $A \dashrightarrow B$, and the assertion follows.

• In case 2 we have $\hat{a}_{2k+2} = \hat{b}_1$, hence the birational map

$$A = ((0_{2k+1}, \alpha_0)) \dashrightarrow B = ((0_{2l+1}, \beta_0, \ldots, \beta_m))$$

restricts to

$$A' = [[0_{2k+1}]] \dashrightarrow B' = [[0_{2l}, \beta_0, \ldots, \beta_m]]\,.$$

By Remark 3.12(2) the standard form of the chain on the right can have at least $2l$ and at most $2l+2$ zeros, therefore $k = l$. Using Lemma 3.3.b repeatedly we get $\hat{a}_{2t} = \hat{b}_{2t+1}$ for $t = 1, \ldots, k$.

If $\hat{a}_1 = \hat{b}_2$ in Γ then by Lemma 3.3.b, $\hat{a}_{2t-1} = \hat{b}_{2t}$ for $t = 1, \ldots, k+1$. In other words, $\hat{A} \subseteq \hat{B} = \Gamma$ and so A is a blowdown of B, which implies that $A = B$. For A and B are both almost standard and B does not admit a nontrivial blowdown to another almost standard circular graph.

Suppose now that $\hat{a}_1 \neq \hat{b}_2$ in Γ. Again by Lemma 3.3.b, either

$$\hat{b}_{2t} \in]\hat{a}_{2t-1}, \hat{a}_{2t} = \hat{b}_{2t+1}[\;\; \forall t = 1, \ldots, k+1 \quad \text{or} \quad \hat{a}_{2t-1} \in]\hat{b}_{2t}, \hat{a}_{2t} = \hat{b}_{2t+1}[\;\; \forall t = 1, \ldots, k+1.$$

In the first case we perform a shift of A by blowing up the edges $[a_{2t-1}, a_{2t}]$ and blowing down a_{2t-1} for $t = 1, \ldots k+1$. Similarly, in the second case we perform a shift in the other direction. In both cases Γ will dominate the corresponding shift $A \dashrightarrow \tilde{A}$. Replacing A by $\tilde{A}$ diminishes the distance between $\hat{a}_1$ and $\hat{b}_2$ in Γ and does not affect the vertices $\hat{a}_{2t} = \hat{b}_{2t+1}$, $t = 1, \ldots, k$ (cf. Remark 3.15(3)). After a finite sequence of shifts we can achieve that $\hat{a}_1 = \hat{b}_2$ and hence $\hat{A} = \hat{B}$. This concludes case 2.

• In case 3 we may suppose that $k \leq l$. Since $\hat{a}_1 = \hat{b}_1$, the map $A \dashrightarrow B$ restricts to

$$A' = [[0_{2k}, \alpha_0, \ldots, \alpha_n]] \dashrightarrow B' = [[0_{2l}, \beta_0, \ldots, \beta_m]]\,.$$

Using Lemma 3.3.a, $\hat{a}_{2s+1} = \hat{b}_{2s+1}$ $\forall s = 1, \ldots, k$; this is true as well if $k = 0$. Hence $A \dashrightarrow B$ further restricts to a birational map

$$A'' = [[\alpha_0, \ldots, \alpha_n]] \dashrightarrow B'' = [[0_{2(l-k)}, \beta_0, \ldots, \beta_m]]\,. \tag{10}$$

Let us consider the cases $k < l$ and $k = l$ separately.

⋄ In the case $k < l$ necessarily $n \geq 1$ and $\alpha_0 = 0$, so A is standard, since otherwise A'' would transform into a standard chain with at most one zero weight, whereas the standard form of B'' has at least 2 zero weights. Comparing the standard models of A'' and B'' yields as well that $l = k+1$. Applying Lemma 3.3.a gives $\hat{a}_{2k+3} = \hat{b}_{2k+3}$. By Lemma 3.3.b, the order in the pair of vertices $\hat{a}_2, \hat{b}_2$ in Γ is inherited by the pairs $\hat{a}_{2t}, \hat{b}_{2t}$ $\forall t = 1, \ldots, k+1$. Hence as before, if $\hat{a}_2 \neq \hat{b}_2$ then Γ dominates a shift of A which diminishes the distance between $\hat{a}_2$ and $\hat{b}_2$. By a finite sequence of such shifts we can transform A into a graph

$$C = ((\eta, 0_{2k+1}, \alpha_1 - \eta, \alpha_2, \ldots, \alpha_n))\,, \quad \eta \in \mathbb{Z}, \tag{11}$$

with vertices $c_1, \ldots, c_{2k+n+2}$ such that $\hat{c}_{2t-1} = \hat{a}_{2t-1} = \hat{b}_{2t-1}$ for $t = 1, \ldots, k+2$ and, moreover, $\hat{c}_2 = \hat{b}_2$. Thus by Lemma 3.3.b, $\hat{c}_{2t} = \hat{b}_{2t}$ for $t = 1, \ldots, k+1$.

We may further assume that the pair of dominations $\Gamma \to C$ and $\Gamma \to B$ is relatively minimal i.e., $\Gamma \ominus (\hat{C} \cup \hat{B})$ does not contain a (-1)-vertex. Since the weights $\alpha_2, \ldots, \alpha_{n-1}$ of C are ≤ -2 and $\hat{c}_t = \hat{b}_t$ $\forall t = 1, \ldots, 2k+3$, no (-1)-vertex in $\hat{C}$ can be blown down in B. Hence $\Gamma = B$ and so C is a blowdown of B. Since the weight of a vertex can only increase under a blowdown, this gives that $\eta, \alpha_1 - \eta \geq 0$ and thus $\alpha_1 \geq 0$, which is a contradiction.

$\diamond$ Finally let us suppose that $k = l$. By assumption, one of the graphs, say A, is standard, see (4). Thus in (10)

$$A'' = [[0, \alpha_1, \dots, \alpha_n]] \sim B'' = [[\beta_0, \dots, \beta_m]] ,$$

unless $A = ((0_{2k+1}, w))$ and $A'' = [[w]]$ with $w \leq -1$.

In the latter case either $B'' = \emptyset$ and interchanging A and B returns us to case 2, or $B'' \neq \emptyset$ contracts to A'' and so B contracts to A. However since $a_1^2 = b_1^2 = a_{2k+1}^2 = b_{2k+1}^2 = 0$ no contraction in B is possible, so $A = B$, as required.

In the former case, if $n = 0$ then after renumbering the vertices of A we are in case 2, which was already treated. If $n \geq 1$ then similarly as above $\beta_0 = 0$ and B is as well standard, cf. Definitions (4) and 3.11. By Lemma 3.3.a we get $\hat{a}_{2i+1} = \hat{b}_{2i+1}$ for $i = 0, \dots, k+1$. Applying shifts to A as in the previous case we can again transform A into a graph C as in (11) which satisfies $\hat{c}_i = \hat{b}_i$ for $i = 1, \dots, 2k+3$. Arguing as before it follows that C is a blowdown of $\Gamma = B$.

If $C = B$ then we are done. If $C \neq B$ then at least one blowdown occurs. Since B is standard, this is only possible if $n = 2$, $B = [[0_{2k+2}, -1, -1]]$, and then C is obtained from B by blowing down the (-1)-vertex b_{2k+4}. Hence $m = 1$ and $\alpha_1 - m = 0$, which gives $\alpha_1 = 1$. This is impossible since A is standard, so indeed $C = B$ as desired. $\square$

3.27. *Proof of Theorem 3.18.* Given a birational morphism $A \dashrightarrow B$ of standard circular graphs, we let $\Gamma \to A$ and $\Gamma \to B$ be a corresponding pair of dominations. If all weights of A or of B are ≤ -1 then the assertion follows by Lemma 3.13. Otherwise, by Lemma 3.23 A can be transformed by a sequence of shifts into an almost standard circular graph A' such that $\hat{A}' \cap \hat{B} \neq \emptyset$. Applying Lemma 3.26 to A' and B the result follows. $\square$

¿From Theorem 3.18 we deduce the following.

Corollary 3.28. *Every standard circular graph $A = ((0_{2k}, a_1, \dots, a_n))$ with $k, n \geq 0$ and $a_1, \dots, a_n \leq -2$ is unique in its birational equivalence class up to cyclic permutation and reversion of the sequence $(a_1, \dots, a_n)$. Moreover the standard circular graphs $((0_l, w))$ ($l \geq 0$, $w \leq 0$) and $((0_{2k}, -1, -1))$ ($k \geq 1$) are also unique in their birational equivalence classes.*

Proof. If A is a graph as in 3.11 then the sequence of numbers $\alpha_0 + \alpha_n, \alpha_1, \dots, \alpha_{n-1}$ remains unchanged by turns and shifts up to a cyclic permutation and reversion. Since by Theorem 3.18 an arbitrary birational transformation is a composition of turns and shifts, the result follows. $\square$

3.4. **Proof of Theorem 3.1.** We need the following stronger form of Proposition 2.9.

Lemma 3.29. *If the weighted graph Γ dominates two non-circular standard graphs A, B then there is a sequence of birational transformations*

$$(*) \qquad \begin{array}{ccccccccccc} & & \Delta_1' & & & & \Delta_2' & \dots & \Delta_n' & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow \; \dots & & \searrow & \\ A = \Delta_0 & & & & \Delta_1 & & & \Delta_2 \; \dots & & & \Delta_n = B , \end{array}$$

where all the Δ_i are semistandard and $\Delta_i' \to \Delta_{i-1}$ and $\Delta_i' \to \Delta_i$ are admissible. Moreover, after a finite sequence of inner blowups of Γ all the Δ_i' are dominated by Γ.

Proof. By Proposition 2.9 we can find a sequence as in $(*)$, where all the $\Delta_i' \to \Delta_{i-1}$ and $\Delta_i' \to \Delta_i$ are admissible. To obtain from such a sequence another one in which all the Δ_i are also semistandard, we proceed as follows. By Theorem 2.15 there is a semistandard graph $\tilde\Delta_i$ which is obtained from Δ_i by an inner transformation. By Lemma 3.8 there are sequences of inner blowups $\tilde\Delta_i' \to \Delta_i'$ and $\tilde\Delta_{i+1}' \to \Delta_i'$ such that $\tilde\Delta_i$ is dominated by $\tilde\Delta_i'$ and $\tilde\Delta_{i+1}'$. Using again Lemma 3.8 after inner blowups Γ will dominate $\tilde\Delta_i'$ and $\tilde\Delta_{i+1}'$. Replacing Δ_i, Δ_i', Δ_{i+1}' by $\tilde\Delta_i$, $\tilde\Delta_i'$, $\tilde\Delta_{i+1}'$, respectively, the new graph Δ_i will be semistandard. After a finite number of steps we arrive at a sequence $(*)$ as required. ☐

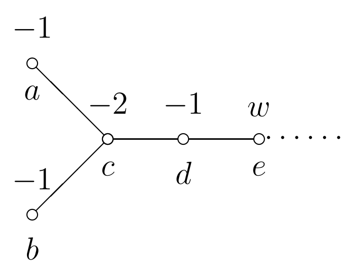
In general, one cannot achieve that all graphs Δ_i as in the lemma above are standard. This can be seen by the following simple example.

Example 3.30. Consider the diagram of dominations

$$\begin{tikzcd}
 & & \Gamma \arrow[dl] \arrow[dr] & & \\
 & \tilde A \arrow[dl] \arrow[dr] & & \tilde B \arrow[dl] \arrow[dr] & \\
A \arrow[rr, dashed, "\alpha"] & & A' \arrow[rr, dashed, "\beta"] & & B\,,
\end{tikzcd}$$

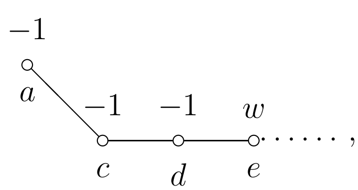
where α, β are elementary transformations, Γ is the non-linear graph

Γ :

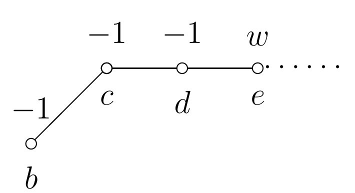


the other graphs are the linear chains

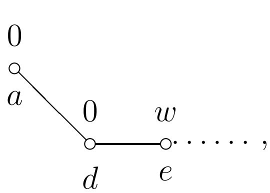
$\tilde A$:



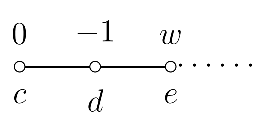
$\tilde B$:



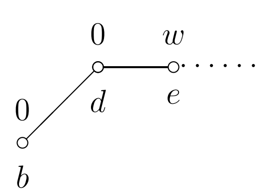
A :



A' :



B :



We notice that $\Gamma \to A$, $\Gamma \to B$ is a relatively minimal pair of dominations. To transform A' into a standard chain an outer blowup at c is required. However, it is not possible to lift this outer blowup to Γ, $\tilde{A}$ and $\tilde{B}$ simultaneously in such a way that $\tilde{A}$ and $\tilde{B}$ remain both linear.

Remark 3.31. It is worthwhile to compare this example with Lemma 3.8. Letting in this lemma $\Gamma = B$, $\Gamma' = \tilde{B}$ and $\tilde{G} = A$ as in Example 3.30 and proceeding as in the proof of the lemma yields a pair of dominations of A and $\tilde{B}$ from a linear chain $\tilde{\Gamma}' = \tilde{B}$. The crucial difference consists in the following. Attributing names to vertices of the graph Γ as in 3.29 or 3.30 gives also names to vertices of A and B. This in general makes the procedure of the proof of 3.8 impossible in the 1-st case of (iii). Actually keeping track of these names, if the outer blowups at v_0 gives two different vertices in $\tilde{\Gamma}$ and in Γ', then we must create a branch point of $\tilde{\Gamma}'$. Fortunately, under the setup as in Lemma 3.8, no names are prescribed in advance.

Proof of Theorem 3.1. Let $A \dashrightarrow B$ be a birational map between two standard graphs as in the theorem. By Lemma 3.29 it suffices to show that for any two admissible dominations $\sigma_i : \Gamma \to \Gamma_i$, where Γ_i are semistandard graphs $(i = 1, 2)$, $\gamma = \sigma_2 \circ \sigma_1^{-1}$ can be decomposed into a sequence of elementary transformations dominated by Γ after, possibly, some inner blowups of Γ.

Since σ_i is admissible, $B(\Gamma_i)\hat{} = B(\Gamma)$, $i = 1, 2$, and so for every segment Σ of Γ, σ_i induces an admissible domination $\sigma_i|\Sigma : \Sigma \to \Sigma_i$ onto the contraction Σ_i of Σ in Γ_i. It suffices to show that $\gamma|\Sigma_1 : \Sigma_1 \dashrightarrow \Sigma_2$ can be decomposed into a sequence of elementary transformations dominated by some inner blowup of Σ. The latter follows from Propositions 3.10 and 3.13[8] for semistandard linear segments Σ_1, Σ_2 and for standard circular graphs, respectively. □

We deduce the following analog of Corollary 3.28.

Corollary 3.32. *The standard form of a linear chain is unique in its birational class up to reversion.*

Proof. If $\Gamma \to A$, $\Gamma \to B$ is a pair of dominations of standard linear chains A, B, then we can factor the corresponding birational transformation $A \dashrightarrow B$ as in Lemma 3.29. Using Remark 2.14(1), each of the semistandard graphs Δ_i in *loc.cit.* can be transformed into a standard linear chain, say A_i, $1 \le i \le n-1$, by a sequence of elementary transformations. It suffices to verify that A_i is equal to A_{i+1} or to its reversion for $i = 0, \ldots, n$. Using Lemma 3.8 the birational map between A_i and A_{i+1} can be dominated by some linear chain, say, L_i[9]. In other words, it suffices to treat the case where A and B are dominated by a linear graph.

In this situation, by virtue of Proposition 3.4, $A = B$ or $A = B^*$ as soon as at least one of the standard chains A and B with $A \sim B$ is different from $[[0_{2k+1}]]$ for all k. Otherwise $A = [[0_{2k+1}]]$ and $B = [[0_{2l+1}]]$, and Proposition 3.7 yields that $k = l$. Alternatively, k and l are equal to the number of positive eigenvalues of the associate bilinear form $I(A)$, $I(B)$, respectively, and this number is a birational invariant, see 4.1 in the Appendix below. □

Finally, Corollaries 3.32 and 3.28 yield the following result.

[8]See also Corollary 3.19.

[9]In general, neither A_i nor L_i are dominated by Γ.

Corollary 3.33. *Every non-circular standard weighted graph is unique in its birational equivalence class up to reversions of its linear segments. Similarly, a circular standard graph is unique in its birational equivalence class up to a cyclic permutation of its nonzero weights and reversion.*

3.5. **The geometric meaning.** In the geometric setup, Γ is the dual graph of a reduced divisor D with normal crossings (an NC-divisor, for short) on the regular part X_{reg} of a normal complete algebraic surface X. If $V = X \setminus D$ then any two such completions $V \hookrightarrow X_1$, $V \hookrightarrow X_2$ can be dominated by a third one $V \hookrightarrow X$:

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X_1 \arrow[rr, dashed] & & X_2 \end{tikzcd} \tag{12}$$

inducing the identity on V. Letting Γ_i, $i = 1, 2$, and Γ be the dual graphs of the corresponding boundary divisors D_1, D_2 and D, respectively, we get a diagram of dominations

$$\begin{tikzcd} & \Gamma \arrow[dl] \arrow[dr] & \\ \Gamma_1 \arrow[rr, dashed] & & \Gamma_2 \end{tikzcd} . \tag{13}$$

Proposition 3.34. *Suppose we are given a diagram (13) and two SN-completions (X_1, D_1) and (X, D) of the same normal algebraic surface V such that (X, D) dominates (X_1, D_1) and induces the given domination $\Gamma \to \Gamma_1$ for the dual graphs of D and D_1, respectively. Then there exists a unique SN-completion (X_2, D_2) of V dominated by (X, D) which fits the diagram (12) and induces the given domination $\Gamma \to \Gamma_2$ for the dual graphs of D and D_2, respectively.*

Proof. To get X_2 from X, it is enough to contract all the irreducible components in D which correspond to the vertices in $\Gamma \ominus \hat{\Gamma}_2$. $\square$

Definition 3.35. We say that an SN-completion (X, D) of V is *standard* if the dual graph Γ of D is.

Corollary 3.36. *Every normal algebraic surface V admits a standard completion (X, D). Moreover, any other such completion can be obtained from (X, D) by a sequence of elementary transformations.*

4. Appendix

4.1. **The adjacency matrix and the discriminant of a weighted graph.** For the sake of simplicity, we restrict in this section to weighted graphs with integral weights without loops and multiple edges. To such a graph Γ one usually associates its adjacency matrix $I(\Gamma) = (v_i.v_j)$, where $v_i^2 = w_i$ is the weight of the vertex v_i, whereas for $i \neq j$, $v_i.v_j = 1$ if v_i and v_j are joined by an edge, and $v_i.v_j = 0$ otherwise. It is well known [Ne, Prop. 1.1], [Ru, Prop. 1.14] that a blowup of Γ just adds a negative eigenvalue to this matrix, see (17) below. In particular, the number of positive (non-negative) eigenvalues is a birational invariant of Γ, and so is its *discriminant* $\delta(\Gamma) = \det(-I(\Gamma))$. If Γ' is a subgraph of Γ then $I(\Gamma')$, being a symmetric submatrix of $I(\Gamma)$, has at most the same number of positive eigenvalues as $I(\Gamma)$.

4.1. For a weighted graph Γ and a vertex v in G of weight $a = v^2$, we let

$$\delta_v(\Gamma) = \delta(\Gamma \ominus \{v\}) = \prod_j \delta(\Gamma_j)\,,$$

where Γ_j runs through the set of branches of Γ at v. If Γ_j is joined to v by a unique edge $[v, v_j]$ then the following holds (see e.g., [DrGo, Ne, OZ]):

$$-\,\delta(\Gamma) = a \cdot \delta_v(\Gamma) + \sum_j \delta_{v_j}(\Gamma_j) \cdot \prod_{i \neq j} \delta(\Gamma_i)\,. \tag{14}$$

4.2. Further, if Γ is contractible i.e., dominates the one-vertex graph with weight -1 then clearly Γ is a tree and $I(\Gamma)$ is negative definite of discriminant $\delta(\Gamma) = 1$. Vice versa, we have the following lemma.

Lemma 4.3. *(a) A weighted tree Γ with negative definite adjacency matrix $I(\Gamma)$ and discriminant $\delta(\Gamma) = 1$ is contractible.*
(b) Consequently, a weighted graph Γ with discriminant 1 that can be transformed into a one-vertex graph is contractible.

Proof. For (a) we refer the reader to [Mu, Hi_2] or [Ru, Prop. 1.20]. (b) is an immediate consequence of (a) due to the fact that the discriminant does not change under birational transformations. □

The following lemma is well known, see e.g., [Fu, 3.8] or [Mi, 3.3.1].

Lemma 4.4. *Let $\frac{m}{e}$ be the continued fraction*

$$\frac{m}{e} = [k_1, \ldots, k_n] = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \frac{1}{k_n}}} \qquad \text{with} \quad 0 \le e < m, \ \ \gcd(e, m) = 1\,,$$

where $k_i \in \mathbb{N}_{\ge 2}\ \forall i \ge 2$, and let Γ, Γ' be the linear chains $[[-k_1, \ldots, -k_n]]$, $[[-k_2, \ldots, -k_n]]$, respectively. Then $m = \delta(\Gamma)$ and $e = \delta(\Gamma')$.

Thus the pair $(\delta(\Gamma), \delta(\Gamma'))$ uniquely determines the linear chain Γ.

4.5. We let $L = [[w_1, \ldots, w_n]]$ $(n \ge 1)$ be a linear branch of a weighted graph Γ at a vertex $v_0 \in \Gamma^{(0)}$, where w_1 is the weight of the neighbor v_1 of v_0 in L and w_n is the weight of the end vertex v_n of L. We let Γ_0 be the graph obtained from Γ by deleting the branch L and changing the weight $w_0 = v_0^2$ to

$$w_0' = -[-w_0, -w_1, \ldots, -w_n] = w_0 + [-w_1, \ldots, -w_n]^{-1} = w_0 + \cfrac{1}{w_1 + \cfrac{1}{\ddots + \frac{1}{w_n}}}\,.$$

We denote by $\langle a \rangle$ the usual Euclidean quadratic form on $\mathbb{R}^1$ multiplied by a.

Lemma 4.6. *In the notation as above suppose that $w_1, \ldots, w_{n-1} \le -2$ and $w_n \le -1$. Then*

$$I(\Gamma) \sim I(\Gamma_0) \oplus \langle w_1' \rangle \oplus \ldots \oplus \langle w_n' \rangle \qquad \text{and} \qquad \delta(\Gamma) = \delta(\Gamma_0) \cdot \prod_{i=1}^{n} (-w_i')\,, \tag{15}$$

where

$$w_i' = -[-w_i, \ldots, -w_n] \le -1 \qquad \forall i = 1, \ldots, n\,. \tag{16}$$

Moreover, $w_i' < -1$ $\forall i = 1, \ldots, n$ if also $w_n \le -2$. Consequently, Γ and Γ_0 have the same number of positive (non-negative) eigenvalues.

Proof. Essentially, our proof repeats an argument in [Mu, p. 20]. We proceed by induction on n. In the case $n = 0$ the linear branch L is empty and the assertion is immediate. Assume now that $n \ge 1$. In the linear space $V(\Gamma)$ spanned over $\mathbb{R}$ by the vertices of Γ, we consider the symmetric bilinear form defined by the adjacency matrix $I(\Gamma)$. The vector $v_{n-1}' = v_{n-1} - w_n^{-1} v_n$ is orthogonal to v_n and satisfies

$$v_{n-1}'^2 = v_{n-1}^2 - \frac{1}{w_n}, \qquad v_{n-1}' v = v_{n-1} v \text{ for all vertices of } \Gamma \text{ different from } v_{n-1}, v_n.$$

Hence deleting the vertex v_n from Γ and changing the weight of v_{n-1} to $v_{n-1}'^2 = w_{n-1}'$ with $w_{n-1}' = w_{n-1} - 1/w_n$, we obtain a new graph Γ_{n-1} which satisfies $I(\Gamma) \sim I(\Gamma_{n-1}) \oplus \langle w_n \rangle$. Since $w_{n-1}' \le -1$, this completes the induction step. ☐

Examples 4.7. 1. If $L = [[-2, \ldots, -2, -1]]$ (of length n) then $w_0' = w_0 + 1$ and $I(\Gamma) = I(\Gamma_0) \oplus \langle -1 \rangle \oplus \ldots \oplus \langle -1 \rangle$ (n times).

2. If $L = [[-1, -2, \ldots, -2]]$ (of length n) then $w_0' = w_0 + n$ and

$$I(\Gamma) \sim I(\Gamma_0) \oplus \left\langle -\frac{1}{n} \right\rangle \oplus \left\langle -\frac{n}{n-1} \right\rangle \oplus \ldots \oplus \left\langle -\frac{r}{r-1} \right\rangle \oplus \ldots \oplus \left\langle -\frac{3}{2} \right\rangle \oplus \langle -2 \rangle \,.$$

In both cases $\delta(\Gamma) = \delta(\Gamma_0)$.

Remarks 4.8. (1) In the geometric setting, Γ is the dual graph of a reduced divisor D on a normal surface X and Γ_0 is the dual graph of the image of D under contraction of the part D' of D with dual graph Γ'. The orthogonal basis of the reduction as in the proof above appears geometrically as follows. For a given i with $0 \le i < n$ we consider the contraction $\sigma_i : X \to X_i$ of all the components C_j of D' with $j > i$. The total transforms $\sigma_i^*(\sigma_i(C_i))$, $i = 1, \ldots, n$, on the original surface X are then mutually orthogonal due to the projection formula. Moreover all of them are orthogonal to the total transform $\sigma_0^*(\sigma_0(\Gamma))$ which has dual graph Γ_0.

(2) More generally, we let $\Gamma \to \Gamma_1$ be a domination of weighted graphs consisting of a sequence of blowdowns

$$\Gamma = \Gamma_{n+1} \to \Gamma_n \to \ldots \to \Gamma_1 \,.$$

Letting also $v_1, \ldots, v_k$ be the vertices of Γ_1 and u_i be the blowdown vertices in $\Gamma_{i+1} \ominus \widehat{\Gamma}_i$, $i = 1, \ldots, n$, we consider as before the vector space $V(\Gamma)$ endowed with the symmetric bilinear form $I(\Gamma)$. It is easily seen that the subspace $V_1 = \operatorname{span}(v_1^*, \ldots, v_k^*)$ has the orthogonal complement $V_1^\perp = \operatorname{span}(u_1^*, \ldots, u_n^*)$, where $u_1^*, \ldots, u_n^*$ are mutually orthogonal eigenvectors of $I(\Gamma)$ corresponding to negative eigenvalues. Moreover the restriction of $I(\Gamma)$ to V_1 is equivalent to the bilinear form $I(\Gamma_1)$ on $V(\Gamma_1)$. Therefore

$$I(\Gamma) \sim I(\Gamma_1) \oplus \langle -1 \rangle^n \,. \tag{17}$$

Proposition 4.9. *For a linear chain $L = [[w_0, \ldots, w_n]]$ with integral weights the following hold.*

(a) If $w_i \le -2$ $\forall i = 0, \ldots, n$ then $I(L)$ is negative definite of discriminant $\delta(L) > 1$.

(b) *If $w_k = -1$ for some $k \in \{0, \dots, n\}$ and $w_i \le -2$ $\forall i \ne k$ then L is contractible if and only if $\delta(L) = 1$ or, equivalently,*

$$\frac{e_1}{m_1} + \frac{e_2}{m_2} = 1 - \frac{1}{m_1 m_2}\,, \tag{18}$$

where

$$\frac{e_1}{m_1} = [-w_{k-1}, \dots, -w_0]^{-1} \quad \textit{and} \quad \frac{e_2}{m_2} = [-w_{k+1}, \dots, -w_n]^{-1}$$

with $m_i > 0$ and $\gcd(e_i, m_i) = 1$, $i = 1, 2$.

(c) *If $w_0 = 0$, $w_1 \in \mathbb{Z}$ and $w_i \le -2$ $\forall i = 2, \dots, n$, or $n = 2$, $w_0 = 0$, $w_1 \in \mathbb{Z}$ and $w_2 \le 0$ then $I(L)$ has exactly one positive eigenvalue, and $\delta(L) < -1$ if $n \ge 3$, $\delta(L) \le 0$ if $n = 2$.*

Proof. (a) follows from Lemma 4.6 above applied to $\Gamma_0 = v_0$.

Similarly, to show (b) we apply the reduction as in Lemma 4.6 to the two branches $L_1 = [[w_0, \dots, w_{k-1}]]$ and $L_2 = [[w_{k+1}, \dots, w_n]]$ at the only (-1)-vertex v_k of Γ. By virtue of Lemma 4.6, Γ and $\Gamma_0 = [[w'_k]]$ have the same number of positive (non-negative) eigenvalues. Hence Γ is negative definite if and only if $w'_k < 0$. Using (4.5)

$$w'_k = -1 + [-w_{k-1}, \dots, -w_0]^{-1} + [-w_{k+1}, \dots, -w_n]^{-1} = -1 + e_1/m_1 + e_2/m_2\,,$$

where by Lemma 4.4, $m_i = \delta(\Gamma_i) > 0$ and $e_i = \delta(\Gamma_i \ominus v_{k\pm1}) \ge 0$. On the other hand, according to (14) and (15),

$$\delta(\Gamma) = (-w'_k)\delta(\Gamma_1)\delta(\Gamma_2) = m_1 m_2 - m_1 e_2 - m_2 e_1 = 1 \iff -w'_k = 1 - \frac{e_1}{m_1} - \frac{e_2}{m_2} = \frac{1}{m_1 m_2}\,.$$

Now (b) is a consequence of Lemma 4.3.a.

In case (c), letting $\langle a, b\rangle = I([[a, b]]) = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$ and applying Lemma 4.6 to $\Gamma_0 = [[w_0, w'_1]] = [[0, w'_1]]$ we obtain

$$I(\Gamma) \sim \langle 0, w'_1\rangle \oplus \langle w'_2\rangle \oplus \dots \oplus \langle w'_n\rangle \sim \langle 0, 0\rangle \oplus \langle w'_2\rangle \oplus \dots \oplus \langle w'_n\rangle\,,$$

where as before $w'_k = -[-w_k, \dots, -w_n] < -1$ $\forall k = 2, \dots, n$ if $n \ge 3$ and $w'_2 = w_2 \le 0$ if $n = 2$. Hence $\delta(\Gamma) = -(-w'_2) \cdot \ldots \cdot (-w'_n) < -1$ if $n \ge 3$ and $\delta(\Gamma) \le 0$ if $n = 2$. Thus $I(\Gamma)$ has exactly one positive eigenvalue, as stated. □

Examples 4.10. According to (b), the equality $\frac{1}{2} + \frac{1}{3} = 1 - \frac{1}{6}$ leads to the contractible linear chain $L = [[-2, -1, -3]]$. Similarly, $\frac{2}{3} + \frac{1}{4} = 1 - \frac{1}{12}$ yields $L = [[-2, -2, -1, -4]]$.

4.2. **Spectra of linear and circular standard graphs.** The spectrum and the inertia indices $i_\pm, i_0$ of a weighted graph Γ are as usual the spectrum, respectively, the inertia indices of the associate symmetric bilinear form $I(\Gamma)$. In the same way as it was done in the proofs of Lemma 4.6 and Proposition 4.9.c, we can find the spectra of the standard graphs. Let us start with the standard linear chains.

Proposition 4.11. (a) *The form $I([[0_m]])$ has $\lfloor \frac{m}{2} \rfloor$ negative and $\lfloor \frac{m}{2} \rfloor$ positive eigenvalues; for an odd m it has in addition a zero eigenvalue. Thus the inertia indices of $\Gamma = [[0_m]]$ are*

$$(i_+, i_-, i_0) = \begin{cases} (k, k, 0), & m = 2k \\ (k, k, 1), & m = 2k + 1 \end{cases}$$

and the discriminant is $\delta([[0_{2k}]]) = (-1)^k$ *and* $\delta([[0_{2k+1}]]) = 0$ $\forall k \geq 0$.

(b) *If* $L = [[0_{2k}, w_1, \ldots, w_n]]$ *and* $L' = [[w_1, \ldots, w_n]]$ *then*

$$I(L) \sim I(L') \oplus I([[0_{2k}]]) \sim I(L') \oplus \bigoplus_{i=1}^{k} \langle 0, 0 \rangle = I(L') \oplus \langle 0, 0 \rangle^k .$$

Consequently, if $w_i \leq -2$ $\forall i = 1, \ldots, n$ *then the intersection form*

$$I(L) \sim \langle w_1' \rangle \oplus \langle w_2' \rangle \oplus \ldots \oplus \langle w_n' \rangle \oplus I([[0_{2k}]]) ,$$

where w_i' *are defined as in (16), has* $n + k$ *negative and* k *positive eigenvalues, and* $\delta(L) = (-1)^k \delta(L')$. [10]

Similarly, for circular graphs the following hold.

Lemma 4.12. (a) *The eigenvalues* λ *of a circular graph* $\Gamma = ((0_m))$ *are*

$$\left\{ 2 \cos \frac{2\pi l}{m} \mid l = 0, \ldots, m - 1 \right\} .$$

All these eigenvalues have multiplicity 2 *except for* $\lambda = \pm 2$ *if* m *is even and* $\lambda = 2$ *if* m *is odd, the latter ones being simple.*

(b) *For* $\Gamma = ((0_{4k}, w_1, \ldots, w_n))$ *with* $n \geq 1$ *we let* $\Gamma' = ((w_1, \ldots, w_n))$ *if* $n \geq 2$ *and* $\Gamma' = [[w_1 + 2]]$ *if* $n = 1$. *Then*

$$I(\Gamma) \sim I(\Gamma') \oplus \bigoplus_{i=1}^{k} I\left([[0_4]]\right) = I(\Gamma') \oplus \langle 0, 0, 0, 0 \rangle^k .$$

Consequently the inertia indices of Γ *and* Γ' *are related via*

$$(i_+, i_-, i_0) = (i_+' + 2k,\, i_-' + 2k,\, i_0') .$$

Proof. To show (a) we notice that $I(\Gamma)$ is the matrix of the linear map $\tau + \tau^{-1}$, where $\tau : v_i \longmapsto v_{i+1}$, $i = 1, \ldots, m$ is the cyclic shift acting on $V(\Gamma)$. Now the statement follows from the Spectral Mapping Theorem. Indeed τ^m =id and so the complex spectrum of τ consists of the m-th roots of unity. Hence

$$\mathrm{spec}\,(\tau + \tau^{-1}) = \{\lambda + \lambda^{-1} \mid \lambda \in \mathbb{C},\, \lambda^m = 1\} .$$

To show (b), in the vector space $V(\Gamma) = \bigoplus_{i=1}^m \mathbb{R} v_i$, where $m = 4k + n$, we perform the following base change:

$$(v_1, \ldots, v_m) \longmapsto (v_1, \ldots, v_4, v_5 + v_1 - v_3, v_6, \ldots, v_{m-1}, v_m - v_2 + v_4) \qquad \text{if} \quad m \geq 6 ,$$

or

$$(v_1, \ldots, v_5) \longmapsto (v_1, \ldots, v_4, v_5 + v_1 - v_2 - v_3 + v_4) \qquad \text{if} \quad m = 5 .$$

It is easily seen that for $k \geq 1$, in the new basis the intersection form $I(\Gamma)$ coincides with those of the disjoint union of the graphs $[[0_4]]$ and $((0_{4(k-1)}, w_1, \ldots, w_n))$ if $m \geq 6$, respectively $[[0_4]]$ and $[[w_1 + 2]]$ if $m = 5$. This allows to single out an orthogonal direct summand $I([[0_4]])$ of $I(\Gamma)$ and so provides a reduction from k to $k - 1$. After k steps we obtain the desired decomposition. Now the second assertion in (b) follows by virtue of (a). Indeed, according to Proposition 4.11.a the inertia indices of the bilinear form $\langle 0_4 \rangle^k$ are $(i_+, i_-, i_0) = (2k, 2k, 0)$. □

[10] For the latter equality, see also [Da_2, L.4.12(1)].

Proposition 4.13. *The inertia indices* (i_+, i_-, i_0) *of the standard circular graphs*[11]

$$((0_l, w)), \qquad ((0_{2k}, -1, -1)) \quad \textit{and} \quad ((0_{2k}, w_1, \ldots, w_n)),$$

where $w \le 0$, $k, l \ge 0$, $n \ge 2$ *and* $w_i \le -2$ $\forall i$, *are given in the following tables:*

Γ	$((0_{4l}))$	$((0_{4l+1}))$	$((0_{4l+2}))$	$((0_{4l+3}))$
(i_+, i_-, i_0)	$(2l-1, 2l-1, 2)$	$(2l+1, 2l, 0)$	$(2l+1, 2l+1, 0)$	$(2l+1, 2l+2, 0)$

Γ	$((0_{2k}, w))$	$((0_{4l}, -1))$	$((0_{4l+1}, w))$	$((0_{4l+3}, w))$
(i_+, i_-, i_0)	$(k, k+1, 0)$	$(2l+1, 2l, 0)$	$(2l+1, 2l+1, 0)$	$(2l+1, 2l+2, 1)$
except for	$((0_{4l}, w))$, $-2 \le w \le 0$	-	-	$w = 0$

Γ	$((0_{4l}, -1, -1))$	$((0_{4l+2}, -1, -1))$	$((0_{2k}, w_1, \ldots, w_n))$	$((0_{4l}, (-2)_n))$
(i_+, i_-, i_0)	$(2l+1, 2l+1, 0)$	$(2l+1, 2l+3, 0)$	$(k, k+n, 0)$	$(2l, 2l+n-1, 1)$
except for	-	-	$((0_{4l}, (-2)_n))$	-

Proof. For the circular graphs $\Gamma = ((0_m))$ the result follows by virtue of Lemma 4.12.a or, alternatively, 4.12.b by an easy computation. In the other cases, according to Lemma 4.12.b it is enough to consider graphs with at most 3 zeros.

• If $\Gamma = ((w))$ then $I(\Gamma) = \langle w+2 \rangle$ and so

$$(i_+, i_-, i_0) = \begin{cases} (0,1,0), & w \le -3 \\ (0,0,1), & w = -2 \\ (1,0,0), & w = 0 \text{ or } w = -1\,. \end{cases}$$

• For the graphs $\Gamma = ((-1,-1))$ and $\Gamma = ((0,w))$, where $w \le 0$, we have $\det I(\Gamma) < 0$ and so $(i_+, i_-, i_0) = (1,1,0)$.

• For graphs $\Gamma_3 = ((0_2, w))$ and $\Gamma_4 = ((0_2, w_1, w_2))$, the respective base changes

$$(v_1, v_2, v_3) \longmapsto (v_1, v_2, v_3 - v_1 - v_2)\,,$$

and

$$(v_1, v_2, v_3, v_4) \longmapsto (v_1, v_2, v_3 - v_1, v_4 - v_2)$$

yield decompositions

$$I(\Gamma_3) \sim \langle 0,0 \rangle \oplus \langle w-2 \rangle \quad \text{and} \quad I(\Gamma_4) \sim \langle 0,0 \rangle \oplus \langle w_1 \rangle \oplus \langle w_2 \rangle\,.$$

Hence their inertia indices are

$$(i_+, i_-, i_0) = (1,2,0) \quad \text{and} \quad (i_+, i_-, i_0) = \begin{cases} (1,3,0), & w_1, w_2 \le -1 \\ (1,2,1), & w_1 = 0, w_2 \le -1 \\ (1,1,2), & w_1 = w_2 = 0\,, \end{cases}$$

respectively.

[11]See (4).

• If $\Gamma = ((w_1, \ldots, w_n))$ is a circular graph with $n \geq 2$ and $w_i \leq -2$ $\forall i$ then for a vector $\vec{v} \in V(\Gamma)$ with coordinates $(x_1, \ldots, x_n)$,

$$I(\Gamma)\vec{v}.\vec{v} = \sum_{i=1}^{n} w_i x_i^2 + 2\sum_{i=1}^{n} x_i x_{i+1} = \sum_{i=1}^{n}(2 + w_i)x_i^2 - \sum_{i=1}^{n}(x_i - x_{i+1})^2\,,$$

where $x_{n+1} := x_1$. Thus $I(\Gamma)$ is negative definite and so $(i_+, i_-, i_0) = (0, n, 0)$ provided that $\Gamma \neq ((\,(-2)_n\,))$. Clearly $(i_+, i_-, i_0) = (0, n-1, 1)$ for the Cartan matrix of $\Gamma = ((\,(-2)_n\,))$.

• Finally in the case where $\Gamma = ((0_2, w_1, \ldots, w_n))$, $n \geq 3$ and $w_i \leq -2$ $\forall i$, we perform the base change

$$(v_1, \ldots, v_{n+2}) \longmapsto (v', v'', v'_1 \ldots, v'_n) := (v_1, v_2, v_3 - v_1, v_4, \ldots, v_{n+1}, v_{n+2} - v_2)\,.$$

Since v' and v'' are perpendicular to v'_i for all i, this results in a decomposition

$$I(\Gamma) \sim \langle 0, 0\rangle \oplus I'_n\,.$$

To compute I'_n, we note first that for $i < j$ we have $v'_i v'_j = v_{i+2}v_{j+2}$ unless $i = 1$ and $j = n$, where $v'_1 v'_n = -1$. Thus similarly as before for a vector $\vec{v} = \sum_{i=1}^{n} x_i v'_i$

$$I'_n\vec{v}.\vec{v} = \sum_{i=1}^{n}(2 + w_i)x_i^2 - \sum_{i=1}^{n}(x_i - x_{i+1})^2 - 4x_1x_n\,.$$

Since $(x_1 - x_n)^2 + 4x_1x_n = (x_1 + x_n)^2$ the form I'_n is negative definite and $(i_+, i_-, i_0) = (1, n+1, 0)$.

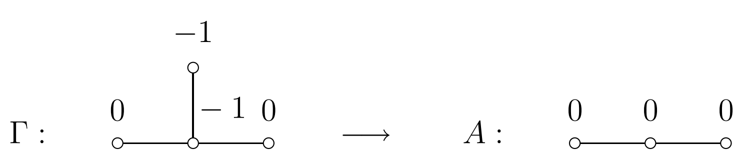
These computations combined with Lemma 4.12.b cover all possible cases in our tables. Now the proof is completed. □

4.3. **Zariski's Lemma.** We recall the following classical fact, see e.g. [Mi, Lemma 2.11.1]. For the sake of completeness we provide a short argument.

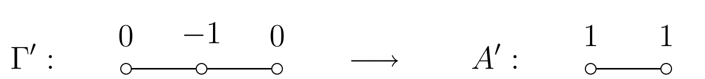
Lemma 4.14. *(*Zariski's Lemma*) If $\Gamma \to A = [[0]]$ is a domination and $\Gamma' \subsetneq \Gamma$ is a proper subgraph then the intersection form $I(\Gamma')$ is negative definite.*

Proof. Letting a be the only vertex of A, the null space of the quadratic form $I(\Gamma) \sim \langle 0\rangle \oplus \langle -1\rangle^{l-1}$ on $V(\Gamma)$ is $\mathbb{R}a^*$, where a^* is the total transform of a and $l = |\Gamma|$, see (17). Since a^* is supported by the whole Γ we have $V(\Gamma') \cap \mathbb{R}a^* = \{0\}$. Hence the quadratic form $I(\Gamma') = I(\Gamma)|V(\Gamma')$ is negative definite. □

Remark 4.15. It is not true in general that the number of non-negative eigenvalues of the intersection form of a graph must strictly diminish when passing to a proper subgraph. Consider for instance the following dominations:



and



For all 4 graphs A, Γ, A' and Γ' we have $(i_+, i_0) = (1, 1)$, whereas Γ' is a proper subgraph of Γ.

However the following partial extension of Zariski's Lemma holds.

Lemma 4.16. *Let $\Gamma \to A = [[0_{2k+1}]]$ be a domination, where Γ is a linear chain. If a proper subgraph Γ' of Γ is either connected or satisfies $|\Gamma \ominus \Gamma'| > k$ then the intersection form $I(\Gamma')$ has at most k non-negative eigenvalues.*

Proof. For $A = [[0_{2k+1}]]$ with vertices $a_0, \ldots, a_{2k}$ the null space of the intersection form $I(A)$ is generated by the vector

$$\vec{v} = a_0 - a_2 + a_4 - \ldots \pm a_{2k} \,.$$

Since $I(\Gamma) \sim I(A) \oplus \langle -1 \rangle^l$ with $l = |\Gamma| - 2k - 1$, see (17), $I(\Gamma)$ has inertia indices $(i_+, i_-, i_0) = (k, k + l, 1)$ and the null space $\mathbb{R}\vec{v}^*$.

Suppose on the contrary that $I(\Gamma') = I(\Gamma)|V(\Gamma')$ has more than k non-negative eigenvalues. Then $\vec{v}^* \in V(\Gamma')$, so all vertices of Γ besides possibly $\hat{a}_{2i+1}$, $i = 0, \ldots, k-1$ must be in Γ'. Thus $|\Gamma \ominus \Gamma'| \le k$ and so, by our assumptions, Γ' is connected. By the same argument as above it must contain the vertices $\hat{a}_0, \hat{a}_{2k}$, hence also $\hat{a}_1, \ldots, \hat{a}_{2k-1}$ since Γ is linear. Therefore $\Gamma' = \Gamma$, contradicting the assumption that $\Gamma' \subsetneq \Gamma$ is a proper subgraph. $\square$

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