Two dimensional quotient singularities deform to quotient singularities

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Reflexive modules on quotient surface singularities

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Let X_0 be a variety over the field $\mathbb{C}$ of complex numbers, having isolated quotient singularities, and let X_η be the general fibre of a deformation $f: X \longrightarrow S$ of X_0. The class of rational singularities is stable under deformations (R. Elkik [1]) and hence X_η belongs to this class. O. Riemenschneider ([9]) conjectured that isolated quotient singularities have a similar property, i.e. that X_η has only quotient singularities. M. Schlessinger ([10]) proved that, as soon as $\dim(X_0) \geq 3$, the isolated singularities are ridgid. Therefore, the only case to consider is the two dimensional one. In this note we give an affirmative answer to Riemenschneider's conjecture (2.5).

The known deformations of quotient singularities often exhibit a bewildering complexity. We are grateful to Kurt Behnke for illustrating this to us by several interesting examples; in particular for showing us deformations for which the order of the group of X_0 is prime to that of X_η. For example, a singularity X_0, whose minimal desingularization is described by the graph

is cyclic of order 5. However, it can be deformed to quotients of order 3 or order 2.

It is well known, that for each rational surface singularity one can construct a cyclic covering, which has Gorenstein singularities (the "canonical covering" described in (1.6)). One approach to study the deformations X_η would be to try to construct the canonical coverings of X_0 and X_η simultaneously. That of X_0 has rational double points (1.7) and the known deformation theory of rational double points would give a proof

of our main result. For this to work one must show that some powe
of the dualizing sheaf of $U_o = Reg(X_o)$ has a trivializing secti
which can be extended to X . In (3.2) we give an example where the obstructions to extending those sections do not vanish (Marc Levine kindly explained the necessary calculations). This means that we can not expect the total space X of our deformation to have only quotient singularities.

Nevertheless, we try to deform as many sections of powers of the dualizing sheaf ω_{U_o} as possible (§ 2) and we try to study the corresponding cyclic coverings (§ 1). The main idea of the proof is explained in (1.9).

We use the usual notations of algebraic geometry as explaine
in [4] ,except that the tensor product ($\otimes$) is always suppose
to be the tensor product of modules over the structure sheaf ($\otimes_{\mathcal{O}_X}$) , and that we denote by $\mathcal{O}_X(D)$ the invertible sheaf associated to a Cartier divisor D on X . We often write $\mathcal{M}(D)$ instead of $\mathcal{M}\otimes\mathcal{O}_X(D)$, for an arbitary sheaf $\mathcal{M}$, and correspondingly $\mathcal{M}^i(D) = \mathcal{M}^i\otimes\mathcal{O}_X(D)$ and $\mathcal{M}(D)^i = \mathcal{M}^i\otimes \mathcal{O}_X(i\cdot D)$.

Some of the methods used in § 1 can be found in [2] , [3] [5] and [11] . There, however, they are discussed in the case of a projective smooth variety. We reformulate them for singular varieties and their desingularization, in the hope that they will have general applications in the theory of singularities.

§1. Cyclic coverings of singularities

Let X be a normal Cohen-Macaulay variety over $\mathbb{C}$ and ω_X its dualizing sheaf. Even, if we don't make this assumption, we are interested in the affine case. We choose a desingularization $\delta: Y \longrightarrow X$, such that the exceptional locus of δ is a normal crossing divisor.

<u>(1.1). As is well known</u> ([6] , p.: 50), X <u>has only rational singularities, if one of the following equivalent conditions is satisfied:</u>

a) $R^q\delta_*\mathcal{O}_Y = 0$ <u>for</u> $q > 0$.

b) $\delta_*\omega_Y = \omega_X$.

We denote the reflexive hull of the N-th tensor power of the dualizing sheaf by $\omega_X^{[N]} = (\omega_X^N)^{\vee\vee}$ and we write $\omega_Y^{[N]} = \delta^*\omega_X^{[N]}/\text{torsion}$. Of course, $\omega_X^{[N]}$ is a coherent sheaf.

(1.2).

Let $\mathcal{N} \longrightarrow \omega_X^{[N]}$ be an inclusion of sheaves, isomorphic outside of the singular locus of X . We assume in the sequel that we have choosen δ such that both $\omega_Y^{[N]}$ and $\mathcal{M} = \delta^*\mathcal{N}/\text{torsion}$ are invertible sheaves. $\mathcal{N}$ and $\mathcal{M}$ are generated by their global sections, at least if we choose X affine. We have the natural inclusions

$$\mathcal{N} \longrightarrow \delta_*\mathcal{M} \longrightarrow \omega_X^{[N]} .$$

We can find effective divisors E and D with support in the exceptional locus of δ such that $\mathcal{M}(D) = \omega_Y(E)^N$.

<u>Definition (1.3).</u> For $0 \le i < N$ we define

$$\mathcal{L}_N^{(i)} = \omega_Y(E)^i \otimes \mathcal{O}_Y(-[\frac{i\cdot D}{N}]) \quad .$$

For simplicity we write $\mathcal{L}^{(i)}$ instead of $\mathcal{L}^{(i)}_{\omega_X^{[N]}}$.

Here $[\frac{i\cdot D}{N}]$ is the largest divisor (with coefficients in $\mathbb{Z}$) satisfying $[\frac{i\cdot D}{N}] \le \frac{i\cdot D}{N}$. Since

$$i\cdot(E+F) - [\frac{i\cdot(D+N\cdot F)}{N}] = i\cdot E - [\frac{i\cdot D}{N}]$$

for all effective divisors F , the sheaf $\mathcal{L}_N^{(i)}$ does not depend on the divisors choosen.

(1.4).

The invertible sheaves $\mathcal{L}_N^{(i)}$ appear in a natural way in the following construction of a cyclic cover. Assume that X is affine and let $t:\mathcal{O}_X \longrightarrow \mathcal{N}$ be a general section. We take $s:\mathcal{O}_X \longrightarrow \omega_X^{[N]}$ to be the induced section and $s^\vee:\omega_X^{[N]^\vee} \longrightarrow \mathcal{O}_X$ to be its dual. We consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} = \bigoplus_{i\ge 0} \omega_X^{[i]^\vee} / \langle s^\vee \rangle = \bigoplus_{i=0}^{N-1} \omega_X^{[i]^\vee}$$

and $X' = \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{A})$.

By construction the zero divisor of s is non singular over $\mathrm{Reg}(X)$. Hence X' is non singular over $\mathrm{Reg}(X)$, as one sees writing down local parameters (see also [2]). Moreover $\mathcal{A}$ is reflexive as an $\mathcal{O}_X$- module and therefore $\mathcal{A}$ and X' must be normal. Let Y' be the normalization of Y in the function field $\mathbb{C}(X')$ and Z be a desingularization of Y' . The induced morphisms are denoted by

$$\begin{array}{ccccc} Z & \xrightarrow{\tau} & Y' & \xrightarrow{\delta'} & X' \\ & \searrow^{\gamma} & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ & & Y & \xrightarrow{\delta} & X \end{array}$$

Lemma (1.5). Using the notations introduced above one knows:

i) Y' has rational singularities.

ii) $\gamma_* \mathcal{O}_Z = \pi'_* \mathcal{O}_{Y'} = \bigoplus_{i=0}^{N-1} L_N^{(i)^{-1}}$

iii) $\gamma_* \omega_Z = \pi'_* \omega_{Y'} = \bigoplus_{i=0}^{N-1} \omega_Y \otimes L_N^{(i)}$.

iv) the higher direct images $R^q \delta_* (\omega_Y \otimes L_N^{(i)}) = 0$ for $q > 0$ and $i = 0, \ldots, N-1$.

v) $\delta_* L_N^{(i)^{-1}}$ is reflexive for $i = 0, \ldots, N-1$.

vi) X' has rational singularities if and only if X' is Cohen-Macaulay and $\delta_* (\omega_Y \otimes L_N^{(i)})$ is reflexive for $i = 0, \ldots, N-1$.

vii) X' has rational singularities if and only if $R^q \delta_* L_N^{(i)^{-1}} = 0$ for $q > 0$ and $i = 0, \ldots, N-1$.

Proof. By construction $\pi'_* \mathcal{O}_{Y'}$ is the normalization of the $\mathcal{O}_Y$-algebra

$$\mathcal{B} = \bigoplus_{i \geq 0} \omega_Y^{[i]^\vee} / \langle \sigma^\vee \rangle$$

where $\sigma : \mathcal{O}_Y \longrightarrow \omega_Y^{[N]}$ is the pullback of s and $\sigma^\vee$ its dual. If we choose the effective divisor E , supported in the exceptional locus of δ , large enough, we have an inclusion $\omega_Y^{[i]} \longrightarrow \omega_Y(E)^i$, and thereby we obtain a section

$$\sigma' : \mathcal{O}_Y \longrightarrow \omega_Y(E)^N = \mathcal{M}(D) \ .$$

The $\mathcal{O}_Y$ - algebra

$$\mathcal{B}' = \bigoplus_{i\geq 0} \omega_Y(E)^{-i} \big/ <\sigma'^{\vee}>$$

is contained in $\mathcal{B}$, and both algebras are isomorphic over an open subvariety. Since $\pi'_*\mathcal{O}_{Y'}$ is normal over $\mathcal{O}_Y$, it must be the normalization of $\mathcal{B}'$ as well.

The section σ' can also be described in the following way: δ^* realizes $H^0(X,\mathcal{N})$ as a subspace of $H^0(Y,\mathcal{M})$. This subspace generates $\mathcal{M}$ and σ' is obtained from a general member of it. Bertini's theorem ([4] , III, 10.9) guaranties that the zero divisor of σ' is of the form $B + D$, where B is non singular, $\mathcal{O}_Y(B) = \mathcal{M}$ and $B + D$ a normal crossing divisor. We may apply [2], lemme 2 , where the normalization of $\mathcal{B}'$ is described. Using that $[\frac{i\cdot(B+D)}{N}] = [\frac{i\cdot D}{N}]$ we obtain

$$\pi'_*\mathcal{O}_{Y'} = \bigoplus_{i=0}^{N-1} \omega_Y(E)^{-i}\otimes\mathcal{O}_Y([\frac{i\cdot D}{N}]) = \bigoplus_{i=0}^{N-1} \mathcal{L}_N^{(i)^{-1}} .$$

Since $B + D$ contains the ramification locus of Y' over Y, we know from [11] or [2], lemme 1 , that Y' has at most rational singularities. Especially

$$R^q(\delta'\cdot\tau)_*\mathcal{O}_Z = R^q\delta'_*\mathcal{O}_{Y'} \quad \text{and} \quad R^q(\delta'\cdot\tau)_*\omega_Z = R^q\delta'_*\omega_{Y'} .$$

iii) follows from ii), using the duality for finite maps, saying

$$\pi'_*\omega_{Y'} = \mathcal{H}om_{\mathcal{O}_Y}(\pi'_*\mathcal{O}_{Y'}, \omega_Y) .$$

iv) is nothing but the Grauert-Riemenschneider vanishing theorem, applied to $\delta'\cdot\tau$, since for $q > 0$

$$0 = \pi_* R^q(\delta'\cdot\tau)_*\omega_Z = \bigoplus_{i=0}^{N-1} R^q\delta_*(\omega_Y\otimes\mathcal{L}_N^{(i)}) .$$

In fact, this also can be obtained from the global vanishing theorem for "integral parts of $\mathbb{Q}$ - divisors " ([5] and [11]) as described in [11] , 2.3.

v) just says that $(\delta'\cdot\tau)_*\mathcal{O}_Z = \mathcal{O}_{Y'}$, and - along the same line - vi) and vii) are nothing but translations of the two equivalent desriptions of rational singularities given in (1.1). For example, from duality for finite morphisms we know that

$$\pi_*\omega_{X'} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \omega_X) = \bigoplus_{i=1}^{N} \omega_X^{[i]}$$

and X' has rational singularities if and only if $\pi_*\delta_*^!\tau_*\omega_Z = \pi_*\delta_*^!\omega_{Y'}$ is equal to $\pi_*\omega_{X'}$. In other words $\delta_*(\mathcal{L}_N^{(i)} \otimes \omega_Y)$ must be equal to $\omega_X^{[i+1]}$ for $i = 0,\dots,N-1$.

Lemma and Definition (1.6). Assume that $\dim(X) = 2$ and that X has only rational singularities.

a) For some $\nu \in \mathbb{N}$ the sheaf $\omega_X^{[\nu]}$ is invertible. The minimal number $\nu > 0$ with this property is denoten by Ind(X) , the index of X .

b) Assume that Ind(X) divides N and choose $\mathcal{N} = \omega_X^{[N]}$. Let X_1 be an affine open subvariety of X . Then the covering $X' \longrightarrow X_1$ considered in (1.4) is ètale over $\mathrm{Reg}(X_1)$ and X' is Gorenstein. We call X' a (local) canonical covering of X of degree N .

Proof. a) In [8] it is shown that for each singular point $p \in X$ the scheme $U = \mathrm{Spec}(\mathcal{O}_{X,p}) - \{p\}$ has only finitly many non-isomorphic invertible sheaves. Therefore some power of ω_U is isomorphic to $\mathcal{O}_U$.

b) We may assume X to be affine and X' to be a covering of X . Then by duality for finite maps $\pi_*\omega_{X'} = \bigoplus_{i=1}^{N} \omega_X^{[i]}$. It contains the A - module generated by $\omega_X^{[N]}$ and since both are reflexive and isomorphic outside of the singular locus, they must be equal. Therefore $\omega_{X'}$ is invertible.

Interpretating (1.5) in the situation described in (1.6), we obtain a characterization of quotient singularities:

Proposition (1.7). Let X be a surface with at most rational singularities. Assume that Ind(X) divides N and choose $N = \omega_X^{[N]}$. Then the following properties are equivalent:

a) X has only quotient singularities.

b) All local canonical coverings X' of X of degree N have rational singularities.

c) $\delta_*(L^{(N-1)} \otimes \omega_Y)$ is reflexive.

d) The divisors E and D (see (1.2)) satisfy $E \leq \{\frac{D}{N}\}$ where $\{\frac{D}{N}\} = -[-\frac{D}{N}]$.

Proof. We may assume X to be affine. The equivalence of a) and b) is well known: If X' has rational singularities, then it has just rational double points. Those are known to be quotient singularities. Therefore - after replacing X and X' by small neighbourhoods of the singularity - we find a non singular cover W of X , unramified outside of the singular locus. Analytically this is just the universal covering of Reg(X) and hence it is a Galois cover. Therefore X has a quotient singularity.

On the other hand, if X has quotient singularities, we may assume that W is a Galois cover, unramified over $\mathrm{Reg}(X)$. The normalization W' of $W\times_X X'$ is a branched cover of W, ètale outside of a finite number of points. By "purity of the branch locus" W' is ètale over W and therefore non singular. By construction of W' the surface X' is obtained as a quotient of W' by a finite group.

From (1.5,vi)) we know that b) implies c). The sheaf $\delta_*(L^{(N-1)}\otimes\omega_Y)$ is reflexive if and only if it is equal to $\omega_X^{[N]}$ or if and only if

$$\omega_Y^{[N]} = \omega_Y^N(N\cdot E - D) \subset L^{(N-1)}\otimes\omega_Y = \omega_Y^N((N-1)\cdot E - [\tfrac{N-1}{N}\cdot D]) .$$

Comparing the divisors on both sides, we get the equivalence of c) and $E \leq D - [\frac{N-1}{N}\cdot D] = \{\frac{D}{N}\}$.

Assume now that c) is satisfied. $\delta_*\pi_*^!\omega_{Y'} = \delta_*(\bigoplus_{i=0}^{N-1} L^{(i)}\otimes\omega_Y)$ is an $A = \pi_*\mathcal{O}_{X'}$ module. The invertible $\mathcal{O}_X$ submodule $\delta_*(L^{(N-1)}\otimes\omega_Y)$ already generates a reflexive A module and therefore $\delta_*\pi_*^!\omega_{Y'}$ must be reflexive itself. Moreover, since X' is a normal surface, it is Cohen-Macaulay, and we can apply (1.5,vi)) to obtain b). Of course,one could also use the inequality d) to show that the assumption of (1.5,vi)) is satisfied.

Corollary (1.8): <u>Assume that</u> X <u>is a surface having at most quotient singularities,</u> $\mathrm{Ind}(X) \mid N$. <u>Let</u> $\mathcal{N} \subset \omega_X^{[N]}$ <u>be any subsheaf, isomorphic to</u> $\omega_X^{[N]}$ <u>outside of the singular locus of</u> X. <u>Then this inclusion factors over</u>

$$\mathcal{N} \longrightarrow \delta_*(L_{\mathcal{N}}^{(N-1)}\otimes\omega_Y) \longrightarrow \omega_X^{[N]} .$$

Proof. Since $\mathcal{N}$ is a subsheaf of $\delta_*\mathcal{M}$ it is enough to construct an inclusion of $\mathcal{M}$ into $\mathcal{L}_N^{(N-1)} \otimes \omega_Y$. We may choose the divisors E and D big enough to obtain a factorization

$$\mathcal{M} \longrightarrow \omega_Y^{[N]} \longrightarrow \omega_Y(E)^N .$$

If we denote the divisor given by the first inclusion by D_1, that of the second inclusion by D_2, we have $D = D_1 + D_2$.

(1.7,d)) guaranties that $E \leq \{\frac{D_2}{N}\}$ and hence $E \leq \{\frac{D}{N}\}$.

We obtain $N \cdot E - D \leq (N-1) \cdot E - [\frac{N-1}{N} \cdot D]$, which just means that $\mathcal{M} = \omega_Y(E)^N \otimes \mathcal{O}_Y(-D)$ is a subsheaf of $\mathcal{L}_N^{(N-1)} \otimes \omega_Y$.

Remark (1.9). Comparing (1.7,c)) and (1.8) one can already guess how we are going to prove the conjecture of Riemenschneider We have found a certain construction, attaching to a subsheaf $\mathcal{N}$ of $\omega_X^{[N]}$ another subsheaf: $\delta_*(\mathcal{L}_N^{(N-1)} \otimes \omega_Y)$.

A) If X has only quotient singularities, then $\delta_*(\mathcal{L}_N^{(N-1)} \otimes \omega_Y)$ is larger than the sheaf $\mathcal{N}$ we started with.

B) If X has rational singularities other than quotient singularities, then $\delta_*(\mathcal{L}_N^{(N-1)} \otimes \omega_Y) \neq \omega_X^{[N]}$, even if we start with $\mathcal{N} = \omega_X^{[N]}$.

In the next section we just have to verify that B) can not happen for the general fibre of a deformation, as soon as A) is true for the special fibre. To this aim we need a method to lift sections from the special fibre to the total space of the deformation. The vanishing theorem (1.5,iv)) turns out to serve this purpose.

§.2. Deformations of quotient singularities

Let $\delta : Y \longrightarrow X$ be a desingularization of the normal Cohen-Macaulay variety X such that the exceptional locus of δ is a normal crossing divisor and such that $\omega_Y^{[N]} = \delta^*\omega_X^{[N]}/\text{torsion}$ is invertible. We consider a reduced Cartier divisor X_o in X and its proper transform Y_o in Y (later X_o will be the special fibre of a deformation with total space X). We assume in addition that we have choosen δ such that $\delta^*(X_o) = Y_o + F$ is a normal crossing divisor. the natural morphisms are denoted by

$$\begin{array}{ccc} Y_o & \xrightarrow{\iota'} & Y \\ {\scriptstyle \delta_o}\downarrow & & \downarrow{\scriptstyle \delta} \\ X_o & \xrightarrow{\iota} & X \end{array}$$

Lemma (2.1). $\mathcal{N}_o = \iota^*\omega_X^{[N]} = \omega_X^{[N]} \otimes \mathcal{O}_{X_o}$ <u>is torsionfree and the sheaf</u> $\mathcal{M}_o = \delta_o^*\mathcal{N}_o/\text{torsion}$ <u>is isomorphic to</u> $\iota'^*\omega_Y^{[N]}$.

<u>Proof.</u> The first statement is true for the restriction to X_o of any reflexive sheaf F on X, which is locally free on a subvariety $i : W \longrightarrow X$ with $\operatorname{codim}(X-W) \geq 2$. In fact, let $i_o : W_o = W \cap X_o \longrightarrow X_o$ and $F_o = \iota^* F$. Since X_o is a Cartier divisor we may use the projection formula to obtain

$$i_*(i^*F(-W_o)) = i_*(i^*F \otimes i^*\mathcal{O}_X(-X_o)) = (i_* i^* F) \otimes \mathcal{O}_X(-X_o) = F(-X_o) .$$

Applying i_* to the exact sequence

$$0 \longrightarrow i^*F(-W_o) \longrightarrow i^*F \longrightarrow i_o^* F_o \longrightarrow 0$$

we obtain

$$0 \longrightarrow F(-X_o) \longrightarrow F \longrightarrow i_{o*} i_o^* F_o \qquad .$$

Hence F_o is a subsheaf of the torsionfree sheaf $i_{o*}i_o^* F_o$.

Now, let K be the torsion part of $\delta^*\omega_X^{[N]}$. We have exact sequences

$$\begin{array}{ccccccc} \iota'^*K & \longrightarrow & \iota'^*\delta^*\omega_X^{[N]} & \longrightarrow & \iota'^*\omega_X^{[N]} & \longrightarrow & 0 \\ & & \| & & & & \\ & & \delta_o^*N_o & \longrightarrow & M_o & \longrightarrow & 0 \end{array} .$$

ι'^*K is supported in the exceptional locus of δ_o and therefore it is a torsion sheaf. The induced map $\iota'^*K \longrightarrow M_o$ has to be the zero map. We obtain a surjection $\iota'^*\omega_Y^{[N]} \longrightarrow M_o$. The first sheaf being invertible, this must be an isomorphism.

(2.2)

Let R be a discrete valuation ring with residue field $\mathbb{C}$, $S = \mathrm{Spec}(R)$ and $f:X \longrightarrow S$ a flat morphism. We write $g = f\circ\delta : Y \longrightarrow S$ and take X_o to be the special fibre of f . Keeping the notations introduced above, the special fibre of g is $Y_o + F$. The general fibres are denoted by X_η and Y_η . Let U be the largest open subvariety of X which is smooth over S . We assume that X-U is proper over S and that X_o is normal. We refer to those conditions by saying that X_η is a <u>deformation</u> of X_o .

S being affine and non singular, we identify ω_S with $\mathcal{O}_S$ and thereby $\omega_{X/S}$ with ω_X and $\omega_{Y/S}$ with ω_Y . The normal sheaves of the special fibres of f and g can as well be identified with the structure sheaves and we can write $\omega_{X_o} = \omega_X \otimes \mathcal{O}_{X_o}$ and $\omega_{Y_o} = \omega_Y(-F)\otimes\mathcal{O}_{Y_o} = \omega_Y(Y_o)\otimes\mathcal{O}_Y(-Y_o-F)\otimes\mathcal{O}_{Y_o}$

Of course, we also have $\omega_{X_\eta} = \omega_X \otimes \mathcal{O}_{X_\eta}$ and $\omega_{Y_\eta} = \omega_Y \otimes \mathcal{O}_{Y_\eta}$.

The sheaf $\mathcal{N}_o = \omega_X^{[N]} \otimes \mathcal{O}_{X_o}$ is torsionfree and restricted to $U_o = U \cap X_o = \mathrm{Reg}(X_o)$ it is isomorphic to $\omega_{U_o}^N$. Therefore it is a subsheaf of $\omega_{X_o}^{[N]}$, isomorphic to it outside of the singular locus of X_o , and we can define $\mathcal{L}_{\mathcal{N}_o}^{(i)}$ on Y_o using (1.3).

<u>Lemma (2.3).</u> $\mathcal{L}^{(i)} \otimes \mathcal{O}_{Y_o} = \mathcal{L}_{\mathcal{N}_o}^{(i)}$.

<u>Proof.</u> As in § 1 we write $\omega_Y^{[N]} = \omega_Y^N(N \cdot E - D)$. By construction Y_o meets E , D and F transversally and therefore the divisors $E_o = E \cap Y_o$, $D_o = D \cap Y_o$ and $F_o = F \cap Y_o$ are normal crossing divisors. Moreover the multiplicities in D_o can not be larger than those occuring in D which implies that $[\frac{i \cdot D_o}{N}] = [\frac{i \cdot D}{N}] \cap Y_o$. We have $\mathcal{M}_o = \omega_{Y_o}^N (N \cdot E_o + N \cdot F_o - D_o)$ and

$$\mathcal{L}_{\mathcal{N}_o}^{(i)} = \omega_{Y_o}^i (i \cdot (E_o + F_o) - [\frac{i \cdot D_o}{N}]) = \omega_Y^i (i \cdot E - [\frac{i \cdot D}{N}]) \otimes \mathcal{O}_{Y_o} .$$

Using the notations introduced in (1.4) the lemma (2.3) is saying that $\pi'^{-1}(Y_o)$ is normal and can also be obtained as the cyclic cover corresponding to a general section of $\mathcal{N}_o$.

<u>Proposition (2.4).</u> <u>Assume that</u> X_o <u>is a surface with at most quotient singularities, and assume that</u> $\mathrm{Ind}(X_o)$ <u>divides</u> N . <u>Then there exists an inclusion</u>

$$\mathcal{N}_o \longrightarrow \delta_*(\mathcal{L}^{(N-1)} \otimes \omega_Y) \otimes \mathcal{O}_{X_o} ,$$

<u>inducing an isomorphism outside of the singular locus of</u> X_o .

<u>Proof.</u> The generalized Grauert-Riemenschneider vanishing

theorem (1.5,iv)) implies that

$$R^1\delta_*(L^{(N-1)}\otimes\omega_Y(-Y_o-F)) = R^1\delta_*(L^{(N-1)}\otimes\omega_Y)\otimes\mathcal{O}_X(-X_o) = 0 .$$

Therefore we have exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow & \delta_*(L^{(N-1)}\otimes\omega_Y(-Y_o-F)) & \longrightarrow & \delta_*(L^{(N-1)}\otimes\omega_Y(-F)) & \longrightarrow & \delta_{o*}(L_{N_o}^{(N-1)}\otimes\omega_{Y_o}) & \longrightarrow 0 \\ & \| & & \cap & & \cap & \\ 0 \longrightarrow & \delta_*(L^{(N-1)}\otimes\omega_Y)\otimes\mathcal{O}_X(-X_o) & \longrightarrow & \delta_*(L^{(N-1)}\otimes\omega_Y) & \longrightarrow & \delta_*(L^{(N-1)}\otimes\omega_{Y_o+F}) & \longrightarrow 0 \end{array}$$

and obtain thereby an inclusion from $\delta_{o*}(L_{N_o}^{(N-1)}\otimes\omega_{Y_o})$ into $\delta_*(L^{(N-1)}\otimes\omega_{Y_o+F}) = \delta_*(L^{(N-1)}\otimes\omega_Y)\otimes\mathcal{O}_{X_o}$. Now (2.4) follows from (1.8).

Proposition (2.4) enables us to prove the main result of this note.

Theorem (2.5). Assume X_o to be a surface with quotient singularities, and let X_η be the general fibre of a deformation of X_o over a discrete valuation ring. Then X_η has quotient singularities.

Remark (2.6). a) In the proof of (2.5) we will also obtain some information about the sheaf N_o and M_o , saying that

$$N_o = \delta_{o*}(L_{N_o}^{(N-1)}\otimes\omega_{Y_o}) .$$

b) Of course, the arguments used in the proof of (2.5) also apply to an analytic deformation of X_o over a disc and show that all "nearby" fibres X_η have quotient singularities.

c) In the proof of (2.5) we will use for simplicity Elkik's result on deformations of rational singularities. However, the arguments given below could be used for $N = 1$ to prove that (1.1) forces X_η to have rational singularities.

<u>Proof of (2.5).</u> We know from [1] that X_η has rational singularities. Hence $\mathrm{Ind}(X_o)$ and $\mathrm{Ind}(X_\eta)$ are defined and we choose some N divisible by both of them. Let C be the cokernel of the inclusion of $\delta_*(L^{(N-1)} \otimes \omega_Y)$ into $\omega_X^{[N]}$: Restricting the corresponding exact sequences to X_o one obtains

$$\delta_*(L^{(N-1)} \otimes \omega_Y) \otimes \mathcal{O}_{X_o} \xrightarrow{\beta} N_o \longrightarrow C \otimes \mathcal{O}_{X_o} \longrightarrow 0 \quad .$$

We know from (2.4) that the left hand side contains N_o and we thereby find a map $\alpha : N_o \longrightarrow N_o$, isomorphic outside of the singular locus. The induced map $N_o^{\vee\vee} \longrightarrow N_o^{\vee\vee}$ between invertible sheaves must be the multiplication with a unit and hence α must be an isomorphism. Therefore β is surjective and $C \otimes \mathcal{O}_{X_o} = 0$.

The support of C is closed in X , since C is coherent, and it is contained in the non-smooth locus $X-U$. Hence the support of C is proper over S . This is only possible for $C = 0$, i.e. if $\delta_*(L^{(N-1)} \otimes \omega_Y)$ is reflexive. Regarding this on the general fibre we find $\delta_*(L^{(N-1)} \otimes \omega_{Y_\eta})$ to be reflexive and (1.7,c)) implies that X_η has only quotient singularities.

The remark (2.6,a)) follows from the fact that the isomorphism α factors by construction over

$$N_o \longrightarrow \delta_{o*}(L_{N_o}^{(N-1)} \otimes \omega_{Y_o}) \longrightarrow N_o$$

and that the sheaf in the middle is torsionfree.

§ 3 Concluding remarks and examples

Keeping the notations introduced in § 2 , we assume $f: X \longrightarrow S$ to be a deformation of the quotient singularity X_o. We denote by U the largest subvariety of X which is smooth over S. If $\mathrm{Ind}(X_o)$ and $\mathrm{Ind}(X_\eta)$ divide N, we may assume - replacing X by an affine neighbourhood of the singularity - that $\omega^N_{U_o} \cong \mathcal{O}_{U_o}$ and $\omega^N_{U_\eta} \cong \mathcal{O}_{U_\eta}$. It seems natural to expect that $\omega^N_U \cong \mathcal{O}_U$, in other words, to expect that the trivializing section of $\omega^N_{U_o}$ can be extended to U. Unfortunatelly this is in general not the case, even if we replace N by some multiple and even if we assume X_η to be non singular.

<u>(3.1). The first order obstruction to deform "trivializing sections" (M. Levine, [7]).</u>

Let N be any multiple of $\mathrm{Ind}(X_o)$ and $\pi_o : X'_o \longrightarrow X_o$ the canonical cover of degree N. We write

$p : \pi_o^{-1}(U_o) = U'_o \longrightarrow U_o$ for the restriction of π_o.

If $s_o : \mathcal{O}_{U_o} \longrightarrow \omega^N_{U_o}$ is an isomorphism, then $p^*(s_o) = t_o^N$

for $t_o : \mathcal{O}_{U'_o} \longrightarrow \omega_{U'_o}$. The deformation f gives an element $\rho \in H^1(U_o, \Theta_{U_o})$ and the evaluation of $p^*(\rho)$ on t_o is

$$\mu_\rho = \langle p^*(\rho), t_o \rangle \in H^1(U'_o, \Omega^1_{U'_o}) .$$

Differentiating and multiplying with t_o^{N-1} we find

$$\nu'_\rho = t_o^{N-1} \cdot d\mu_\rho \in H^1(U'_o, \omega^N_{U'_o}) \quad \text{and}$$

$$\nu_\rho = \mathrm{trace}_{U'_o/U_o}(\nu'_\rho) \in H^1(U_o, \omega^N_{U_o}) .$$

In [7] it is shown that ν_ρ is the obstruction wanted.

Especially $\nu_\rho \neq 0$ implies that s_o can not extend to a trivializing section of ω_U^N .

Let $i: X_o \longrightarrow \mathbb{A}^n$ be an embedding and $\tau: \mathbb{A}^2 \longrightarrow X_o$ be the Galois cover with Galois group G , étale over U_o . Then the first order deformations of X_o are described by $T^1_{X_o}$, the kernel of the map

$$H^1(U_o, \Theta_{U_o}) = H^1(\mathbb{A}^2-\{0\}, \Theta_{\mathbb{A}^2-\{0\}})^G \longrightarrow H^1(\mathbb{A}^2-\{0\}, \tau^* i^* \Theta_{\mathbb{A}^n}) .$$

The fibre product $U_o' \times_{U_o} (\mathbb{A}^2-\{0\})$ is the disjoint union of several copies of $\mathbb{A}^2-\{0\}$. In order to calculate ν_ρ , we can consider μ_ρ and ν_ρ' on one of those $\mathbb{A}^2-\{0\}$ and identify t_o with $dx \wedge dy$, where (x,y) denotes a coordinate system on $\mathbb{A}^2$.

<u>Example (3.2).</u> Let $G = \langle\sigma\rangle$ be the cyclic group of order three acting on $\mathbb{C}[x,y]$ by $\sigma(x) = e \cdot x$ and $\sigma(y) = e \cdot y$ for a third root of unit e . Let $X_o = \mathrm{Spec}(\mathbb{C}[x,y]^G)$. The inclusion $i: X_o \longrightarrow \mathbb{A}^4$ is defined by the invariants x^3 , $x^2 \cdot y$, $x \cdot y^2$ and y^3 .

$T^1_{X_o}$ is generated as an $\mathcal{O}_{X_o}$ - module by $\rho_1 = x^{-1} \cdot y^{-1} \cdot \frac{\partial}{\partial x}$ and $\rho_2 = x^{-1} \cdot y^{-1} \cdot \frac{\partial}{\partial y}$. We find

$$\nu_{\rho_1} = (dx \wedge dy)^{N-1} \cdot d\langle x^{-1} \cdot y^{-1} \cdot \frac{\partial}{\partial x}, dx \wedge dy\rangle =$$

$$= (dx \wedge dy)^{N-1} \cdot d(x^{-1} \cdot y^{-1} \cdot dy) = -x^{-2} \cdot y^{-1} \cdot (dx \wedge dy)^N$$

and similary

$$\nu_{\rho_2} = x^{-1} \cdot y^{-2} \cdot (dx \wedge dy)^N .$$

Both are independent elements of $H^1(U_o, \omega_{U_o}^N)$ and therefore $\nu_\rho \neq 0$ for all nontrivial $\rho \in T^1_{X_o}$. Of course, we may choose ρ to be the infinitesimal deformation corresponding to a smoothing of X_o.

Remark (3.3). It is not too surprising that ν_ρ may be non zero. If we return to the notations of (2.2) we see that the sheaf $\omega_{Y_o}^{[N]}$ is larger that $\omega_{Y_o}^N$ in general. The arguments given in [7] in order to show that $\nu_\rho = 0$ can only work for sections of powers of dualizing sheaves. Nevertheless, our calculation in § 2 gives some conditions for sections to be deformable. The sheaf $\mathcal{N}_o$ is the sheaf generated by all deformable sections of $\omega_{X_o}^{[N]}$ and (2.6,a)) gives the conditio that $H^o(X_o, \mathcal{N}_o) = H^o(X_o, \delta_*(\mathcal{L}_{\mathcal{N}_o}^{(N-1)} \otimes \omega_{Y_o}))$.

Even if we don't see at the moment how to interpretate this condition, it seems to say that $\mathcal{N}_o$ can not be too small compared to $\omega_{X_o}^{[N]}$.

If one tries to use the methods of our note , i.e. the use of vanishing theorems for integral parts of $\mathbb{Q}$ -divisors (and related results), to the global problem considered in [7] , one finds a similar description of the sheaf of deformable sectio of powers of dualizing sheaves. It would be interesting to reprove the results from [7] using this description.

On the other hand the obstruction classes explained in (3.1) seem to contain some information on $\mathcal{N}_o$. May be, if one is able to define and to calculate those classes not only for the fibres X_o but also for fibrecomponents Y_o and their infinitesimal neighbourhoods, this could give another more direct approach to describe N_o and to reprove (2.5).

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