

# **HIGGS BUNDLES AND CHARACTERISTIC CLASSES**

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- Kähler manifold  $M$ ,  $w_2 = 0$
- spin structure:  $K^{1/2}$
- Dirac operator  $D$

$$D = \frac{1}{\sqrt{2}}(\bar{\partial} + \bar{\partial}^*) : \Omega^{0,ev}(M, K^{1/2}) \rightarrow \Omega^{0,od}(M, K^{1/2})$$

The polynomials  $T_k$  are called TODD polynomials. The identity

$$\frac{x}{1 - e^{-x}} = \exp\left(\frac{1}{2}x\right) \frac{\frac{1}{2}x}{\sinh \frac{1}{2}x} \quad (\text{we write } \exp(a) = e^a)$$

is useful for the calculation of the first few TODD polynomials. It implies, using Lemma 1.3.1, formula (6<sub>m</sub>) in the  $(c_i, x, \gamma_i)$  formulation and the relations (7), that

$$T_k(c_1, \dots, c_k) = \sum \frac{1}{2^{4s} r!} \left(\frac{1}{2}c_1\right)^r A_s(p_1, \dots, p_s) \quad (12)$$

where the sum is over all non-negative integers  $r, s$  with  $r + 2s = k$ .

F.Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer Verlag, Berlin-Heidelberg-New York, (1966)

certain divisibility properties modulo a power of 2 in the case  $w_2$  vanishes. The Riemann-Roch theorem ( $M_{n, 1}$ ) of algebraic geometry<sup>4</sup> seems to indicate that a generalization of Rohlin's theorem to higher dimensions must have something to do with the  $A$ -genus. It can be proved<sup>4</sup> for an *algebraic variety*  $V_{2k}$  that the vanishing of the second Stiefel-Whitney class  $w_2$  implies:  $A(V_{2k}) = 0 \pmod{2^{4k}}$ , and that for  $k$  *odd*, the vanishing of  $w_2$  implies:  $A(V_{2k}) = 0 \pmod{2^{4k+1}}$ . It is easy to calculate that for the non-singular hypersurface  $V_{2k}^{(2r)}$  of degree  $2r$  imbedded in the complex-projective space  $\mathbf{P}_{2k+1}$  the Stiefel-Whitney class  $w_2$  vanishes and that

$$A(V_{2k}^{(2r)}) = 2^{4k+1} \binom{r+k}{2k+1}.$$

F.Hirzebruch, *Problems on Differentiable and Complex Manifolds*, Ann.of Math. **60** (1954) 213–236.

- compact 2-manifold  $\Sigma$ , real Lie group  $G^r$
- character variety  $\text{Hom}(\pi_1(\Sigma), G^r)/G^r$

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- flat  $G^r$ -bundle, maximal compact  $K$

- characteristic classes in

$$H^1(\Sigma, \pi_0(K)), H^2(\Sigma, \pi_1(K))$$

... help to distinguish components of the character variety

## EXAMPLE $G^r = SL(2, \mathbf{R})$

- maximal compact  $SO(2)$ , Chern class  $c \in H^2(\Sigma, \mathbf{Z}) = \mathbf{Z}$
- $|c| \leq 2g - 2$  (Milnor-Wood)
- $|c| < 2g - 2$  determines a connected component
- $|c| = 2g - 2 \Rightarrow 2^{2g}$  connected components

## PLAN

- Higgs bundles and real forms
- Characteristic class  $w_2$  for  $SL(n, \mathbf{R})$
- (B,A,A)-branes and (B,B,B)-branes

# HIGGS BUNDLES

- compact Riemann surface  $\Sigma$
- principal  $G$ -bundle  $P$
- $\mathcal{A}$  affine space of connections on  $P$
- infinite-dimensional flat Kähler manifold

- $\mathcal{G}$  group of gauge transformations
- $(A, \Phi) \in T^*\mathcal{A} = \mathcal{A} \times \Omega^{1,0}(\Sigma, \mathfrak{g})$  flat hyperkähler manifold
- moment map  $(F_A + [\Phi, \Phi^*], \bar{\partial}_A \Phi)$
- quotient moduli space of Higgs bundles
- $\bar{\partial}_A \Phi = 0 \quad F_A + [\Phi, \Phi^*] = 0$

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- complex structure  $I$ : moduli space of (stable) pairs  $(A, \Phi)$

$G = U(n)$  vector bundle  $V$ ,  $\Phi \in H^0(\Sigma, \text{End } V \otimes K)$

- complex structure  $J$ : flat  $G^c$ -connection

$\nabla_A + \Phi + \Phi^*$  (representations  $\pi_1(\Sigma) \rightarrow G^c$ )

- complex structure  $K$ : flat  $G^c$ -connection

$\nabla_A + i\Phi - i\Phi^*$

## REAL FORM $G^r$

- $K \subset G^r$  maximal compact
- principal  $K^c$ -bundle
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$
- Higgs field  $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$
- holonomy of  $\nabla + \Phi + \Phi^* \in G^r$

## EXAMPLE $G^r = SL(n, \mathbf{R})$

- orthogonal vector bundle  $V$
- $\Lambda^n V \cong \mathcal{O}$
- $\Phi = \Phi^T \in H^0(\Sigma, \text{End } V \otimes K)$

# ABELIANIZATION

## THE FIBRATION

- hyperkähler moduli space  $\mathcal{M}(G)$

$$\dim_{\mathbf{R}} = 4(g - 1) \dim G$$

- principal  $G^c$ -bundle,  $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$

- invariant polynomials  $p_1, \dots, p_\ell$  on  $\mathfrak{g}$

$$p_m(\Phi) \in H^0(\Sigma, K^{d_m})$$

- fibration  $\mathcal{M}^{2k}(G) \rightarrow \mathbf{C}^k$

- integrable system
- generic fibre abelian variety  $A$
- $G^c = GL(n, \mathbf{C})$   $\det(x - \Phi) = 0$  spectral curve  $S$
- fibre =  $\text{Jac}(S)$

- spectral curve  $S$ :  $\det(x - \Phi) = 0$
- curve in the cotangent bundle  $\pi : K \rightarrow \Sigma$
- $\pi : S \rightarrow \Sigma$   $n$ -fold cover

- $V = \pi_* L$
- $\Phi = \pi_*(L \xrightarrow{\eta} L \otimes \pi^* K)$
- $\eta \in \pi^* K$  canonical section on total space

- $L$  trivial bundle

- $\pi_* L = \mathcal{O} \oplus K^{-1} \oplus \dots \oplus K^{-n-1}$

- $\Phi =$  companion matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & \dots & \dots & a_1 \end{pmatrix}$$

- section of  $\mathcal{M}^{2k}(G) \rightarrow \mathbf{C}^k$

- $G^c = SL(n, \mathbf{C})$
- $L = U \otimes \pi^* K^{(n-1)/2}$ ,  $\deg U = 0$
- $\text{Nm} : \text{Pic}^0(S) \rightarrow \text{Pic}^0(\Sigma)$
- $U \in \mathcal{P}(S, \Sigma) = \text{Prym variety} = \text{kernel}$

- $L = \pi^* K^{(n-1)/2}$

- $\pi_* L = K^{(n-1)/2} \oplus K^{(n-3)/2} \oplus \dots \oplus K^{-(n-1)/2}$

- $\Phi =$  companion matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & \dots & \dots & 0 \end{pmatrix}$$

- section of  $\mathcal{M}^{2k}(SL(n, \mathbf{C})) \rightarrow \mathbf{C}^k$

## REAL FORM $G^r$

- real form/compact  $\sim$  involution  $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$
- $\pm 1$  eigenspaces  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$
- Higgs field  $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$ :  $\sigma^* \Phi = -\Phi$
- $G^r$  character variety  $\sim$  fixed points  $\mathcal{M}^\sigma$

- $\sigma$  respects fibration  $\mathcal{M}^\sigma \rightarrow (\mathbf{C}^k)^\sigma$
- fibre over  $p \in (\mathbf{C}^k)^\sigma =$
- $=$  fixed points  $A^\sigma$  in an abelian variety  $A$
- $0 \rightarrow A_0^\sigma \rightarrow A^\sigma \rightarrow \mathbf{Z}_2^{2N} \rightarrow 0$

THE CASE  $G^r = SL(n, \mathbb{R})$

- involution  $\sigma(a) = -a^T$

- $\sigma^*(\Phi) = -\Phi^T$

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- $\sigma^*(\Phi) = -\Phi^T$
- $\det(x - \Phi^T) = \det(x - \Phi)$
- $\Rightarrow$  action on  $\mathbf{C}^k$  trivial
- $\Rightarrow$  fibre  $\sim \mathbf{Z}_2^{2k}$

- fixed points in fibre  $\sim \{U \in \mathbf{P}(S, \Sigma) : U^2 = \mathcal{O}\}$
- $L = U \otimes \pi^* K^{(n-1)/2}$
- vector bundle  $V = \pi_* L$  where  $L^2 \cong \pi^* K^{n-1}$

- $V$  is an orthogonal bundle

- $\Lambda^n V \cong \mathcal{O} \Rightarrow w_1(V) = 0$

- What is  $w_2(V)$ ?

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• What is  $w_2(V)$ ?

.... as a function of  $\mathbf{Z}_2^{2k}$

## ORTHOGONAL STRUCTURE

- spectral curve  $S \subset$  canonical bundle of  $\Sigma$
- $K_S \cong \pi^* K^n$
- $d\pi \in H^0(S, K_S \otimes \pi^* K^*) = H^0(S, \pi^* K^{n-1}) \cong H^0(S, L^2)$
- inner product on  $V = \pi_* L$  at  $y \in S$

$$(s, s)_y = \sum_{\pi(x)=y} \frac{s_x^2}{d\pi_x}$$

## *KO* THEORY

- $L^2 \cong K_S \otimes \pi^* K^* = KO\text{-orientation of } \pi : S \rightarrow \Sigma$
- $\pi_! : KO(S) \rightarrow KO(\Sigma)$
- $\pi_!(1) = [V]$

- $KO(\Sigma) \cong \mathbf{Z} \oplus H^1(\Sigma, \mathbf{Z}_2) \oplus H^2(\Sigma, \mathbf{Z}_2)$
- $w_1(V) = 0 \Rightarrow V = [n] + cu$
- $u = [L \oplus L^*] - [2]$ ,  $L$  degree one complex line bundle

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- $w_1(V) = 0 \Rightarrow V = [n] + cu$
- $u = [L \oplus L^*] - [2]$ ,  $L$  degree one complex line bundle
- $w_2(u) = 1 \in \mathbf{Z}_2 \cong H^2(\Sigma, \mathbf{Z}_2) \Rightarrow w_2(V) = c$

M.F. Atiyah, *Riemann surfaces and spin structures*, Ann.Sci. ENS  
 4 (1971) 47–62.

## THE MOD 2 INDEX

- $M^{8k+2}$  spin manifold,  $E$  real vector bundle
- Dirac operator  $D : E \otimes S^+ \rightarrow E \otimes S^-$
- $\dim \ker D \pmod 2$  is a  $KO$ -theory characteristic number
- $p : M \rightarrow pt.$ ,  $\varphi(E) = p_!(E) \in \mathbf{Z}_2$

- $L$  holomorphic degree one line bundle on  $\Sigma$
- Dirac operator  $\sqrt{2}D = \bar{\partial} : (L+L^*) \otimes K^{1/2} \rightarrow (L+L^*) \otimes K^{1/2}\bar{K}$

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- Dirac operator  $\sqrt{2}D = \bar{\partial} : (L+L^*) \otimes K^{1/2} \rightarrow (L+L^*) \otimes K^{1/2}\bar{K}$
- $\dim H^0(\Sigma, LK^{1/2}) + \dim H^0(\Sigma, L^*K^{1/2}) =$   
 $= \dim H^0(\Sigma, LK^{1/2}) + \dim H^1(\Sigma, LK^{1/2})$   
 $= (\dim H^0(\Sigma, LK^{1/2}) - \dim H^1(\Sigma, LK^{1/2})) \bmod 2 = 1$   
(Riemann-Roch)
- $1 = \varphi(L + L^*) = \varphi[2] + \varphi(u) = \varphi(u) = w_2(u)$

- spin structure  $K^{1/2}$ ,  $L^2 \cong K_S \pi^* K^{-1}$  take  $K_S^{1/2} = L \pi^* K^{1/2}$
- $\varphi_S(1) = \varphi(\pi_!(1)) = \varphi(V) = \varphi([n] + w_2(V)u) = n\varphi(1) + w_2(V)$

- $n$  even  $w_2(V) = \varphi_S(1)$
- $n$  odd  $w_2(V) = \varphi_S(1) + \varphi(1)$

## EXAMPLE $SL(2, \mathbf{R})$

- Higgs bundle  $V = M \oplus M^*$   $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

$$w_2(V) = c_1(M) \bmod 2$$

- spectral curve  $0 = \det(x - \Phi) = x^2 - q$  involution  $\sigma(x) = -x$
- Prym variety  $P(S, \Sigma) = \{U : \sigma^*U \cong U^*\}$
- $U^2 \cong \mathcal{O} \Rightarrow \sigma^*U \cong U$

- $L = U\pi^*K^{1/2}$
- $\pi_*L = M^+ \oplus M^-$ : invariant/anti-invariant local sections
- $c_1(M^+) = g - 1 - k$  where  $2k =$  no of fixed points where action of  $\sigma$  on  $L$  is  $-1$
- $w_2(V) = (g - 1 - k) \bmod 2$

- How many elements of order 2 give  $w_2(V) = 0$ ?

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- fixed points  $x = 0 \sim 4g - 4$  zeros of quadratic differential  $q$
- $0 = w_2(V) = g - 1 - k$ :  $g$  odd  $\Rightarrow k$  even

$$\sum_k \binom{4g-4}{4k} =$$

$$= \frac{1}{4} \left( (1+1)^{4g-4} + (1-1)^{4g-4} + (1+i)^{4g-4} + (1-i)^{4g-4} \right)$$

$$= 2^{4g-6} + 2^{2g-4} \left( (e^{i\pi/4})^{4g-4} + (e^{-i\pi/4})^{4g-4} \right) = 2^{4g-6} + 2^{2g-3}$$

- No:  $= 2^{4g-7} + 2^{2g-4}$

THE CASE  $G^r = SU(m, m)$

- $SL(2, \mathbf{R}) \cong SU(1, 1)$
- maximal compact  $S(U(m) \times U(m))$
- bundle  $V = V_+ \oplus V_-$  Higgs field  $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$
- $\Lambda^m V_+ \cong \Lambda^m V_-^*$ ,
- characteristic class  $c_1(V_+) \in H^2(\Sigma, \mathbf{Z})$

L.Schaposnik, *Spectral data for G-Higgs bundles*, arXiv:1301.1981

- $\det(x - \Phi) = x^{2m} + a_1 x^{2m-2} + \dots + a_m$
- involution  $\sigma(x) = -x$  on  $S$
- $L = U\pi^*K^{(2m-1)/2}$ ,  $U \in \mathbf{P}(S, \Sigma)$
- $\sigma^*U \cong U$

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- $L = U\pi^*K^{(2m-1)/2}$ ,  $U \in \mathbf{P}(S, \Sigma)$
- $\sigma^*U \cong U$
- $c_1(V_+) = m(g - 1) - k$   
 $2k =$  no of fixed points where action is  $-1$

BRANES

## BRANES

- symplectic geometry

- A-brane = Lagrangian submanifold ... + flat bundle

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- symplectic geometry

• A-brane = Lagrangian submanifold ... + flat bundle

- complex geometry

• B-brane = complex submanifold... + holomorphic bundle

# HYPERKÄHLER MANIFOLDS

- complex structures  $I, J, K$
- symplectic structures  $\omega_1, \omega_2, \omega_3$
- $(B, A, A)$ -brane: cx wrt  $I$ , totally real wrt  $J, K$
- $(B, B, B)$ -brane: HK submanifold + hyperholomorphic bundle

- Mirror symmetry:
- the mirror of a  $(B, A, A)$ -brane....
- .... is a  $(B, B, B)$ -brane

- What is the mirror of the  $G^r$ -character variety?

## SYZ MIRROR SYMMETRY

- mirror = fibration by dual abelian varieties
- mirror of  $\mathcal{M}(G) = \mathcal{M}({}^L G)$
- $SU(m, m) \subset SL(2m, \mathbf{C})$
- ${}^L SL(2m, \mathbf{C}) = PSL(2m, \mathbf{C})$
- Need a hyperkähler submanifold of  $\mathcal{M}(PSL(2m, \mathbf{C}))$

- spectral curve  $S: x^{2m} + a_1x^{2m-2} + \dots + a_m = 0$

quotient curve  $\bar{S} = S/\sigma$

- connected component of fibre  $P(\bar{S}, \Sigma) \subset P(S, \Sigma)$

- spectral curve  $S: x^{2m} + a_1x^{2m-2} + \dots + a_m = 0$   
 quotient curve  $\bar{S} = S/\sigma$
- connected component of fibre  $P(\bar{S}, \Sigma) \subset P(S, \Sigma)$
- “annihilator” in dual of  $P(S, \Sigma)$   
 = line bundles on  $P(S, \Sigma)$  trivial on  $P(\bar{S}, \Sigma)$

- connected component of fibre  $P(\bar{S}, \Sigma) \subset P(S, \Sigma)$

- dual of  $P(S, \Sigma) = \text{Jac}(S) / \text{Jac}(\Sigma)$

- $P(S, \bar{S}) / P(S, \bar{S}) \cap \text{Jac}(\Sigma) \rightarrow \text{Jac}(S) / \text{Jac}(\Sigma) \rightarrow \text{Jac}(\bar{S}) / \text{Jac}(\Sigma)$

- $P(S, \bar{S}) \cap \text{Jac}(\Sigma) = \{U \in \text{Jac}(\Sigma) : \sigma^* \pi^* U \cong \pi^* U^*\} = H^1(\Sigma, \mathbf{Z}_2)$

- spectral curve  $S: x^{2m} + a_1x^{2m-2} + \dots + a_m = 0$
- Prym variety  $P(S, \bar{S})$
- = moduli space of  $Sp(2m, \mathbf{C})$  Higgs bundles
- modulo  $H^1(\Sigma, \mathbf{Z}_2) = PSp(2m, \mathbf{C})$  bundles
- hyperkähler submanifold of  $PSL(2m, \mathbf{C})$  Higgs bundles

# HYPERHOLOMORPHIC BUNDLE

- Higgs bundle equations: dimensional reduction of ASDYM

- ASD connection  $A_1 dx_1 + A_2 dx_2 + \phi_1 dx_3 + \phi_2 dx_4$

- $$D^* = \nabla_1 + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \nabla_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_1 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \phi_2$$

- Dirac operator 
$$D^* = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix}$$

$$D^* = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix} : \begin{pmatrix} V \otimes K \\ V \otimes \bar{K} \end{pmatrix} \rightarrow \begin{pmatrix} V \otimes K \bar{K} \\ V \otimes K \bar{K} \end{pmatrix}$$

- $D^*D \sim -\nabla_1^2 - \nabla_2^2 - \phi_1^2 - \phi_2^2 \Rightarrow \ker D = 0$
- index theorem  $\Rightarrow \dim \ker D^* = (2g - 2) \operatorname{rk} V$
- $\mathcal{L}^2$  connection is hyperholomorphic

- complex structure  $I$

$$\begin{array}{ccc} \bullet \Omega^{0,p}(V) & \xrightarrow{\Phi} & \Omega^{0,p}(V \otimes K) \\ \bar{\partial} \downarrow & & \bar{\partial} \downarrow \end{array}$$

$$\Omega^{0,p+1}(V) \xrightarrow{\Phi} \Omega^{0,p+1}(V \otimes K)$$

- total differential  $\bar{\partial} \pm \Phi$

- Hodge theory:  $\ker D^* \cong$  hypercohomology  $\mathbf{H}^1$

## HYPERCOHOMOLOGY

- $\mathcal{O}(V) \xrightarrow{\Phi} \mathcal{O}(V \otimes K)$
- $\ker D^* \cong$  hypercohomology  $\mathbf{H}^1(V)$
- $\pi_*[\mathcal{O}(L) \xrightarrow{\eta} \mathcal{O}(L\pi^*K)]$

$$\mathbf{H}^1(V) \cong \bigoplus_{\{x:\eta(x)=0\}} (L\pi^*K)_x$$

- connected components of  $SU(m, m)$ -moduli space  
 $\sim 2k$ -element subsets of  $\{x : \eta(x) = 0\}$

$$\Lambda^{2k} \mathbf{H}^1(V) \cong \bigoplus_{\{x_i : \eta(x_i) = 0\}} (L\pi^* K)_{x_1} (L\pi^* K)_{x_2} \dots (L\pi^* K)_{x_{2k}}$$

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- **CLAIM:** This is the required hyperholomorphic bundle.

- What about  $SL(n, \mathbf{R})$ ?
- general spectral curve  $\Rightarrow$  HK manifold is the full  $PSL(2m, \mathbf{C})$ -moduli space
- components  $w_2 = 0, w_2 = 1$
- What is the hyperholomorphic bundle?

## NONABELIANIZATION

- other real forms  $U^*(m), Sp(2m, 2m), SO^*(4m)$
- $\det(x - \Phi) = p(x)^2$
- $V = \pi_* U, U$  rank 2
- What is the mirror here?