Bound of automorphisms of surfaces of general type, I

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The aim of this two-parted work is to prove

Theorem 1. *Let S be a minimal smooth projective surface of general type over $\mathbb{C}$, $G = \mathrm{Aut}(S)$. Then*

$$|G| \leq 42^2 K_S^2 \ . \qquad (1)$$

Theorem 1 is a natural generalisation of the classical theorem of Hurwitz that the automorphism group of a curve of genus $g \geq 2$ is of order $\leq 84(g-1) = 42 \deg K$.

Note that it is easy to construct examples with equality in (1) and with K_S^2 arbitrarily large, by taking for S the product of a Hurwitz curve with itself: see e.g. [X2, Example 4] for details.

As long as to the knowledge of the author, the fact that the automorphism group of a surface of general type is finite and bounded by a function of K^2 goes back to Andreotti [A], who also gives there an effective bound of $|G|$ which grows exponentially with K^2. Then Howard/Sommese [Ho-So] exhibited several polynomial bounds for special subgroups of G or for $|G|$ with extra conditions on S. Recently, using quite different methods, Corti [C] and Huckleberry/Sauer [Hu-Sa] have shown independently that $|G|$ is bounded by a polynomial function of small degree in K_S^2. At the same time, we obtained [X2] a linear bound for abelian subgroups of G.

Our approach here to this problem starts from a very natural consideration: we just look at the quotient space of S by the action of G (or rather the minimal resolution X of it), and get an estimate on K_S via the intersection form on X.

In fact, for any $\mathbf{Q}$-divisor L contained in the sum of K_X and the $\mathbf{Q}$-branch locus $\mathfrak{B}$ of the projection of (some blow-up of) S onto X, if $L^2 > 0$ and some high multiple of L is effective, then $|G|/K_S^2$ is bounded by $1/L^2$ (Lemma 1). So the bulk of our proof consists of finding a good candidate for L, under different possibilities of X and $\mathfrak{B}$. This method turns out to be much more precise than those used in the previous attempts, against a drawback of being long and computational, especially when X is a rational surface, where complicated case-to-case analyses are required in order to get the correct coefficient in Theorem 1.

In view of this complicatedness, we present in this first part the technical tools which are generally needed (§1), and prove Theorem 1 for the cases where the quotient is not rational (§2, §3). As for the rational case, we give in §4 the proof of a weaker version:

Theorem 2. *There exists a universal constant c such that $|G| \leq cK_S^2$.*

The arguments of §4 can also be considered as a sketch of the complete proof of Theorem 1, to be found in the second part of this paper. The author hopes that such a presentation could spare some unnecessary efforts from the reader trying to unearth the conceptual insight from complicated computations (as well as insignificant technical errors which are often unavoidable in such computations).

Theorem 2 is related to the canonical ring of S in the following way. Let

$$R = \bigoplus_{i=0}^{\infty} H^0\left(S, \omega_S^{\otimes i}\right)$$

be the canonical ring of S, R_G the subring of elements fixed under the induced action of G. Then

Corollary. *There is a linear function $f(x)$ such that for any minimal surface of general type S, R_G contains a non-zero element of degree at most $f(K_S^2)$.*

Proof. As $H^0(S, \omega_S^{\otimes 2}) \neq 0$, we may take a non-sero section s in $H^0(S, \omega_S^{\otimes 2})$. Then the element

$$\bigotimes_{\gamma \in G} \gamma(s) \in H^0\left(S, \omega_S^{\otimes 2|G|}\right)$$

is G-invariant. **QED**

On the other hand, if there is a universal constant c such that for any S, there exists an $i \leq c$ such that $H^0(S, \omega_S^{\otimes i})$ contains a G-invariant subspace of dimension ≥ 2, it will result easily that $|G|$ is bounded by a linear function in K_S^2. Therefore it is an interesting question whether the function $f(x)$ in the above corollary can be replaced by a constant.

Another interesting problem is to give a classification of surfaces with large automorphism groups, as well as the automorphism groups themselves in such cases.

Finally, we are tempted by the resemblance of the situations for curves and surfaces to advance the following bet.

Conjecture. *Let V be a smooth complex projective variety of general type, of dimension d. Suppose that K_V is nef. Then*

$$|\operatorname{Aut}(V)| \leq 42^d K_V^d \quad .$$

§1. Technical preparations

We fix some notations which will be followed thoughout this paper. Let X be the minimal resolution of singularities of the quotient surface S/G. Note that S/G is a normal surface, hence has only a finite number of isolated singularities. Let $\Phi : S \dashrightarrow X$ be the rational map induced by the projection of S onto S/G, $\sigma : \tilde{S} \to S$ the minimal blow-ups of S such that we have an induced morphism $\tilde{\Phi} : \tilde{S} \to X$, with the following commutative diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\Phi}} & X \\ \sigma \downarrow & & \downarrow \\ S & \overset{\Phi}{\dashrightarrow} & S/G \end{array}$$

By construction, the action of G lifts to an action on $\tilde{S}$, such that X is a contraction of the quotient $\tilde{S}/G$, or in some sense the map $\tilde{\Phi}$ is "almost Galois".

We let $\tilde{B}$ to be the reduced divisor on X over which $\tilde{\Phi}$ is ramified.

In what follows, it is convenient to carry out computations on $\mathbb{Q}$-divisors on X. By definition, a *$\mathbb{Q}$-divisor* D is a symbolic sum

$$D = \sum_{i=1}^{k} c_i \Gamma_i \ ,$$

where the Γ_i's are reduced irreducible curves on X, and c_i's are rational numbers. The *integral part* of D, $[D]$, is by definition

$$[D] = \sum_{i=1}^{k} [c_i]\, \Gamma_i \ ,$$

where $[c_i]$ is the greatest integer less than or equal to c_i. We say that two $\mathbb{Q}$-divisors D_1 and D_2 are linearly (resp. numerically) equivalent, written as $D_1 \equiv D_2$ (resp. $D_1 \sim D_2$), only when $D_1 - [D_1] = D_2 - [D_2]$, and the integral parts of D_1 and D_2 are linearly (resp. numerically) equivalent. We also write $D_1 \geq D_2$ if $D_1 - D_2$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. In this case D_2 is also called a *subdivisor* of D_1.

Let $\tilde{S}'$ be the complement in $\tilde{S}$ of all the curves contracted by $\tilde{\Phi}$. For a $\mathbb{Q}$-divisor $D = \sum c_i \Gamma_i$ on X, its inverse image on $\tilde{S}'$ is well defined, and we can define $\tilde{\Phi}^*(D)$

to be the closure of this inverse image in $\tilde{S}$, then define $\Phi^*(D) = \sigma\tilde{\Phi}^*(D)$. As a curve contracted by $\tilde{\Phi}$ is also contracted by σ, for any irreducible curve Γ on S, if the generic point of Γ is mapped onto that of Γ_i by Φ, then the coefficient of Γ in $\Phi^*(D)$ is $r_i c_i$, where r_i is the ramification number of $\tilde{\Phi}$ over Γ_i. Note that if $r_i c_i$ is an integer for all i, $\Phi^*(D)$ is an integral divisor on S.

Let $\tilde{B} = \sum_{i=1}^{k} B_i$ be the decomposition of $\tilde{B}$ into irreducible components, r_i the ramification number of $\tilde{\Phi}$ over B_i. Let

$$\tilde{\mathfrak{B}} = \sum_{i=1}^{k} \left(1 - \frac{1}{r_i}\right) B_i \ .$$

It is the $\mathbb{Q}$-*branch divisor* of $\tilde{\Phi}$.

More useful in our computations is the following $\mathbb{Q}$-divisor. For each irreducible component B_i of $\tilde{B}$, let z_i be the multiplicity of its strict transform in the effective divisor $K_{\tilde{S}} - \sigma^* K_S$, and let s_i be the smallest positive integer such that $1/s_i \leq (1+z_i)/r_i$. Let

$$\bar{\mathfrak{B}} = \tilde{\mathfrak{B}} - \sum_{i=1}^{k} \frac{z_i}{r_i} B_i = \sum_{i=1}^{k} \frac{r_i - z_i - 1}{r_i} B_i \ , \quad \mathfrak{B} = \sum_{s_i \geq 2} \left(1 - \frac{1}{s_i}\right) B_i \ ,$$

and let B be the support of $\mathfrak{B}$. We have $\tilde{B} \geq B$, $\tilde{\mathfrak{B}} \geq \mathfrak{B} \geq \bar{\mathfrak{B}}$.

The fact we will use most frequently (and tacitly) is that $n\tilde{\mathfrak{B}} \geq \tilde{B} \geq B$ for $n \geq 2$.

The following lemma is obvious.

Lemma 1. *I) If D_1 and D_2 are linearly (resp. numerically) equivalent $\mathbb{Q}$-divisors on X, then their inverse images on S are linearly (resp. numerically) equivalent divisors.*

II) Let K_X be a canonical divisor of X, r a positive integer, L_r a $\mathbb{Q}$-divisor on X linearly (resp. numerically) equivalent to $r(K_X + \tilde{\mathfrak{B}})$. Then $\tilde{\Phi}^*(L_r)|_{\tilde{S}'}$ is linearly (resp. numerically) equivalent to $rK_{\tilde{S}}|_{\tilde{S}'}$, hence $\Phi^*(L_r)$ is linearly (resp. numerically) equivalent to rK_S.

The starting point of our proof is the following.

Lemma 2. *Let r be a positive integer, and suppose that there is a $\mathbb{Q}$-divisor L on X such that $L \leq r(K_X + \tilde{\mathfrak{B}})$, that $LA > 0$ for some effective divisor A on X such that $|A|$ has no fixed part, and that $d = L^2 > 0$. Then*

$$|G| \leq \frac{r^2}{d} K_S^2 \ .$$

In particular, when L is integral, we have $|G| \leq r^2 K_S^2$.

Proof. Let $N \gg 0$ be a large integer such that NL is an integral divisor, $s = h^0(\mathcal{O}_X(NL))$. Consider the map $\psi : X - - \to \mathbb{P}^{s-1}$ defined by the linear system $|NL|$, and let Σ be the image of this map. As s increases quadratically with N by Riemann-Roch (note that $h^2(NL) = 0$ because of $LA > 0$), Σ is a surface. Let M be the strict transform on S of a general hyperplane section on Σ, via the composition map $\psi \circ \Phi : S - - \to \mathbb{P}^{s-1}$. We have $NrK_S \equiv M + Z$ for some effective divisor Z, therefore

$$\begin{aligned} N^2r^2K_S^2 &= M^2 + MZ + NrK_SZ \geq M^2 \\ &\geq \deg(\psi \circ \Phi) \deg \Sigma \\ &= \deg \psi \deg \Sigma \, |G| \quad . \end{aligned}$$

Lemma 2 is then a direct consequence of the following lemma, by letting N go to ∞.

Lemma 3. *Let L be a divisor on a smooth projective surface X with $LA > 0$ for some effective divisor A such that $|A|$ has no fixed part, and $d = L^2 > 0$. For each integer N with $h^0(NL) > 0$, let $\psi_N : X - - \to \mathbb{P}^{h^0(NL)-1}$ be the map associated to $|NL|$, $\Sigma_N = \mathrm{Im}\psi_N$. Then*

$$\lim_{N\to\infty} \frac{\deg \psi_N \deg \Sigma_N}{N^2 d} \geq 1 \quad . \tag{2}$$

Proof. Let λ be the left-hand side of (2). We have $\lim\limits_{N\to\infty} \dfrac{h^0(NL)}{N^2} \geq \dfrac{1}{2}d$ by Riemann-Roch and $\lim\limits_{N\to\infty} \dfrac{\deg \Sigma_N}{h^0(NL)} \geq 1$, therefore we can assume $\deg \psi_N = 1$ for $N \gg 0$. And if $\lambda < 1$, we will have $\lim\limits_{N\to\infty} \dfrac{\deg \Sigma_N}{h^0(NL)} < 2$, hence by [X1, Lemma 1], we have a constant integer l such that for every $N \gg 0$, Σ_N has a pencil Λ_N of rational curves of degree $\leq l$.

Fix an n such that Σ_n is a surface and $\deg \psi_n = 1$, and let $N = tn$, where t is an integer with $t > l$. Let F be the strict transform on X of a general element of Λ_N. As the image of F in Σ_n is a curve, we have $\deg \psi_n(F) \geq 1$. But it follows then that $\deg \psi_N(F) \geq t$, contradiction. **QED**

In order to compute the intersection on X, we generally have to contract X to a minimal model. Therefore Lemma 2 will be used in conjunction with the following definition.

Definition 1. Suppose that there is a birational morphism $\rho : X \to Y$, to a smooth surface Y. We decompose ρ into a series of blow-ups $\rho_i : X_{i+1} \to X_i$, $i = 1, \dots, k$, with $X_1 = Y$, $X = X_{k+1}$. Let p_i be the center of ρ_i, and $\mathfrak{E}_i$ the algebraic inverse image in X of the exceptional curve of ρ_i. We have $\mathfrak{E}_i^2 = -1$, $\mathfrak{E}_i\mathfrak{E}_j = 0$ for $i \neq j$.

Let $\tilde{D}$ be an effective $\mathbb{Q}$-divisor on X, $D = \rho(\tilde{D})$. Write

$$\tilde{D} \equiv \rho^* D - \sum_{i=1}^{k} a_i \mathfrak{E}_i \ .$$

a_i is called the *order* of $\tilde{D}$ at the point p_i (it may be negative *a priori*). Let r be a positive integer. Then the r-*saturation* of $\tilde{D}$ is by definition the divisor

$$\begin{aligned} L &= r\rho^* K_Y + \tilde{D} + \sum_{a_i<r} a_i \mathfrak{E}_i + \sum_{a_i \geq r} r\mathfrak{E}_i \\ &\equiv \rho^* (rK_Y + D) - \sum_{a_i \geq r} (a_i - r)\, \mathfrak{E}_i \end{aligned}$$

on X. When $\tilde{D}$ is the strict transform of D, L is also called the r-*saturated pullback* of D. We have

$$L^2 = (rK_Y + D)^2 - \sum_{a_i \geq r} (a_i - r)^2 \ , \tag{3}$$

and the following properties are immediate.

I) If $\tilde{D} \leq r\tilde{\mathfrak{B}}$, then $L \leq r(K_X + \tilde{\mathfrak{B}})$. In particular, this is the case when $D \leq r\rho(\tilde{\mathfrak{B}})$ and L is the r-saturated pullback of D.

II) If $K_Y + D$ is numerically equivalent to a non-zero effective $\mathbb{Q}$-divisor, then we can find an A to meet the condition $AL > 0$ in Lemma 2.

III) If L is the r-saturated pullback of D and the singularities of D have order $\leq r$, then $L \equiv \rho^*(rK_Y + D)$, and $L^2 = (rK_Y + D)^2$.

IV) When $\tilde{D}$ is an integral divisor, we have

$$\sum_{a_i \geq r} a_i\,(a_i - 1) \leq (K_Y + D)\,D - \left(2p_a\left(\tilde{D}\right) - 2\right) \tag{4}$$

by the formula for arithmetic genus. This will be used to give an upper bound of the term $\sum_{a_i \geq r} (a_i - r)^2$ in (3).

Remark. When considering divisors on Y, (3) allows us to ignore blow-ups of singularities of D of order $\leq r$ (note that the strict transform of D is always a subdivisor of $\tilde{D}$). For example, if $r \geq 2$ and D is integral and reduced, we can assume that the only blow-ups are on singular points of D of order ≥ 3. We will often do this in the future without explicit mention.

Definition 2. We denote by $\mathfrak{D}$ the subdivisor of $K_X + \tilde{\mathfrak{B}}$ such that $\tilde{\Phi}^*\mathfrak{D}|_{\tilde{S}'} = \sigma^* K_S|_{\tilde{S}'}$. It is clear by definition that $\mathfrak{D} \leq K_X + \bar{\mathfrak{B}} \leq K_X + \mathfrak{B} \leq K_X + \tilde{\mathfrak{B}}$, and the difference $\mathfrak{Z} = K_X + \tilde{\mathfrak{B}} - \mathfrak{D}$ is a uniquely determined effective $\mathbb{Q}$-divisor, whose support is contained in the total inverse images Z of the singular points of S/G. Hence the components of Z are rational, $\pi_1(Z)$ is trivial, and the restriction of the intersection form on the subspace of $NS(X)$ generated by the classes of components of Z is negative-definite.

Lemma 4. *I) $\mathfrak{D}$ equals the maximal nef subdivisor of $K_X+\tilde{\mathfrak{B}}$ (and also of $K_X+\mathfrak{B}$).*

II) $\bar{\mathfrak{B}}$ is effective, and $\mathfrak{D} = K_X + \bar{\mathfrak{B}}$. In particular, $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ has the same support B.

Proof. I) Take a general divisor H in $|5K_S|$, which does not pass through the points blown up by σ. Then as $\tilde{\Phi}$ is "almost Galois", $\tilde{\Phi}(\sigma^*H)$ is linearly equivalent to $N\mathfrak{D}$, where $N = 5|G|$. As $|5K_S|$ has no fixed part, $N\mathfrak{D}$, hence $\mathfrak{D}$, is nef.

It follows also that $\mathfrak{D}Z = 0$, hence as the intersection form on Z is negative-definite, if a subdivisor D' of $K_X+\tilde{\mathfrak{B}}$ is not contained in $\mathfrak{D}$, we can find an irreducible component Z_1 of Z such that $D'Z_1 < 0$.

II) Let $\bar{\mathfrak{B}} = \bar{\mathfrak{B}}_1 - \bar{\mathfrak{B}}_2$, where $\bar{\mathfrak{B}}_1$ and $\bar{\mathfrak{B}}_2$ are effective $\mathbb{Q}$-divisors with no common components. Let $\bar{\mathfrak{Z}} = K_X + \bar{\mathfrak{B}}_1 - \mathfrak{D}$, $\bar{Z}$ its support. $\bar{\mathfrak{Z}}$ is effective by definition, with no common components with $\bar{\mathfrak{B}}_1$. Let $\mathfrak{D}' = \mathfrak{D} + \bar{\mathfrak{Z}} = K_X + \bar{\mathfrak{B}}_1$. $\mathfrak{D}'$ is a subdivisor of $K_X + \tilde{\mathfrak{B}}$.

Let C be a curve on X. If C is in $\bar{Z}$, then C maps to a point in S/G, hence $K_XC \geq 0$ as X is the minimal resolution of S/G, and then $\mathfrak{D}'C \geq 0$; otherwise $\mathfrak{D}'C \geq \mathfrak{D}C \geq 0$. Therefore $\mathfrak{D}'$ is nef, and we can use part I) to conclude that $\mathfrak{D}' = \mathfrak{D}$. **QED**

Corollary 1. *For any (-1)-curve E on X, we have $E(K_X+\mathfrak{B}) \geq 0$, and E has the same coefficient in $\tilde{\mathfrak{B}}$, $\mathfrak{B}$ and $\bar{\mathfrak{B}}$.*

Proof. We just note that by construction, $\sigma\tilde{\Phi}^{-1}(E)$ is a curve on S. Hence Z does not contain E, and $E(K_X+\mathfrak{B}) \geq E\mathfrak{D} \geq 0$. **QED**

Corollary 2. *Let $\rho : X \to Y$ be a contraction to a smooth Y. Then ρ only blows up singular points of $\rho(B)$.*

Proof. Suppose $\rho_1 : Y_1 \to Y$ is the blow-up of one point p on Y, which ρ factors through, with $\rho' : X \to Y_1$. Assume that the order of $\rho(B)$ at p is ≤ 1, hence that of $\rho(\bar{\mathfrak{B}})$ equals $a < 1$.

Let E be the exceptional curve of ρ_1 in Y_1, and let c be the coefficient of E in $\rho'(\bar{\mathfrak{B}})$. Then as $-c > a-c-1 = \rho'(\mathfrak{D})E \geq 0$, we would get $c < 0$, contradicting the effectiveness of $\bar{\mathfrak{B}}$. **QED**

Corollary 3. *If some irreducible component Γ of B has different coefficients in $\mathfrak{B}$ and $\bar{\mathfrak{B}}$, then $\Gamma\mathfrak{D} = 0$, $\Gamma \cong \mathbb{P}^1$, and $\Gamma^2 < 0$.*

Proof. The condition means that Γ is in Z. Furthermore, if $p_a(\Gamma) > 0$ then $K_X\Gamma \geq -\Gamma^2$. But the coefficient of Γ in $\bar{\mathfrak{B}}$ is strictly less than 1, so $\bar{\mathfrak{B}}\Gamma > \Gamma^2$, which violates the requirement $\Gamma\mathfrak{D} = 0$. **QED**

Lemma 5. *Let X be a smooth surface, $E = \sum_{i=1}^{l} E_i$ a curve on X with negative-definite intersection form, $x_1, \ldots, x_l$ a set of non-negative rational numbers. Then there is a unique effective $\mathbb{Q}$-divisor D supported in E, with the property that $DE_i = -x_i$ for $i = 1, \ldots, l$.*

Suppose that D' is another $\mathbb{Q}$-divisor supported in E, such that $D'E_i \geq -x_i$ for $i = 1, \ldots, l$. Then $D' \leq D$.

Proof. Let H be the subspace of $H^2(X, \mathbb{Q})$ generated by the classes of E_i, v a vector in H. Then the equation $vE_i = -x_i$ corresponds to a hyperplane H_i in H. The negative-definiteness means that these hyperplanes $H_1, \ldots, H_l$ are in general position, so their intersection is a unique point, corresponding to D.

As the zero-divisor 0 satisfies $0E_i \geq x_i$, it suffices to show the last statement. And by considering $D - D'$, we may assume $x_1 = \cdots = x_l = 0$. Then if D' is not anti-effective, we may write $D' = D_1 - D_2$, where D_1 and D_2 are anti-effective without common components, and $D_2 \neq 0$. As $-D_2^2 > 0$, there exists an E_i with $D_2E_i > 0$. Hence $D'E_i > 0$, contradiction. **QED**

Remark. Let p be a singular point on S/G, $E = \sum_{i=1}^{l} E_i$ the intersection of its inverse image with B, D' (resp. D) the part of $\mathfrak{B}$ (resp. $\bar{\mathfrak{B}}$) supported on E. Letting $x_i = (K_X + \mathfrak{B} - D')E_i$, we find by Lemma 5 that the coefficients of E_i in $\bar{\mathfrak{B}}$ is uniquely determined by $\mathfrak{B} - D'$ and the numerical configuration of E.

Another direct consequence of Lemma 5 is

Lemma 6. *Let $\rho : X \to Y$ be a contraction, E a negative-definite configuration of (-2)-curves on Y, which is contained in $\rho(B)$. Then $\rho(B) - E$ intersects E positively.*

Proof. Suppose the contrary, then $\rho(\bar{\mathfrak{B}})$ intersects every component of E non-negatively, hence its part on E is anti-effective by Lemma 5. **QED**

The following observation will also be useful.

Lemma 7. *Suppose that X has a fibration $f : X \to C$. Let F be a general fibre of f. Then:*

If F is elliptic, $BF > 0$; if F is rational, $\mathfrak{B}F > 2$.

Proof. Let $\tilde{f} : \tilde{S} \to \tilde{C}$ be the fibration on $\tilde{S}$ pulled back from f. Let $\tilde{F}$ be a general fibre of $\tilde{f}$ over F. As $\tilde{S}$ is of general type, we must have $\sigma^*(K_S)\tilde{F} > 0$. This implies $(K_X + \mathfrak{B})F \geq \mathfrak{D}F > 0$. **QED**

§2. First reductions

Proposition 1. *I) If X is a surface of general type, $|G| \leq K_S^2$;*
II) if $\kappa = 1$, $|G| \leq 3K_S^2$;
III) if X has a fibration onto a curve of genus ≥ 2, $|G| \leq 10.5K_S^2$.

Proof. I) Let $\rho : X \to Y$ be the contraction of X onto its uniquely determined minimal model Y, and let L be the 1-saturated pullback of the 0 divisor on Y. Then $L^2 = K_Y^2 > 0$, hence Lemma 2 gives the estimate.

II) Let $\rho : X \to Y$ be as above. We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\rho} & Y \\ f\downarrow & & \downarrow f_Y \\ C & \xrightarrow{\sim} & C \end{array},$$

where f and f_Y are elliptic fibrations over a curve C. The exceptional curves of ρ are all contained in fibres of f.

Let E be a general fibre of f. By Lemma 7, there is an irreducible component Γ in B with $\Gamma E > 0$. Let Γ_Y be the image of Γ in Y, and L the 3-saturated pullback of Γ_Y. As $K_Y^2 = 0$ and $p_a(\Gamma) \geq 0$, we have

$$\begin{aligned} L^2 &= 6K_Y\Gamma_Y + \Gamma_Y^2 - \sum_{a_i>3} (a_i - 3)^2 \\ &\geq 5 + (2p_a(\Gamma) - 2) \geq 3 \ . \qquad \text{by (4)} \end{aligned}$$

This allows us to use Lemma 2 for $r = 3$, to get $|G| \leq 3K_S^2$.

III) Let $\phi : X \to C$ be such a fibration. ϕ pulls back to a fibration $\tilde{f} : \tilde{S} \to \tilde{C}$. As $g(\tilde{C}) \geq g(C) \geq 2$, $\tilde{f}$ descends to a fibration $f : S \to \tilde{C}$. We have $K_S^2 \geq 8(g-1)(g(\tilde{C})-1)$, where g is the genus of a general fibre F of f [B]. On the other hand, let $H \subset G$ be the normal subgroup consisting of elements with trivial induced action on $\tilde{C}$. Then we have an injective homomorphism $H \to \mathrm{Aut}(F)$ (therefore $|H| \leq 84(g-1)$), and an injection $G/H \to \mathrm{Aut}(\tilde{C})$. The latter inplies $[G : H] \leq g(\tilde{C}) - 1$, because the quotient C is of genus ≥ 2. **QED**

For the rest of this section, we suppose $\kappa(X) = 0$. As in the proof of Proposition 1, let $\rho : X \to Y$ be the contraction to the unique minimal model Y. Let B_1 be a connected component of $\rho(B)$.

Lemma 8. *Let D be a connected effective divisor on a minimal surface Y of Kodaira dimension 0. Then:*

I) if $D^2 < 0$ and D is reduced and irreducible, then D is a (-2)-curve and Y is a K3 surface or an Enriques surface;

II) if $D^2 = 0$ and the intersection form restricted to the classes of the irreducible components of D is negative semi-definite, then there is an elliptic fibration $f_Y : Y \to C$ such that D contains a subdivisor D' which is some multiple of a fibre of f_Y.

Proof. I) is well-known (and easy). Also II) is well-known if Y is an abelian surface or a bielliptic surface (in these cases D_{red} is a smooth elliptic curve). If Y is K3, we have $\dim|D| \geq 1$, and the condition that subdivisors of D have self-intersection ≤ 0 means that the moving part of $|D|$ has self-intersection 0, hence it is associated to a fibration of Y. Then use Zariski's Lemma to get what we need. Finally, for Enriques Y, see [BPV, §VIII.17]. **QED**

There are several possibilities for the configuration of B_1:

Case I) B_1 contains a component Γ with $\Gamma^2 > 0$.

Let L (resp. $\tilde{\Gamma}$) be the 2-saturated pullback (resp. the strict transform) of Γ. We have

$$L^2 = \Gamma^2 - \sum_{a_i > 2} (a_i - 2)^2 > 0$$

because $-2 \leq 2p_a(\tilde{\Gamma}) - 2 = \Gamma^2 - \sum_i a_i(a_i - 1)$. Hence $|G| \leq 4K_S^2$ by Lemma 2.

Case II) There exists a connected effective divisor B_2 supported in B_1, with $B_2^2 = 0$ and with negative semi-definite intersection form.

Let $f_Y : Y \to C$ be as in Lemma 8, such that there is an effective divisor B_3 supported in B_1, which is a multiple of a fibre F of f_Y. We may suppose that B_3 is not an integral multiple of another effective divisor. Then by Kodaira's table of singular fibres in an elliptic fibration (e.g. [BPV, p. 150]), we see that the components of B_3 have multiplicity ≤ 6.

According to Lemma 7, there exists an irreducible component Γ of B_2 such that $B_3\Gamma > 0$. We may suppose $\Gamma^2 \leq 0$ in view of Case I), and $\Gamma^2 = -2$ (hence Γ is smooth) unless B_3 is irreducible, modulo replacement of B_3 by Γ.

Note that the singularities of the divisor $2B_3 + \Gamma$ are of order at most 23, therefore the 24-saturated pullback L of $2B_3 + \Gamma$ has $L^2 = (2B_3 + \Gamma)^2 \geq 2$, and Lemma 2 gives $|G| \leq 288K_S^2$.

Case III) There are 2 (-2)-curves in B_1, say Γ_1, Γ_2, such that $\Gamma_1\Gamma_2 \geq 3$.

As Γ_1 and Γ_2 are smooth, the divisor $\Gamma = \Gamma_1 + \Gamma_2$ has at most double points, therefore the 2-saturated pullback L satisfies $L^2 = (\Gamma_1 + \Gamma_2)^2 \geq 2$, and consequently $|G| \leq 2K_S^2$.

Now the only possibility left for B_1 is that all the components are (-2)-curves, and the intersection of any two different components is at most 1. As the intersection form is not negative definite due to Lemma 6, it is easy to see that B_1 contains a subconfiguration with negative semi-definite intersection, and we get a divisor B_2 for Case II). It results that the above 3 cases are exhaustive, and

Proposition 2. *If* $\kappa(X)=0$, $|G|\leq 288K_S^2$.

According to the above two propositions and the classification of algebraic surfaces, we have

Corollary. *Theorem 1 is true unless* X *is either a rational surface, or a surface birationally ruled over an elliptic curve.*

§3. The case of elliptic ruled X

For the reader's convenience, we include a full proof of the following elementary fact (compare [H1, Proposition 3.1]).

Lemma 9. *Let* X *be a smooth surface with a fibration* $f: X \to C$ *to a smooth curve* C*, such that a general fibre* F *of* f *is a smooth rational curve. Let* B *be a reduced effective divisor on* X*, and let* $n = BF > 0$*. Then there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\rho} & Y \\ f\downarrow & & \downarrow f_Y \\ C & \xrightarrow{\sim} & C \end{array},$$

where ρ *is a birational morphism,* Y *is geometrically ruled over* C *via* f_Y*, such that:*

Let B_Y *be the image of* B *in* Y*. Then the singularities of* B_Y *introduced by* ρ *all have order* $\leq n/2+1$*, and if* p *is a singular point of order* $> n/2$*, then* B_Y *contains the fibre of* f_Y *passing through* p*.*

Proof. Let B_1 be the sum of components of B which are not contained in fibres of f, and let F_0 be a singular fibre of f. Take a (-1)-curve Γ in F_0. If Γ has multiplicity ≥ 2 in F_0, we have $B_1\Gamma \leq \frac{1}{2}B_1F_0 = \frac{n}{2}$, therefore we can blow down Γ, introducing a singularity of the image of B_1 of order only $B_1\Gamma$.

Suppose Γ is a simple component of F_0. Then as $K_XF_0=-2$, we must have another (-1)-curve Γ' in F_0. Now as $B_1(\Gamma+\Gamma')\leq B_1F_0 = n$, we can suppose $B_1\Gamma\leq\frac{n}{2}$, and blow down Γ. The lemma follows by induction on the Picard number of X. **QED**

Definition 3. Let $f_Y: Y\to C$ be a geometrically ruled surface, D a $\mathbb{Q}$-divisor on Y. Let $K_{Y/C} = K_Y - f_Y^*(K_C)$ be a relative canonical divisor on Y, and F a general fibre of f_Y. We define the *vertical degree* and *horizontal degree* to be respectively DF and $-DK_{Y/C}$. D is also called a divisor of *bidegree* $(DF, -DK_{Y/C})$.

If D_1, D_2 are 2 divisors of bidegrees (n_1, m_1) and (n_2, m_2) respectively, then $D_1D_2 = (n_1m_2+n_2m_1)/2$. Note that this definition of horizontal degree differs from some conventional definitions by a factor of 2. The advantage is that this horizontal degree is an integer when D is integral.

Now we suppose that X is birationally ruled over an elliptic curve C, with B as defined in §1, Y, F as in Lemma 9.

Let B_1 be the sum of components in $B_Y = \rho(B)$ which are not contained in fibres of f_Y, $\tilde{B}_1$ the strict transform of B_1 on X. Note that $B \geq \tilde{B}_1$, and $p_a(\tilde{B}_1) \geq 1$ as $\tilde{B}_1$ has no rational components.

Let (n, m) be bidegree of B_1. We have $n \geq 3$ (Lemma 7), which implies that $m \geq 0$ (cf. [H2, §V.2]).

Now we separate different possibilities for n.

Case I) $m = 0$.

As $p_a(B_1) = 1$, the map $f_Y|_{B_1} : B_1 \to C$ is étale, in particular B_1 is smooth.

Let $\pi : \tilde{C} \to C$ be a finite étale Galois cover such that if $\tilde{f} : \tilde{Y} \to \tilde{C}$ is the pull-back of f_Y by π with $\Pi : \tilde{Y} \to Y$ the induced cover, $\Pi^{-1}(B_1)$ is composed of n sections of $\tilde{f}$. Now a ruled surface with 3 disjoint sections is trivially ruled, hence we have $\tilde{Y} \cong \tilde{C} \times \mathbb{P}^1$, with $\tilde{f} = p_1$. Then as the Galois group of Π respects p_2, p_2 descends to an elliptic fibration $\phi : Y \to \mathbb{P}^1$, such that B_1 is contained in a finite number of fibres of ϕ. In particular we have $\Gamma^2 = 0$ for any irreducible component Γ of B_1.

Now by Lemma 7 applied to ϕ, there is a component Γ' in B_Y with $B_1\Gamma' > 0$. By definition, Γ' is a fibre of f_Y, hence $\Gamma'^2 = 0$, and the divisor $B_1 + \Gamma'$ has at most ordinary double points.

Suppose that $B_1 = \sum_{i=1}^{l} \Gamma_i$ is the decomposition into irreducible components, and let $\mathfrak{B}_1 = \sum_{i=1}^{l}(1 - 1/s_i)\Gamma_i$ be the part of $\rho(\mathfrak{B})$ supported on B_1.

Lemma 10. *Let $s_1, \ldots, s_l$ be l integers with $2 \leq s_1 \leq \cdots \leq s_l$, such that*

$$\Sigma = \sum_{i=1}^{l}(1 - \frac{1}{s_i}) > 2 \quad . \tag{5}$$

Then:

I) $\Sigma - 2 \geq 1/42$, with equality iff $l = 3$, $(s_1, s_2, s_3) = (2, 3, 7)$;

II) there exists an integer $N \leq 42$ such that

$$\sum_{i=1}^{l}\left[N - \frac{N}{s_i}\right] > 2N \ .$$

Furthermore, if we take $\{s_i\}$ to be a minimal set satisfying the condition (5), then $[N - N/s_i] \leq \frac{6}{7}N$ for $i = 1, \ldots, l$.

Proof. As $1 - 1/s_i \geq 1/2$, the lemma is obvious when $l \geq 4$ (we can take $N = 2$ when $l > 4$, and $N = 6$ when $l = 4$). So we assume $l = 3$. Then it is easy to see that the triplet (s_1, s_2, s_3) dominates one of the following three:

$$(2,3,7) \ , \ (2,4,5) \ , \ (3,3,4) \ .$$

We may replace (s_1, s_2, s_3) by the triplet it dominates, and take $N = 42, 20, 12$ respectively. **QED**

Let N be as in Lemma 10, such that $N\mathfrak{B}_1$ contains an integral divisor B_N, such that for a general fibre F of f_Y, $B_N F > -NK_Y F = 2N$. We minimalize B_N as in Lemma 10.

Let $t = N - [\frac{6}{7}N]$, and let L be the N-saturated pullback of $B_N + t\Gamma'$. Then as the components of B_N have multiplicities $\leq \frac{6}{7}N$, singular points of $B_N + t\Gamma'$ have order $\leq N$, and hence $L = \rho^*(NK_Y + B_N + t\Gamma')$. Now Lemma 2 gives

$$|G| \leq \frac{N^2}{2t} K_S^2 \leq 147 K_S^2 \ .$$

Case II) $m > 0$, $n > 4$.

Let L be the 2-saturated pullback of B_1. We have

$$L^2 = m(n-4) - \sum_{a_i \geq 2} (a_i - 2)^2$$

by (3). On the other hand, the fact $2p_a(\tilde{B}_1) - 2 \geq 0$ and (4) gives

$$\sum_{a_i \geq 2} a_i (a_i - 1) \leq m(n-1) \ .$$

We conclude from these $L^2 > 0$, as $a_i \leq n/2$ implies $\frac{(a_i-2)^2}{a_i(a_i-1)} < \frac{n-4}{n-1}$. Hence $|G| \leq 4K_S^2$ by Lemma 2.

Case III) $m > 0$, $n = 4$.

Let Γ_i, s_i, F be as in Case I). According to Lemma 7, there is an i with $s_i > 2$. Let $B_6 = \sum_{i=1}^{l} a_i \Gamma_i$, where $a_i = 3$ if $s_i = 2$, and $a_i = 4$ otherwise. We have $B_6 \leq 6\mathfrak{B}_1$, and let $t = (B_6 + 6K_Y)F$, we have $1 \leq t \leq 4$.

Now B_1 has at most double points. And modulo elementary transformations if necessary, we may suppose that the order of B_6 at such a double point is at most $6 + [\frac{1}{2}t]$.

Let L be the 6-saturated pullback of B_6. Then

$$L^2 = tm_6 - \sum_{a_i \geq 6} (a_i - 6)^2 \geq tm_6 - \left[\frac{1}{2}t\right]^2 \delta \ ,$$

where $m_6 > 0$ is the horizontal degree of B_6, and $\delta \leq 3m$ is the number of double points of B_1 blown up.

We now show that $L^2 > 0$, hence $|G| \leq 36K_S^2$:

In fact, this is clear when $t = 1$; when $t = 2$ or 3, it is easy to see that $m_6 \geq 3m$, hence $tm_6 \geq 6m > 3m$; and if $t = 4$, we have $m_6 = 4m$.

Case IV) $m > 0$, $n = 3$.

In this case B_1 is smooth, and as in Case I), there exists an N such that $N\mathfrak{B}_1$ contains an integral divisor B_N with $B_N F > 2N$. One verifies easily that the vertical degree m_N of B_N is strictly positive. Therefore the N-saturated pullback L of B_N, which equals $\rho^*(NK_Y + B_N)$, has $L^2 \geq 2$. Moreover we may assume $N \leq 12$ because B_1 has at most 2 irreducible components, hence

$$|G| \leq 72K_S^2$$

as in Case I).

Summing up this section,

Proposition 3. *If X is ruled over an elliptic curve,* $|G| \leq 147K_S^2$.

§4. The case of rational X: proof of Theorem 2

We suppose that X is a rational surface in this section. First note that if $X \cong \mathbb{P}^2$ and the degree of B is at least 7, then

$$|G| \leq 4K_S^2$$

by Lemma 2, letting $L = 2K_X + B$, $r = 2$.

Now we suppose that X is not isomorphic to $\mathbb{P}^2$, therefore there is a fibration $f: X \to \mathbb{P}^1$ whose general fibres are $\cong \mathbb{P}^1$.

Consider the set of all such fibrations f and contractions $\rho: X \to Y$ with commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\rho} & Y \\ f\downarrow & & \downarrow f_Y \\ \mathbb{P}^1 & \xrightarrow{\sim} & \mathbb{P}^1 \end{array}$$

where all the fibres of f_Y are $\cong \mathbb{P}^1$. For such a contraction, let C_0 be a section of f_Y with minimal self-intersection, $e = -C_0^2$. Then minimize ρ by first restricting to those such that the vertical degree n of $B_Y = \rho(B)$ is minimal, then further restricting to those with B_Y^2, and finally e, minimal. We fix a ρ in the final subset of contractions, and let m be the horizontal degree of B_Y. Note that m has the same parity as en, and $n \geq 3$ by Lemma 7.

Definition 4. We write $B = \rho^* B_Y - \sum_{i=1}^k a_i \mathfrak{E}_i$, where $\mathfrak{E}_i, p_i$ are as in Definition 1. Let also E_i be the strict transform in X of the exceptional curve of ρ_i, F_i the fibre of f containing E_i.

Definition 5. Let $\rho : X \to Y$ be a series of blow-ups of a smooth surface Y, $\rho_1 : X_1 \to Y_1$ one step among them, centered on $p \in Y_1$. Then the *level* of ρ_1 (or of p, or its exceptional curve E in X_1), $\lambda(\rho_1)$ or $\lambda(p)$ or $\lambda(E)$, with respect to ρ, is the level of infinitely nearness of p on Y plus 1, which can be defined inductively:

If p is an ordinary point of Y, then $\lambda(p) = 1$; otherwise $\lambda(p)$ equals 1 plus the maximal level of the exceptional curves on Y_1 passing through p.

We can also define a curve E on X not contracted by ρ to be of level 0.

In order to simplify notations in our future study of singularities, when we consider one singularity p_i individually, we will always assume that p_1 is the image of p_i on Y, and p_j is immediately infinitely near p_{j-1} for $j = 2, \dots, i$. (In particular, $\lambda(p_i) = i - 1$.)

Lemma 11. *Let p_i be a point with $i = \lambda(p_i) > 1$, and suppose that p_1 is worse than a double point of B_Y. Then we have $a_1 \geq a_i - 1$, and if $a_1 = a_i - 1$, $E_1, \dots, E_{i-1}$ form an A_{i-1}-configuration of (-2)-curves in B, which intersects other components of B only once.*

Proof. Let $\tilde{B}$ be the strict transform of B_Y in X, and write

$$\tilde{B} = \rho^*(B_Y) - \sum_{i=1}^k b_i \mathfrak{E}_i \ .$$

As $b_j \leq b_{j-1}$ for $j = 2, \dots, i$ and $b_1 - 1 \leq a_1 \leq b_1$, $a_i \leq b_i + 2$, the inequality $a_i > a_1 + 1$ is satisfied only if either E_1 and E_{i-1} are in B, but E_i is not; or p_i is a singular point in the inverse image of p_1 in X_i. In the first case we get an isolated (-2)-chain, which is excluded by Lemma 6; in the second case we have $b_1 \geq 2b_i$, so $a_i \leq b_i + 2 \leq b_1/2 + 2 \leq a_1 + 1$ as $b_1 > 2$. Note that equality cannot occur without introducing isolated (-2)-configuration in the second case, hence if $a_i = a_1 + 1$, p_j is a simple point on the inverse image of p_1 in X_j, for $j = 2, \dots, i$. The rest is straightforward. **QED**

Remark. In the situation of Lemma 11, it is clear that if E_j is the component which meets other parts of B, then $a_1 = a_{j+1} = a_{j+2} = \cdots = a_{i-1} = a_2 - 1 = a_3 - 1 = \cdots = a_j - 1 = a_i - 1$.

Lemma 12. *If $n \geq 5$, we have $a_i \leq \min\{n/2, m/4\} + 1$.*

Proof. First note that $a_1 \leq n/2$: otherwise an elementary transformation centered at p_1 would contradict the minimality of B_Y^2. Hence by Lemma 11, we can assume $m < 2n$. Then the condition $n \geq 5$ gives $e \leq 2$, also $e > 0$ due to the minimality of n.

Assume $e = 2$. Then B_Y must contain C_0, which is a (-2)-curve. So Lemma 6 says that $(B_Y - C_0)C_0 > 0$, or $m = 2n - 2$. Then as a_1 is not on C_0, the minimality of e implies further $a_1 < n/2$, and we are done.

Let $e = 1$. We have $B_Y C_0 = (m - n)/2$, hence if p_1 is on C_0, we may contract C_0 and apply Lemma 11, to get

$$a_i \leq (m - n)/2 + 1 < m/4 + 1 \ . \tag{6}$$

Otherwise, we may consider the inverse image of the pencil of curves of bidegree (1,1) passing through p_1. Then by the minimality of n, we get $a_1 \leq (m - n)/2$, so (6) also holds. **QED**

Definition 6. We factorise the map ρ into $\tilde{\rho} : X \to \hat{Y}$ and $\hat{\rho} : \hat{Y} \to Y$, with $\hat{K} = K_{\hat{Y}}, \hat{B} = \tilde{\rho}(B)$, such that $\hat{K}^2 = \max\{K_X^2, 0\}$, and that $\hat{B}^2$ is minimal among all such choices of $\hat{Y}$. When $K_X^2 < 0$, we may suppose that $\hat{\rho}$ consists of the first 8 blowing-ups $\rho_1, \ldots, \rho_8$. Then the minimality of $\hat{B}^2$ means that these blow-ups are chosen in a way that $a_1, \ldots, a_8$ are maximal in the obvious sense.

Lemma 13. *When $n \gg 0$, we have $\hat{K}\hat{B} < 0$.*

Proof. Suppose $\hat{K}\hat{B} \geq 0$. By Riemann-Roch, there is an effective divisor $\hat{D}$ on $\hat{Y}$, linearly equivalent to $-\hat{K}$. Its image $D = \hat{\rho}(\hat{D})$ on Y is a divisor of bidegree $(2, 4)$, with

$$\hat{D} = \hat{\rho}^* (D) - \sum_{i=1}^{8} \mathfrak{E}_i \ .$$

Suppose first that D has an irreducible component Γ whose strict transform

$$\hat{\Gamma} = \hat{\rho}^* (\Gamma) - \sum_{i=1}^{8} b_i \mathfrak{E}_i$$

in $\hat{Y}$ has $\hat{\Gamma}\hat{K} > 0$. As $h^0(-\Gamma - K_Y) > 0$, Γ is either a section of f_Y passing through at least $\Gamma^2 + 3$ p_i's (hence $\Gamma^2 \leq 5$), a fibre passing through at least 3 p_i's, a divisor of bidegree (2,2) passing through at least 5 p_i's, or that of bidegree (2,4) passing through all the 8 points blown up by $\hat{\rho}$, with a double point on one of them.

Let $l = \min\{n/2, m/4\}$. As

$$\hat{K}\hat{B} = -m - 2n + \sum_{i=1}^{8} a_i \leq 8 - |m - 2n|$$

by Lemma 12, we have $|m - 2n| \leq 8$. This gives $e \leq 2$ when $n \gg 0$, or $\Gamma^2 \geq -2$ when Γ is a section. It also gives $a_i > l - 8$ for $i = 1, \ldots, 8$.

Now $\hat{\Gamma}^2 < 0$, and $\hat{B}\hat{\Gamma} \geq \hat{\Gamma}^2$ as $\hat{B}$ is reduced. Checking this condition with the above possibilities for Γ, one sees that this cannot be satisfied for $n \gg 0$.

Now we can assume that for every irreducible component Γ, $\hat{\Gamma}\hat{K} \leq 0$. Then it is immediate that $\hat{D}$ equals the strict transform of D, so that $\hat{K}\Gamma = 0$ for every irreducible component Γ of $\hat{B}$, hence $\hat{K}\hat{B} = 0$.

Let $\hat{B} = \sum_{i=1}^{l} \Gamma_i$ be the decomposition into connected components. The intersection form restricted on any Γ_i is negative semidefinite, for otherwise Hodge Index Theorem would give triviality of $\hat{K}$. And it cannot be negative definite by Lemma 6, therefore there is an effective divisor B_i supported on each Γ_i, with $B_i^2 = 0$. Due to the negative semidefiniteness of Γ_i, B_i contains every irreducible component of Γ_i.

Let $\bar{B} = \sum_i B_i$, and note that $\chi(\hat{Y}, \bar{B} + n\hat{K}) > 0$ for any $n \in \mathbf{Z}$. Let n be the largest integer such that $H^0(\hat{Y}, -\bar{B} - (n-1)\hat{K}) = 0$. Then $h^0(\bar{B} + n\hat{K}) = h^2(-\bar{B} - (n-1)\hat{K}) > 0$, $h^0(-\bar{B} - n\hat{K}) > 0$, which means that $\bar{B}$ is linearly equivalent to $n\hat{D}$. As the vertical degree of B_Y is greater than 2, we have $\dim |n\hat{D}| \geq 1$.

Let $|M|$ and Z be respectively the moving and fixed parts of $|n\hat{D}|$. Then due to $M^2 \geq 0$, $Z\hat{D} = M\hat{D} = 0$, we get $M^2 = Z^2 = MZ = 0$. In other words $|M|$ is associated to an elliptic fibration $\hat{f} : \hat{Y} \to \mathbb{P}^1$, and Z is composed of multiples of fibres of $\hat{f}$. But then all the components of $\hat{B}$ are contained in fibres of $\hat{f}$, which contradicts Lemma 7.

QED

As our estimate of L^2 is based on the inequality (4) and an estimate on $p_a(\tilde{D})$, the following lemmas are useful.

Lemma 14. *Let $\rho' : X \to Y'$ be any contraction of X, $B' = \rho'(B)$. Then $p_a(B') \geq K_{Y'}^2 - 9 = 1 - d$, where $d = \rho(Y')$ is the Picard number of Y'.*

Proof. Let $B' = \sum_{i=1}^{l} \Gamma_i$ the decomposition of B' into connected components. As Γ_i is reduced, we have $h^0(\mathcal{O}_{\Gamma_i}) = 1$, hence $1 - p_a(B')$ is greater than or equal to the number of Γ_i with $p_a(\Gamma_i) = 0$.

Suppose $p_a(B') < 1 - d$. Then there are $d + 1$ components, say $\Gamma_1, \ldots, \Gamma_{d+1}$, with $p_a = 0$. This leads to a numerical (hence linear as Y' is rational) equivalence relation

$$D = \sum_i a_i \Gamma_i \equiv \sum_j b_j \Gamma_j$$

among them, where Γ_i and Γ_j are mutually different components among $\Gamma_1, \ldots, \Gamma_{d+1}$, $a_i > 0, b_j > 0$. It implies that $|D|$ is a linear system without fixed component with $D^2 = 0$, hence it is associated to a pencil without base point. And it is immediate that $K_{Y'}D < 0$, so this is a pencil of rational curves. Now by hypothesis, all the components of B' are contained in fibres of this pencil, which contradicts Lemma 7. **QED**

Combining Lemma 14 and (4), we get

$$\sum_{i=1}^{j} a_i(a_i-1) \leq mn - m - 2n + 2j + 4 \tag{7}$$

for any $j \leq k$.

Lemma 15. *Fix k real constants $h_1, \ldots, h_k$, and let $a_1, \ldots, a_k$ be a series of variable real numbers, with $a_1 \geq \cdots \geq a_k$, $h_1 \geq \cdots \geq h_k$, and $0 \leq a_i \leq h_i$ for $i = 1, \ldots, k$. Let $r \geq 2$ be an integer. Under the condition $\sum_i a_i(a_i-1) = \text{Const.}$, the sum $\sum_{a_i \geq r}(a_i - r)^2$ attains the maximum when there is an index $j \leq k$ such that $a_i = h_i$ for $i \leq j$, and $a_i = 0$ for $i > j+1$.*

Proof. By induction, we have only to consider the case $k = 2$, and we can obviously assume $a_1 \geq a_2 \geq r$. But in this case

$$\sum a_i(a_i-1) - \sum (a_i - r)^2 = (2r-1)(a_1+a_2) - 2r^2 \ ,$$

therefore the problem is equivalent to minimizing $a_1^2 + a_2^2$ for constant $a_1 + a_2$. **QED**

Proposition 4. *If $m \gg 0$ or $n \gg 0$, we have $|G| \leq 4K_S^2$.*

Proof. Let $h = m - 2n$. We consider the following cases separately.

Case I) $|h| \leq 16$, $n \gg 0$.

Due to the symmetry between $2n$ and m in Lemma 12, we may assyme $h \geq 0$ to simplify notations.

Let L be the 2-saturation of B. Our aim is to show $L^2 > 0$ for $n \gg 0$.

Arrange the sequences of blow-ups of ρ according to Definition 6, then rearrange the numbers $a_1, \ldots, a_8$ such that $a_1 \geq \cdots \geq a_8$ (if $k < 8$, just let $a_k + 1 = \cdots = a_8 = 1$). By Lemma 11, we have $a_i \leq a_8 + 1$ for $i > 8$, and Lemma 15 allows us to further assume $a_9 = \cdots = a_{k-1} = a_8 + 1$ (now the a_i's are only real numbers). Then we can assume $\sum_{i=1}^{8} a_i = m + 2n - 1 = 4n + 2h - 1$ according to Lemma 13. And maximize $\{a_1, \ldots, a_8\}$ by Lemma 15, we can assume $a_1 = \cdots = a_7 = n/2 + 1$, $a_8 = n/2 + h - 8$. Now use (7) with $j = 9$, we get

$$a_9(a_9 - 1) \leq n - (h-8)^2 + 14 \ ,$$

in particular $a_9 < a_8 + 1$ when $n \gg 0$, or $k \leq 9$. In particular, we have by (3)

$$L^2 = n - (h-8)^2 - 11 - (a_9 - 2)^2 > 0$$

when $n > 140$, hence $|G| \leq 4K_S^2$ by Lemma 2.

Case II) $|h| > 16$, $\min\{m, 2n\} \gg 0$; or $n > 6$, $h \gg 0$.

We may assume $m > 2n$ as in the previous case, and let L be the 2-saturation of B. Assuming $a_1 = \cdots = a_{k-1} = n/2 + 1$ using Lemma 15, we get

$$k < \frac{8n(n-2) + 4h(n-1) + 16}{(n+4)(n-2)} + 1$$

by (7) with $j = k - 1$. Then using (3), we have

$$(n+4)L^2 \geq (n-6)(3h-44) - 128 \ ,$$

hence $L^2 \gg 0$ under the condition of this case.

Case III) $n = 5$ or 6, $m \gg 0$.

Let B_h be the part of B not contained in fibres of f. In view of Lemma 9, we can construct a new contraction $\rho' : X \to Y'$, where Y' still has a fibration $f'_Y : Y' \to \mathbb{P}^1$ induced by f, such that the divisor $B'_1 = \rho'(B_h)$ has only singularities of order $\leq n/2$. Note that by the minimality of B_Y^2, the horizontal degree of the divisor $B'_Y = \rho'(B)$ is at least m. Therefore either the horizontal degree m_1 of B'_1 is $\gg 0$, or B'_Y contains enough fibres of f'_Y which do not pass through singular points of B'_1. In any case, we get a divisor B'_2 with $B'_1 \leq B'_2 \leq B'_Y$ and with horizontal degree $m_2 \gg 0$, having only singularities of order $\leq n/2$. Moreover, if we blow up only the triples points of B'_2, the arithmetic genus of the strict transform of B'_2 is at least -5.

Let L be the 2-saturated pullback of B'_2. When $n = 5$, B'_2 has only double points, so $L^2 = B'^2_2 \gg 0$ by the remark following Definition 1; while if $n = 6$, we can have at most $j = \frac{5}{6}m_2$ triple points, hence $L^2 \geq B'^2_2 - j = \frac{7}{6}m_2 - 8 \gg 0$.

Case IV) $n = 4$, $m \gg 0$.

Again let $B_h = \sum_{i=1}^{l} \Gamma_i$ be the non-vertical part of B, and let $1 - 1/s_i$ be the coefficient of Γ_i in $\mathfrak{B}$, as in the definition preceding Lemma 1. By Lemma 7, there is an i with $s_i > 2$, hence letting B_6 be the integral part of B_h, we have $13 \leq n_6 \leq 16$ where n_6 is the vertical degree of B_6.

As in the above case, construct $\rho' : X \to Y'$ such that the divisor $B'_6 = \rho'(B_6)$ has only singularities of order $\leq n_6/2$. When the horizontal degree of B'_6 is bounded, add a sufficient number of (reduced) fibres not passing through its singularities. We get a divisor B'' of arbitrarily large horizontal degree m", and let L be its 6-saturated pullback. Note that B" has singularities reducing L^2 only when $n_6 \geq 14$, and in this case the number of such singularities is bounded by the requirement that the reduced part of the strict transform of B" is at least -3. Hence a computation as in Case III) gives $L^2 \gg 0$.

Case V) $n = 3,\ m \gg 0$.

In this case we may get a contraction ρ' so that the non-vertical part of B'_Y is smooth, and that B'_Y contains enough fibres. So we may proceed as in Case I) of §3, to get an arbitrarily small $|G|/K_S^2$. Details are left to the reader. **QED**

For the remaining cases, it is convenient to break ρ as follows:

Definition 7. Let $\rho = \rho'' \circ \rho'$, where $\rho' : X \to Y'$ (resp. $\rho'' : Y' \to Y$) consists of blowing-ups of ordinary double points on the image of B (resp. that of singularities worse than ordinary double points). We denote the image by ρ' of a $\mathbf{Q}$-divisor on X by adding a prime to it.

Now Theorem 1 (hence Theorem 2) is proved except if B_Y is of bounded bidegree in Y (hence Y has bounded invariant e), or if $X = Y \cong \mathbf{P}^2$ and $B = B_Y$ is of bounded degree. There are only a finite number of combinations of these degrees and invariant of Y; for each such combination, a finite number of numerical decompositions of B_Y into irreducible components; for each decomposition, a finite number of possibilities for the numerical characteristics of singularities of B_Y; and for each such possibility, a finite number of numerical possibilities for Y' and B'. Here a *numerical possibility* means an abelian group $NS(Y')$ together with the intersection form and the canonical class, the classes of irreducible components B'_i ($i = 1, \ldots, l$) of B', and a series of symbols $p_1, \ldots, p_k$ (representing ordinary double points of B'), each associated to a pair of indices (i, j) (the branches of B' passing through the double point, and we may have $i = j$).

Consequently, Theorem 2 results from the following proposition.

Proposition 5. *For a fixed numerical possibility of Y' and B' as above, there is a constant c such that $|G| \le cK_S^2$.*

Proof. Assume the contrary. Then there exist a series of surfaces ${}_1S, {}_2S, \ldots$, all associated to the numerical classes of Y' and B', such that ${}_1c < {}_2c < \cdots$, where we denote by adding a left subscript j an object corresponding to ${}_jS$, and ${}_jc = |{}_jG|/K^2_{{}_jS}$. By a slight abuse of language, we identify ${}_jY'$ (resp. ${}_jB'$) with Y' (resp. B') as only the numerical behavior is involved.

Define the set of positive integers $\{{}_js_i\}_{i,j}$ by letting

$$ {}_j\mathfrak{B}' = \sum_{i=1}^{l} (1 - \frac{1}{{}_js_i}) B'_i \quad . $$

By extracting a subseries from $\{{}_jS\}$, we may assume that for each B'_i, either ${}_js_i = s_i$ is a constant for all j (in this case B'_i is called *stationary*), or ${}_js_i \ge j$.

As the blow-ups in ${}_j\rho' : {}_jX \to Y'$ are centered only on ordinary double points, for any fixed λ there are only a finite number of possibilities of blow-ups of level $\le \lambda$. Hence by extracting a subseries if necessary, we may assume that for all $j \geq \lambda$, blow-ups of level $\leq \lambda$ in ${}_j\rho'$ have the same numerical behavior.

Let $\rho'(j) : X(j) \to Y'$ the part of ${}_j\rho'$ consisting of blow-ups of level up to λ. Then symbolically, for any pairs of indices j, j' with $j < j'$, we can write $\tilde{\rho}_{j,j'} : X(j') \to X(j)$, such that $\rho'(j') = \rho'(j) \circ \tilde{\rho}_{j,j'}$. This makes sense because we will only be dealing with intersection numbers in the inverse image of B'. We denote the image in $X(j)$ of a $\mathbb{Q}$-divisor in ${}_{j'}X$ by adding a parenthesis (j) to the right. Then as before, we may assume that for any component E in $\rho'(j)^{-1}(B')$, if its coefficient in ${}_{j'}\mathfrak{B}(j)$ is $1 - 1/{}_{j'}s$, then either ${}_{j'}s$ is constant for all $j' > j$ (i.e. E is *stationary*), or ${}_{j'}s \geq j'$. This allows us to pass to the limit, obtaining a symbolic "surface" $X(\infty)$ with $\rho'(\infty) : X(\infty) \to Y'$ and a $\mathbb{Q}$-divisor $\mathfrak{B}(\infty)$ on it, supported on $B(\infty) \leq \rho'(\infty)^{-1}(B')$. The coefficient of a component in $\mathfrak{B}(\infty)$ is either 1 or $1 - 1/s$ for some integer $s > 0$. We will denote the image of $B(\infty), \mathfrak{B}(\infty)$, etc. in $X(j)$ by $B(j), \mathfrak{B}(j)$, etc.

Moreover, for any irreducible curve E in $X(j)$, let ${}_{j'}x$ be its coefficient in ${}_{j'}\bar{\mathfrak{B}}(j)$, for $j' = j, j+1, \ldots$ Again by extracting subseries, we may assume that the series ${}_jx, {}_{j+1}x, \ldots$ increases or decreases monotonously for every curve E in $\rho'(j)^{-1}(B')$, thus defining by passing to the limit an $\mathbb{R}$-divisor $\bar{\mathfrak{B}}(\infty)$ with $0 \leq \bar{\mathfrak{B}}(\infty) \leq \mathfrak{B}(\infty)$, as well as a $\mathfrak{D}(\infty)$ which is the limit of ${}_{j'}\mathfrak{D}(j)$. We have $\mathfrak{D}(\infty)E \geq 0$ for a curve E in $B(\infty)$ whenever the intersection makes sense (i.e. whenever $E^2 > -\infty$), and on any intermediate surface $X(j)$, the image of $\mathfrak{D}(\infty)$ intersects any curve in $\rho'(j)^{-1}(B')$ non-negatively.

Note that by the definition of $\mathfrak{B}$, a curve in $X(\infty)$ is stationary iff its coefficient in $\bar{\mathfrak{B}}(\infty)$ is less than 1.

A stationary curve E is called *superstationary* if its coefficients in ${}_{j'}\bar{\mathfrak{B}}(j)$ and ${}_{j'}\mathfrak{B}(j)$ coincide for $j \gg 0$. In particular, its coefficients in $\mathfrak{B}(\infty)$ and $\bar{\mathfrak{B}}(\infty)$ coincide. By applying Corollary 3 of Lemma 4 then passing to the limit, we have $\mathfrak{D}(\infty)E = 0$ unless E is superstationary.

Also for a non-superstationary E, we may assume that E has different coefficients in ${}_{j'}\bar{\mathfrak{B}}(j)$ and ${}_{j'}\mathfrak{B}(j)$ for $j \gg 0$.

In the rest of the proof, a *point* will be a point of level $\lambda \geq 1$ (i.e. an ordinary point on $X(\lambda-1)$), which is an ordinary double point of $B(\lambda-1)$. Let p be such a point. p is called *stationary* (resp. *semistationary, nonstationary*), if the two branches of $B(\lambda - 1)$ passing through p are both stationary (resp. one stationary and one not, or both not stationary). Here the *level* of a point always refers to the map $\rho'(\infty)$.

The following elementary fact will be called repeatedly, so we emphasize it by putting it into a lemma.

Lemma 16. *Let p be a point, x_1, x_2, x (resp. y_1, y_2, y) the coefficients in $\bar{\mathfrak{B}}(\infty)$ (resp. in $\mathfrak{B}(\infty)$) of the two branches passing through it and the exceptional curve of its blow-up. Then $(1 - x) \geq (1 - x_1) + (1 - x_2)$, and $y \leq \min\{y_1, y_2\}$.*

Proof. Let λ be the level of p, E the (-1)-curve in $X(\lambda)$ corresponding to the blow-up of p. Then

$$0 \leq \mathfrak{D}(\lambda)E = x_1 + x_2 - x - 1 \ ,$$

which is just the inequality for $1-x$. That for y is then a direct consequence of this one and the minimality of coefficients in the definition of $\mathfrak{B}$. **QED**

Corollary. *I) If p is stationary (resp. semistationary), then points above p are stationary (resp. stationary or semistationary).*

II) If E is a stationary component of $B(\infty)$, then $E^2 > -\infty$.

III) If p is stationary, then the relative levels (with respect to $\lambda(p)$) of the points above p are bounded above.

IV) If p is nonstationary and if the inverse image of p in $X(\infty)$ has a stationary component, then the exceptional curve of the blow-up of p is stationary. In particular, all the points above p are stationary or semistationary.

Proof. I) is obvious.

II) Let $x < 1$ be the coefficient of E in $\mathfrak{B}(\infty)$, N an integer such that $N(1-x) > 1$. It suffices to show that there are no blow-ups of level $> N + \lambda$ on the image of E, where λ is the level of E.

Assume the contrary. Then there is a series of points $p_1, \ldots, p_N$, where p_i is of level $\lambda + i$, which are centers of blow-ups contained in $\rho'(\infty)$, such that:

p_1 is on the image of E in $X(\lambda)$, and for $i > 1$, p_i is the intersection point of the exceptional curve E_{i-1} of the blow-up of p_{i-1} and the image of E, in $X(\lambda + i - 1)$.

Let x_i be the coefficient of (the strict transform in $X(\infty)$ of) E_i in $\mathfrak{B}(\infty)$. Then by Lemma 16, we have $1 - x_1 \geq 1 - x$, and $(1 - x_i) \geq (1 - x) + (1 - x_{i-1}) \geq i(1 - x)$ by induction. This gives $x_N < 0$, which is impossible.

III) Let x_1, x_2 be the coefficients in $\bar{\mathfrak{B}}(\infty)$ of the two branches passing through p, with $x_1 \leq x_2 < 1$. Let q be a point above p with $\lambda = \lambda(q) - \lambda(p)$, and let x'_λ be the coefficient of the exceptional curve of the blow-up of q. Then by Lemma 16 and induction, we have $1 \geq (1 - x'_\lambda) \geq (1 - x_1) + \lambda(1 - x_2)$, or

$$\lambda \leq \frac{x_1}{1 - x_2} \ . \tag{?}$$

IV) Let q be a point above p with minimal level, such that its exceptional curve is stationary, and let $\lambda = \lambda(q) - \lambda(p)$. Assume $\lambda > 1$. Then one of the branches passing through q is the exceptional curve E' of a point q' above p with level $\lambda(q') = \lambda(q) - 1$.

Consider the intermediate surface X_q between $X(\infty)$ and Y', on which q is blown up to a (-1)-curve E, and (the image of) E' is a (-2)-curve. Then the image of $\mathfrak{D}(\infty)$ on X_q intersects E' by at most $x - 1 < 0$, where x is the coefficient of E in $\bar{\mathfrak{B}}(\infty)$, which is a contradiction. **QED**

Now let p be a nonstationary point of level 1 on Y'. If the exceptional curve of p is stationary, we may incorporate the blow-up of p into ρ'', to assume that the inverse image of a nonstationary point does not contain stationary components.

Our main difficulty comes from semistationary points. Consider therefore such a point p, with $\lambda(p) = 1$. Let Γ be the nonstationary branch passing through p. If the blow-ups on (the strict transform of) Γ have bounded level, then everything in the inverse image of p is bounded (hence stationary) by the above Corollary, and we can incorporate the blow-ups above p into ρ'' and forget about them. So we assume that there is an infinite series of points $p_1 = p, p_2, p_3, \ldots$, such that $\lambda(p_i) = i$, and for $i \geq 2$, p_i is the intersection point of the exceptional curve E_{i-1} of p_{i-1} with Γ. Let q_i $(i = 3, 4, \ldots)$ be the intersection point of E_{i-1} and E_{i-2}. Our aim is to prove that there is no blow-up centered at q_i for $i \gg 0$.

Let x_i (resp. $1 - 1/s_i$) be the coefficient of E_i in $\bar{\mathfrak{B}}(\infty)$ (resp. $\mathfrak{B}(\infty)$). We have $x_1 \geq x_2 \geq \cdots$, $s_1 \geq s_2 \geq \cdots$ by Lemma 16, and $s_N = s_{N+1} = \cdots = s$ for $N \gg 0$.

Suppose that q_i is blown up. Then because q_i is stationary, its inverse image Q_i in $X(\infty)$ is supported on a chain of smooth rational curves $E_{i,1}, \cdots, E_{i,k_i}$, such that

$$E_{i-1}E_{i,1} = E_{i,1}E_{i,2} = \cdots = E_{i,k_i-1}E_{i,k_i} = E_{i,k_i}E_{i-2} = 1 \ ,$$

with no other mutual intersections. We define the *type* of q_i, τ_i, to be the set of integers

$$\{k_i, n_{i,0}, n_{i,1}, \ldots, n_{i,k_i}, n_{i,k_i+1}, s_{i,1}, \ldots, s_{i,k_i}\} \quad ,$$

where $n_{i,j} = E_{i,j}^2$ (we let $E_{i,0} = E_{i-1}$, $E_{i,k_i+1} = E_{i-2}$), and $1 - 1/s_{i,j}$ is the coefficient of $E_{i,j}$ in $\mathfrak{B}(\infty)$. We also let $\tau_i = 0$ if q_i is not blown up.

Lemma 17. *We have* $x_i = x_{i+1} = \cdots$ *for* $i \gg 0$.

Proof. We assume that E_i is not superstationary for $i \gg 0$, for otherwise there is nothing to prove.

The lemma is easy when q_i is not blown up for $i \gg 0$: E_i is a (-2)-curve in $X(\infty)$, and the condition $\mathfrak{D}(\infty)E_i = 0$ gives $x_{i-1} - x_i = x_i - x_{i+1}$, hence because the series $\{x_i\}$ is bounded below, we must have $x_{i+1} = x_i$ for $i \gg 0$.

So assume that there are infinitely many q_i's which are blown up. According to the part III) of the Corollary to Lemma 16, there are only a finite number of mutually different types over p. It results that there is a subseries $i_1 < i_2 < \cdots$, such that

$$\tau_{i_1} = \tau_{i_2} = \cdots = \tau' \neq 0 \ , \ \tau_{i_1+1} = \tau_{i_2+1} = \cdots = \tau'' \ .$$

Case I) $\tau'' \neq 0$.

Let $i = i_t$. Note that both Q_i and Q_{i+1} contain (-1)-curves, which are superstationary by Corollary 1 of Lemma 4. Hence we may consider the maximal connected divisor M of $B(\infty)$ containing E_{i-1} and composed of non-superstationary curves. As the intersection form on M is negative definite, the Remark following Lemma 5 says that x_{i-1} is determined only by the numerical property of M and the two curves touching it in the outside world, hence only by the types τ' and τ''.

Case II) $\tau'' = 0$.

For each t, let u_t be the least index such that E_{i_t,u_t} is superstationary. By extracting a subseries, we may assume that u_t equals a constant u.

Fix a $t \gg 0$. Let $d_{-1} = x_{i_1} - x_{i_t}$, d_j $(j = 0, \ldots, u)$ be the difference of the coefficients in $\mathfrak{B}(\infty)$ of $E_{i_1,j}$ and $E_{i_t,j}$ (so $d_0 = x_{i_1-1} - x_{i_t-1}$, $d_u = 0$). We may also assume $d_{-1} \leq d_0$ for the series $\{x_1, x_2, \ldots\}$ is lower-bounded.

From the condition $\mathfrak{D}(\infty)E_{i_1,j} = \mathfrak{D}(\infty)E_{i_t,j} = 0$, we get $d_{j-1} + n_{i_1,j}d_j + d_{j+1} = 0$ for $j = 0, \ldots, u-1$. Now $n_{i_1,j} \leq -2$ because $E_{i_1,j}$ is not superstationary (Corollary 1 of Lemma 4), hence $d_i \geq d_{i-1}$ by induction, and $0 \leq d_0 \leq d_u = 0$. **QED**

Corollary. *q_i is not blown up for $i \gg 0$.*

Proof. Assume the contrary, and let i be an index such that $x_i = x_{i+1}$, and that there exist i', i'' with $i' < i$, $i+1 < i''-1$, such that both $q_{i'}$ and $q_{i''}$ are blown up. Then as in Case I) of the proof of the Lemma, x_i and x_{i+1} are determined by the non-superstationary block containing E_i and E_{i+1}, which are confined at most by superstationary (-1)-curves in the inverse images of $q_{i'}$ and $q_{i''}$.

Let j be large enough such that ${}_jX$ has points q_k blown up the same way as in $X(\infty)$, for $k = 1, \ldots, i''$. Then the coefficients of E_i and E_{i+1} in ${}_j\bar{\mathfrak{B}}$ are also $x_i = x_{i+1}$. But this is impossible: let X' be an intermediate surface between ${}_jX$ and Y', on which the image E' of E_{i+1} is a (-1)-curve. Then E' would intersect the image of ${}_j\mathfrak{D}$ negatively, for the coefficient of Γ in ${}_j\mathfrak{D}$ is strictly less than 1. **QED**

Now it follows from this Corollary that $\rho'(\infty)$ contains only a finite number of stationary blow-ups. Forgetting some beginning terms of the series $\{{}_jS\}$ and incorporating enough blow-ups into ρ'', we may assume that there are no stationary blow-ups, and that for every semistationary point p, we have $s_1 = s_2 = \cdots =$ the corresponding number for the stationary branch passing through p.

And our proposition follows readily from the following easy lemma:

Lemma 18. *Let $\rho : X \to Y$ be a contraction, $\mathfrak{B}$ an effective $\mathbf{Q}$-divisor on X, $\mathfrak{B}_Y = \rho(\mathfrak{B})$, and p an ordinary double point of $\mathfrak{B}_Y$ blown up by ρ. Let x_1, x_2 be the* *coefficients in* $\mathfrak{B}_Y$ *of the two branches passing through* p *with* $x_1 \leq x_2 < 1$, *and let* $\lambda_p = [x_1/(1-x_2)]$.

Suppose that for every blow-up of level $\leq \lambda_p$ *above* p, *the coefficient of the exceptional curve in* $\mathfrak{B}$ *is* $\geq x_1$. *Then* $\mathfrak{B}$ *contains a subdivisor* $\mathfrak{B}'$, *whose 1-saturation is equal to the algebraic inverse image of* $\mathfrak{B}_Y$, *on the inverse image of* p.

Proof. Blow up p and replace $\mathfrak{B}$ by the subdivisor $\mathfrak{B}_1$ which equals $\mathfrak{B}$ except on the exceptional curve, letting the coefficient of the latter in $\mathfrak{B}_1$ to be $x_1 + x_2 - 1$. The lemma then follows from induction on λ_p. **QED**

Indeed, for $j \geq 1$, define ${}_j\mathfrak{C}$ to be a Q-divisor on ${}_jX$ with the same support as ${}_j\mathfrak{B}$, such that for any component Γ of ${}_jB$, if Γ is contracted by ${}_j\rho'$, then its coefficient in ${}_j\mathfrak{C}$ equals that of ${}_j\mathfrak{B}$; otherwise this coefficient equals that of the corresponding component in ${}_1\bar{\mathfrak{B}}$.

Now there exists an N_p for every nonstationary or semistationary point p, such that for $j \geq N_p$, the morphism ${}_j\rho' : {}_jX \to Y'$ and the Q-divisor ${}_j\mathfrak{C}$ satisfies the condition of Lemma 18 on p, with

$$\mathfrak{B}_{Y'} = {}_j\rho'({}_j\mathfrak{C}) = {}_1\bar{\mathfrak{B}}' \quad .$$

Let $N = \max_p\{N_p\}$, $j \geq N$. Then the 1-saturation ${}_jL$ of ${}_j\mathfrak{C}$ equals ${}_j\rho'^*({}_1\bar{\mathfrak{B}}')$, hence

$$\frac{1}{{}_jc} \geq {}_jL^2 = ({}_1\bar{\mathfrak{B}}')^2 \geq \frac{1}{{}_1c} \quad .$$

This contradiction with the hypothesis on the series $\{{}_jc\}_j$ completes the proof of this proposition, and accordingly that of Theorem 2. **QED**

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