A mean value estimate for Rankin-Selberg Zeta functions

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1 Introduction

Let $\{\phi_j\}$ denote an orthonormal basis of L^2-eigenfunctions of the Laplacian on $\mathrm{PSL}(2,\mathbf{Z})\backslash\mathbf{H}$ and $R_j(s)$ the corresponding Rankin–Selberg zeta function. In [1] Iwaniec stated a "mean Lindelöf" conjecture for these zeta functions which was recently proven by Luo and Sarnak in [3] if ϕ_j is assumed a Hecke eigenform. Indeed they consider the second symmetric power L–function to which they can relate the Rankin–Selberg zeta function because of the multiplicative properties of the Fourier coefficients of a Hecke eigenform. Also they use bounds of Iwaniec and Hoffstein–Lockhart for the first Fourier coefficient as well as the large sieve inequality due to Deshoulliers and Iwaniec.

In the present paper we want to give a proof for Iwaniec's conjecture in the half–integral case, see the theorem below. This proof carries over to the zero weight case. It differs from that given by Luo and Sarnak in that it does not need multiplicative properties of the Fourier coefficients. We do not consider the symmetric square L–function, also no bounds for the first Fourier coefficients are needed. Instead of a large sieve inequality we utilize a spectral mean square theorem which we proved in [6]. For the weight zero case such a theorem was proved by Kuznecov [2].

A difficulty arising for nonvanishing weight is the fact that one has to treat two Rankin–Selberg zeta functions, corresponding to the Fourier coefficients at the nonnegative and negative indices respectively. In [4] we have given an analytic continuation for both zeta functions together with a functional equation. In the present paper we give a more elaborate version, see lemma 3.1, of the factors occuring in the functional equation. This we need for the proof of the theorem.

We do this elaboration for arbitrary real weight since there is no need to restrict to half integral weight here.

2 Definitions

Let Ξ be a unitary m-dimensional multipliersystem of weight r for $\Gamma := \mathrm{SL}(2,\mathbf{Z})$ and $\vec{f_1},\dots,\vec{f_m}$ be an orthonormal set of eigenvectors of $\Xi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ with eigenvalues $e^{2\pi i\alpha_j}$, $0 \le \alpha_j < 1$, $j = 1,\dots,m$. A real analytic automorphic form for Γ with respect to Ξ is a function $F(z)$ on the upper halfplane $\mathcal{H} = \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$ with values in $\mathbf{C}^m$ which satisfies

i) $F(Mz) = \Xi(M) e^{ir \arg(cz+d)} F(z), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

ii) $-\Delta_r F = \lambda F$,
where $\Delta_r = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iry \frac{\partial}{\partial x}$ is the so-called *Laplace-Beltrami operator*,

iii) $F(z) = \mathcal{O}(y^c)$ for $y \to \infty$ with some constant $c > 0$.

Automorphic forms as just defined are eigenfunctions of an elliptic differential operator and hence are real analytic functions. In the sequel we shall especially be interested in the subspace of cusp forms, i.e. forms, that are of exponential decay at infinity. There is the usual Hilbert space $L^2(\Gamma \backslash \mathcal{H}, r, \Xi)$ with the scalar product

$$(F, G) = \int_{D_\Gamma} F(z) \bar{G}(z) \frac{dx dy}{y^2},$$

where D_Γ is a fundamental domain for Γ. Let $L_0^2(\Gamma \backslash \mathcal{H}, r, \Xi)$ denote the corresponding subspace of cusp forms and let $\{u_k\}$ be an orthonormal basis of this space.

For simplicity we consider only cusp forms with eigenvalue $\geq 1/4$.

It is then convenient to write the eigenvalue of u_k in the form

$$\lambda_k = (\frac{1}{2} + it_k)(\frac{1}{2} - it_k)$$

with $t_k \geq 0$. Since

$$u_k(z+1) = \Xi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) u_k(z)$$

we can expand u_k in a Fourier series

$$u_k(z) = \sum_{j=1}^{m} \psi_j(z) \vec{f}_j,$$

with

$$\psi_j(z) = \sum_{\substack{n \equiv \alpha_j \mathrm{mod} 1, \\ n \neq 0}} \rho_{j,k}(n) W_{\mathrm{sgn}(n)\frac{r}{2}, it_k}(4\pi |n| y) e^{2\pi i n x}.$$

The *Rankin-Selberg zeta functions* belonging to u_k are given by the Dirichlet series

$$R_{\pm,k}(s) = \sum_{j=1}^{m} \sum_{\substack{n \equiv \alpha_j \mathrm{mod} 1, \\ \pm n > 0}} \frac{|\rho_{j,k}(n)|^2}{|n|^{s-1}}.$$

In [4] we showed, that it is sufficient to restrict to $r \in [-1, 1)$, since all possible Rankin-Selberg zeta functions already occur for these weights. We introduce the the Mellin transform

$$\mathcal{M}_{\kappa, it}(s) = \int_0^\infty W_{\kappa, it}^2(y) y^{s-2} dy, \quad \kappa \text{ real, } \mathrm{Re}(s) > |2\mathrm{Re}(it)|,$$

where $W_{\kappa, it}$ is the exponentially decreasing Whittakerfunction. We write for short $\mathcal{M}_{\pm,k}(s)$ instead of $\mathcal{M}_{\pm r/2, it_k}(s)$.

3 Analytic properties of the Rankin–Selberg Zeta functions

In [4] we proved several analytic properties of the Rankin–Selberg zeta functions which we are going to state in this section. In the sequel always write $s = \sigma + i\tau$. Then we have shown in [4]

Proposition 3.1 *i) The abscissas of convergence of $R_{\pm,k}(s)$ are 1. Moreover we can continue $R^*_{\pm,k} = \zeta(2s)R_{\pm,k}(s)$ as meromorphic functions over the whole complex plane with the only pole at $s = 1$. Denote the residues at $s = 1$ by $b_{\pm,k}$. Then*

$$b_{\pm,k} = \frac{3(\mathcal{M}_{\mp,k}(2)\pm r\mathcal{M}_{\mp,k}(1)}{\pi\Gamma(1+2it_k)\Gamma(1-2it_k)}.$$

ii) We write

$$\mathbf{R}^*_k(s) = \begin{pmatrix} R^*_{+,k}(s) \\ R^*_{-,k}(s) \end{pmatrix}$$

and have a functional equation

$$\mathbf{R}^*_k(s) = \mathbf{C}_k(s)\mathbf{R}^*_k(1-s)$$

with

$$\mathbf{C}_k(s) = \frac{4^s\pi^{4s-\frac{3}{2}}\Gamma(1-s)\Gamma(2s-1)}{\Gamma^2(s)\Gamma(s-\frac{1}{2})\Gamma(s+2it_k)\Gamma(s-2it_k)}\begin{pmatrix} sc_{-+,k}(s) & sc_{--,k}(s) \\ sc_{++,k}(s) & sc_{+-,k}(s) \end{pmatrix},$$

where

$$c_{xy,k}(s) = \mathcal{M}_{x,k}(s+1)\mathcal{M}_{y,k}(1-s) + \delta(x,y)(s^2+4t_k^2)\mathcal{M}_{x,k}(s)\mathcal{M}_{y,k}(-s) \tag{1}$$

and $\delta(x,y) = 1$ if $x = y$ and -1 else.

Concerning the Mellin transform $\mathcal{M}_{\kappa,it}(s)$ we remark that for weight zero this is a Γ-factor which it fails to be for generic weight. Nevertheless we could prove in [4] the following recurrence relation

$$(s+1)\mathcal{M}_{\kappa,it}(s+2) - 2\kappa(2s+1)\mathcal{M}_{\kappa,it}(s+1) = s(s^2+4t^2)\mathcal{M}_{\kappa,it}(s). \tag{2}$$

From this follows, that $\mathcal{M}_{\kappa,it}(s)$ is a meromorphic function with simple poles at most at $s = 0, -1, \ldots$ and $s = \pm 2it, -1 \pm 2it, \ldots$.

A growth estimate for $\mathbf{C}_k$ was given in [5]. We have uniformly in $(0<)\delta \le \sigma \le 1-\delta$

i) if $\tau \le 1 + t_k$

$$\mathbf{C}_k(\pm\sigma + i\tau) \ll \delta^{-2}(1+|\tau|)^{2\mp 2\sigma}(1+t_k)^{1\mp 2\sigma}\begin{pmatrix} 1 & (1+t_k^{-1} \\ 1+t_k & 1 \end{pmatrix}$$

ii) if $|\tau| > 1 + t_k$

$$\mathbf{C}_k(\pm\sigma + i\tau) = \left(\frac{|\tau|}{\sqrt{2\pi}}\right)^{2\mp 4\sigma}$$
$$\cdot \begin{pmatrix} \mathcal{O}\left((1+\frac{t_k}{\sqrt{|\tau|}})^2\right) & \mathcal{O}\left((1+t_k)^{-1}e^{-\pi(|\tau|)}\right) \\ \mathcal{O}\left((1+t_k)e^{-\pi(|\tau|)}\right) & \mathcal{O}\left((1+\frac{t_k}{\sqrt{|\tau|}})^2\right). \end{pmatrix}$$

We are now able to give the explicit form of the transformation matrix $\mathbf{C}_k(s)$.

Lemma 3.1

$$sc_{\pm\pm,k}(s) = -\frac{\pi^5 b_{\pm,k}^2 \sin 2\pi s}{18\sin^2\pi s \sin\pi(s-2it_k)\sin\pi(s+2it_k)},$$

$$sc_{\pm\mp,k}(s) = \frac{\frac{\pi^5}{9}b_{+,k}b_{-,k} - 8\pi\cos\pi r + 8\pi\cos 2\pi(s\mp\frac{r}{2})}{\sin\pi s\sin\pi(s-2it_k)\sin\pi(s+2it_k)}.$$

Proof. We put

$$T_{\pm,k}(s) = sc_{\pm\mp,k}(s)\sin(\pi(s-2it_k))\sin(\pi(s+2it_k))\sin^2(\pi s).$$

It follows immediately from the recurrence relation that $c_{xy,k}(s) = c_{yx,k}(-s)$. Further By equation (2) we obtain

$$\begin{aligned} sc_{xy,k}(s) &= s\mathcal{M}_{x,k}(1+s)\mathcal{M}_{y,k}(1-s) + \delta(x,y)\mathcal{M}_{x,k}(s) \\ &\quad \cdot((s-1)\mathcal{M}_{y,k}(2-s) - yr(2s-1)\mathcal{M}_{y,k}(1-s)). \end{aligned} \tag{3}$$

Replace s by $1-s$. This yields

$$\begin{aligned} (1-s)c_{xy,k}(1-s) &= (1-s)\mathcal{M}_{x,k}(2-s)\mathcal{M}_{y,k}(s) + \delta(x,y)\mathcal{M}_{x,k}(1-s) \\ &\quad \cdot((-s)\mathcal{M}_{y,k}(1+s) + yr(2s-1)\mathcal{M}_{y,k}(s)) \\ &= -\delta(x,y)s\mathcal{M}_{x,k}(1-s)\mathcal{M}_{y,k}(1+s) - (s-1)\mathcal{M}_{x,k}(2-s)\mathcal{M}_{y,k}(s) \\ &\quad + yr(2s-1)\delta(x,y)\mathcal{M}_{x,k}(1-s)\mathcal{M}_{y,k}(s) \\ &= -\delta(x,y)sc_{yx,k}(s) \end{aligned} \tag{4}$$

since $y\delta(x,y) = x$. Especially $c_{+-,k}(1/2) = -c_{-+,k}(1/2)$ and

$$(s-1)c_{xy,k}(s-1)) = \delta(x,y)sc_{xy,k}(s).$$

Notice further that $sc_{xy,k}(s)$ has simple poles at $s=0$ and $s=\pm 2it_k$. From these properties together with the growth estimate we find that $T_{\pm,k}(s)$ is an entire function with $T_{\pm,k}(s+1) = T_{\pm,k}(s)$ and $T_{\pm,k}(s) \ll_k \tau^2 e^{3\pi\tau}$. Therefore $T_{\pm,k}(s)$ is a trigonometric polynomial

$$T_{\pm,k}(s) = a_{-1}^{\pm}e^{2\pi is} + a_0^{\pm} + a_1^{\pm}e^{2\pi is}.$$

Since $T_{\pm,k}(0) = 0$ we find $a_0^{\pm} = 0$. Further

$$\begin{aligned}\lim_{s\to 0} s^2 c_{\pm\pm,k}(s) &= \lim_{s\to 0} s^2(s^2+4t_k^2)\mathcal{M}_{\pm,k}(s)\mathcal{M}_{\pm,k}(-s)\\ &= -\left(\frac{\pi\Gamma(1+2it_k)\Gamma(1-2it_k)b_{\pm,k}}{12t_k}\right)^2.\end{aligned}$$

From this we obtain, since $\Gamma(1+z)\Gamma(1-z) = \pi z/\sin\pi z$

$$\lim_{s\to 0}\frac{T_{\pm,k}(s)}{\sin\pi s} = -\frac{\pi^5 b_{\pm,k}^2}{9}$$

and thus

$$a_{-1}^{\pm} = i\frac{\pi^5 b_{\pm,k}^2}{36}.$$

Now put

$$P_{\pm,k}(s) = sc_{\pm\mp,k}(s)\sin(\pi(s-2it_k))\sin(\pi(s+2it_k))\sin(\pi s).$$

Then with the arguments as above

$$P_{\pm,k}(s) = b_{-1}^{\pm}e^{-2\pi is} + b_0^{\pm} + b_{-1}^{\pm}e^{2\pi is}.$$

Since $P_{\pm,k}(s) = P_{\mp,k}(-s)$ we obtain $b_{-1}^{+} = b_1^{-}$, $b_1^{+} = b_{-1}^{-}$, $b_0^{-} = b_0^{+} := b_0$.

Now the asymptotic behaviour for $\tau \to \pm\infty$ which can be obtained via Stirlings formula

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}}e^{s(\log s-1)}\left(1+\mathcal{O}\left(|s|^{-1}\right)\right),\quad |\arg s| < \pi,$$

and

$$\mathcal{M}_{\kappa,it}(s) = \frac{\Gamma(s-\frac{1}{2}+\kappa-it)\Gamma(s-\frac{1}{2}+\kappa+it)}{\Gamma(s)}\left(1+\mathcal{O}\left(|\tau|^{-\frac{1}{2}t}\right)\right)$$

valid for $0 < \sigma \le 1000$ and $|\tau| \ge 1+t$ which we showed in [5] gives

$$b_{-1}^{+} = 2\pi e^{r\pi},\quad b_1^{+} = 4\pi e^{-r\pi}$$

hence

$$P_{\pm}(s) = b_0 + 8\pi\cos 2\pi(s\mp\frac{r}{2}).$$

Observe that

$$\begin{aligned}\lim_{s\to 0} s^2 c_{\pm\mp,k}(s) &= -\lim_{s\to 0} s^2(s^2+4t_k^2)\mathcal{M}_{\pm,k}(s)\mathcal{M}_{\mp,k}(-s)\\ &= \left(\frac{\pi\Gamma(1+2it_k)\Gamma(1-2it_k)}{12t_k}\right)^2 b_{+,k}b_{-,k}.\end{aligned}$$

This gives

$$b_0 + 8\pi\cos\pi r = \frac{\pi^5 b_{+,k}b_{-,k}}{9}$$

and the proof of the lemma is complete. □

The last thing we need is a growth estimate for the Rankin Selberg zeta function. This is given as

Proposition 3.2

$$R^*_{\pm,k}(s) \ll_\epsilon |\tau|^{-2\sigma+2+2\epsilon}(1+t_k)^{-\sigma+1+\epsilon} S_{\pm,k}(1+\varepsilon),$$

for $0 < \sigma < 1$ *where*

$$S_{\pm,k}(1+\varepsilon) := \max\left(R_{\pm,k}(1+\varepsilon), (1+t_k)^{\mp 1} R_{\mp,k}(1+\varepsilon)\right).$$

The implied constants do not depend on k.

4 The Theta case

Now we leave the general case and focus our attention to the space $L^2(\Gamma\backslash\mathbf{H}, \frac{1}{2}, \mathcal{V})$ of real analytic automorphic forms of half integral weight with values in $\mathbf{C}^3$, where $\mathcal{V}$ is the multiplier system coming from the holomorphic Θ–function

$$\Theta(z) = (\vartheta_2(z), \vartheta_3(z), \vartheta_4(z))^t$$

with the three classical theta series

$$\vartheta_2(z) = \sum_{n\in\mathbf{Z}} e^{\pi i\left(n+\frac{1}{2}\right)^2 z} \quad , \quad \vartheta_3(z) = \sum_{n\in\mathbf{Z}} e^{\pi i n^2 z}, \vartheta_4(z) = \vartheta_3(z+1)$$

It is at once seen, that

$$\mathcal{V}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} e\left(\frac{1}{8}\right) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Put $\alpha_1 = 0, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{1}{8}$, then $\mathcal{V}(U)$ has eigenvalues $e(\alpha_1), e(\alpha_2), e(\alpha_3)$, the corresponding orthonormal set of eigenvectors is given by

$$\vec{f_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \vec{f_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad , \quad \vec{f_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For the corresponding Rankin–Selberg zeta functions we want to prove the following mean value estimate

Theorem 4.1

$$\sum_{t_k<T} \frac{R^*_{+,k}(\frac{1}{2}+iv)}{\mathrm{ch}\pi t_k} \ll_\epsilon (vT)^{3/2+\epsilon},$$

$$\sum_{t_k<T} \frac{R^*_{-,k}(\frac{1}{2}+iv)}{\mathrm{ch}\pi t_k} \ll_\epsilon v^{1+\epsilon}T^{5/2+\epsilon}.$$

Remark. Of course one would suppose that for $R_{+,k}$ one should also obtain in v-aspect $v^{1+\epsilon}$ but unfortunately our results for R_+ are (relatively) worse than those for R_-. Indeed a better result can be obtained when using a middling over n in the asymptotic formula of proposition 5.1 below. But since our main interest was in the k–aspect we didn't go further into this.

5 Proof of theorem

Instead of a large sieve inequality à la Iwaniec and Deshoulliers, which is used by Luo and Sarnak in [3] we shall use the following two propositions. For a proof, see [6] and [7].

Proposition 5.1 *For $T > 2$ and $j = 1, 2, 3$*

$$\sum_{0<t_k\leq T} \frac{|\rho_{k,j}(\pm|n|)|^2}{\mathrm{ch}\pi t_k} = \frac{T^{2\mp\frac{1}{2}}}{4|n|} + \mathcal{O}_\varepsilon\left(T^{1\mp\frac{1}{2}+\varepsilon}|n|^{-1+\varepsilon} + T^{\frac{1}{4}\mp\frac{1}{4}}|n|^{-\frac{1}{2}+\varepsilon}\right),$$

valid for any $\varepsilon > 0$.

Proposition 5.2 *Let $T > 2$, $\gamma > -3/2$. Then for any small ε, $\delta > 0$ we have*

$$\sum_{0<t_k\leq T} \frac{R_{\pm,k}(1+\delta)}{\mathrm{ch}\pi t_k} t_k^\gamma \ll_\varepsilon \frac{T^{2\mp\frac{1}{2}+\gamma+\varepsilon}}{\delta^2}. \tag{5}$$

Throughout this section we write $w = u + iv$. Introduce for $\nu \in \mathbf{N}$ and $x > 0$ the notation $bf \int_{\nu,x} f(t) := \int_0^x \int_0^{t_{\nu-1}} \dots \int_0^{t_1} f(t)dtdt_1 \dots dt_{\nu-1}$ (it is understood that $\int_{1,x} = \int_0^x$) and for $0 < u < 1$ and $c \in \mathbf{R}$ the Perron Integral

$$J_{\pm,k}(\nu, x, c, w) = \frac{1}{2\pi i}\int_{\nu,x}\int_{(c)} \frac{t^s}{s} R^*_{\pm,k}(s+w)ds = \frac{1}{2\pi i}\int_{(c)} \frac{x^{s+\nu}}{s(s+1)\dots(s+\nu)} R^*_{\pm,k}(s+w)ds.$$

By Perrons formula we obtain for any $c + u > 1$ and $x > 0$ such, that $x/d \not\equiv \alpha_j \mathrm{mod} 1$ for any positive integer $d \leq x$

$$J_{\pm,k}(\nu, x, c, w) = \int_{\nu,x} B_{\pm,k}(t, w)$$

where $B_{\pm,k}(x) = \sum_{0<d\leq x} b(d)d^{-w}A_{\pm,k}(x/d, w)$, with $b(d) = 1$, if d is a square and 0 else and

$$A_{\pm,k}(x, w) := \sum_{j=1}^{m} \sum_{\substack{n \equiv \pm\alpha_j \mathrm{mod} 1, \\ 0 < n \leq x}} |\rho_{j,k}(n)|^2 n^{1-w}.$$

To see this one should observe, that for $\sigma > 1$ we can write

$$R^*_{\pm,k}(s) = \sum_{j=1}^{3} \sum_{\substack{n \in \mathbf{N}_o, \\ n \pm \alpha_j > 0}} \frac{e_{j,k}(n, s)}{n^s}$$

with $e_{j,k}(n, s) = \sum_{d^2|n} \left(1 + \alpha_j \frac{d^2}{n}\right)^{-s} \left|\rho_{j,k}(\frac{n}{d^2})\right|^2 (\frac{n}{d^2} + \alpha_j)$. Remark, that $e_{j,k}(n, s)n^{-s}$ is also meaningfull for $n = 0$ and gives $|\rho_{j,k}(0)|^2\zeta(2s)(\alpha_j)^{1-s}$.

From proposition 3.2 we find that $J_{\pm,k}(\nu,x,d,w)$ is defined for $d>1/2-u-\nu/4$. Our central identity is

$$\begin{aligned}&J_{\pm,k}(\nu-1,x,c+1,w-1)+(w-1)J_{\pm,k}(\nu,x,c,w)\\ &= -J_{\pm,k}(\nu-1,x,d+1,w-1)-(w-1)J_{\pm,k}(\nu,x,d,w)+(w-1)R^*_{\pm,k}(w)\frac{x^\nu}{\nu!} \quad (6)\end{aligned}$$

for some $0>d>1/2-u-\nu/4$ which follows simply by the Cauchy residue theorem.

Observe that

$$\begin{aligned}&J_{\pm,k}(\nu-1,x,c+1,w-1)+(w-1)J_{\pm,k}(\nu,x,c,w)\\ &= \frac{1}{2\pi i}\int_{(c+1)}\frac{x^{s+\nu-1}}{s(s+1)\dots(s+\nu-1)}R^*_{\pm,k}(s+w-1)ds\\ &\quad+\frac{w-1}{2\pi i}\int_{(c)}\frac{x^{s+\nu}}{s(s+1)\dots(s+\nu)}R^*_{\pm,k}(s+w)ds\\ &= \frac{1}{2\pi i}\int_{(c)}\frac{x^{s+\nu}}{s(s+1)\dots(s+\nu)}(s+(w-1))R^*_{\pm,k}(s+w)ds\end{aligned}$$

and hence there is no pole coming from $s=1-w$ corresponding to the pole at $z=1$ of $R^*_{\pm,k}(z)$.

We prove the following

Lemma 5.1 *For $T>2$ and $0<u<1$ we have*

$$\begin{aligned}&\sum_{t_k<T}\frac{J_{\pm,k}(\nu-1,x,c+1,w-1)+(w-1)J_{\pm,k}(\nu,x,c,w)}{\operatorname{ch}\pi t_k}\\ &\ll_\epsilon (Tx)^\epsilon x^\nu\left(T^{2\mp1/2}+T^{1\mp1/2}x^{1-u}+T^{1/4\mp1/4}x^{3/2-u}\right).\end{aligned}$$

Proof. Let $Z(s)=\sum_{n>0}a(n)n^{-s}$ denote a Dirichlet series convergent for $\sigma>1$ and $S(t,w)$ its partial sums

$$S(t,w)=\sum_{n\le t}a(n)n^{-w},\quad 0<u<1,$$

where as usual the summand with $n=t$ has to be multiplied by $\frac{1}{2}$. To begin with, one easily computes that

$$\begin{aligned}\int_{(c+1)}\frac{x^s}{s}Z(s+w-1) &= x\int_{(c)}\frac{x^s}{s}Z(s+w)ds-\int_{(c)}\frac{x^{s+1}}{s(s+1)}Z(s+w)ds\\ &= xS(x,w)-\int_0^x S(t,w)dt=\int_0^x t\,d_tS(t,w).\end{aligned}$$

If we use this with $Z(s)=R^*_{\pm,k}(s)$ we obtain the Stieltjes integral

$$\sum_{t_k<T}\frac{J_{\pm,k}(\nu-1,x,c+1,w-1)}{\operatorname{ch}\pi t_k}=\int_{\nu-1,x}\int_0^t r\,d_r\mathcal{B}_T(r,w)$$

with

$$\mathcal{B}_T(r,w) = \sum_{t_k<T} \frac{B_{\pm,k}(r,w)}{\text{ch}\pi t_k}.$$

From proposition 5.1 we discover that

$$\begin{aligned}
\mathcal{B}_T(r,w) &= \sum_{t_k<T} \sum_{0<d^2\leq r} d^{-2w} \frac{A_{\pm,k}(\frac{r}{d^2},w)}{\text{ch}\pi t_k} \\
&= T^{2\mp 1/2} \sum_{0<d^2<r} \sum_{n\leq r/d^2} n^{-w} + \mathcal{O}_\epsilon\left((Tx)^\epsilon \left(T^{1\mp 1/2} r^{1-u} + T^{1/4\mp 1/4} r^{3/2-u}\right)\right) \\
&= T^{2\mp 1/2} \sum_{0<d^2<r} \int_0^r \xi^{-w} d\xi + \mathcal{O}_\epsilon\left((Tx)^\epsilon \left(T^{2\mp 1/2} + T^{1\mp 1/2} r^{1-u} + T^{1/4\mp 1/4} r^{3/2-u}\right)\right) \\
&= \frac{T^{2\mp 1/2}}{1-w} \sum_{0<d^2<r} r^{1-w} + \mathcal{O}_\epsilon\left((Tx)^\epsilon \left(T^{2\mp 1/2} + T^{1\mp 1/2} r^{1-u} + T^{1/4\mp 1/4} r^{3/2-u}\right)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{t_k<T} \frac{J_{\pm,k}(\nu-1,x,c+1,w-1)}{\text{ch}\pi t_k} &= \int_{\nu-1,x} \left(\frac{T^{2\mp 1/2}}{1-w} \int_0^t \sum_{0<d^2<r} r\, d_r r^{1-w}\right) \\
&+ \mathcal{O}_\epsilon\left((Tx)^\epsilon x^\nu \left(T^{2\mp 1/2} + T^{1\mp 1/2} x^{1-u} + T^{1/4\mp 1/4} x^{3/2-u}\right)\right) \\
&= \int_{\nu-1,x} \left(T^{2\mp 1/2} \int_0^t \sum_{0<d^2<r} r^{1-w} dr\right) \\
&+ \mathcal{O}_\epsilon\left((Tx)^\epsilon x^\nu \left(T^{2\mp 1/2} + T^{1\mp 1/2} x^{1-u} + T^{1/4\mp 1/4} x^{3/2-u}\right)\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\sum_{t_k<T} (1-w) \frac{J_{\pm,k}(\nu,x,c,w)}{\text{ch}\pi t_k} = (1-w) \int_{\nu-1,x} \int_{(c)} \frac{t^{s+1}}{s(s+1)} \sum_{t_k<T} \frac{R^*_{\pm,k}(s+w)}{\text{ch}\pi t_k} ds \\
&= (1-w) \int_{\nu,x} \left(\int_0^t \sum_{t_k<T} \sum_{0<d^2\leq r} d^{-2w} \frac{A_{\pm,k}(\frac{r}{d^2},w)}{\text{ch}\pi t_k} dr\right) \\
&= \int_{\nu-1,x} \left(T^{2\mp 1/2} \int_0^t \sum_{0<d^2<r} r^{1-w} dr\right) + \mathcal{O}_\epsilon\left((Tx)^\epsilon x^\nu \left(T^{2\mp 1/2} + T^{1\mp 1/2} x^{1-u} + T^{1/4\mp 1/4} x^{3/2-u}\right)\right).
\end{aligned}$$

This completes the proof of the lemma. □

When using the functional equation we obtain for $\frac{1}{2} - u - \frac{\nu}{2} < d < -u$

$$J_{\pm,k}(\nu,x,d,w) = \sum_{j=1}^{3} \sum_{n>0} \frac{1}{2\pi i} \int_{(d)} n^{s+w-1} D_{\pm,k}(n,s+w) \frac{x^{s+\nu}}{s(s+1)\dots(s+\nu)} ds$$

where

$$D_{+,k}(n,z) = \mathbf{C}_{11,k}(z) e_{j,k}(n,1-z) + \mathbf{C}_{12,k}(z) e_{j,k}(-n,1-z)$$

$$D_{-,k}(n,z) = \mathbf{C}_{21,k}(z) e_{j,k}(n,1-z) + \mathbf{C}_{22,k}(z) e_{j,k}(-n,1-z)$$

with $\mathbf{C}_{ij,k}(s)$ the entries of the transformation matrix $\mathbf{C}_k(s)$.

From lemma 3.1 we further obtain that

$$\begin{aligned}&\int_{(d)} n^{s+w-1} D_{\pm,k}(n,s+w)\frac{x^{s+\nu}}{s(s+1)\dots(s+\nu)}ds\\ &\ll\ Q_k^{1-2d-2u} x^{\nu+d} v^{2-4d-4u}\left(\frac{e_{j,k}(n,1-d-u)+Q_k^{-1}e_{j,k}(-n,1-d-u)}{n^{1-d-u}}\right).\end{aligned}$$

Hence for $\varepsilon>0$

$$\begin{aligned}J_{\pm,k}(\nu,x,d,w) &= \sum_{j=1}^{3}\sum_{0<n<N}\frac{1}{2\pi i}\int_{(d)} n^{s+w-1}D_{\pm,k}(n,s+w)\frac{x^{s+\nu}}{s(s+1)\dots(s+\nu)}ds \quad (7)\\ &+\ \mathcal{O}_\varepsilon\left(Q_k^{1-2d-2u}x^{\nu+d}v^{2-4d-4u}N^{u+d+\varepsilon}S_{+,k}(1+\varepsilon)\right).\end{aligned}$$

For this we used the fact that we can replace $1-d-w$ in $e_{j,k}(\pm n,.)$ by $1+\varepsilon$, without making a considerable error.

Analogous considerations lead to

$$\begin{aligned}&J_{\pm,k}(\nu-1,x,d+1,w-1) \qquad (8)\\ &= \sum_{j=1}^{3}\sum_{0<n<N}\frac{1}{2\pi i}\int_{(d+1)} n^{s+w-2}D_{\pm,k}(n,s+w-1)\frac{x^{s+\nu-1}}{s(s+1)\dots(s+\nu-1)}ds\\ &+\ \mathcal{O}_\varepsilon\left(Q_k^{1-2d-2u}x^{\nu+d}v^{2-4d-4u}N^{u+d+\varepsilon}S_{+,k}(1+\varepsilon)\right).\end{aligned}$$

Now for the '-' case we can complete the proof of the theorem on choosing $x=Q_k^2$, $u=1/2$, $\nu=2$, $N=1/2$ and $d=-1/2-\varepsilon$. From our central identity (6) we thus obtain

$$\begin{aligned}\sum_{t_k<T}(w-1)\frac{R^*_{-,k}(w)}{\mathrm{ch}\pi t_k} &\ll_\varepsilon\ T^{5/2+\varepsilon}+v^{2+4\varepsilon}\sum_{t_k<T}\frac{S_{-,k}(1+\varepsilon)}{\mathrm{ch}\pi t_k}\\ &\ll_\varepsilon\ v^{2+4\varepsilon}T^{5/2+\varepsilon}\end{aligned}$$

hereby using proposition 5.2.

Unfortunately the '+'-case is more involved, since the $\mathcal{O}$-term in proposition 5.1 is worse in this case. The problem is that we cannot choose $x=Q_k^2$ then. Hence we proceed differently. Fix $w=1/2+iv$.

We split up the sum

$$\sum_{t_k<T}(w-1)\frac{R^*_{+,k}(w)}{\mathrm{ch}\pi t_k}$$

into

$$\sum_{t_k<|v|/2}+\sum_{|v|/2\le t_k<T}.$$

The first sum is easily estimated using the functional equation and Stirlings formula

$$R^*_{+,k}(1/2+iv)\ll b_{-,k}^{-2}\left(e^{2\pi t_k}R^*_{-,k}(1/2+iv)+\left(\mathrm{sh}\pi v+\frac{b_{+,k}b_{-,k}}{\mathrm{ch}\pi v}\right)R^*_{-,k}(1/2-iv)\right).$$

In [7] we showed that

$$b_{\pm,k} = k_\pm t_k^{\mp\frac{1}{2}} e^{\pi t_k} \left(1 + \mathcal{O}(t_k^{-1})\right)$$

which yields

$$R^*_{+,k}(1/2 + iv) \ll t_k^{-1} |R^*_{-,k}(1/2 + iv)|$$

since $R^*_{-,k}(1/2 + iv) = \bar{R}^*_{-,k}(1/2 - iv)$.

Using the above established mean value estimate for $R^*_{-,k}$ we therefore find by partial summation

$$\sum_{t_k<T} \frac{R^*_{+,k}(w)}{\mathrm{ch}\pi t_k} \ll_\epsilon v^{1+4\epsilon} T^{3/2+\epsilon} + \sum_{2t_k<v} \frac{R^*_{+,k}(w)}{\mathrm{ch}\pi t_k}$$

For estimating the sum with $2t_k < v$ we return to equations (7) and (8). We move the integral in the sum over n to the line $\sigma = 1$ (resp. to $\sigma = 2$ for $J_k(\nu-1, x, d+1, w-1)$). We thereby pass a pole in (7) at $s = 0$ giving as residue

$$\frac{x^\nu}{\nu!} D_{+,k}(n, w) n^{w-1}.$$

From Stirlings formula and proposition 4.1 one discovers that

$$\sum_{2t_k<v} \frac{x^\nu D_{+,k}(n, w) n^{w-1}}{\nu! \mathrm{ch}\pi t_k} \ll v^{3/2} N^{1/2} + N.$$

There is a pole of $D_k(n, z)$ at $z = 1$. It is a first order pole since as we have already shown $|z c_{xy,k}(z)|$ has period 1 and has a simple pole at $z = 0$. This gives for $J_{+,k}(\nu, x, d, w)$ the contribution

$$\frac{x^{1-w+\nu}}{(1-w)(2-w)\ldots(1+\nu-w)} \mathrm{Res}_{z=1} D_k(n, z)$$

at $s = 1 - w$ and for $J_k(\nu - 1, x, d + 1, w - 1)$ the contibution

$$\frac{x^{1-w+\nu}}{(2-w)\ldots(1+\nu-w)} \mathrm{Res}_{z=1} D_k(n, z)$$

at $s = 2 - w$. So it contributes 0 for $J_k(\nu - 1, x, d + 1, w - 1) + (w - 1) J_k(\nu, x, d, w)$.

Now for the integral in (7) we obtain for $\varepsilon > 0$

$$\begin{aligned}\int_{(1)} n^{s+w-1} D_{\pm,k}(n, s+w) \frac{x^{s+\nu}}{s(s+1)\ldots(s+\nu)} ds &\ll_\epsilon \frac{x^{\nu+1}}{\nu!} N^{3/2+\epsilon} Q_k^{-1} v^{-2} S_{+,k}(1+\varepsilon) \\ &\ll_\epsilon \frac{x^{\nu+1}}{\nu!} N^{3/2+\epsilon} Q_k^{-3} S_{+,k}(1+\varepsilon).\end{aligned}$$

The proof can now be completed on choosing $x = Q_k^{3/2}$, $N = Q_k$ by the same considerations as inj the '-' case.

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