Automorphism of Enriques surfaces which act trivially on the cohomology groups

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Let S be a smooth surface over the complex number field $\mathbb{C}$. An automorphism φ of S is cohomologically trivial (resp. numerically trivial) if the induced automorphism φ^* of the cohomology ring $H^*(S,\mathbb{Z})$ (resp. $H^*(S,\mathbb{Q})$) is trivial. We denote by $\Delta_0(S)$ (resp. $\Delta(S)$) the quotient of the group of cohomologically (resp. numerically) trivial automorphisms of S by its connected component. It is known that $\Delta(S)$ is a finite group if S is Kähler, [4], [3]. $\Delta(S)$ is even trivial, for example, in the case of rational surfaces, abelian surfaces and K3 surfaces, [9], [5]. In this article, we shall study $\Delta(S)$ for an Enriques surface S. In this case, it is no more true that $\Delta_0(S)$ is always trivial. So far two examples are known.

Example 1. (Liebermann, cf. [9]) Let A be the product of two elliptic curves E_1 and E_2 and a a 2-torsion point of A not lying on E_1 nor E_2. Let σ_R (resp. σ_K) be the involution of the Kummer surface $\tilde{S}$ of A induced by the endomorphism $(-1_{E_1}, 1_{E_2})$ (resp. the translation by a). Then $\varepsilon = \sigma_R\sigma_K$ is a fixed point free involution of $\tilde{S}$ and the quotient

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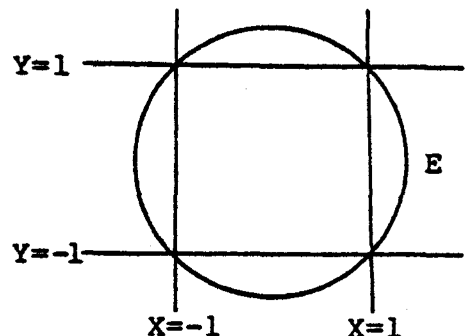
$S = \tilde{S}/\varepsilon$ is an Enriques surface. The involution σ of S induced by σ_R is cohomologically trivial (see Proposition 4.8 for the proof).

Example 2. (Barth-Peters [1]) Let E be an elliptic curve of bidegree (2.2) in $\mathbb{P}^1 \times \mathbb{P}^1$ which passes through the 4 points $(\pm 1, \pm 1)$. There is the unique involution $\varphi = (\varphi_1, \varphi_2)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\varphi(E) = E$ and $\varphi(\pm 1, \pm 1) = (\mp 1, \mp 1)$. Let $\tilde{S}$ be the minimal resolution of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is the union of E and 4 lines $X = \pm 1$ and $Y = \pm 1$. We denote by σ_R the involution of $\tilde{S}$ induced by the covering involution. There are two involutions of $\tilde{S}$ which covers φ. One of them, which we denote by ε, has no fixed points and the quotient $S = \tilde{S}/\varepsilon$ is an Enriques surface. S has 10 smooth rational curves, 8 of which come from 4 D_4-singularities of the double cover and 2 of which come from the 4 lines in the branch locus. S has two elliptic fibration induced by two $\mathbb{P}^1$-bundle structures of $\mathbb{P}^1 \times \mathbb{P}^1$. As we will see in Proposition 4.5, these 10 rational curves and the reduced parts of the multiple fibres of the two elliptic fibrations generate the cohomology group $H^2(S, \mathbb{Z})$. It will be clear that the involution σ of S induced by σ_R fixes all these curves. Hence σ is a cohomologically trivial involution.



Once Ueno claimed that $\Delta(S)$ is always trivial [11] but his proof contained a mistake, which was pointed out by Peters. The first example of a numerically trivial involution was constructed by Liebermann. We note that our Example 2 is described in a slightly different way from [1].

Main Theorem 0.1. $\Delta_0(S)$ is trivial or a group of order 2 for every Enriques surface S. Moreover, every pair of an Enriques surface S and a cohomologically trivial involution σ of S is obtained in the way of Example 1 or Example 2.

Both families of Enriques surfaces in Example 1 and Example 2 form two disjoint 3 dimensional subvarieties in the period space of Enriques surfaces.

For almost all Enriques surfaces S, $\Delta(S)$ coincides with $\Delta_0(S)$ but $\Delta(S)$ is a cyclic group of order 4 for special Enriques surfaces S in Example 2.

Example 3. Let E and $\varphi = (\varphi_1, \varphi_2)$ be as in Example 2. E is defined by a φ-invariant bihomogeneous polynomial P of bidegree $(2,2)$. Assume that P is anti-invariant by φ_1, or equivalently by φ_2. Then there is an automorphism ρ_R of $\tilde{S}$ such that $\rho_R^2 = \sigma_R$. The automorphism ρ of S induced by ρ_R has order 4. Moreover, ρ is numerically trivial. For ρ_R fixes 10 rational curves on S described in Example 2 and the classes of these 10 rational curves generate $H^2(S,\mathbb{Q})$. But ρ is not cohomologically trivial: Let $f : S \to \mathbb{P}^1$ be an elliptic fibration induced by a $\mathbb{P}^1$-bundle structure of

$\mathbb{P}^1 \times \mathbb{P}^1$. f has two multiple fibres and the reduced parts of them are not cohomologous. It is easy to see that ρ interchange the two multiple fibres. Hence ρ is not cohomologically trivial.

Our second result is the following:

Theorem 0.2. $\Delta(S)$ is a cyclic group of order 4 generated by ρ for every Enriques surface S in Example 3. $\Delta(S)$ coincides with $\Delta_0(S)$ for all other Enriques surfaces S.

Notation. For an Enriques surface S, we denote the universal covering by $\pi : \tilde{S} \to S$ and the covering involution by ε. Let ε^* be the isometry of $H^2(\tilde{S}, \mathbb{Z})$ induced by ε. We denote the $(+1)$ (resp. (-1)) eigenspace of ε^* by M (resp. N). $H^0(\tilde{S},\Omega^2)$ is a subspace of $N \otimes \mathbb{C}$ and N has a natural polarized Hodge structure induced by that on $H^2(\tilde{S},\mathbb{Z})$ Hence an Enriques surface S determines a point of D/Γ, D being a bounded symmetric domain of type IV and dimension 10 and $\Gamma = O(N)$ a discrete subgroup, which we call the period of the Enriques surface S. For details on this subject we refer the reader to [6]. In this article, E_8 is an even unimodular <u>negative</u> definite lattice of rank 8. U denotes a hyperbolic lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of rank 2. If L is a lattice, we denote by $L(a)$ the lattice obtained by multiplying the form with the integer a. For an R-module M, we denote the dual $\mathrm{Hom}_R(M,R)$ by $M^\vee$.

§1. Preliminary

We first prove the following:

Proposition 1.1. For every Enriques surface S, the group $\Delta(S)$ of numerically trivial automorphisms of S is a cyclic group of finite order.

For the proof we begin with the following two lemmas.

Lemma 1.2. Let τ be an automorphism of $\tilde{S}$. If some power of τ is equal to ε, then τ itself is equal to ε.

Proof. Since ε has no fixed points, neither does τ. Hence the order of τ divides $\chi(O_{\tilde{S}}) = 2$. Therefore τ is an involution and equal to ε. q.e.d.

Let $\Lambda(\tilde{S})$ be the group of automorphisms of the K3 surface $\tilde{S}$ which act trivially on the sublattice M of $H^2(S, \mathbb{Z})$ and G the subgroup of $\Lambda(\tilde{S})$ consisting of automorphisms which act trivially on both M and $H^0(S,\Omega^2)$. Every automorphism in $\Lambda(\tilde{S})$ commutes with ε and induces a numerically trivial automorphism of S. Hence we have the exact sequence

$$(1.3) \qquad 1 \to \langle\varepsilon\rangle \to \Lambda(\tilde{S}) \to \Delta(S) \to 1.$$

The following is our first step to the classification of $\Delta(S)$.

Lemma 1.4. G is trivial or a group of order 2.

Proof. For a group H of automorphisms of $\tilde{S}$, we denote by S_H the orthogonal complement to the lattice of H -

invariants in $H^2(\tilde{S}, \mathbb{Z})$. By definition, S_G is contained in $H^{1,1}(\tilde{S})$ and orthogonal to M. Hence the rank of S_G is at most $h^{1,1} - \mathrm{rk}\, M = 10$. If τ belongs to G, then $S_{<\tau>}$ is a sublattice of S_G and hence rank $S_{<\tau>} \le 10$. It follows from [8] that $\tau^2 = \mathrm{id}$ for every $\tau \in G$. Hence G is a 2 elementary abelian group. Since rank $S_G \le 10$, again by [8], G is at most a group of order 2. q.e.d.

Proof of Proposition 1.1: The quotient $\Lambda(\tilde{S})/G$ is a subgroup of $\mathbb{C}^*$ and hence a cyclic group. Therefore $\Lambda(\tilde{S})$ is an abelian group by Lemma 1.4. If the order $\Delta(S)$ is odd, then both $\Lambda(\tilde{S})$ and $\Lambda(S)$ are cyclic. If the order of $\Delta(S)$ is even, then the order of $\Lambda(\tilde{S})$ is divisible by 4 and $\Lambda(\tilde{S})$ is not cyclic by Lemma 1.2. Hence G is not trivial and the exact sequence

$$1 \to G \to \Lambda(\tilde{S}) \to \Lambda(\tilde{S})/G \to 1$$

splits. By Lemma 1.2, $<\varepsilon>$ is a direct summand of $\Lambda(\tilde{S})$. Hence by the exact sequence (1.3), $\Delta(S)$ is a cyclic group.

q.e.d.

A numerically trivial automorphism of S induces a Hodge isometry of N, or an automorphism of polarized Hodge structure of N. Next we determine when a Hodge isometry of N is induced by a numerically trivial automorphism of S.

By definition, every $\tau \in \Lambda(\tilde{S})$ acts trivially on M. Hence τ induces an isometry τ_N of N which acts trivially on the descriminant form (A_N, q_N) of N. τ_N is an automorphism of the polarized Hodge structure on N and we have the sequence

$$(1.5) \qquad 1 \to \Lambda(\tilde{S}) \to \mathrm{Aut}_{\substack{\text{polarized}\\ \text{Hodge str.}}}(N) \to \mathrm{Aut}(A_N, q_N).$$

Proposition 1.6. The exact sequence (1.5) is exact.

Proof. Let t be an automorphism of the polarized Hodge structure on N which acts trivially on A_N. By [7], there exists an isometry T of $H^2(S,\mathbb{Z})$ which is identity on M and equal to t on N. It suffices to show that the isometry T is induced by an automorphism of $\tilde{S}$. By virtue of the Torelli theorem for K3 surfaces it suffices to show the following:

Claim: $T(\alpha)$ is effective for every class α of smooth rational curve on S.

Let M_T be the lattice of T-invariants of $H^2(\tilde{S}, \mathbb{Z})$ and N_T the orthogonal complement of M_T. Then the difference $\beta = \alpha - T(\alpha)$ of α and $T(\alpha)$ belongs to N_T. Since N_T is contained in N, $\varepsilon(\beta)$ is equal to $-\beta$. If $\beta = 0$, then there is nothing to prove. So we may assume that $\beta \neq 0$. Since ε is an automorphism of S, β is not effective.
Therefore, $-T(\alpha)$ is not effective. Since $(T(\alpha)^2) = (\alpha^2) = -2$, $T(\alpha)$ is effective by the Riemann Roch theorem.

q.e.d.

§2. Classification of the numerically trivial involutions.

In this section, we shall prove that the moduli of Enriques surfaces with numerically trivial involutions has exactly two components of dimension 3.

For an involution σ of an Enriques surface S, there are two involutions of $\tilde{S}$ which are liftings of σ (Lemma 1.2). One acts trivially on $H^0(\Omega^2_S)$ and another not, which we denote by σ_K and σ_R, respectively. The (+1) eigenspace of $H^2(\tilde{S}, \mathbb{Z})$ with respect to the action of σ_K (resp. σ_R) is denoted by M_K (resp. M_R) and (-1) eigenspace by N_K (resp. N_R). N_K (resp. N_R) is the orthogonal complement to M_K (resp. M_R) in the K3 lattice $H^2(\tilde{S}, \mathbb{Z})$ and vice versa. If σ is numerically trivial, then $H^2(\tilde{S}, \mathbb{Z})$ contains $M \perp N_K \perp N_R$ as a sublattice of finite index and the involutions ε, σ_K and σ_R act on it as follows:

	M	N_K	N_R
ε	1	-1	-1
σ_K	1	-1	1
σ_R	1	1	-1

$H^0(\Omega^2_S)$ is a subspace of $N_R \otimes \mathbb{C}$. N_K is a primitive sublattice of $N = M^{\perp}$ and orthogonal to the period.

Lemma 2.1. The lattice N_K is isomorphic to $E_8(2)$.

Proof. N_K is the orthogonal complement $S_{<\sigma_K>}$ to the lattice of σ_K-invariants in $H^2(\tilde{S}, \mathbb{Z})$. σ_K is an involution of $\tilde{S}$ and acts trivially on $H^0(\Omega^2)$. By [8], the moduli of pairs of a K3 surface X and such an involution τ of X is connected. Hence it suffices to show that $S_{<\tau>} \simeq E_8(2)$ for an example of such an involution τ. For that purpose, we use the K3 surface constructed in Shioda-Inose [10]. They constructed an elliptic K3 surface $f : X \to \mathbb{P}^1$ with two singular fibres of type II*. f has an involution τ and τ interchanges the two singular fibres. τ acts trivially on $H^0(\Omega^2)$ and the resolution of the quotient X/τ is the Kummer surface of the product of two elliptic curves. $H^2(X, \mathbb{Z})$ contains a sublattice K isomorphic to $E_8 \perp E_8$ coming from the two singular fibres of type II*. τ is trivial on the orthogonal complement to K in $H^2(X, \mathbb{Z})$. Hence $S_{<\tau>}$ is isomorphic to $E_8(2)$. q.e.d.

Conversely, we have the following:

Proposition 2.2. If N_0 is a primitive sublattice of N isomorphic to $E_8(2)$ and orthogonal to $H^0(\Omega^2_{\tilde{S}})$, then there is a numerically trivial involution σ such that N_K with respect to σ coincides with N_0. In particular, an Enriques surface S has a numerically trivial involution if and only if the lattice N contains $E_8(2)$ as a primitive sublattice orthogonal to $H^0(\Omega^2_{\tilde{S}})$.

Proof. Since N_0 is a 2-elementary lattice, there is an isometry t of N which is -1 on N_0 and 1 on the orthogonal complement to N_0 in N. It will be obvious that t

is an isomorphism of the Hodge structure on N and acts trivially on the discriminant form of N. Therefore, our proposition follows from Proposition 1.6. q.e.d.

Next we study the lattice $N_R = M_R^{\perp}$.

Lemma 2.3. The discriminant group $A_{M_R} = M_R^{\vee}/M_R$ of M_R is a 2-elementary abelian group and the discriminant form q_{M_R} is even, i.e., $q_{M_R}(a)$ is an integer for every $a \in A_{M_R}$.

Proof. The lattice M_R contains $M \perp N_K$ as a sublattice of finite index. Hence it suffices to prove our assertion for M and N_K. The lattice M is isomorphic to $L_S(2)$ via the map π^*, where L_S is the torsion free part of $H^2(S, \mathbb{Z})$. The discriminant group of M is canonically isomorphic to $L_S/2L_S$ and the discriminant form is equivalent to the quadratic form q, $q(a) = \frac{1}{2}(a^2) \bmod 2$, on $L_S/2L_S$. Since L_S is an even lattice, the quadratic form q is even. In the same way, we have, by Lemma 2.1, that the discriminant group of N_K is 2-elementary and that the discriminant form is even.

q.e.d.

The signatures of $H^2(\tilde{S}, \mathbb{Z})$, M and N_K are equal to $(3,19)$, $(1,9)$ and $(0,8)$, respectively. Hence the signature of N_R is equal to $(2,2)$. Since N_R is the orthogonal complement to M_R, we have by Lemma 2.3, that the discriminant group of N_R is 2-elementary and the discriminant form is even.

Lemma 2.4. The discriminant group A_{N_R} of N_R is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ or $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$.

Proof. Let n be the rank of A_{N_R} over $\mathbb{F}_2$. Since N_R has rank 4, n is at most 4. Since M and N_K have different discriminant groups, M_R is not unimodular. Hence neither is N_R. Therefore, we have $n \neq 0$. Since A_{N_R} has an even quadratic form, n must be even. Hence we have our lemma. q.e.d.

These data for N_R are sufficient to determine N_R as a lattice. In fact, by [7] we have

Proposition 2.5. The lattice N_R is isomorphic to $U \perp U(2)$ or $U(2) \perp U(2)$.

More generally, the above argument shows that if N contains a lattice T isomorphic to $E_8(2)$ as a primitive sublattice, then the orthogonal complement T' to T in N is isomorphic to $U \perp U(2)$ or $U(2) \perp U(2)$. In the case $T' \simeq U \perp U(2)$, N is isomorphic to the orthogonal sum of T and T'. In the case $T' \simeq U(2) \perp U(2)$. these exist $\alpha \in A_T$ and $\alpha' \in A_{T'}$ such that $q_T(\alpha) = q_{T'}(\alpha') \neq 0$ and N coincides with

$$T + T' + \mathbb{Z}(a,a') \subset (T \oplus T') \otimes \mathbb{Q},$$

where a (resp. a') is a vector of $T^\vee \subset T \otimes \mathbb{Q}$ (resp. $T'^\vee \subset T' \otimes \mathbb{Q}$) which represents the class α (resp. α').

Proposition 2.6. Let T_1 and T_2 be a primitive sublattice of N isomorphic to $E_8(2)$. If their orthogonal complements T_1' and T_2' are isomorphic to each other, then there exists an isometry of N which maps T_1 and T_1' onto T_2 and T_2', respectively.

Proof. In the case $T_1' \cong T_2' \cong U \perp U(2)$, our assertion is obvious since $N \cong T_1 \perp T_1' \cong T_2 \perp T_2'$. In the case $T_1' \cong T_2' \cong U(2) \perp U(2)$, take $\alpha_2 \in A_{T_i}$ and $\alpha_i' \in A_{T_i'}$ so that $N = T_i + T_i' + \mathbb{Z}(a_i, a_i')$, $i = 1, 2$, as above. Since T_i and T_i' are 2-elementary lattices, there exist isometries $\varphi : T_1 \to T_2$ and $\varphi' : T_1' \to T_2'$ and the isometries of the discriminant group induced by φ and φ' map α_1 and α_1' to α_2 and α_2' respectively, [7]. By [7], the isometry $(\varphi, \varphi') : T_1 \perp T_1' \to T_2 \perp T_2'$ can be extended to an isometry Φ of N. It is obvious that Φ satisfies our requirement. q.e.d.

Let P be the subset of period domain consisting of the periods of Enriques surfaces such that N contains $E_8(2)$ as a primitive sublattice orthogonal to $H^0(\Omega^2_{\tilde{S}})$. P is the disjoint union of P_1 and P_2 for which the orthogonal complements to the $E_8(2)$ in N are isomorphic to $U \perp U(2)$ and $U(2) \perp U(2)$, respectively. It is easy to see that both P_1 and P_2 are closed subsets of dimension 3 of the period domain. By the proposition we have the following:

Corollary 2.7. Both P_1 and P_2 are irreducible.

§3. Classification of the numerically trivial automorphisms of higher order.

(3.1) In this section we classify the numerically trivial automorphisms of Enriques surfaces of order greater than 2. In fact we prove that there exists only one such automorphism (unique up to deformation, which is of order 4.)

(3.2) Let g be a numerically trivial automorphism of an Enriques surface S. Denote by $\tilde{g}$ a lifting of g to the automorphism of the covering K3 surface $\tilde{S}$ and by $\rho = \tilde{g}^*$ the induced isometry on the K3 lattice $L = H^2(\tilde{S}, \mathbb{Z})$. Note that $\tilde{g}$ (hence ρ) has the same order as g by Lemma (1.2).

A) Automorphism of order 4.

(3.3) Now let us assume that g is of order 4 (hence so is ρ).

Since $g^2 = h$ is a numerically trivial involution of S, it is one of two types in the previous section. Denoting by $\sigma = \rho^2$ the induced isometry on the K3 lattice, we observe that ρ preserves the ± 1-eigenspaces M_σ, N_σ with respect to σ because they are generated by the eigenvectors corresponding to the eigenvalues ± 1 or $\pm i$ with respect to ρ respectively. Moreover since ρ is an involution on M_σ, it acts trivially on the discriminant $M_\sigma^\vee/M_\sigma \simeq N_\sigma^\vee/N_\sigma$.

From these observations combined with the next wellknown lemma Proposition (3.5) follows, which is our first step.

Lemma (3.4). The group $GL(n, \mathbb{Z})(2) = \{M \in GL(n, \mathbb{Z});\ M \equiv 1_n \pmod 2\}$ contains no element of order 4.

Proof. For $M \in GL(n, \mathbb{Z})(2)$ one sees immediately that $M^2 \equiv 1 \pmod 4$. Hence for M with $M^4 = 1$ its minimal polynomial, a factor of $X^4 - 1$, cannot contain the factor $X^2 + 1$.

Proposition (3.5). Under the notations (3.2), (3.3) N_σ cannot be $E_8(2)$ nor $U(2) \perp U(2)$.

(3.6) Keeping the situation (3.2) and the notation (3.3) we may now assume moreover that $N_\sigma = N_R \simeq U \perp U(2)$ on which ρ operates as $\rho^2 = -1$.

We write

$$U = \mathbb{Z}e_1 + \mathbb{Z}f_1$$
$$U(2) = \mathbb{Z}e_2 + \mathbb{Z}f_2$$

with $\langle e_k, e_k \rangle = \langle f_k, f_k \rangle = 0$ and $\langle e_k, f_k \rangle = k$ for $k = 1,2$. An elementary calculation shows that a matrix $A \in O(U \perp U(2))$ with $A^2 = -1$ is in the form

$$A = \begin{pmatrix} a & 0 & -2\delta & -2\beta \\ 0 & -a & -2\gamma & -2\alpha \\ \alpha & \beta & d & 0 \\ \gamma & \delta & 0 & -d \end{pmatrix}$$

with the condition that either i) $a + d = 0$, $\beta = \gamma = 0$, $a^2 - 2\alpha\delta = -1$ or ii) $a - d = 0$, $\alpha = \delta = 0$, $a^2 - 2\beta\gamma = -1$.

Clearly such matrices are contained in the image of two embeddings of $\Gamma_0 = \{\begin{pmatrix} a & b \\ c' & d \end{pmatrix} \in SL(2, \mathbb{Z});\ c' \equiv 0 \pmod 2\}$ into $O(U \perp U(2))$ defined as

$$\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & -2b & 0 \\ 0 & d & 0 & 2c \\ -c & 0 & d & 0 \\ 0 & b & 0 & a \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & 0 & -2b \\ 0 & d & 2c & 0 \\ 0 & b & a & 0 \\ -c & 0 & 0 & d \end{pmatrix}.$$

These two embeddings are conjugate to each other by the element

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore in order to classify A as above it suffices to classify the elements of order 4 in Γ_0. (Recall that $N = N_K \perp N_R$ in this case, hence any isometry in $O(U \perp U(2))$ extends to that in $O(N)$.) But the latter is, up to conjugate and ± 1, unique and written as

$$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

as one sees easily from the shape of a fundamental domain of the action of Γ_0 on the upper half plane. Hence we may assume that one of $\pm\sigma|_{N_R}$ is in the form

$$(*)\ A_0 = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Our conclusion in A) is

Proposition (3.7). Let ρ be the isometry induced from a numerically trivial Enriques automorphism of order 4 (3.2). Then up to Enrique involution (i.e. replacing ρ by $\rho\varepsilon^*$ if necessary), $\rho|_{M_R} = \mathrm{id}$ and $\rho|_{N_R} = \pm A_0$ as above with $N_R \simeq U \perp U(2)$.

For the proof we observe the following:

Lemma (3.8). Let ρ be an isometry of L which preserves the subspaces M, N_K, N_R (hence also $M_R = N_R^{\perp}$ = the primitive hull of $M \perp N_K$). If ρ acts trivially on $M^{\vee}/M$, then it acts trivially also on $N_K^{\vee}/N_K$ and $M_R^{\vee}/M_R$.

Proof. By [7] Proposition 1.15.1 the mutually orthogonal primitive embeddings of M and N_K into M_R are determined by an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & N_K^{\vee}/N_K & \to & M^{\vee}/M & \to & M_R^{\vee}/M_R & \to & 0 \\ & & \wr\wr & & \wr\wr & & \wr\wr & & \\ & & \overset{4}{\perp} u(2) & & \overset{5}{\perp} u(2) & & \perp u(2) & & \end{array}$$

($u(2) = U(2)^{\vee}/U(2)$) which preserves the discriminant forms. By the assumption the action of σ is compatible with this exact sequence, from which the lemma follows trivially.

(3.9) Proof of (3.7). We know already that $\rho|_{N_R} = \pm A_0$ and the invariant subspace M_ρ of ρ is subject to the inclusions

$$M \subset M_\rho \subset M_R.$$

What we should prove is hence that either $M = M_\rho$ or $M_\rho = M_R$, or equivalently that $\rho|_{N_K} = \pm 1$ (since $\rho^2|_{M_R} = 1$).

By the above Lemma (3.8) ρ acts trivially on $N_K^{\vee}/N_K$. On the other hand, as we have seen in (2. 1), $\rho|_{N_K} \in O(N_K) \simeq O(E_8)$ and $N_K^{\vee}/N_K \simeq (\mathbb{Z}/2\mathbb{Z})^8$. Hence ρ/N_K is contained in $\mathrm{Ker}(O(E_8) \to GL(8, \mathbb{F}_2))$, which is known to be $\{\pm 1\}$ (see [2], p.228 for example). q.e.d.

(3.10) The periods of Enriques surfaces having the automorphism as above is easy to discribe. We have a (unique) primitive embedding of N_R into the K3 lattice L (and in N). Then the period ω should be in the $\pm i$-eigenspaces $N_{R,\mathbb{C}}(\pm i)$ which have dimension 2.

If the space $\mathbb{C}\omega \oplus \mathbb{C}\bar{\omega}$ is not rational (i.e. not defined over

$\mathbb{Q}$), then the transcendental lattice $T_{\tilde{S}}$ of $\tilde{S}$ is N_R. In this case $T_{\tilde{S}}^{\perp}$ (in N) $\simeq E_8(2)$ does not contain -2 vectors, hence all such vectors come from the periods of Enriques surfaces.

If the space $\mathbb{C}\omega \oplus \mathbb{C}\bar{\omega}$ is rational, then $T_{\tilde{S}} \subsetneq N_R$ and rank $T_{\tilde{S}} = 2$. The vector ω is realized as the period of an Enriques surface if and only if the orthogonal complement of $T_{\tilde{S}}$ in N does not contain a vector of length -2.

B) Automorphism of order 8.

Our goal of this paragraph is to prove

Proposition (3.11). There exists no numerically trivial automorphism of an Enriques surface of order 8.

(3.12) We shall prove this by contradiction. Suppose that there exists such an automorphism g. We use the notation in (3.2).

By the assumption g^2 in a numerically trivial automorphism of order 4, as (3.7). By the same reason as before in (3.3) ρ preserves N_R.

Consider $B = \rho|_{N_R} \in O(U \perp U(2))$. We decompose B into $\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ according to the decomposition $U \oplus U(2)$ (3.6).

By Lemma (3.8) the numerical triviality of ρ (i.e. $\rho|M = id$) implies that $\rho|_{N_K} = \pm 1$ (3.9) and B acts trivially on $N_R^{\vee}/N_R$, i.e. $B_4 \equiv 1 \pmod 2$.

Note that ρ preserves a sequence of subgroups of $(U \oplus U(2)) \otimes \mathbb{Q}$:

$$\frac{1}{2}(U \oplus U(2)) \supset N_R^*(= U \oplus \frac{1}{2}U(2)) \supset N_R$$

where $\frac{1}{2}M = M \otimes_{\mathbb{Z}} (\frac{1}{2}\mathbb{Z})$ for a $\mathbb{Z}$-module M, hence in particular $B_2 \equiv 0 \pmod 2$.

Moreover $B^2 = \pm A_0$ by (3.7), hence $B_1^2 \equiv 1 \pmod 2$.

Summing up, we have obtained

Facts. $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \pmod 2$ is in the form

i) $B_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2$,

ii) $B_2 \equiv 0 \pmod 2$,

iii) $B_4 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2$.

Now consider the equality

$$\pm A_0 = B^2 = \begin{pmatrix} B_1^2+B_2B_3 & B_1B_2+B_2B_4 \\ B_3B_1+B_4B_3 & B_3B_2+B_4^2 \end{pmatrix},$$

in particular

$$1 = B_3B_1 + B_4B_3.$$

Taking the reduction modulo 2, we have

$$1 \equiv B_3B_1 + B_3 \pmod 2$$

$$= B_3(B_1+1),$$

hence $B_1+1 \pmod 2$ is invertible, which contradicts to the fact that $B_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2$.

The proof of Proposition (3.11) is now complete.

Remark (3.13). The group $O(U \perp U(2))$ itself contains elements of order 8 such as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

hence the use of Lemma (3.8) is inevitable.

C) Automorphism of odd order.

Proposition (3.14). There exists no numerically trivial automorphism of an Enriques surface of odd order other than the identity.

Proof. Suppose that there is such an automorphism g. We keep the notation in (3.2). Consider the restriction ρ_N of ρ to $N \simeq E_8(2) \perp U(2) \perp U$.

First note that it preserves the following inclusions of subgroups of $N_{\mathbb{Q}}$ (cf. (3.12)):

$$\tfrac{1}{2}(E_8(2) \oplus U(2) \oplus U) \supset \underbrace{\tfrac{1}{2}(E_8(2) \oplus U(2)) \oplus U}_{\displaystyle \| \atop \displaystyle N^\vee} \supset N,$$

hence it induces isomorphisms of finite groups

$$\bar{\rho}_1 \in GL(N^\vee/N),$$

$$\bar{\rho}_2 \in GL(\tfrac{1}{2}U/U)$$

and the characteristic polynomial of ρ_N is subject to a relation

$$\det(t1 - \rho_N) \bmod 2 = \det(t1 - \bar{\rho}_1)\det(t1 - \bar{\rho}_2).$$

Since ρ is the identity on M, it acts trivially on the discriminant form, namely $\bar{\rho}_1 = 1$.

On the other hand, using a basis $\langle e,f\rangle$ of U with $\langle e,e\rangle = \langle f,f\rangle = 0$, and $\langle e,f\rangle = 1$, we have $GL(\frac{1}{2}U/U) \cong GL(2,\mathbb{F}_2)$. If $\bar{\rho}_2$ has an odd order, it is therefore either $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$. But if $\bar{\rho}_2 = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$, then $\rho_N(e) = ae+bf+x$ with $a \equiv b \equiv 1 \pmod 2$ and $x \in E_8(2) \perp U(2)$, which is impossible because

$$\begin{aligned} 0 = \langle e,e\rangle &= \langle \rho_N(e), \rho_N(e)\rangle \\ &= 2ab + \langle x,x\rangle \\ &\equiv 2 \pmod 4. \end{aligned}$$

By the similar reason $\bar{\rho}_2 \neq \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$, hence $\bar{\rho}_2 = 1$.

Therefore we have

$$\det(t1 - \rho_N) \equiv (t-1)^{12} \pmod 2,$$

hence the minimal polymonial of ρ_N has also similar reduction modulo 2 (i.e. it decomposes into linear terms), which cannot occur for the cyclotomic polynomial for the odd order.

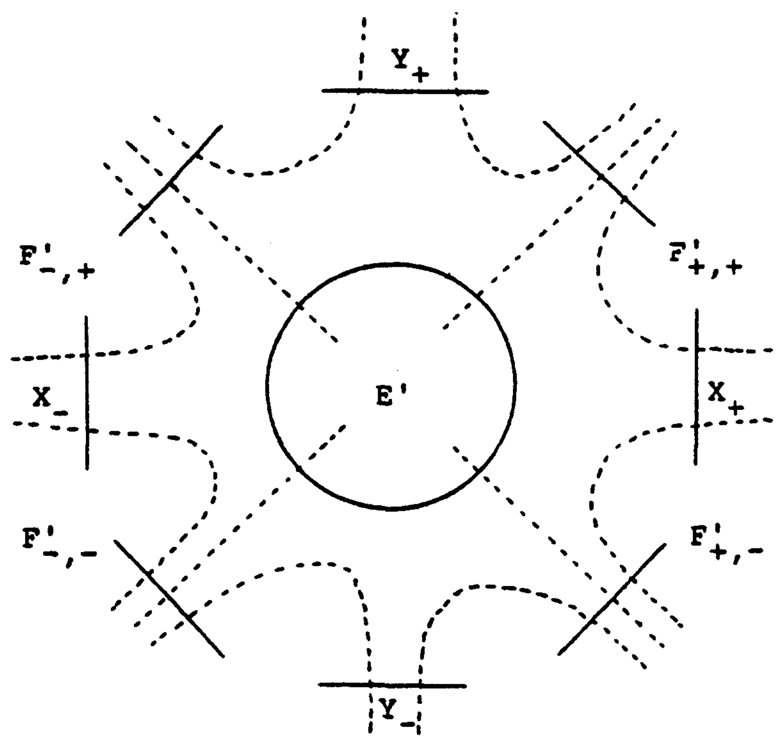
Thus we have contradiction and the proof of (3.14) is complete.

§4. Cohomologically trivial involutions

In this section, we shall compute the lattice N_R for the numerically trivial involutions of Example 1 and Example 2 and show that those involutions are, in fact, cohomologically trivial.

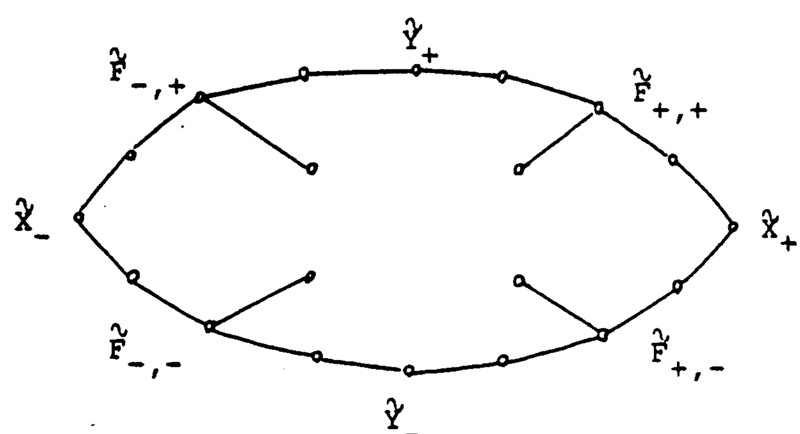
Let the situation be as in Example 2. Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the 4 points $(\pm 1, \pm 1)$ and let $F_{\pm,\pm}$ be the exceptional curve over $(\pm 1, \pm 1)$. Blow up again at the 12 intersection points of $F_{\pm,\pm}$ and the strict transforms of E and the 4 lines $X = \pm 1$ and $Y = \pm 1$. We denote the blown up rational surface by R. R has the following configuration of curves on it:

(4.1)



where $F'_{\pm,\pm}$, E', $X_\pm$ and $Y_\pm$ are the strict transforms of $F_{\pm,\pm}$, E and the 4 lines $X = \pm 1$ and $Y = \pm 1$, respectively. 12 dotted lines are the exceptional curves of the second blowing up. The divisor $D = X_+ + X_- + Y_+ + Y_- + \Sigma F'_{\pm,\pm} + E'$ belongs to the linear system $|-2K_R|$. The K3 surface $\tilde{S}$ is the double cover of R whose branch locus is D. We denote by $\tilde{X}_\pm$, $\tilde{Y}_\pm$, $\tilde{F}_{\pm,\pm}$ and $\tilde{E}$ the reduced parts of the inverse images of $X_\pm$, $Y_\pm$, $F'_{\pm,\pm}$ and E', respectively. All are smooth rational curves on $\tilde{S}$. The inverse images of the 12 exceptional curves on R are also smooth rational curves on $\tilde{S}$. Hence $\tilde{S}$ has 20 smooth rational curves. The dual graph of their configuration is as follows:

(4.2)



The covering involution σ_R of $\tilde{S}/R$ fixes these 20 rational curves. Hence their cohomology classes are contained in M_R.

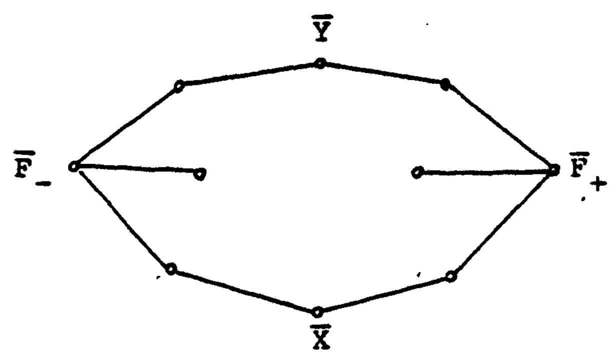
Proposition 4.3. These 20 rational curves generate M_R and $|\det M_R| = 4$.

Proof. Removing $\tilde{Y}_+$ and $\tilde{Y}_-$ from the diagram (4.2), we obtain the disjoint union of two $\tilde{D}_8$ diagrams. The two $\tilde{D}_8$'s and $\tilde{Y}_+$ generate a lattice M'' isomorphic to $D_8 \perp D_8 \perp U$. $\tilde{Y}_-$ does not belong to M'' but $2\tilde{Y}_-$ does. Hence the lattice M' generated by the 20 rational curves has $|\det M'| = |\det M''|/4 = 4$. Since $\det M_R = 4$ or 8 by Proposition 2.5, M' coincides with M_R and $\det M_R = 4$.

q.e.d.

Let ε be the fixed point free involution of $\tilde{S}$ such that $S = \tilde{S}/\varepsilon$. ε acts on the graph (4.2) by 180^0 rotation. Hence the Enriques surface S has the 10 rational curves with the following dual graph:

(4.4)



where $\bar{X}, \bar{Y}$ and $\bar{F}_\pm$ are the images of $\tilde{X}_+, \tilde{Y}_+$ and $\tilde{F}_{+,\pm}$. Obviously the involution σ of S induced by σ_R fixes these 10 rational curves.

Proposition 4.5. σ is cohomologically trivial.

Proof. The 10 rational curves in the diagram (4.4) generate a rank 10 lattice E' with $|\det E'| = 4$. Let $f : S \to \mathbb{P}^1$ be the elliptic fibration of S induced by a $\mathbb{P}^1$-bundle structure of $\mathbb{P}^1 \times \mathbb{P}^1$. Let G_1 and G_2 be the reduced parts of the two multiple fibres of f. The class of G_1 does not belong to the lattice E' but the twice of it does. Hence the 10 rational curves and G_1 generate the Enriques lattice $H^2(S,\mathbb{Z})$/torsion. Since the difference of G_1 and G_2 is a canonical divisor, the 10 rational curves and two elliptic curves G_1 and G_2 generate the cohomology group $H^2(S,\mathbb{Z})$. Since σ fixes G_1 and G_2, too, σ is cohomologically trivial.

q.e.d.

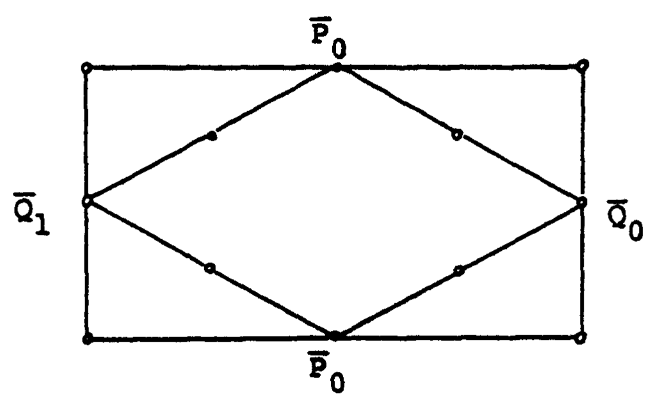
Next we consider Example 1. In this case, the surface $\tilde{S}$ is a Kummer surface. If the two elliptic curves E_1 and E_2 are not isogenous, then the lattice N_R coincides with the transcendental lattice of $\tilde{S}$. By [5], N_R is isomorphic to $N'(2)$ for an even unimodular lattice N'. Therefore, by Proposition 2.5, we have

Proposition 4.6. The lattice N_R is isomorphic to $U(2) \perp U(2)$ for every Enriques surface S and numerically trivial involution σ in Example 1.

The quotient $E_i/(-1_{E_i})$ is isomorphic to $\mathbb{P}^1$ and the branch points are the images of 4 2-torsion points of E_i.

Then the rational surface $R = \tilde{S}/\sigma_R$ is isomorphic to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 16 points $(\overline{x}_1, \overline{x}_2)$, where x_i runs the 4 2-torsion points of E_i and $\overline{x}_i$ is the image of x_i for $i = 1,2$. We denote by F_{x_1,x_2} the exceptional curve over $(\overline{x}_1, \overline{x}_2)$. Let P_{x_1} and Q_{x_2} be the strict transforms of $\overline{x}_1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \overline{x}_2$, respectively. Then the branch locus of $\tilde{S}/R$ is the disjoint union of P_{x_1}'s and Q_{x_2}'s, where x_1 and x_2 are the 2-torsion points of E_1 and E_2, respectively. The involution σ_R fixes all these 24 rational curves F_{x_1,x_2}, P_{x_1} and Q_{x_2}. The involution of R induced by ε sends F_{x_1,x_2}, P_{x_1} and Q_{x_2} to $F_{x_1+a_1,x_2+a_2}, P_{x_1+a_1}$, and $Q_{x_2+a_2}$, respectively. Hence the Enriques surface S has 12 rational curves and the dual graph of their configuration is as follows:

(4.7)



where $\overline{P}_0$ and $\overline{P}_1$ (resp. $\overline{Q}_0$ and $\overline{Q}_1$) are the images of P_0

and P_{b_1} (resp. Q_0 and Q_{b_2}) and b_1 and b_2 are nonzero 2 - torsion points of E_1 and E_2 other than a_1 and a_2, respectively.

Removing $\overline{P}_0$ and $\overline{P}_1$ from the above diagram, we obtain the disjoint union of two $\tilde{D}_4$'s. These two $\tilde{D}_4$'s and $\overline{P}_0$ generate a lattice E" isomorphic to $D_4 \perp D_4 \perp U(2)$. The class of $\overline{P}_1$ does not belong to E" but the twice of it does. Hence the 12 rational curves in the graph(4.7) generate a lattice E' with $|\det E'| = 16$.

Proposition 4.8. The numerically trivial involution σ in Example 1 is cohomologically trivial.

Proof. S has two elliptic fibrations f_1 and f_2 induced by the two $\mathbb{P}^1$-fibrations of $\mathbb{P}^1 \times \mathbb{P}^1$. Let A_i be the reduced part of a multiple fibre of f_i, i=1,2. Then the lattice generated by E', A_1 and A_2 is unimodular. Hence E', A_1, A_2 and the canonical class K_S generate the cohomology group $H^2(S,\mathbb{Z})$. It is clear that the involution σ acts trivially on E' and fixes A_1 and A_2. Hence σ is cohomologically trivial.

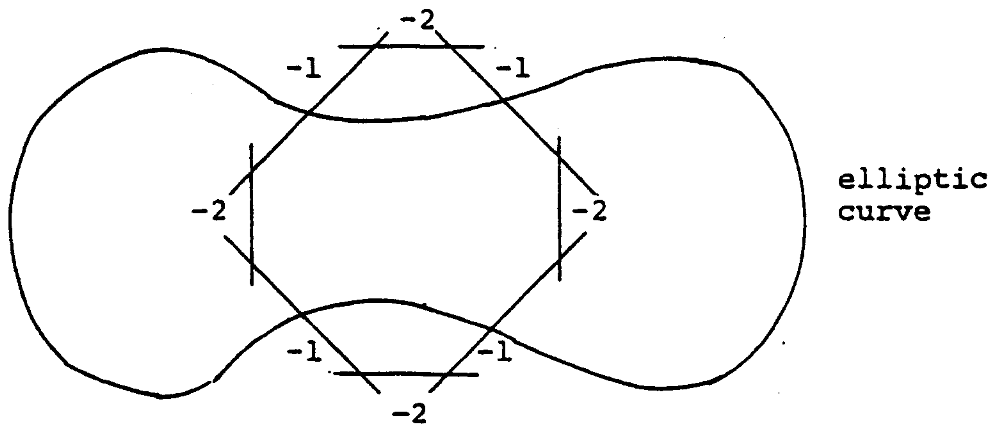
q.e.d.

§5. Proof of Main Theorem

In this section we shall prove that every numerically trivial involution is cohomologically trivial and obtained in the way of Example 1 or 2. Main Theorem 0.1. follows from this and the result in §3.

Let σ be a numerically trivial involution of an Enriques surface S. By Corollary 2.7, the pair (S,σ) is a deformation of that of Example 1 or 2 according as $N_R \simeq U(2) \perp U(2)$ or $U \perp U(2)$. Hence, in the case $N_R \simeq U(2) \perp U(2)$, the branch locus of $\tilde{S}/R$ is the disjoint union of 8 smooth rational curves and the rational surface $R = \tilde{S}/\sigma_R$ has 16 exceptional curves of the first kind. The configuration of these 24 rational curves is same as Example 1. Hence the surface obtained from R by contracting 16 exceptional curves is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ because it has two $\mathbb{P}^1$-fibrations. The image of the branch locus of $\tilde{S}/R$ is the union of 4 fibres of one $\mathbb{P}^1$-fibration and 4 fibres of another $\mathbb{P}^1$-fibration. Hence the pair (S,σ) is obtained in the way of Example 1.

In the case $N_R \simeq U \perp U(2)$, the pair (S,σ) is a deformation of Example 2. The branch locus of $\tilde{S}/R$ is the disjoint union of an elliptic curve and 8 smooth rational curves. The rational surface R has 12 exceptional curves of the first kind and the configuration of the elliptic curve and these 20 rational curves is same as Example 2. Contract the 12 exceptional curves. Then the configuration of the 21 curves becomes as follows:



4 rational curves in the branch locus become exceptional curves of the first kind and other 4 become (-2) rational curves. Contract the 4 exceptional curves of the first kind. Then we obtain a surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The image of the branch locus in $\mathbb{P}^1 \times \mathbb{P}^1$ is same as Example 2. Hence (S, σ) is obtained in the way of Example 2, which completes our proof of Main Theorem 0.1.

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