FINITE-DIMENSIONAL REPRESENTATIONS OF HYPER LOOP ALGEBRAS

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Abstract: We study finite-dimensional representations of hyper loop algebras, i.e., the hyperalgebras over a field of positive characteristic associated to non-twisted affine Kac-Moody algebras, or rather, to the underlying loop algebras. The main results are the classification of the irreducible modules, a version of Steinberg's Tensor Product Theorem, and the construction of positive characteristic analogues of the Weyl modules as defined by Chari and Pressley in the characteristic zero case. Assuming a natural conjecture on a tensor product decomposition for these Weyl modules, we describe the blocks of the underlying abelian category. We also start the study of reduction modulo p and prove that every irreducible module of the hyper loop algebras can be constructed as a quotient of a module obtained by a certain reduction modulo p process applied to suitable characteristic zero modules.

Introduction

Let G be a semisimple connected algebraic group over an algebraically closed field $\mathbb{F}$. One can associate to G its Lie algebra $L(G)$ and its algebra of distributions $U(G)$, which we prefer to call the hyperalgebra of G. If $\mathbb{F}$ is of characteristic zero, the hyperalgebra coincides with the universal enveloping algebra $U(L(G))$ of $L(G)$, but this is not so in positive characteristic. $U(G)$ acts naturally on any G-module and it turns out that, as conjectured originally by Verma and proved by Sullivan [28], every finite-dimensional $U(G)$-module can be "lifted" to a rational finite-dimensional G-module. We will restrict our attention to the case when G is a simple Chevalley group of adjoint type. In this case the algebra $U(G)$ is isomorphic to the algebra $U(\mathfrak{g})_{\mathbb{F}}$ constructed by considering Kostant's integral form of $U(\mathfrak{g})$ and tensoring with $\mathbb{F}$ over $\mathbb{Z}$, where $\mathfrak{g}$ is the complex simple Lie algebra corresponding to G. It will suffice, for our purposes, to work over the purely algebraic setting of $U(\mathfrak{g})_{\mathbb{F}}$.

Let $\mathfrak{g}$ be as above and $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the loop algebra over $\mathfrak{g}$. The finite-dimensional representation theory of $\tilde{\mathfrak{g}}$ is a very active research topic in the last decades. It is related, for instance, to integrable models and the Bethe ansatz in statistical mechanics. In [14], Garland introduced an integral form of $U(\tilde{\mathfrak{g}})$ which can be used to construct what we call the hyper loop algebra $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ of $\mathfrak{g}$ over $\mathbb{F}$. The hyperalgebra $U(\mathfrak{g})_{\mathbb{F}}$ is naturally a subalgebra of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$.

The purpose of the present paper is to study some basic aspects of the category $\tilde{\mathcal{C}}_{\mathbb{F}}$ of finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules such as the classification of its simple objects and its block decomposition. When $\mathbb{F} = \mathbb{C}$, thus $U(\tilde{\mathfrak{g}})_{\mathbb{F}} = U(\tilde{\mathfrak{g}})$, these questions were studied in [3, 6, 8]. It turns out that the simple finite-dimensional $\tilde{\mathfrak{g}}$-modules are highest-weight modules with respect to the triangular decomposition of $\tilde{\mathfrak{g}}$ obtained by "looping" the usual triangular decomposition of $\mathfrak{g}$. As usual, we will say ℓ-highest-weight to differ from the triangular decomposition coming from the Chevalley generators of $\tilde{\mathfrak{g}}$ (highest-weight representations with respect to the later decomposition were studied in [14]). Moreover, all the simple modules are isomorphic to suitable tensor products of the so-called evaluation representations (obtained by pulling back the simple $\mathfrak{g}$-modules by the evaluation map $t \mapsto a$ for some nonzero $a \in \mathbb{C}$).

Keywords: Loop algebras, finite-dimensional representations, hyperalgebras

We prove that these two results hold in positive characteristic, as well. This is done in Corollary 3.2 and Theorem 3.8, the later being a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-version of Steinberg's Tensor Product Theorem.

The set of ℓ-highest weights can be identified with rank($\mathfrak{g}$)-tuples of polynomials in $\mathbb{F}[u]$ with constant term 1. For $\mathbb{F} = \mathbb{C}$, it was shown in [9] that there exists a family of finite-dimensional ℓ-highest-weight modules, called the Weyl modules, which are universal in the sense that any finite-dimensional ℓ-highest-weight module is isomorphic to a quotient of the corresponding Weyl module. Furthermore, these Weyl modules can be decomposed as a tensor product of smaller Weyl modules according to the decomposition of simple modules into tensor products of evaluation representations. This fact was one of the key ingredients for proving that the blocks of $\tilde{\mathcal{C}}_{\mathbb{C}}$ are parametrized by functions of finite support $\chi : \mathbb{C}^\times \to \Gamma$, where Γ is the quotient of the weight lattice of $\mathfrak{g}$ by its root lattice. We prove that the Weyl modules for $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ can be defined in a similar fashion. However, the techniques used to prove the tensor product decomposition in the characteristic zero situation resisted adaptations to the positive characteristic setting so far. Still, we conjecture that the Weyl modules admit a tensor product decomposition corresponding to the minimal way of decomposing its irreducible quotient into a tensor product of evaluation representations (Conjecture 3.11). Assuming this conjecture we then prove that the block decomposition of $\tilde{\mathcal{C}}_{\mathbb{F}}$ is described similarly to that of $\tilde{\mathcal{C}}_{\mathbb{C}}$. The proof uses a few results from the theory of reduction modulo p for finite-dimensional $U(\mathfrak{g})$-modules which enables us to obtain a positive characteristic version of [6, Proposition 3.4], a key ingredient to construct some useful indecomposable modules.

The reason that the universal ℓ-highest-weight modules discussed above were called Weyl modules comes from a conjecture in [9] stating that these modules can be obtained as the classical limit of some irreducible finite-dimensional modules for the corresponding quantum loop algebra, resembling the process of obtaining the Weyl modules for $U(\mathfrak{g})_{\mathbb{F}}$ by reduction modulo p of simple $\mathfrak{g}$-modules. This conjecture has been recently proved when $\mathfrak{g}$ is of type A in [5] using Gelfand-Tsetlin filtrations and when $\mathfrak{g}$ is of type ADE in [13] using Demazure modules. Moreover, H. Nakajima has pointed out that the general case can be deduced using the crystal and global basis results from [1, 18, 19, 23, 24]. Other interesting related references include [4, 12, 21, 22]. We have an analogous conjecture for the Weyl modules for $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$, stating that they can be obtained from the Weyl modules for $U(\tilde{\mathfrak{g}})$ by reduction modulo p (Conjecture 4.8). However, we consider more general lattices than $\mathbb{Z}$-lattices since they allow us to obtain Conjecture 3.11, mentioned above, as a corollary of Conjecture 4.8. This would not be possible if we consider $\mathbb{Z}$-lattices only. We also prove that all finite-dimensional ℓ-highest-weight $U(\tilde{\mathfrak{g}})$-modules whose roots of the ℓ-highest-weights are units in some discrete valuation ring $\mathbb{A}$ contain an admissible $\mathbb{A}$-lattice and, thus, we obtain all of the irreducible modules as quotients of a module coming from a reduction modulo p process. This is done in Theorem 4.6 and Corollary 4.7. Combining Conjecture 4.8 with the one in [9], which is now a theorem as remarked above, we have a bridge connecting the Weyl modules for $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ with certain irreducible representations for quantum loop algebras (at generic quantization parameter).

The paper is organized as follows. In section 1 we fix the basic notation on finite-dimensional complex simple Lie algebras and their loop algebras, define the hyperalgebras and collect some important Lemmas. Section 2 is dedicated to a review of the relevant facts about finite-dimensional $U(\mathfrak{g})_{\mathbb{F}}$-modules. The main part of the paper consists of sections 3 and 4. In 3.1 we define ℓ-highest-weight modules and obtain the necessary relations satisfied by the finite-dimensional ones. The construction of the Weyl modules and consequent characterization of the simple finite-dimensional representations in terms of ℓ-highest-weights is done in 3.2. The aforementioned tensor product results are established in 3.3, while the block decomposition is described in 3.4. Section 4 ends the paper with the results and the conjecture on reduction modulo p.

Acknowledgements: D.J. is pleased to thank FAPESP (processo 2006/00609-1) for supporting her visit to the University of Campinas, when part of this paper was developed, and the Max-Planck-Institut für Mathematik in Bonn for its hospitality. The research of A.M. is partially supported by CNPq (processo 303349/2005-0) and FAPESP (processo 2006/00833-9). We thank V. Chari for turning our attention to this subject and for very helpful questions and suggestions. We also thank A. Engler and P. Brumatti for useful discussions about discrete valuation rings.

1. Hyperalgebras

Throughout the paper $\mathbb{C}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ will denote, respectively, the sets of complex numbers, integers, non-negative integers, and positive integers. Given a ring $\mathbb{A}$, the underlying multiplicative group of units will be denoted by $\mathbb{A}^\times$. The dual of a vector space V will be denoted by V^*.

1.1. **Preliminaries on Lie Algebras over $\mathbb{C}$.** Let I be the set of vertices of a finite-type connected Dynkin diagram and $\mathfrak{g}$ be the associated simple complex Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$ and corresponding nilpotent subalgebras $\mathfrak{n}^\pm$. We will denote by R the root system and by R^+ the set of positive roots so that

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}, \quad \text{where} \quad \mathfrak{g}_{\pm\alpha} = \{x \in \mathfrak{g} : [h, x] = \pm\alpha(h)x \ \forall\ h \in \mathfrak{h}\}.$$

The simple roots will be denoted by α_i, the fundamental weights by ω_i, while Q, P, Q^+, P^+ will be the root and weight lattices with corresponding positive cones, respectively. We equip $\mathfrak{h}^*$ with the partial order $\lambda \le \mu$ iff $\mu - \lambda \in Q^+$. The Weyl group will be denoted by $\mathcal{W}$, its longest element by w_0, and the maximal positive root is denoted by θ. Let $\langle , \rangle$ be the bilinear form on $\mathfrak{h}^*$ induced by the Killing form on $\mathfrak{g}$ and, for $\alpha \in R^+$, set $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ and $d_\alpha = \frac{1}{2}\langle\alpha,\alpha\rangle$. Then $\{\alpha_i^\vee : i \in I\}$ is the set of simple roots of the simple Lie algebra $\mathfrak{g}^\vee$ whose Dynking diagram is obtained from that of $\mathfrak{g}$ by reversing the arrows and $R^\vee = \{\alpha^\vee : \alpha \in R\}$ is its root system. Moreover, if $\alpha = \sum_i m_i\alpha_i$ and $\alpha^\vee = \sum_i m_i^\vee \alpha_i^\vee$, then

$$m_i^\vee = \frac{d_{\alpha_i}}{d_\alpha} m_i. \tag{1.1}$$

If $\mathfrak{a}$ is a Lie algebra (over any field $\mathbb{F}$), define its loop algebra $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]$ with bracket given by $[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$. Clearly $\mathfrak{a} \otimes 1$ is a subalgebra of $\tilde{\mathfrak{a}}$ isomorphic to $\mathfrak{a}$ and, by abuse of notation, we will continue denoting its elements by x instead of $x \otimes 1$. In case $\mathfrak{a} = \mathfrak{g}$, we have $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{h}}$ is an abelian subalgebra. Let $U(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. Then $U(\mathfrak{a})$ is a subalgebra of $U(\tilde{\mathfrak{a}})$ and, for $\mathfrak{a} = \mathfrak{g}$, multiplication establishes isomorphisms

$$U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+) \qquad \text{and} \qquad U(\tilde{\mathfrak{g}}) \cong U(\tilde{\mathfrak{n}}^-) \otimes U(\tilde{\mathfrak{h}}) \otimes U(\tilde{\mathfrak{n}}^+).$$

The assignments $\triangle : \mathfrak{a} \to U(\mathfrak{a}) \otimes_{\mathbb{F}} U(\mathfrak{a}), x \mapsto x \otimes 1 + 1 \otimes x$, $S : \mathfrak{a} \to \mathfrak{a}, x \mapsto -x$, and $\epsilon : \mathfrak{a} \to \mathbb{F}, x \mapsto 0$, can be uniquely extended so that $U(\mathfrak{a})$ becomes a Hopf algebra with comultiplication $\triangle$, antipode S, and counit ϵ. We shall denote by $U(\mathfrak{a})^0$ the augmentation ideal, i.e., $U(\mathfrak{a})^0$ is the kernel of ϵ.

1.2. **Reduction Modulo p.** As usual, given any associative algebra A over a field of characteristic zero, $a \in A$, and $k \in \mathbb{Z}_+$, we set $a^{(k)} = \frac{a^k}{k!}$, $\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} \in A$.

Let $\Phi = \{x_\alpha^\pm, h_{\alpha_i} : \alpha \in R^+, i \in I\}$ be a Chevalley basis for $\mathfrak{g}$, where $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$, $h_\alpha = [x_\alpha^+, x_\alpha^-]$, and let $x_{\alpha,r}^\pm = x_\alpha^\pm \otimes t^r, h_{\alpha,r} = h_\alpha \otimes t^r$. When $r = 0$ we may just write $x_\alpha^\pm$ and h_α and, if $\alpha = \alpha_i$, we may write $x_{i,r}^\pm$ and $h_{i,r}$. Notice that the set $\tilde{\Phi} = \{x_{\alpha,r}^\pm, h_{i,r} : \alpha \in R^+, i \in I, r \in \mathbb{Z}\}$ is a basis for $\tilde{\mathfrak{g}}$ and define $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ to be the $\mathbb{Z}$-span of $\tilde{\Phi}$. The $\mathbb{Z}$-span of Φ is a Lie $\mathbb{Z}$-subalgebra of $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ which we denote by $\mathfrak{g}_{\mathbb{Z}}$.

If $\mathbb{F}$ is any field, set
$$\mathfrak{g}_{\mathbb{F}} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F} \qquad \text{and} \qquad \tilde{\mathfrak{g}}_{\mathbb{F}} = \tilde{\mathfrak{g}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F},$$
so that $\mathfrak{g}_{\mathbb{F}}$ and $\tilde{\mathfrak{g}}_{\mathbb{F}}$ are Lie algebras over $\mathbb{F}$.

Given $\alpha \in R^+, r \in \mathbb{Z}$, define elements $\Lambda_{\alpha,r} \in U(\tilde{\mathfrak{h}})$ by the following equality of formal power series in u:
$$\Lambda^{\pm}_{\alpha}(u) = \sum_{r=0}^{\infty} \Lambda_{\alpha,\pm r} u^r = \exp\left(- \sum_{s=1}^{\infty} \frac{h_{\alpha,\pm s}}{s} u^s \right). \tag{1.2}$$
If $\alpha = \alpha_i$ we may write $\Lambda_{i,r}$ in place of $\Lambda_{\alpha_i,r}$. It follows from (1.1) that, if $\alpha = \sum_i m_i \alpha_i \in R^+$, then $h_\alpha = \sum_i m_i^\vee h_i$ and
$$\Lambda^{\pm}_{\alpha}(u) = \prod_{i \in I} (\Lambda^{\pm}_{\alpha_i}(u))^{m_i^\vee}. \tag{1.3}$$
Let $z \in \mathbb{C}^\times$ and consider the evaluation map $\mathrm{ev}_z : \tilde{\mathfrak{g}} \to \mathfrak{g}$. We also denote by ev_z the induced map $U(\tilde{\mathfrak{g}}) \to U(\mathfrak{g})$. We have (cf. [14, Lemma 5.1]):

$$\mathrm{ev}_z(\Lambda_{\alpha,r}) = (-z)^r \binom{h_\alpha}{|r|}. \tag{1.4}$$

For $k \in \mathbb{Z}, k \neq 0$, consider also the endomorphism τ_k of $U(\tilde{\mathfrak{g}})$ extending $t \mapsto t^k$ and set $\Lambda_{\alpha,r;k} = \tau_k(\Lambda_{\alpha,r}), \Lambda^{\pm}_{\alpha;k}(u) = \sum_{r=0}^{\infty} \Lambda_{\alpha,\pm r;k} u^r$. Notice that $\binom{h_i}{k}$ is a polynomial in h_i of degree k. Hence, the set $\{\binom{h_1}{k_1} \cdots \binom{h_\ell}{k_\ell} : k_j \in \mathbb{Z}_+\}$, where $\ell = |I|$, is a basis for $U(\mathfrak{h})$. Similarly, observe that $\Lambda_{i,\pm r;k}, r, k \in \mathbb{N}$, is a polynomial in $h_{i,\pm sk}, 1 \leq s \leq r$, whose leading term is $(-h_{i,\pm k})^{(r)}$. Finally, given an order on $\tilde{\Phi}$ and a PBW monomial with respect to this order, we construct an ordered monomial in the elements
$$(x^{\pm}_{\alpha,r})^{(k)}, \quad \Lambda_{i,r;k}, \quad \binom{h_i}{k}, \qquad r, k \in \mathbb{Z}, k > 0, \alpha \in R^+, i \in I,$$
using the correspondence just discussed for the basis elements of $U(\tilde{\mathfrak{h}})$ as well as the obvious correspondence $(x^{\pm}_{\alpha,r})^k \leftrightarrow (x^{\pm}_{\alpha,r})^{(k)}$. The set of monomials thus obtained is then a basis for $U(\tilde{\mathfrak{g}})$ while the monomials involving only $(x^{\pm}_{\alpha})^{(k)}, \binom{h_i}{k}$ form a basis for $U(\mathfrak{g})$. Let $U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$ (resp. $U(\mathfrak{g})_{\mathbb{Z}}$) be the $\mathbb{Z}$-span of these monomials. The following crucial theorem was proved in [20] ($U(\mathfrak{g})$ case) and [14] ($U(\tilde{\mathfrak{g}})$ case).

Theorem 1.1. *$U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$ (resp. $U(\mathfrak{g})_{\mathbb{Z}}$) is the $\mathbb{Z}$-subalgebra of $U(\tilde{\mathfrak{g}})$ generated by $\{(x^{\pm}_{\alpha,r})^{(k)}, \alpha \in R^+, r, k \in \mathbb{Z}, k \geq 0\}$ (resp. $\{(x^{\pm}_{\alpha})^{(k)}, \alpha \in R^+, k \in \mathbb{Z}_+\}$).* □

For $\mathfrak{a} \in \{\mathfrak{g}, \mathfrak{n}^{\pm}, \mathfrak{h}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{n}}^{\pm}, \tilde{\mathfrak{h}}\}$, let $U(\mathfrak{a})_{\mathbb{Z}}$ denote the corresponding $\mathbb{Z}$-subalgebra of $U(\tilde{\mathfrak{g}})$. Given a field of characteristic $\mathbb{F}$, the $\mathbb{F}$-hyperalgebra of $\mathfrak{a}$ is
$$U(\mathfrak{a})_{\mathbb{F}} = U(\mathfrak{a})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}.$$
Then the PBW theorem gives
$$U(\mathfrak{g})_{\mathbb{F}} = U(\mathfrak{n}^-)_{\mathbb{F}} U(\mathfrak{h})_{\mathbb{F}} U(\mathfrak{n}^+)_{\mathbb{F}}, \qquad U(\tilde{\mathfrak{g}})_{\mathbb{F}} = U(\tilde{\mathfrak{n}}^-)_{\mathbb{F}} U(\tilde{\mathfrak{h}})_{\mathbb{F}} U(\tilde{\mathfrak{n}}^+)_{\mathbb{F}}.$$
Clearly, if $\mathbb{F} = \mathbb{C}$, $U(\mathfrak{a})_{\mathbb{F}} = U(\mathfrak{a})$. If no confusion arises, we will write x, instead of $x \otimes 1$, for the image of an element $x \in U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$ in $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$.

The Hopf algebra structure on $U(\tilde{\mathfrak{g}})$ induces a Hopf algebra structure on $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ with comultiplication given by
$$\triangle\left((x^{\pm}_{\alpha,r})^{(k)} \right) = \sum_{l+m=k} (x^{\pm}_{\alpha,r})^{(l)} \otimes (x^{\pm}_{\alpha,r})^{(m)}, \tag{1.5}$$

$$\triangle\left(\binom{h_i}{k}\right) = \sum_{l+m=k} \binom{h_i}{l} \otimes \binom{h_i}{m}, \quad \text{and} \quad \triangle(\Lambda_{\alpha,\pm k}) = \sum_{l+m=k} \Lambda_{\alpha,\pm l} \otimes \Lambda_{\alpha,\pm m}. \tag{1.6}$$

1.3. **Some Lemmas.** We now collect some essential identities on $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$, when $\mathbb{F}$ is a field of characteristic $p > 0$. We begin with the following trivial observation:

$$(x^{\pm}_{\alpha,r})^{(k)}(x^{\pm}_{\alpha,r})^{(l)} = \binom{k+l}{k}(x^{\pm}_{\alpha,r})^{(k+l)}. \tag{1.7}$$

From this, one easily deduces

$$((x^{\pm}_{\alpha,r})^{(k)})^p = 0. \tag{1.8}$$

It is well known (see [16]) that the elements $\binom{h_i}{k}$ satisfy

$$\binom{h_i}{k}^p = \binom{h_i}{k} \tag{1.9}$$

and it is easy to see that we have

$$\binom{h_i}{l}(x^{\pm}_{\alpha,r})^{(k)} = (x^{\pm}_{\alpha,r})^{(k)}\binom{h_i \pm k\alpha(h_i)}{l}. \tag{1.10}$$

Given $\alpha \in R^+, s \in \mathbb{Z}$, define

$$X^-_{\alpha;s,\pm}(u) = \sum_{r \geq 1} x^-_{\alpha,\pm(r+s)} u^r.$$

Lemma 1.2. We have

$$(x^+_\alpha)^{(l)}(x^-_\alpha)^{(k)} = \sum_{m=0}^{\min\{k,l\}} (x^-_\alpha)^{(k-m)}\binom{h_\alpha - k - l + 2m}{m}(x^+_\alpha)^{(l-m)} \tag{1.11}$$

and

$$(x^+_{\alpha,\mp s})^{(l)}(x^-_{\alpha,\pm(s+1)})^{(k)} \in (-1)^l\left((X^-_{\alpha;s,\pm}(u))^{(k-l)}\Lambda^{\pm}_\alpha(u)\right)_k + U(\tilde{\mathfrak{g}})_{\mathbb{F}}U(\tilde{\mathfrak{n}}^+)^0_{\mathbb{F}}. \tag{1.12}$$

In (1.12) $k \geq l \geq 1$ and the subindex k means the coefficient of u^k of the above power series.

Proof. In both cases the strategy is to commute the elements on the left hand side. The proof of (1.11) can be found in [15, Lemma 26.2].

Equation (1.12) was proved in [14, Lemma 7.5] for $s = 0$ and "+". Consider the subalgebra of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ generated by $(x^{\pm}_{\alpha,r})^{(k)}$ for a fixed $\alpha \in R^+$. It is easy to see that, for each $s \in \mathbb{Z}$, the assignment $(x^{\pm}_{\alpha,r})^{(k)} \mapsto (x^{\pm}_{\alpha,r\pm s})^{(k)}$ extends uniquely to an algebra automorphism of this subalgebra which is the identity when restricted to $U(\tilde{\mathfrak{h}})_{\mathbb{F}}$. The general case of (1.12) (with "+") follows easily from the case $s = 0$ using these automorphisms (see also [9, Lemma 1.3]). For "−", just apply the automorphism determined by the assignment $(x^{\pm}_{\alpha,r})^{(k)} \mapsto (x^{\pm}_{\alpha,-r})^{(k)}$. □

In the proof of Theorem 3.6 we will also need the following lemmas. Consider monomials involving only the elements $(x^-_{\alpha,r})^{(k)}$, which span $U(\tilde{\mathfrak{n}}^-)_{\mathbb{F}}$. Define the degree of $(x^-_{\alpha,r})^{(k)}$ to be k and extend it additively.

Lemma 1.3. Let $\alpha, \beta \in R^+, k, l \in \mathbb{Z}_+, r, s \in \mathbb{Z}$. Then $(x^-_{\alpha,r})^{(k)}(x^-_{\beta,s})^{(l)}$ is in the span of $(x^-_{\beta,s})^{(l)}(x^-_{\alpha,r})^{(k)}$ together with monomials of degree strictly smaller than $k + l$.

Proof. Immediate from the proof of [15, Lemma 26.3.C] using that $U(\tilde{\mathfrak{n}}^-)_{\mathbb{Z}}$ is $(Q^+ \times \mathbb{Z})$-graded. □

Lemma 1.4. In $U(\tilde{\mathfrak{h}})_{\mathbb{Z}}$, for all $k, s \in \mathbb{N}$ and $\alpha \in R^+$, we have

$$h_{\alpha,\pm s} = (-1)^s \Lambda_{\alpha,\pm s} + \sum_{(\vec{r},\vec{n})} m_{\vec{r},\vec{n}} \Lambda^{n_1}_{\alpha,\pm r_1} \cdots \Lambda^{n_l}_{\alpha,\pm r_l},$$

$$\Lambda_{\alpha,\pm s;k} = k\Lambda_{\alpha,\pm sk} + \sum_{(\vec{r},\vec{n})} m'_{\vec{r},\vec{n}} \Lambda^{n_1}_{\alpha,\pm r_1} \cdots \Lambda^{n_l}_{\alpha,\pm r_l},$$

for some $m_{\vec{r},\vec{n}}, m'_{\vec{r},\vec{n}} \in \mathbb{Z}$. The sums are over the pairs $(\vec{r},\vec{n})$, where $\vec{r} = (r_1, \cdots, r_l), \vec{n} = (n_1, \cdots, n_l)$, are such that $l, r_j, n_j \in \mathbb{N}$, $r_i \neq r_j, l \sum_j n_j > 1$, and $\sum_j n_j r_j = s$. In particular, the image of $h_{\alpha,\pm s}$ in $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ is not zero.

Proof. The first statement is obtained by comparing the coefficients of u^s in the identity of formal power series in u:

$$-\sum_{s\geq 1} \frac{h_{\alpha,\pm s}}{s} u^s = \ln\left(1 + \sum_{r\geq 1} \Lambda_{\alpha,\pm r} u^r\right) = \sum_{k\geq 0} \frac{(-1)^k}{k+1} \left(\sum_{r\geq 1} \Lambda_{\alpha,\pm r} u^r\right)^{k+1}.$$

The second statement is part of [14, Lemma 5.11]. Write $\alpha = \sum_i m_i \alpha_i$ and observe that the greatest common divisor of all $m_i^\vee$ is 1. Hence, there always exists $i_0 \in I$ such that p does not divide $m_{i_0}^\vee$. The last statement now follows from the first together with $h_{\alpha,s} = \sum_i m_i^\vee h_{i,s}$. $\square$

1.4. **Frobenius Homomorphism.**

Lemma 1.5. $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ (resp. $U(\mathfrak{g})_{\mathbb{F}}$) is generated as an algebra by the elements $(x^\pm_{\alpha,r})^{(p^k)}$ (resp. $(x^\pm_\alpha)^{(p^k)}$), $\alpha \in R^+, k, r \in \mathbb{Z}, k \geq 0$. Moreover, $U(\mathfrak{h})_{\mathbb{F}}$ is generated as an algebra by $\binom{h_i}{p^k}, i \in I, k \in \mathbb{Z}_+$. $\square$

Proof. The first statement is immediate from Theorem 1.1 and (1.7). The second is a statement on $U(\mathfrak{g})_{\mathbb{F}}$ and is well known. $\square$

Let $\bar{U}(\tilde{\mathfrak{g}})_{\mathbb{F}}$ be the quotient of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ by the 2-sided ideal generated by $x^\pm_{\alpha,s}$ for $\alpha \in R^+, s \in \mathbb{Z}$ and denote by $(\bar{x}^\pm_{\alpha,s})^{(p^k)}$ the image of $(x^\pm_{\alpha,s})^{(p^k)}$ in $\bar{U}(\tilde{\mathfrak{g}})_{\mathbb{F}}$ under the canonical projection. It is not difficult to see that the assignment $(\bar{x}^\pm_{\alpha,s})^{(p^k)} \mapsto (x^\pm_{\alpha,s})^{(p^{k-1})}$ extends uniquely to a Hopf algebra isomorphism $\bar{U}(\tilde{\mathfrak{g}})_{\mathbb{F}} \to U(\tilde{\mathfrak{g}})_{\mathbb{F}}$. Let $\tilde{\phi} : U(\tilde{\mathfrak{g}})_{\mathbb{F}} \to U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ be the composition of this isomorphism with the canonical projection and let ϕ be the restriction of $\tilde{\phi}$ to $U(\mathfrak{g})_{\mathbb{F}}$. We have:

$$\phi\left(\binom{h_i}{p^k}\right) = \binom{h_i}{p^{k-1}} \qquad \text{and} \qquad \tilde{\phi}(\Lambda_{i,r}) = \begin{cases} \Lambda_{i,r/p}, & \text{if } p \text{ divides } r \\ 0, & \text{otherwise.} \end{cases} \tag{1.13}$$

We call $\tilde{\phi}$ and ϕ the Frobenius homomorphisms. Given a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module (resp. $U(\mathfrak{g})_{\mathbb{F}}$-module) V, we denote by $V^{\tilde{\phi}^m}$ (resp. V^{ϕ^m}) the pull-back of V by $\tilde{\phi}^m$ (resp. ϕ^m).

2. Review on finite-dimensional $U(\mathfrak{g})_{\mathbb{F}}$-modules

In this section we review some results on finite-dimensional representations of $U(\mathfrak{g})_{\mathbb{F}}$ which will be relevant for our purposes. In the first subsection we consider the case $\mathbb{F} = \mathbb{C}$, where we summarize the basic results without proofs. The literature for this subsection is vast and well known (all the results we mention can be found in [15] to name but one reference). In the other subsections $\mathbb{F}$ will be an algebraically closed field of characteristic $p > 0$. Essentially all of the results can be found in [17] (see also [2]), although the approach there is heavily geometric. Our approach follows that of [15, Chapter VII] and [16]. Since some proofs are relevant for section 3, we consider it appropriate to sketch them.

2.1. **Characteristic Zero and Lattices.** Given a $U(\mathfrak{g})$-module V, a vector $v \in V$ is called a weight vector if $hv = \mu(h)v$ for some $\mu \in \mathfrak{h}^*$ and all $h \in \mathfrak{h}$. The subspace consisting of weight vectors of weight μ will be denoted by V_μ. If v is a weight vector such that $\mathfrak{n}^+ v = 0$, then v is called a highest-weight vector. If V is generated by a highest-weight vector of weight λ, then V is said to be a highest-weight module of highest weight λ.

The following theorem summarizes the basic facts about finite-dimensional $U(\mathfrak{g})$-modules.

Theorem 2.1. Let V be a finite-dimensional $U(\mathfrak{g})$-module.

(a) $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ and $\dim V_\mu = \dim V_{w\mu}$ for all $w \in \mathcal{W}$.
(b) V is completely reducible.
(c) For each $\lambda \in P^+$ the $U(\mathfrak{g})$-module $V^0(\lambda)$ generated by a vector v satisfying
$$x_i^+ v = 0, \quad h_i v = \lambda(h_i)v, \quad (x_i^-)^{\lambda(h_i)+1} v = 0, \quad \forall\, i \in I,$$
is irreducible and finite-dimensional. If V is irreducible, then V is isomorphic to $V^0(\lambda)$ for some $\lambda \in P^+$.
(d) If $\lambda \in P^+$ and $V \cong V^0(\lambda)$, then $V_\mu \neq 0$ iff $w\mu \le \lambda$ for all $w \in \mathcal{W}$. □

An admissible lattice for a $U(\mathfrak{g})$-module V is the $\mathbb{Z}$-span of a basis for V which is invariant under the action of $U(\mathfrak{g})_\mathbb{Z}$. The basic results about lattices can be summarized in the following Theorem (see [15]).

Theorem 2.2. Let V, W be finite-dimensional $U(\mathfrak{g})$-modules.

(a) If L is an additive subgroup of V which is invariant under the action of $U(\mathfrak{g})_\mathbb{Z}$, then $L = \bigoplus_{\mu \in P} L_\mu$, where $L_\mu = L \cap V_\mu$.
(b) There exists an admissible lattice for V.
(c) If L, M are admissible lattices for V, W, respectively, then $L \otimes_\mathbb{Z} M$ is an admissible lattice for $V \otimes W$.
(d) If V is an irreducible module and v is a highest-weight vector of weight λ, then $L = U(\mathfrak{n}^-)_\mathbb{Z} v$ is minimal in the set of admissible lattices for V satisfying $L_\lambda = \mathbb{Z}v$. □

2.2. **Classification of Irreducible Modules in Positive Characteristic.** From now on, $\mathbb{F}$ is an algebraically closed field of characteristic $p > 0$ and $\mathbb{F}_p$ denotes its prime field. In the present subsection we recall the methods used to classify the irreducible representations of $U(\mathfrak{g})_\mathbb{F}$ up to isomorphism. Although the classification is the same as in the case of $U(\mathfrak{g})$, the methods are quite different and will be used when we treat the case of $U(\tilde{\mathfrak{g}})_\mathbb{F}$.

Let V be a $U(\mathfrak{h})_\mathbb{F}$-module. A nonzero vector $v \in V$ is called a weight vector if there exists $\boldsymbol{z} = (z_{i,k}), z_{i,k} \in \mathbb{F}, i \in I, k \in \mathbb{Z}_+$, called the weight of v, such that $\binom{h_i}{p^k} v = z_{i,k} v$. Notice that (1.9) implies that $z_{i,k}$ must be in $\mathbb{F}_p$. We say that $\boldsymbol{z}$ is integral (resp. dominant integral) if $z_{i,k} = \binom{\mu(h_i)}{p^k}$ for some $\mu \in P$ (resp. $\mu \in P^+$). In that case we identify $\boldsymbol{z}$ with μ and say that v has weight μ. If V is a $U(\mathfrak{g})_\mathbb{F}$-module and v is weight vector such that $(x_\alpha^+)^{(k)} v = 0$ for all $\alpha \in R^+, k \in \mathbb{N}$, then v is said to be a highest-weight vector. If V is generated by a highest-weight vector, V is called a highest-weight module.

Since the $\binom{h_i}{p^k}$ commute, we can decompose any finite-dimensional representation V of $U(\mathfrak{g})_{\mathbb{F}}$ in a direct sum of generalized eigenspaces for the action of $U(\mathfrak{h})_{\mathbb{F}}$:

$$V = \bigoplus_{\boldsymbol{z}} V_{\boldsymbol{z}}.$$

We say that $\boldsymbol{z}$ is a weight of V if $V_{\boldsymbol{z}} \neq 0$ and, in that case, $V_{\boldsymbol{z}}$ is called a weight space of V. In the case $\boldsymbol{z}$ is integral we write V_μ instead of $V_{\boldsymbol{z}}$.

Given $\boldsymbol{z} = (z_{i,k})$ and $\mu \in P$ define $\boldsymbol{z} + \mu = \boldsymbol{y}$ by the equality $y_{i,k} v = \binom{h_i + \mu(h_i)}{p^k} v$, where v is some weight vector of weight $\boldsymbol{z}$. It follows from (1.10) that if v has weight $\boldsymbol{z}$ then $(x_\alpha^\pm)^{(k)} v$ is either zero or has weight $\boldsymbol{z} \pm k\alpha$. Hence, if v is a highest-weight vector for a highest-weight representation V, we have $\dim(V_{\boldsymbol{z}}) = 1$ and $V_{\boldsymbol{y}} \neq 0$ only if $\boldsymbol{y} \leq \boldsymbol{z}$, where $\boldsymbol{y} \leq \boldsymbol{z}$ iff $\boldsymbol{y} = \boldsymbol{z} - \eta$ for some $\eta \in Q^+$. Standard arguments then show:

Proposition 2.3. Every highest-weight module is indecomposable and has a unique maximal proper submodule, hence, also a unique irreducible quotient. □

Recall that any nonnegative integer m can be written uniquely as $m = \sum_{j \geq 0} m_j p^j$, where $0 \leq m_j < p$, so that $\binom{m}{p^r} = m_r \pmod p$ for all $r \geq 0$. We shall write $\overline{m}$ for the image of $m \in \mathbb{Z}$ in $\mathbb{F}_p$.

Theorem 2.4. If V is an irreducible finite-dimensional $U(\mathfrak{g})_{\mathbb{F}}$-module, then V is a highest-weight representation with dominant integral highest weight.

Proof. Since V is irreducible, the weight spaces $V_{\boldsymbol{z}}$ are in fact eigenspaces. Moreover, since V is finite-dimensional, it also follows that there exists a maximal weight $\boldsymbol{z}$ and, hence, V is a highest-weight module. It remains to prove that $\boldsymbol{z}$ is dominant integral. Let v be a highest-weight vector for V. As we have already observed above, $(x_\alpha^-)^{(k)} v$ is either zero or has weight $\boldsymbol{z} - k\alpha$. This implies that, for every $i \in I$, there exists $N_i \in \mathbb{Z}_+$ minimal such that $(x_i^-)^{(p^k)} v = 0$ for all $k \geq N_i$. Moreover, we conclude from (1.11) with $k = l \geq p^{N_i}$ that $\binom{h_i}{p^r} v = 0$ for all $r \geq N_i$. Now we easily see that $\boldsymbol{z}$ coincides with $\lambda \in P^+$ defined by $\lambda(h_i) = \sum_{j=0}^{N_i - 1} m_{i,j} p^j$ with $0 \leq m_{i,j} < p$ such that $\overline{m}_{i,r} = z_{i,r}$. □

In order to complete the classification of the irreducible $U(\mathfrak{g})_{\mathbb{F}}$-modules (in terms of highest weights), it remains to prove that for every $\lambda \in P^+$, there exists an irreducible $U(\mathfrak{g})_{\mathbb{F}}$-module having λ as highest weight. This is an easy consequence of the discussion preceding Theorem 2.8 below. We will denote by $V(\lambda)$ any representative of the isomorphism class of the irreducible finite-dimensional $U(\mathfrak{g})_{\mathbb{F}}$-modules of highest weight λ.

We end this subsection remarking that the following statement remains true in positive characteristic.

Proposition 2.5. Let V be a finite-dimensional $U(\mathfrak{g})_{\mathbb{F}}$-module. The generalized eigenspaces V_μ are in fact eigenspaces and $\dim V_\mu = \dim V_{w\mu}$ for all $w \in \mathcal{W}$. □

2.3. **Reduction Modulo p and Weyl Modules.** If V is a finite-dimensional $U(\mathfrak{g})$-module and L is an admissible lattice for V, let $L_{\mathbb{F}} = L \otimes_{\mathbb{Z}} \mathbb{F}$. Then $L_{\mathbb{F}}$ is a $U(\mathfrak{g})_{\mathbb{F}}$-module and $\dim_{\mathbb{F}}(L_{\mathbb{F}}) = \dim_{\mathbb{C}}(V)$. The following result was proved in [25].

Proposition 2.6. Let V be a finite-dimensional $U(\mathfrak{g})$-module and L an admissible lattice for V. The simple constituents of $L_{\mathbb{F}}$ depend only on V (not on L). □

Definition 2.7. Given $\lambda \in P^+$, let $W(\lambda)$ be the $U(\mathfrak{g})_{\mathbb{F}}$-module generated by a vector v satisfying

$$(x_\alpha^+)^{(p^k)} v = 0, \quad \binom{h_i}{p^k} v = \binom{\lambda(h_i)}{p^k} v, \quad (x_\alpha^-)^{(l)} v = 0, \quad \forall\, \alpha \in R^+, i \in I, k, l \in \mathbb{Z}_+, l > \lambda(h_\alpha). \tag{2.1}$$

The modules $W(\lambda)$ are called Weyl modules. It is not difficult to see that every finite-dimensional highest-weight $U(\mathfrak{g})_{\mathbb{F}}$-module is a quotient of some $W(\lambda)$. If L is a minimal admissible lattice for $V^0(\lambda)$, Theorem 2.2(d) implies that $L_{\mathbb{F}}$ is isomorphic to a quotient of $W(\lambda)$ and a comparison between the definition of $W(\lambda)$ and Theorem 2.1(c) hints that we have:

Theorem 2.8. Let $\lambda \in P^+$ and L be a minimal admissible lattice for $V^0(\lambda)$. Then $W(\lambda)$ is isomorphic to $L_{\mathbb{F}}$. □

We will need the following results in the proof of Theorem 3.15 below.

Proposition 2.9. The assignment $(x_\alpha^{\pm})^{(k)}(y \otimes 1) \mapsto ((\mathrm{ad}(x_\alpha^{\pm}))^{(k)}(y)) \otimes 1$, for all $y \in \mathfrak{g}_{\mathbb{Z}}$, extends to a structure of $U(\mathfrak{g})_{\mathbb{F}}$-module on $\mathfrak{g}_{\mathbb{F}}$. Furthermore, we have an isomorphism of $U(\mathfrak{g})_{\mathbb{F}}$-modules $W(\theta) \to \mathfrak{g}_{\mathbb{F}}$.

Proof. The first statement is [15, Proposition 25.5]. For the second, it suffices to show that $\mathfrak{g}_{\mathbb{Z}}$ is a minimal admissible lattice for the adjoint representation. This was proved in [27] (see also [2, Proposition A.2.6]). □

2.4. **Tensor Products.** We now recall some results on tensor products of $U(\mathfrak{g})_{\mathbb{F}}$-modules. We begin with Steinberg's Tensor Product Theorem [26] (our proof is essentially following the one given in [10]). Let $P_p^+ = \{\lambda \in P^+ : \lambda(h_i) < p, \forall\, i \in I\}$. We shall need the following lemma.

Lemma 2.10. Let $\lambda, \mu \in P_p^+ - \{0\}$. Then $V(\lambda)$ is irreducible as $\mathfrak{g}_{\mathbb{F}}$-module and $V(\lambda) \otimes V(\mu)$ is reducible as $U(\mathfrak{g})_{\mathbb{F}}$-module. □

Theorem 2.11. For $\lambda \in P^+$, let λ_k be the unique elements of P_p^+ such that $\lambda = \sum_{k=0}^m p^k \lambda_k$. Then $V(\lambda) \cong \otimes_k V(p^k\lambda_k)$. Moreover, if $\mu_j \in P_p^+ - \{0\}$ and $l_j \in \mathbb{Z}_+, j = 0, \cdots, n$, are such that $\otimes_{j=0}^n V(p^{l_j}\mu_j) \cong V(\lambda)$, then $m = n$ and (up to reordering) $\mu_k = \lambda_k$ and $l_k = k$ for all k.

Proof. First observe that for any $\mu \in P_p^+$ and $k \in \mathbb{Z}_+$ we have $V(p^k\mu) \cong V(\mu)^{\phi^k}$ (see section 1.4). Therefore, $(x_\alpha^{\pm})^{(p^l)}V(p^k\mu) = 0$ if $l < k$. Now let v_k be highest-weight vectors for $V(p^k\lambda_k)$, $V' = \otimes_{k=1}^m V(p^k\lambda_k)$, and $v = \sum_i w_i \otimes w_i' \in V(\lambda_0) \otimes V'$, where w_i' are linearly independent. Then $x_\alpha^+ v = \sum_i (x_\alpha^+ w_i) \otimes w_i'$. Since $V(\lambda_0)$ is irreducible as $\mathfrak{g}_{\mathbb{F}}$-module, it follows that $x_\alpha^+ v = 0$ only if $v = v_0 \otimes v'$ for some $v' \in V'$. Now let $V'' = \otimes_{k=2}^m V(p^k\lambda_k)$, and $v = v_0 \otimes (\sum_i w_i' \otimes w_i'') \in V(\lambda_0) \otimes V(p\lambda_1) \otimes V''$, where w_i'' are linearly independent. Then $(x_\alpha^+)^{(p)} v = v_0 \otimes (\sum_i ((x_\alpha^+)^{(p)} w_i') \otimes w_i'')$. Since $V(\lambda_1)$ is irreducible as $\mathfrak{g}_{\mathbb{F}}$-module, it follows that $(x_\alpha^+)^{(p)} v = 0$ only if $v = v_0 \otimes v_1 \otimes v''$ for some $v'' \in V''$. Continuing like this we see that $\otimes_{k=0}^m V(p^k\lambda_k)$ is irreducible. Since it is clearly a highest-weight module with highest-weight λ, the first statement is proved. On the other hand, we must have $\lambda_k = \sum_{j \in J_k} \mu_j$ where $J_k = \{j : l_j = k\}$. Therefore, if $\{\mu_j\}$ were not as stated, there would clearly exist $j \neq j'$ such that $l_j = l_{j'}$. The lemma above would then imply $V(p^{l_j}\mu_j) \otimes V(p^{l_{j'}}\mu_{j'})$ is reducible and, hence, also $\otimes_{j=0}^n V(p^{l_j}\mu_j)$. □

Remark. One of the reasons Theorem 2.11 is important comes from the fact that the modules $V(\lambda_k)$ are irreducible as modules for the subalgebra of $U(\mathfrak{g})_{\mathbb{F}}$ generated by $x_\alpha^{\pm}$ (since they are irreducible as $\mathfrak{g}_{\mathbb{F}}$-modules). Hence, we are left to study finitely many irreducible modules for a finite-dimensional algebra (which is isomorphic to a quotient of $U(\mathfrak{g}_{\mathbb{F}})$ called the restricted universal enveloping algebra of $\mathfrak{g}_{\mathbb{F}}$).

The next proposition will be used in the proof of Theorem 3.15 below.

Lemma 2.12. Let L, M be admissible lattices for some $U(\mathfrak{g})$-modules. Then $(L \otimes_{\mathbb{Z}} M)_{\mathbb{F}}$ is isomorphic to $L_{\mathbb{F}} \otimes M_{\mathbb{F}}$ as $U(\mathfrak{g})_{\mathbb{F}}$-module. □

Proposition 2.13. Let $\lambda, \mu, \nu \in P^+$ be such that $\mathrm{Hom}_{\mathfrak{g}}(V^0(\lambda) \otimes V^0(\mu), V^0(\nu)) \neq 0$. Then $V(\nu)$ is a simple constituent of $W(\lambda) \otimes W(\mu)$.

Proof. Since all finite-dimensional $\mathfrak{g}$-modules are completely reducible, we have that $V^0(\nu)$ is a direct summand of $V^0(\lambda) \otimes V^0(\mu)$. We are done using the previous Lemma, Theorem 2.8, and Proposition 2.6. □

3. Finite-Dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules

In this section we establish some basic results about the category of finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules such as the classification of the irreducible ones, characterization of the universal highest-weight modules and the block decomposition.

3.1. **ℓ-highest-weight Modules.** Let V be a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module. We say $v \in V$ is an ℓ-weight vector if it is an eigenvector for the action of $U(\tilde{\mathfrak{h}})_{\mathbb{F}}$, i.e., if there exist $z_{i,k}, \varpi_{i,r} \in \mathbb{F}$ such that

$$\binom{h_i}{p^k} v = z_{i,k} v, \qquad \Lambda_{i,r} v = \varpi_{i,r} v, \tag{3.1}$$

for all $i \in I$ and all $r, k \in \mathbb{Z}, k \geq 0$. In that case the corresponding functional $\boldsymbol{\varpi} \in (U(\tilde{\mathfrak{h}})_{\mathbb{F}})^*$ is called the ℓ-weight of v. If v is an ℓ-weight vector and $(x^+_{\alpha,r})^{(k)} v = 0$ for all $\alpha \in R^+$ and all $r, k \in \mathbb{Z}, k > 0$, we say v is an ℓ-highest-weight vector. If V is generated by an ℓ-highest-weight vector, we say V is an ℓ-highest-weight module.

Given a finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module V we know from section 2 that V can be written as the direct sum of its weight-spaces when regarded as $U(\mathfrak{g})_{\mathbb{F}}$-module:

$$V = \bigoplus_{\mu \in P} V_\mu.$$

Moreover, since $U(\tilde{\mathfrak{h}})_{\mathbb{F}}$ is a commutative algebra we can also write the following decomposition of V into direct sum of generalized eigenspaces for the action of $U(\tilde{\mathfrak{h}})_{\mathbb{F}}$:

$$V = \bigoplus_{\boldsymbol{\varpi} \in (U(\tilde{\mathfrak{h}})_{\mathbb{F}})^*} V_{\boldsymbol{\varpi}}.$$

The next proposition establishes a set of relations which are necessarily satisfied by all finite-dimensional ℓ-highest-weight modules. Given a polynomial $f(u) = \prod_j (1 - a_j u) \in \mathbb{F}[u]$, set $f^-(u) = \prod_j (1 - a_j^{-1} u)$.

Proposition 3.1. Let V be a finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module, $\lambda \in P^+$, and $v \in V_\lambda$ be such that

$$(x^+_{\alpha,s})^{(k)} v = 0 \qquad \text{and} \qquad \Lambda_{i,s} v = \omega_{i,s} v,$$

for all $\alpha \in R^+, i \in I, k, s \in \mathbb{Z}, k > 0$, and some $\omega_{i,s} \in \mathbb{F}$. Then

$$(x^-_{\alpha,s})^{(k)} v = \Lambda_{i,\pm r} v = 0 \quad \text{for all } k > \lambda(h_\alpha), r > \lambda(h_i).$$

Moreover, $\omega_{i,\pm\lambda(h_i)} \neq 0$ and, if we set

$$\boldsymbol{\omega}_i(u) = 1 + \sum_{r=1}^{\lambda(h_i)} \omega_{i,r} u^r,$$

then $\boldsymbol{\omega}_i^-(u) = 1 + \sum_{r=1}^{\lambda(h_i)} \omega_{i,-r} u^r$. In particular, $\lambda = \sum_{i \in I} \deg(\boldsymbol{\omega}_i)\, \omega_i$.

Proof. For each $r \in \mathbb{Z}, \alpha \in R^+$, the elements $(x^{\pm}_{\alpha,\pm r})^{(k)}, k \in \mathbb{Z}_+$, generate a subalgebra of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ isomorphic to $U(\mathfrak{sl}_2)_{\mathbb{F}}$. Hence, the equality $(x^-_{\alpha,r})^{(k)}v = 0$ for $k > \lambda(h_\alpha)$ follows from the fact that v generates a (finite-dimensional) highest-weight module for this subalgebra, which is then isomorphic to a quotient of the Weyl module $W(\lambda(h_\alpha))$.

Setting $s = 0, l = k = r$ in (1.12) we get $\Lambda_{i,\pm r}v = 0$ for $r > |\lambda(h_i)|$. Similarly, choosing $r = \lambda(h_i)$, we see that $\omega_{i,\pm\lambda(h_i)} \neq 0$. The statement about $\boldsymbol{\omega}^-_i(u)$ is proved exactly as in [9, Proposition 1.1(v)] after using the last statement of Lemma 1.4. ☐

The element $(\boldsymbol{\omega}_i)_{i\in I}$ given by the Proposition is called the Drinfeld polynomial of the ℓ-highest-weight module generated by v. We denote by $\mathcal{P}^+_{\mathbb{F}}$ the multiplicative monoid consisting of all $|I|$-tuples of the form $(\boldsymbol{\omega}_i)_{i\in I}$ where each $\boldsymbol{\omega}_i$ is a polynomial in $\mathbb{F}[u]$ with constant term 1. The corresponding multiplicative group will be denoted by $\mathcal{P}_{\mathbb{F}}$. We let $\mathrm{wt} : \mathcal{P}_{\mathbb{F}} \to P$ be the unique group homomorphism such that $\mathrm{wt}(\boldsymbol{\omega}) = \sum_{i\in I} \deg(\boldsymbol{\omega}_i)\omega_i$ for all $\boldsymbol{\omega} \in \mathcal{P}^+_{\mathbb{F}}$.

We also have an injective group homomorphism $\mathcal{P}_{\mathbb{F}} \to (U(\tilde{\mathfrak{h}})_{\mathbb{F}})^*$ given as follows. Any element $\boldsymbol{\varpi} \in \mathcal{P}_{\mathbb{F}}$ can be written uniquely as $\boldsymbol{\omega}\boldsymbol{\pi}^{-1}$, where $\boldsymbol{\omega}, \boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{F}}$ are such that $\boldsymbol{\omega}_i, \boldsymbol{\pi}_i$ are coprime for all $i \in I$. Then define $\overline{\boldsymbol{\varpi}} \in (U(\tilde{\mathfrak{h}})_{\mathbb{F}})^*$ by

$$\overline{\boldsymbol{\varpi}}\left(\binom{h_i}{p^k}\right) = \binom{\mathrm{wt}(\boldsymbol{\varpi})(h_i)}{p^k}, \qquad \overline{\boldsymbol{\varpi}}(\Lambda^{\pm}_i(u)) = \boldsymbol{\varpi}^{\pm}_i(u),$$

for all $k \in \mathbb{Z}_+$, and where $\boldsymbol{\varpi}^+_i = \boldsymbol{\varpi}_i$, $\boldsymbol{\varpi}^-_i = \boldsymbol{\omega}^-_i(\boldsymbol{\pi}^-_i)^{-1}$. The second equality is that of power series in u, obtained by expanding $(\boldsymbol{\pi}^{\pm}_i)^{-1}$ as a product of geometric power series. We shall identify $\mathcal{P}_{\mathbb{F}}$ with its image in $(U(\tilde{\mathfrak{h}})_{\mathbb{F}})^*$ and refer to its elements as the integral ℓ-weights. Similarly, the elements in $\mathcal{P}^+_{\mathbb{F}}$ will be referred to as the dominant integral ℓ-weights.

If V is a finite-dimensional irreducible $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module, proceeding as in the proof of Theorem 2.4, we see that V is generated by a vector v satisfying

$$(x^+_{\alpha,r})^{(p^k)}v = 0, \quad \binom{h_i}{p^k}v = \binom{\lambda(h_i)}{p^k}v, \quad \Lambda_{i,r}v = \omega_{i,r}v,$$

for all $\alpha \in R^+, i \in I, r, k \in \mathbb{Z}, k \geq 0$ and some $\lambda \in P^+$, $\omega_{i,r} \in \mathbb{F}$. In particular, we have:

Corollary 3.2. Every finite-dimensional irreducible $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module is an ℓ-highest-weight module whose ℓ-highest-weight lies in $\mathcal{P}^+_{\mathbb{F}}$. ☐

Let us now introduce an important class of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules called evaluation representations. For $a \in \mathbb{F}^\times$, consider the evaluation map $\mathrm{ev}_a : U(\tilde{\mathfrak{g}})_{\mathbb{F}} \to U(\mathfrak{g})_{\mathbb{F}}$ which is the $\mathbb{F}$-algebra homomorphism given by $(x^{\pm}_{\alpha,r})^{(k)} \mapsto a^{rk}(x^{\pm}_\alpha)^{(k)}$ (in particular, $\Lambda_{\alpha,r} \mapsto (-a)^r\binom{h_\alpha}{|r|}$). Given any $U(\mathfrak{g})_{\mathbb{F}}$-module V, let $V(a)$ be the pull-back of V by ev_a. $V(a)$ is called the evaluation representation with spectral parameter a corresponding to V. If V is a $U(\mathfrak{g})_{\mathbb{F}}$-highest-weight module of highest-weight $\lambda \in P^+$, it is easy to see that $V(a)$ is an ℓ-highest-weight module and its Drinfeld polynomial is the element $\boldsymbol{\omega}_{\lambda,a} \in \mathcal{P}^+_{\mathbb{F}}$ whose i-th entry is $(1-au)^{\lambda(h_i)}$, $i \in I$. We shall denote this evaluation representation by $V(\lambda, a)$ when $V = V(\lambda)$ and by $W(\lambda, a)$ when $V = W(\lambda)$.

3.2. **The Weyl Modules.**

Definition 3.3. Given $\boldsymbol{\omega} = (\boldsymbol{\omega}_i)_{i\in I} \in \mathcal{P}_{\mathbb{F}}^+$, let $W(\boldsymbol{\omega})$ be the $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module generated by a vector v satisfying

$$(x_{\alpha,r}^+)^{(p^k)}v = 0, \quad \binom{h_i}{p^k} v = \binom{\mathrm{wt}(\boldsymbol{\omega})(h_i)}{p^k} v, \quad \Lambda_{i,\pm s} v = (\boldsymbol{\omega}_i^{\pm}(u))_s v, \tag{3.2}$$

$$(x_{\alpha,r}^-)^{(l)}v = 0, \tag{3.3}$$

for all $\alpha \in R^+, i \in I, k, l, r, s \in \mathbb{Z}, s, k \geq 0, l > \mathrm{wt}(\boldsymbol{\omega})(h_\alpha)$. Here, as before, $(\boldsymbol{\omega}_i^{\pm}(u))_s$ means the coefficient of u^s. We call $W(\boldsymbol{\omega})$ the Weyl module with ℓ-highest-weight $\boldsymbol{\omega}$.

We have the following corollary of (1.10) and Proposition 3.1.

Corollary 3.4. $W(\boldsymbol{\omega})$ has a unique maximal submodule and, hence, a unique irreducible quotient. Moreover, every finite-dimensional ℓ-highest weight module of ℓ-highest-weight $\boldsymbol{\omega}$ is isomorphic to a quotient of $W(\boldsymbol{\omega})$. □

We shall denote by $V(\boldsymbol{\omega})$ the irreducible quotient of $W(\boldsymbol{\omega})$, $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^+$.

We now proceed to prove that $W(\boldsymbol{\omega})$ is finite-dimensional. This was proved in [9] for characteristic zero and for quantum groups (the later under the assumption that $\mathfrak{g}$ is simply laced). In particular, it follows that the isomorphism classes of irreducible finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules are in one-to-one correspondence with elements of $\mathcal{P}_{\mathbb{F}}^+$. The next proposition is proved as in the characteristic zero case.

Proposition 3.5. If $W(\boldsymbol{\omega})_\mu \neq 0$, then $W(\boldsymbol{\omega})_{w\mu} \neq 0$ for all $w \in \mathcal{W}$. In particular, $W(\boldsymbol{\omega})_\mu \neq 0$ only if $w_0\mathrm{wt}(\boldsymbol{\omega}) \leq \mu \leq \mathrm{wt}(\boldsymbol{\omega})$. □

Theorem 3.6. $W(\boldsymbol{\omega})$ is finite-dimensional.

Proof. Let $\lambda = \mathrm{wt}(\boldsymbol{\omega})$ and v be an ℓ-highest-weight vector of $W(\boldsymbol{\omega})$. It suffices to prove that $W(\boldsymbol{\omega})$ is spanned by the elements

$$(x_{\beta_1,s_1}^-)^{(k_1)} \cdots (x_{\beta_m,s_m}^-)^{(k_m)}v,$$

with $m, s_j, k_j \in \mathbb{Z}_+, \beta_j \in R^+$ such that $s_j < \lambda(h_{\beta_j})$ and $\sum_j k_j\beta_j \leq \lambda - w_0\lambda$. The last condition is immediate from the previous proposition. Moreover, the elements $(x_{\beta_1,s_1}^-)^{(k_1)} \cdots (x_{\beta_m,s_m}^-)^{(k_m)}v$ with no restriction on s_j clearly span $W(\boldsymbol{\omega})$.

Let $\mathcal{R} = R^+ \times \mathbb{Z} \times \mathbb{Z}_+$ and Ξ be the set of functions $\xi : \mathbb{N} \to \mathcal{R}$ given by $j \mapsto \xi_j = (\beta_j, s_j, k_j)$, such that $k_j = 0$ for all j sufficiently large. Let also Ξ' be the subset of Ξ consisting of the elements ξ such that $0 \leq s_j < \lambda(h_{\beta_j})$. Given $\xi \in \Xi$ we associate an element $v_\xi \in W(\boldsymbol{\omega})$ as above in the obvious way, i.e., if $k_j = 0$ for $j > m$, then $v_\xi = (x_{\beta_1,s_1}^-)^{(k_1)} \cdots (x_{\beta_m,s_m}^-)^{(k_m)}v$. Define the degree of ξ to be $d(\xi) = \sum_j k_j$ and the maximal exponent of ξ to be $e(\xi) = \max\{k_j\}$. Clearly $e(\xi) \leq d(\xi)$ and $d(\xi) \neq 0$ implies $e(\xi) \neq 0$. Since there is nothing to be proved when $d(\xi) = 0$ we assume from now on that $d(\xi) > 0$. Thus, let $\Xi_{d,e}$ be the subset of Ξ consisting of those ξ satisfying $d(\xi) = d$ and $e(\xi) = e$, and set $\Xi_d = \bigcup\limits_{1\leq e\leq d} \Xi_{d,e}$.

We prove by induction on d and sub-induction on e that if $\xi \in \Xi_{d,e}$ is such that there exists j with either $s_j < 0$ or $s_j \geq \lambda(h_{\beta_j})$, then v_ξ is in the span of vectors associated to elements in Ξ'. More precisely, given $0 < e \leq d \in \mathbb{N}$, we assume, by induction hypothesis, that this statement is true for every ξ which belongs either to $\Xi_{d,e'}$ with $e' < e$ or to $\Xi_{d'}$ with $d' < d$.

Observe that (1.12) implies

$$\left((X^-_{\beta;r,+}(u))^{(k-l)}\Lambda^+_\beta(u)\right)_k v = 0 \qquad \forall\, \beta\in R^+, k,l,r\in\mathbb{Z}, k>\lambda(h_\beta), 1\le l\le k. \tag{3.4}$$

We split the proof in 2 cases according to whether $e=d$ or $e<d$.

When $e=d$, it follows that $v_\xi=(x^-_{\beta,s})^{(e)}v$ for some $\beta\in R^+$ and $s\in\mathbb{Z}$. Suppose first that $e=1$ and let $l=\lambda(h_\beta)$ and $k=l+1$ in (3.4) to get

$$(x^-_{\beta,r+1}\Lambda_{\beta,l}+x^-_{\beta,r+2}\Lambda_{\beta,l-1}+\cdots+x^-_{\beta,r+l+1})v=0. \tag{3.5}$$

We consider the cases $s\ge l$ and $s<0$ separately and prove the statement by a further induction on s and $|s|$, respectively. If $s\ge l$ this is easily done by setting $r=s-l-1$ in (3.5). Similarly, after observing that $\Lambda_{\beta,l}v\ne0$, the case $s<0$ is dealt with by setting $r=s-1$ in (3.5). If $e>1$ let $l=e\lambda(h_\beta)$ and $k=l+e$ in (3.4) to obtain

$$\sum_{n=0}^{\lambda(h_\beta)}(x^-_{\beta,r+1+n})^{(e)}\Lambda_{\beta,l-en}v+\text{ other terms}=0, \tag{3.6}$$

where the other terms belong to the span of elements $v_{\xi'}$ with $\xi'\in\Xi_{e,e'}$ for $e'<e$. As before we argue by induction on s and $|s|$ by setting $r=s-1-\lambda(h_\beta)$ and $r=s-1$ in (3.6), respectively.

For the case $e<d$ we can assume, by inductions hypothesis, that $0\le s_j<\lambda(h_{\beta_j})$ for $j>1$. An easy application of Lemma 1.3 completes the argument in this case. □

We end the subsection recording an interesting fact about the Jordan-Holder series of Weyl modules.

Proposition 3.7. The simple constituents of $W(\boldsymbol{\omega}_{\lambda,a})$ (counted with multiplicities) are the evaluation representations at a of the simple constituents of $W(\boldsymbol{\omega}_{\lambda,a})$ when regarded as $U(\mathfrak{g})_\mathbb{F}$-modules.

Proof. This can be done as in the characteristic zero case [6, Proposition 3.3]. □

3.3. **Tensor Products.** We elaborate further on the classification of the irreducible finite-dimensional $U(\tilde{\mathfrak{g}})_\mathbb{F}$-modules using tensor products (cf. [3, 8, 9]). If $\lambda\in P^+_p$, it is easy to see that $V(p^k\lambda,a)$ is isomorphic to $V(\lambda,a^{p^k})^{\tilde\phi^k}$. Moreover, for any $\lambda\in P^+$, Theorem 2.11 implies

$$V(\lambda,a)\cong\otimes_k V(p^k\lambda_k,a),\quad\text{ where }\quad \lambda_k\in P^+_p \text{ are such that } \lambda=\sum_k p^k\lambda_k. \tag{3.7}$$

Theorem 3.8. If $\mu_j\in P^+_p-\{0\}, a_j\in\mathbb{F}^\times$, and $l_j\in\mathbb{Z}_+, j=0,\cdots,n$, then $V=\otimes_j V(p^{l_j}\mu_j,a_j)$ is irreducible if and only if $a_j\ne a_{j'}$ whenever $l_j=l_{j'}$.

Proof. The proof is a combination of the arguments used in Theorem 2.11 and [8, Theorem 1.7]. First consider the case $V=V(p^l\lambda,a)\otimes V(p^l\mu,b)$, where $\lambda,\mu\in P^+_p$, and let $v=\sum_j v_j\otimes w_j\in V$ be such that w_j are linearly independent. Using (1.5) we get

$$(x^+_{\alpha,r})^{(k)}v=\sum_j\sum_{l+m=k}a^{rl}b^{rm}\left((x^+_\alpha)^{(l)}v_j\right)\otimes\left((x^+_\alpha)^{(m)}w_j\right).$$

Hence, if $a=b$, this implies $(x^+_{\alpha,r})^{(k)}v=a^{rk}(x^+_\alpha)^{(k)}v$, and it follows that, if v generates a $U(\mathfrak{g})_\mathbb{F}$-submodule of V, it also generates a $U(\tilde{\mathfrak{g}})_\mathbb{F}$-submodule of V. This proves the "only if" part.

Conversely, for each $l\in\mathbb{Z}_+$, let $J_l=\{j:l_j=l\}$ and $V_l=\otimes_{j\in J_l}V(p^l\mu_j,a_j)$, so that $V\cong\otimes_{l:J_l\ne\emptyset}V_l$. Now observe that $V_l\cong\left(\otimes_{j\in J_l}V(\mu_j,a_j^{p^l})\right)^{\tilde\phi^l}$. The same arguments used in [8, Theorem 1.7] show that $\otimes_{j\in J_l}V(\mu_j,a_j^{p^l})$ is irreducible as $\tilde{\mathfrak{g}}_\mathbb{F}$-module and, therefore, V_l is irreducible as $U(\tilde{\mathfrak{g}})_\mathbb{F}$-module.

Now let $\{l_1, \cdots, l_m\} = \{l : J_l \neq \emptyset\}$, $\mathcal{I} = \mathcal{I}_{l_1} \times \cdots \times \mathcal{I}_{l_m}$, and suppose $l_1 < l_2 < \cdots < l_m$. Set $V' = \otimes_{j=2}^m V_{l_j}$ and let $v = \sum_i w_i \otimes w_i' \in V_{l_1} \otimes V'$ such that w_i' are linearly independent. Then $(x_{\alpha,r}^+)^{(p^{l_1})} v = \sum_i \left((x_{\alpha,r}^+)^{(p^{l_1})} w_i\right) \otimes w_i'$ and we see that v is an ℓ-highest-weight vector only if $v = v_1 \otimes v'$ where v_1 is an ℓ-highest-weight vector for V_{l_1}, since we already know that V_{l_j} is irreducible. Proceeding inductively, similarly to the proof of Theorem 2.11, we conclude that v must be a multiple of the tensor product of the ℓ-highest-weight vectors. □

It is easy to see that every element $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^+$ can be uniquely decomposed as $\boldsymbol{\omega} = \prod_j \boldsymbol{\omega}_{\lambda_j, a_j}$ with $a_i \neq a_j$ (see last paragraph of section 3.1 for the definition of $\boldsymbol{\omega}_{\lambda,a}$).

Corollary 3.9. If $\boldsymbol{\omega} = \prod \boldsymbol{\omega}_{\lambda_j, a_j} \in \mathcal{P}_{\mathbb{F}}^+$ with $a_i \neq a_j, i \neq j$, and $\lambda_j = \sum_k p^k \lambda_{j,k}$ with $\lambda_{j,k} \in P_p^+$, then $V(\boldsymbol{\omega}) \cong \otimes_{j,k} V(p^k \lambda_{j,k}, a_j) \cong \otimes_j V(\lambda_j, a_j)$. □

Corollary 3.10. If V is a finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module, then $V_{\boldsymbol{\varpi}} \neq 0$ only if $\boldsymbol{\varpi} \in \mathcal{P}_{\mathbb{F}}$ and $V_\mu = \bigoplus\limits_{\boldsymbol{\varpi}\,:\,\mathrm{wt}(\boldsymbol{\varpi}) = \mu} V_{\boldsymbol{\varpi}}$.

Proof. It suffices to prove the corollary when $V = V(\lambda, a)$ with $\lambda \in P_p^+$. But this case is dealt with similarly to the characteristic zero situation. □

Contrary to what happens in characteristic zero, we have more than one way to decompose $V(\boldsymbol{\omega})$ into a tensor product of evaluation representations. The maximal such decomposition is given by $V(\boldsymbol{\omega}) \cong \otimes_{j,k} V(p^k \lambda_{j,k}, a_j)$ and the minimal by $V(\boldsymbol{\omega}) \cong \otimes_j V(\lambda_j, a_j)$. We conjecture that the Weyl modules also decompose as tensor products according to the minimal decomposition of its irreducible quotient. The analogous statements corresponding to the other decompositions are false.

Conjecture 3.11. If $\boldsymbol{\omega} = \prod \boldsymbol{\omega}_{\lambda_j, a_j} \in \mathcal{P}_{\mathbb{F}}^+$ with $a_i \neq a_j, i \neq j$, then $W(\boldsymbol{\omega}) \cong \otimes W(\boldsymbol{\omega}_{\lambda_j, a_j})$.

The characteristic zero counterpart of this conjecture was proved in [9, Section 3]. So far we did not manage to adapt or compliment those techniques. Another approach for proving this result is discussed in Section 4 below. In particular, the above conjecture follows from Conjecture 4.8.

In the next subsection we will assume Conjecture 3.11 in order to obtain the block decomposition of the category of finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules. We remark that what is really needed in section 3.4 is the next corollary of the above conjecture.

For each $a \in \mathbb{F}^\times$ and $i \in I$, define the ℓ-fundamental weight $\boldsymbol{\omega}_{i,a}$ to be the element of $\mathcal{P}_{\mathbb{F}}^+$ whose j-th entry is $1 - \delta_{ij} a u$, and the ℓ-simple root $\alpha_{i,a}(u) = \prod_j (\boldsymbol{\omega}_{j,a}(u))^{\alpha_i(h_j)} \in \mathcal{P}_{\mathbb{F}}$. Let $\mathcal{Q}_{\mathbb{F}}$ (resp. $\mathcal{Q}_{\mathbb{F}}^+$) be the subgroup (resp. submonoid) of $\mathcal{P}_{\mathbb{F}}$ generated by all $\alpha_{i,a}(u)$. We call $\mathcal{Q}_{\mathbb{F}}$ the ℓ-root-lattice.

Corollary 3.12. If V is a finite-dimensional ℓ-highest-weight $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module with ℓ-highest-weight $\boldsymbol{\omega}$, then $V_{\boldsymbol{\varpi}} \neq 0$ only if $\boldsymbol{\varpi} \in \boldsymbol{\omega}(\mathcal{Q}_{\mathbb{F}}^+)^{-1}$.

Proof. First, as in the proof of Corollary 3.10, one sees that the present statement is true for the evaluation representations $V(\lambda, a)$. Then we are done using Conjecture 3.11 and Proposition 3.7. □

3.4. **Block Decomposition.**

Definition 3.13. A spectral character is a function $\chi : \mathbb{F}^\times \to P/Q$ with finite support. Equipping the space of all spectral characters $\Xi_{\mathbb{F}}$ with the usual abelian group structure, one sees that the assignment $\boldsymbol{\omega}_{i,a} \mapsto \chi_{i,a}$, where $\chi_{i,a}(b) = \delta_{a,b} \omega_i$, determines a group homomorphism $\mathcal{P}_{\mathbb{F}} \to \Xi_{\mathbb{F}}, \boldsymbol{\varpi} \mapsto \chi_{\boldsymbol{\varpi}}$, with kernel $\mathcal{Q}_{\mathbb{F}}$. We say that a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module V has spectral character χ if $\chi_{\boldsymbol{\varpi}} = \chi$ whenever $V_{\boldsymbol{\varpi}} \neq 0$. Let $\tilde{\mathcal{C}}_\chi$ be the category of all finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules with spectral character χ.

The next proposition is immediate from Corollary 3.12 and the proof of Theorem 3.8 (cf. [6, Corollary 4.3 and Lemma 5.1]).

Proposition 3.14.

(a) $W(\boldsymbol{\omega}) \in \tilde{\mathcal{C}}_{\chi_{\boldsymbol{\omega}}}$ for all $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^+$.
(b) $\widetilde{\mathcal{C}}_{\chi_1} \otimes \widetilde{\mathcal{C}}_{\chi_2} \subseteq \widetilde{\mathcal{C}}_{\chi_1+\chi_2}$ for all $\chi_1, \chi_2 \in \Xi_{\mathbb{F}}$. □

Let $\tilde{\mathcal{C}}$ be the category of all finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules. In the rest of the section we prove that the block decomposition of $\tilde{\mathcal{C}}$ is described just as in the characteristic zero case [6] and quantum group case [7, 11]. Namely:

Theorem 3.15. The categories $\tilde{\mathcal{C}}_\chi, \chi \in \Xi_{\mathbb{F}}$, are the blocks of $\tilde{\mathcal{C}}$.

The same arguments used in [6, section 5] show that every indecomposable object from $\tilde{\mathcal{C}}$ belongs to some $\tilde{\mathcal{C}}_\chi$, proving that we have the decomposition

$$\tilde{\mathcal{C}} = \bigoplus_{\chi \in \Xi_{\mathbb{F}}} \tilde{\mathcal{C}}_\chi.$$

It remains to see that $\tilde{\mathcal{C}}_\chi$ are indecomposable abelian subcategories. The idea for proving this is exactly the same as in [6, section 4]. However, one of the key ingredients ([6, Proposition 3.4]) does not make sense (a priori) in the present context. In order to fix it, we make use of Propositions 2.9, 2.13, and the following version of [6, Proposition 3.4].

Proposition 3.16. Let $\lambda, \mu \in P^+$ be such that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V^0(\lambda), V^0(\mu)) \neq 0$. Then there exists a surjective $U(\mathfrak{g})_{\mathbb{F}}$-module map $\psi : W(\theta) \otimes W(\lambda) \to M$ for some indecomposable $U(\mathfrak{g})_{\mathbb{F}}$-module M having $V(\mu)$ as simple constituent. Moreover, for each $a \in \mathbb{F}^\times$ and such ψ, the following formula defines a structure of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module on $W(\lambda) \oplus M$:

$$(x_{\alpha,r}^{\pm})^{(k)}(v, w) = \left(a^{kr}(x_\alpha^{\pm})^{(k)}v, \quad a^{kr}(x_\alpha^{\pm})^{(k)}w + ra^{kr-1}\psi(\iota(x_\alpha^{\pm}) \otimes ((x_\alpha^{\pm})^{(k-1)}v))\right).$$

Here $\iota : \mathfrak{g}_{\mathbb{F}} \to W(\theta)$ is an isomorphism given by Proposition 2.9. In particular, if we denote this module by $W(\lambda, \mu, a)$, we have a non-split short exact sequence of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules:

$$0 \to M(a) \to W(\lambda, \mu, a) \to W(\lambda, a) \to 0.$$

Proof. The first statement is a rephrasing of Proposition 2.13, where M is set to be an indecomposable direct summand of $W(\theta) \otimes W(\lambda)$ having $V(\mu)$ as simple constituent. For the second statement, recall that the formula

$$(x \otimes t^r)(v^0, w^0) = (a^r x v^0, a^r x w^0 + ra^{r-1}\varphi(x \otimes v^0)),$$

where $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V^0(\lambda), V^0(\mu)) - \{0\}$ and $x \in \mathfrak{g}$, defines a $U(\tilde{\mathfrak{g}})$-module structure on $V^0(\lambda) \oplus V^0(\mu)$. Then, one easily computes that

$$(x_{\alpha,r}^{\pm})^{(k)}(v^0, w^0) = \left(a^{kr}(x_\alpha^{\pm})^{(k)}v^0, \quad a^{kr}(x_\alpha^{\pm})^{(k)}w^0 + ra^{kr-1}\varphi(x_\alpha^{\pm} \otimes ((x_\alpha^{\pm})^{(k-1)}v^0))\right).$$

Hence, the second statement follows from the definition of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ together with Propositions 2.13 and 2.9. The remaining statement is proved exactly as in [6, Proposition 3.4]. □

The remainder of the proof of Theorem 3.15 is done as in the characteristic zero case [6, section 4].

Remark. We give an informal reasoning to justify why it should be expected that the block decomposition of $\tilde{\mathcal{C}}_{\mathbb{F}}$ is described similarly to that of $\tilde{\mathcal{C}}_{\mathbb{C}}$, contrary to what happens with the block decompositions of $\mathcal{C}_{\mathbb{F}}$ and $\mathcal{C}_{\mathbb{C}}$ (the categories of finite-dimensional representations for $U(\mathfrak{g})_{\mathbb{F}}$ and $U(\mathfrak{g})$, respectively). While the blocks of $\mathcal{C}_{\mathbb{C}}$ are as small as possible ($\mathcal{C}_{\mathbb{C}}$ is a semisimple category), the blocks of $\tilde{\mathcal{C}}_{\mathbb{C}}$ are as large as one can expect (for instance, when P/Q is trivial, $\tilde{\mathcal{C}}_{\mathbb{C}}$ is itself an indecomposable abelian category). Hence, while the blocks of $\mathcal{C}_{\mathbb{F}}$ have space to become larger (and they indeed become so, but still not as large as possible [17, Chapter II.7]), that is not the case for $\tilde{\mathcal{C}}_{\mathbb{F}}$.

4. Reduction Modulo p

In this section we start the theory of reduction modulo p for $U(\tilde{\mathfrak{g}})$-modules. In the case of $U(\mathfrak{g})$ it suffices to prove the existence of admissible lattices for the irreducible modules because the underlying abelian category is semisimple. The category $\widetilde{\mathcal{C}}_{\mathbb{C}}$ is not semisimple so, even if it is possible to obtain a nice lattice theory for all irreducible modules, one could not guarantee that all of the objects in $\widetilde{\mathcal{C}}_{\mathbb{C}}$ would contain such a lattice. In fact, even for irreducible modules the story is more subtle than the one in the $U(\mathfrak{g})$-case since, for instance, one cannot expect that the evaluation representations $V^0(\lambda, 3)$ would be reducible modulo 3. Still, we will prove that all of the ℓ-highest-weight modules whose roots of their Drinfeld polynomials are "good" with respect to p can indeed be reduced modulo p. In particular, it will follow that every irreducible $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module can be constructed as a quotient of a module obtained by a reduction modulo p process.

We consider two kinds of lattice theories. The first one is a natural generalization of the one reviewed in Theorem 2.2 for $U(\mathfrak{g})$ and, simultaneously, of the one used in [9, Definition 4.2] for quantum loop algebras. Namely, in subsection 4.1, we consider modules which contain finitely generated $\mathbb{Z}$-lattices which are invariant under the action of $U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$. The modules $V^0(\lambda, a)$ with $a \in \mathbb{Z}, a \neq \pm 1$, are easily seen not to contain such a lattice. Then, in subsection 4.2, we consider lattices over rings other than $\mathbb{Z}$, namely, over discrete valuation rings. We think these lattices are more suitable for studying reduction modulo p in the present context. In particular, we give an example showing that this approach may provide a proof for Conjecture 3.11, as well.

Let us fix some general notation to be used below. If $\mathbb{A}$ is any commutative ring with 1, let $\mathcal{P}_{\mathbb{A}}, \mathcal{P}_{\mathbb{A}}^+$ be defined in the obvious way (cf. definition of $\mathcal{P}_{\mathbb{F}}$). Define also $\mathcal{P}_{\mathbb{A}}^{++}$ as the subset of $\mathcal{P}_{\mathbb{A}}^+$ consisting of elements $\boldsymbol{\omega}$ such that the coefficient of the leading term of $\boldsymbol{\omega}_i$ belongs to $\mathbb{A}^\times$ for all $i \in I$. Recall that $\mathbb{A}$ is a discrete valuation ring if it is a local principal ideal domain which is not a field and that its residue field is the quotient of $\mathbb{A}$ by its unique maximal ideal. If $\mathbb{A}$ is a discrete valuation ring with residue field $\mathbb{K}$, $a \in \mathbb{A}$, and $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{A}}^+$, we let $\bar{a}$ and $\overline{\boldsymbol{\omega}}$ be the images of a in $\mathbb{K}$ and of $\boldsymbol{\omega}$ in $\mathcal{P}_{\mathbb{K}}^+$, respectively. As before, $\mathbb{F}$ denotes an algebraically closed field of characteristic $p > 0$. We shall also denote by $\overline{\boldsymbol{\omega}}$ the image of $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{Z}}^+$ in $\mathcal{P}_{\mathbb{F}}^+$. Given $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{C}}^+$, we denote by $W^0(\boldsymbol{\omega})$ the corresponding $U(\tilde{\mathfrak{g}})$-Weyl module and by $V^0(\boldsymbol{\omega})$ its irreducible quotient.

4.1. $\mathbb{Z}$-Lattices.

We will make use of the automorphisms $\psi_a, a \in \mathbb{F}^\times$, of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ determined by the assignment $(x_{\alpha,r}^{\pm})^{(k)} \mapsto a^{rk}(x_{\alpha,r}^{\pm})^{(k)}$ for all $\alpha \in R^+, k, r \in \mathbb{Z}, k > 0$.

Definition 4.1. If V is a finite-dimensional $\mathbb{C}$–vector space we say that a finitely generated free $\mathbb{Z}$–submodule L of V is an ample lattice for V if L spans V over $\mathbb{C}$. If the rank of L is equal to the dimension of V, then L is a lattice for V. If V is a $U(\tilde{\mathfrak{g}})$-module, we say that an (ample) lattice for V is admissible if L is invariant under the action of $U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$.

If L is an ample admissible lattice for a $U(\tilde{\mathfrak{g}})$–module V, we set $L_{\mathbb{F}} = L \otimes_{\mathbb{Z}} \mathbb{F}$. Thus, $L_{\mathbb{F}}$ is a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module and $\operatorname{rank}(L) = \dim_{\mathbb{F}}(L_k) \geq \dim_{\mathbb{C}}(V)$. It is trivial to see that the modules $V^0(\lambda, a)$ contain a finitely generated $\mathbb{Z}$-submodule invariant under the action of $U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$ only if $a = \pm 1$.

Proposition 4.2. Let V be a finite–dimensional ℓ-highest weight $U(\tilde{\mathfrak{g}})$-module with ℓ-highest-weight $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{Z}}^{++}$ and ℓ-highest-weight vector v. Then $L = U(\tilde{\mathfrak{g}})_{\mathbb{Z}} v$ is an ample admissible lattice for V and $L_{\mathbb{F}}$ is isomorphic to a quotient of $W(\overline{\boldsymbol{\omega}})$. Moreover, if $V = W^0(\boldsymbol{\omega})$, then L is a lattice.

Proof. It is easy to see from Lemma 1.4 that $U(\tilde{\mathfrak{h}})_{\mathbb{Z}}v = \mathbb{Z}v$ and, therefore, $L = U(\tilde{\mathfrak{n}}^-)_{\mathbb{Z}}v$. Also, L is quite clearly a torsion free $\mathbb{Z}$–submodule of V which is invariant under the action of $U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$. The proof of Theorem 3.6 together with the hypothesis $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{Z}}^{++}$ shows that L is a finitely generated $\mathbb{Z}$-module which spans V over $\mathbb{C}$ (the hypothesis $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{Z}}^{++}$ is used replacing the remark $\Lambda_{\beta,l}v \neq 0$ by $\Lambda_{\beta,l}v = av$ with $a \in \mathbb{Z}^\times$). This completes the proof that L is an ample admissible lattice. Since the image of v in $L_{\mathbb{F}}$ is clearly an ℓ-highest-weight vector with ℓ-highest-weight $\overline{\boldsymbol{\varpi}}$, the second statement follows immediately. The last statement is clear since $L \otimes_{\mathbb{Z}} \mathbb{C}$ is an ℓ-highest-weight $U(\tilde{\mathfrak{g}})$-module of ℓ-highest weight $\boldsymbol{\omega}$ and dimension at least that of $W^0(\boldsymbol{\omega})$, thus $L \otimes_{\mathbb{Z}} \mathbb{C} \cong W^0(\boldsymbol{\omega})$. □

It is easy to see that the only irreducible $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules which can be obtained as a quotient of some $L_{\mathbb{F}}$ where L is as in the proposition are precisely those whose Drinfeld polynomials $\boldsymbol{\omega}$ lie in $\mathcal{P}_{\mathbb{F}_p}^+$ and the coefficient of the leading term of $\boldsymbol{\omega}_i$ is ± 1 for all $i \in I$. However, all of the ℓ-highest-weight $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules V whose Drinfeld polynomial is of the form $\boldsymbol{\omega}_{\lambda,1}$ can be obtained in this way and quite clearly the pull-back of such V by ψ_a is an ℓ-highest-weight module with Drinfeld polynomial $\boldsymbol{\omega}_{\lambda,a}$. Hence, up to twisting by ψ_a we obtain all of the evaluation modules $V(\lambda, a)$. The other irreducible modules are then obtained using tensor products.

4.2. **Lattices Over Discrete Valuation Rings.** We now proceed with the construction of lattices over discrete valuation rings. Given a $\mathbb{C}$–vector space V, a subring $\mathbb{A}$ of $\mathbb{C}$ and a subset L of V, the $\mathbb{A}$–span of L will be denoted by $\mathbb{A}L$.

We begin by giving some motivation. Let $\mathbb{A} = \mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at $\mathbb{Z} - p\mathbb{Z}$ and $U(\tilde{\mathfrak{g}})_{\mathbb{A}} = \mathbb{A}U(\tilde{\mathfrak{g}})$. Then $\mathbb{A}$ is a discrete valuation ring with residue field $\mathbb{F}_p$ and $U(\tilde{\mathfrak{g}})_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbb{F} \cong U(\tilde{\mathfrak{g}})_{\mathbb{F}}$. Suppose $a \in \mathbb{A}^\times$, let v be an ℓ-highest-weight vector for $V = V^0(\lambda, a)$, and set $L = U(\tilde{\mathfrak{g}})_{\mathbb{A}}v$. It is easy to see that $L = U(\tilde{\mathfrak{n}}^-)_{\mathbb{A}}v = \mathbb{A}\,(U(\tilde{\mathfrak{n}}^-)_{\mathbb{Z}}v) = \mathbb{A}L'$ where $L' = U(\mathfrak{n}^-)_{\mathbb{Z}}v$. Since L' is the $\mathbb{Z}$-span of a basis for V by Theorem 2.2, it follows that L is the $\mathbb{A}$-span of the same basis. Thus, setting $L_{\mathbb{F}} = L \otimes_{\mathbb{A}} \mathbb{F}$, we obtain a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module isomorphic to $W(\lambda, \bar{a})$, where $\bar{a}$ is the image of a in $\mathbb{F}_p$. This way we are able to obtain all evaluation representations of the form $V(\lambda, b), b \in \mathbb{F}_p$, as a quotient of the reduction modulo p of the irreducible $U(\tilde{\mathfrak{g}})$-modules $V^0(\lambda, a)$, where a is such that $\bar{a} = b$. For $b \notin \mathbb{F}_p$ we will have to use more general discrete valuation rings given by the following result from number theory explained to us by A. Engler and P. Brumatti.

Theorem 4.3. For every $b_1, \cdots, b_m \in \mathbb{F}^\times$, there exist a discrete valuation ring $\mathbb{A}$ which is a subring of $\mathbb{C}$ and $a_1, \cdots, a_m \in \mathbb{A}^\times$ such that the residue field of $\mathbb{A}$ is isomorphic to a subfield of $\mathbb{F}$ and $\bar{a}_j = b_j$ for all $j = 1, \cdots, m$. □

Definition 4.4. Let $\mathbb{A} \subseteq \mathbb{C}$ be a discrete valuation ring with residue field isomorphic to a subfield of $\mathbb{F}$. If V is a finite-dimensional $\mathbb{C}$–vector space we say that a finitely generated free $\mathbb{A}$–submodule L of V is an ample $\mathbb{A}$–lattice for V if L spans V over $\mathbb{C}$. If the rank of L is equal to the dimension of V, then L is an $\mathbb{A}$–lattice for V. If V is a $U(\tilde{\mathfrak{g}})$-module, we say that an (ample) lattice for V is admissible if L is invariant under the action of $U(\tilde{\mathfrak{g}})_{\mathbb{A}} = \mathbb{A}U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$.

If L is an ample $\mathbb{A}$–lattice for a $U(\tilde{\mathfrak{g}})$–module V as in the definition we set $L_{\mathbb{F}} = L \otimes_{\mathbb{A}} \mathbb{F}$. We have $U(\tilde{\mathfrak{g}})_{\mathbb{F}} \cong U(\tilde{\mathfrak{g}})_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbb{F}$ and $L_{\mathbb{F}}$ is a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module. From now on a discrete valuation ring will mean one that is a subring of $\mathbb{C}$ and whose residue field is isomorphic to a subfield of $\mathbb{F}$. The next lemma is immediate.

Lemma 4.5. Let $\mathbb{A}$ be a discrete valuation ring, V and W finite-dimensional $U(\tilde{\mathfrak{g}})$-modules, L and M (ample) admissible lattices for V and W, respectively. Then $L \otimes_{\mathbb{A}} M$ is an (ample) admissible lattice for $V \otimes W$ and $(L \otimes_{\mathbb{A}} M)_{\mathbb{F}} \cong L_{\mathbb{F}} \otimes M_{\mathbb{F}}$. □

Theorem 4.6. Let $\mathbb{A}$ be a discrete valuation ring, V a finite-dimensional $U(\tilde{\mathfrak{g}})$-ℓ-highest-weight module with Drinfeld polynomial $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{A}}^{++}$ and ℓ-highest-weight vector v, and $L = U(\tilde{\mathfrak{g}})_{\mathbb{A}}v$. We have:

(a) L is an ample admissible $\mathbb{A}$-lattice for V and $L_{\mathbb{F}}$ is isomorphic to a quotient of $W(\overline{\boldsymbol{\omega}})$.
(b) If either $V = V^0(\boldsymbol{\omega})$ or $V = W^0(\boldsymbol{\omega})$, then L is a lattice.

Proof. We first prove (b) in the case $V = V^0(\boldsymbol{\omega})$. When V is an evaluation representation we proceed similarly to the discussion at the beginning of this subsection. In general, writing $\boldsymbol{\omega} = \prod_{j=1}^m \boldsymbol{\omega}_{\lambda_j,a_j}$ with $a_j \neq a_{j'}$, we have $V = V^0(\lambda_1, a_1) \otimes \cdots \otimes V^0(\lambda_m, a_m)$. Let v_j be ℓ-highest-weight vectors of $V^0(\lambda_j, a_j)$ so that $v = v_1 \otimes \cdots \otimes v_m$ and set $L' = L_1 \otimes_{\mathbb{A}} \cdots \otimes_{\mathbb{A}} L_m$, where $L_j = U(\tilde{\mathfrak{g}})_{\mathbb{A}} v_j$. Then L_j are admissible lattices for $V^0(\lambda_j, a_j)$ and, by Lemma 4.5, L' is an admissible lattice for V. It is clear from (1.5) that L is an $\mathbb{A}$-submodule of L'. Moreover, similarly as in the proof of Proposition 4.2, it follows from the proof of Theorem 3.6 that L is a finitely generated free $\mathbb{A}$-module which spans V. This completes the proof of (b) in this case.

Now let V be any finite-dimensional $U(\tilde{\mathfrak{g}})$-ℓ-highest-weight module with Drinfeld polynomial $\boldsymbol{\omega}$. Since $V^0(\lambda_1, a_1) \otimes \cdots \otimes V^0(\lambda_m, a_m)$ is isomorphic to a quotient of V, it follows from (1.4), Lemma 1.4, and the previous paragraph that $U(\tilde{\mathfrak{h}})_{\mathbb{A}} v = \mathbb{A} v$. Hence, $L = U(\tilde{\mathfrak{n}}^-)_{\mathbb{A}} v$ and we complete the proof in the same way we proved Proposition 4.2. ☐

The next corollary tells us that we have accomplished the task of constructing all of the irreducible $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules directly as quotients of some $U(\tilde{\mathfrak{g}})$-modules by a reduction modulo p process.

Corollary 4.7. For every $\boldsymbol{\omega}' \in \mathcal{P}_{\mathbb{F}}^+$ there exist a discrete valuation ring $\mathbb{A}$ and $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{A}}^{++}$ such that $\overline{\boldsymbol{\omega}} = \boldsymbol{\omega}'$ and $V(\boldsymbol{\omega}')$ is isomorphic to a quotient of $L_{\mathbb{F}}$, where $L = U(\tilde{\mathfrak{g}})_{\mathbb{A}} v$ and v is an ℓ-highest-weight vector for $W^0(\boldsymbol{\omega})$.

Proof. Write $\boldsymbol{\omega}' = \prod_j \boldsymbol{\omega}_{\lambda,b_j}, b_j \in \mathbb{F}^\times, b_i \neq b_j$ for $i \neq j$, let $\mathbb{A}$ and $a_j \in \mathbb{A}^\times$ be as in Theorem 4.3, and set $\boldsymbol{\omega} = \prod_j \boldsymbol{\omega}_{\lambda_j,a_j}$. The corollary now follows from Theorem 4.6. ☐

Now let $\mathbb{A}, \boldsymbol{\omega}, v$, and L be as in the previous theorem, suppose $V = W^0(\boldsymbol{\omega})$ and write $\boldsymbol{\omega} = \prod_{j=1}^m \boldsymbol{\omega}_{\lambda,a_j}$ with $a_i \neq a_j, i \neq j$, so that $W^0(\boldsymbol{\omega}) \cong \otimes_j W^0(\boldsymbol{\omega}_{\lambda_j,a_j})$. Choose ℓ-highest-weight vectors v_j of $W(\boldsymbol{\omega}_{\lambda_j,a_j})$ such that $v = v_1 \otimes \cdots \otimes v_m$ and set $L_j = U(\tilde{\mathfrak{g}})_{\mathbb{A}} v_j, L = U(\tilde{\mathfrak{g}})_{\mathbb{A}} v$, and $L' = L_1 \otimes_{\mathbb{A}} \cdots \otimes_{\mathbb{A}} L_m$. As before, it follows from (1.5) that $L \subseteq L'$.

Conjecture 4.8. In the notation above we have:

(a) $W(\overline{\boldsymbol{\omega}}) \cong L_{\mathbb{F}}$.
(b) If $\bar{a}_i \neq \bar{a}_j$ for $i \neq j$, then $L = L'$.

Part (a) is the analogous statement of the conjecture in [9] mentioned in the introduction of the paper. Notice that Theorem 4.6 implies that $\dim_{\mathbb{F}}(L_{\mathbb{F}}) = \dim_{\mathbb{C}}(W^0(\boldsymbol{\omega}))$. Hence, for proving (a), it suffices to prove that $\dim_{\mathbb{F}}(W(\overline{\boldsymbol{\omega}})) \leq \dim_{\mathbb{C}}(W^0(\boldsymbol{\omega}))$.

Now part (b) is rather unusual since for $\mathbb{Z}$-lattices the appropriate analogous statement is false (as a counter-example one can take $\mathfrak{g} = \mathfrak{sl}_2, p \neq 2$, and $\boldsymbol{\omega} = (1-u)(1+u)$). Below we give an example showing that equality indeed can happen when working with $\mathbb{A}$-lattices. This is actually the main point behind the choice of working with discrete valuation rings: they have plenty of units. Quite clearly Conjecture 3.11 follows from this conjecture together with Lemma 4.5.

Before starting with the example, let us note that, proceeding exactly as in [15, Theorem 27.1], one can show that every additive subgroup of V which is invariant under the action of $U(\tilde{\mathfrak{g}})_{\mathbb{A}}$ is the direct sum of its intersection with the weight-spaces of V.

For the example, we let $\mathfrak{g} = \mathfrak{sl}_2$. Since I is a singleton, we shall drop the index referring to the roots and write $x_r^\pm, h_r$, and Λ_r instead of $x_{1,r}^\pm$, etc., and shall also identify P with $\mathbb{Z}$. We will verify part (b) of Conjecture 4.8 for the Weyl module $V = W^0((1-au)^2(1-bu))$ where $a, b \in \mathbb{A}^\times$ for some discrete valuation ring $\mathbb{A}$ such that $\bar{a} \neq \bar{b}$. In particular $W^0((1-au)^2(1-bu)) \cong W^0((1-au)^2) \otimes W^0(1-bu)$. Let v_0, w_0 be ℓ-highest weight vectors of $W^0((1-au)^2)$ and $W^0(1-bu)$, respectively.

$W^0(1-bu)$ is isomorphic to the evaluation representation $V^0(1,b)$. It is then easy to see that $x_s^- w_0 = b^s x_0^- w_0$ for all $s \in \mathbb{Z}$. Thus, letting $w_1 = x_0^- w_0$, the set $\{w_0, w_1\}$ is an $\mathbb{A}$-basis for $L_2 = U(\tilde{\mathfrak{g}})_{\mathbb{A}} w_0$.

Now consider $W^0((1-au)^2)$ and let $L_1 = U(\tilde{\mathfrak{g}})_{\mathbb{A}} v_0$. Since $\mathrm{wt}((1-au)^2) = 2$, letting $k > 2$ in (1.12) we get

$$\left((X^-_{\alpha;s,+}(u))^{(k-l)} \Lambda^+_\alpha(u)\right)_k v_0 = 0 \qquad \forall\, l, s \in \mathbb{Z}, 1 \le l \le k. \tag{4.1}$$

Setting $k = 3, l = 2$ above, we get $(x^-_{s+1}\Lambda_2 + x^-_{s+2}\Lambda_1 + x^-_{s+3})v_0 = 0$. Since $\Lambda_2 v_0 = a^2 v_0$ and $\Lambda_1 v_0 = -2av_0$, one easily proves inductively that

$$x_s^- v_0 = s a^{s-1} x_1^- v_0 - (s-1)a^s x_0^- v_0, \quad \text{for all} \quad s \in \mathbb{Z}. \tag{4.2}$$

Let $v_1 = x_0^- v_0$ and $v_3 = x_1^- v_0$. Thus we see that $\{v_1, v_3\}$ is an $\mathbb{A}$-basis for the zero-weight space of $W^0((1-au)^2) \cap L_1$. Now setting $k = 3, l = 1$ and $s = 1$ in (4.1), we get $(x_1^-)^{(2)}\Lambda_1 v_0 + x_1^- x_2^- v_0 = 0$. Applying $h_{r-1}, r \in \mathbb{Z}$, gives $x_1^- x_{r+1}^- v_0 = a^2 x_0^- x_r^- v_0$. Proceeding inductively we finally get

$$x_r^- x_s^- v_0 = 2a^{r+s}(x_0^-)^{(2)} v_0 \quad \forall r, s \in \mathbb{Z}. \tag{4.3}$$

Hence, $v_2 = (x_0^-)^{(2)} v_0$ completes an $\mathbb{A}$-basis for L_1, i.e., L_1 is the $\mathbb{A}$-span of $\{v_0, v_1, v_2, v_3\}$.

Clearly the set $A = \{v_i \otimes w_j : i = 0,1,2,3 \text{ and } j = 0,1\}$ is an $\mathbb{A}$-basis for $L' = L_1 \otimes_{\mathbb{A}} L_2$. Since $L = U(\tilde{\mathfrak{g}})_{\mathbb{A}}(v_0 \otimes w_0) \subseteq L'$, we are left to show that $A \subseteq L$. Using (4.2), (4.3) and $x_s^- w_0 = b^s x_0^- w_0$ we compute:

$$\begin{aligned}
x_0^-(v_0 \otimes w_0) &= v_1 \otimes w_0 + v_0 \otimes w_1,\\
x_1^-(v_0 \otimes w_0) &= v_3 \otimes w_0 + bv_0 \otimes w_1,\\
x_2^-(v_0 \otimes w_0) &= 2av_3 \otimes w_0 - a^2 v_1 \otimes w_0 + b^2 v_0 \otimes w_1.
\end{aligned}$$

Recording the coordinates of these vectors with respect to the basis $\{v_1 \otimes w_0, v_3 \otimes w_0, v_0 \otimes w_1\}$ of $L' \cap V_1$ we get the following matrix

$$\begin{bmatrix} 1 & 0 & -a^2 \\ 0 & 1 & 2a \\ 1 & b & b^2 \end{bmatrix}$$

whose determinant is $(a-b)^2$. Since $\bar{a} \neq \bar{b}$ iff $a - b \in \mathbb{A}^\times$, we see that the vectors $x_0^-(v_0 \otimes w_0)$, $x_1^-(v_0 \otimes w_0)$, and $x_2^-(v_0 \otimes w_0)$ also form an $\mathbb{A}$-basis for $L' \cap V_1$. Now we compute

$$\begin{aligned}
(x_0^-)^{(2)}(v_0 \otimes w_0) &= v_2 \otimes w_0 + v_1 \otimes w_1,\\
x_1^- x_0^-(v_0 \otimes w_0) &= 2av_2 \otimes w_0 + bv_1 \otimes w_1 + v_3 \otimes w_1,\\
(x_1^-)^{(2)}(v_0 \otimes w_0) &= a^2 v_2 \otimes w_0 + bv_3 \otimes w_1,
\end{aligned}$$

and, recording the coordinates of these vectors with respect to the basis $\{v_2 \otimes w_0, v_1 \otimes w_1, v_3 \otimes w_1\}$ of $L' \cap V_{-1}$ we get the matrix

$$\begin{bmatrix} 1 & 2a & a^2 \\ 1 & b & 0 \\ 0 & 1 & b \end{bmatrix}.$$

The determinant is again $(a-b)^2$ and we are done with this weight-space as before. Finally one easily sees that $(x_0^-)^{(3)}(v_0 \otimes w_0) = v_2 \otimes w_1$, showing that $A \subseteq L$ as claimed.

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