Dimensions of locally and asymptotically self-similar spaces

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Abstract

We obtain two in a sense dual to each other results: First, that the capacity dimension of every compact, locally self-similar metric space coincides with the topological dimension, and second, that the asymptotic dimension of a metric space, which is asymptotically similar to its compact subspace coincides with the topological dimension of the subspace. As an application of the first result, we prove the Gromov conjecture that the asymptotic dimension of every hyperbolic group G equals the topological dimension of its boundary at infinity plus 1, $\operatorname{asdim} G = \dim \partial_\infty G + 1$. As an application of the second result, we construct Pontryagin surfaces for the asymptotic dimension, in particular, those are examples of metric spaces X, Y with $\operatorname{asdim}(X \times Y) < \operatorname{asdim} X + \operatorname{asdim} Y$. Other applications are also given.

Contents

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1 Introduction

We say that a map $f : Z \to Z'$ between metric spaces is *quasi-homothetic* with coefficient $R > 0$, if for some $\lambda \geq 1$ and for all z, $z' \in Z$, we have

$$R|zz'|/\lambda \leq |f(z)f(z')| \leq \lambda R|zz'|.$$

In this case, we also say that f is λ-quasi-homothetic with coefficient R.

A metric space Z is *locally similar* to a metric space Y, if there is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\operatorname{diam} A \leq \Lambda_0/R$, where $\Lambda_0 = \min\{1, \operatorname{diam} Y/\lambda\}$, there is a λ-quasi-homothetic map $f : A \to Y$ with coefficient R (note that the condition $\operatorname{diam} A \leq \Lambda_0/R$ implies $\operatorname{diam} f(A) \leq \operatorname{diam} Y$). If a metric space Z is locally similar to itself then we say that Z is *locally self-similar*.

The notion of the capacity dimension of a metric space Z, $\operatorname{cdim} Z$, is introduced in [Bu1], and turns out to be useful in many questions, [Bu2]. The capacity dimension is larger than or equal to the topological dimension, $\dim Z \leq \operatorname{cdim} Z$ for every metric space Z, and it is important to know for which spaces the equality holds. Our first main result is the following

Theorem 1.1. *Assume that a metric space Z is locally similar to a compact metric space Y. Then* $\operatorname{cdim} Z < \infty$ *and* $\operatorname{cdim} Z \leq \dim Y$.

Corollary 1.2. *The capacity dimension of every compact, locally self-similar metric space Z is finite and coincides with its topological dimension,* $\operatorname{cdim} Z = \dim Z$.

On the other hand, we also prove a proposition (see Proposition 4.2), which allows to construct examples of compact metric spaces with the capacity dimension arbitrarily larger than the topological dimension.

Now, consider in a sense the dual situation. A metric space X is *asymptotically similar* to a metric space Y, if there are Λ_0, $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset X$ with $\operatorname{diam} A \leq R/\Lambda_0$ there is a λ-quasi-homothetic map $f : Y \to X$ with coefficient R, whose image contains an isometric copy $A' \subset X$ of A, $A' \subset f(Y)$. If a metric space X is asymptotically similar to a bounded subset then we say that X is *asymptotically self-similar*. Taking a copy A' instead of A provides an additional flexibility of this definition, which is necessary for applications, see sect. 7.4.

We recall the well established notion of the asymptotic dimension, asdim, in sect. 2.1.

Our second main result is the following

Theorem 1.3. *Assume that a metric space X is asymptotically similar to a compact metric space Y. Then the both dimensions,* $\operatorname{asdim} X$, $\dim Y$ *are finite and coincide,* $\operatorname{asdim} X = \dim Y$.

As applications of Theorem 1.1, we obtain the following results.

Theorem 1.4. *The capacity dimension of the boundary at infinity of any hyperbolic group G (taken with any visual metric) coincides with the topological dimension,* $\operatorname{cdim} \partial_\infty G = \dim \partial_\infty G$.

Theorem 1.4 together with the main result of [Bu1] leads to the following result which proves a Gromov conjecture, see [Gr, $1.\mathrm{E}_1'$].

Theorem 1.5. *The asymptotic dimension of any hyperbolic group G equals topological dimension of its boundary at infinity plus 1,*

$$\operatorname{asdim} G = \dim \partial_\infty G + 1.$$

Another application of Theorem 1.4 is the following embedding result, obtained from the main result of [Bu2].

Theorem 1.6. *Every hyperbolic group G admits a quasi-isometric embedding $G \to T_1 \times \cdots \times T_n$ into the n-fold product of simplicial metric trees $T_1, \ldots, T_n$ with* $n = \dim \partial_\infty G + 1$.

The group structure plays no role in the proof of Theorems 1.4 – 1.6. Actually, we prove more general Theorems 6.3, 6.4, 6.6, and have chosen the statements above for simplicity of formulations.

Theorem 6.3 has applications also to nonembedding results, which are discussed in sect. 6.3, see Theorem 6.7. Using Corollary 1.2, we give examples of strict inequality in the product theorem for the capacity dimension. These are famous Pontryagin surfaces self-similar construction of which is discussed in sect. 7, see Theorem 7.1. Finally, we construct metric spaces asymptotically similar to self-similar Pontryagin surfaces. As a corollary of Theorem 1.3, we give examples of strict inequality in the product theorem for the asymptotic dimension, that is, we construct metric spaces X, Y with

$$\operatorname{asdim}(X \times Y) < \operatorname{asdim} X + \operatorname{asdim} Y,$$

see Corollary 7.7.

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2 Preliminaries

Here, we recall notions and facts necessary for the paper.

Let Z be a metric space. For U, $U' \subset Z$ we denote by $\operatorname{dist}(U, U')$ the distance between U and U', $\operatorname{dist}(U, U') = \inf\{|uu'| : u \in U,\ u' \in U'\}$ where $|uu'|$ is the distance between u, u'. For $r > 0$ we denote by $B_r(U)$ the open r-neighborhood of U, $B_r(U) = \{z \in Z : \operatorname{dist}(z, U) < r\}$, and by $\overline{B}_r(U)$ the closed r-neighborhood of U, $\overline{B}_r(U) = \{z \in Z : \operatorname{dist}(z, U) \le r\}$. We extend these notations over all real r putting $B_r(U) = U$ for $r = 0$, and defining $B_r(U)$ for $r < 0$ as the complement of the closed $|r|$-neighborhood of $Z \setminus U$, $B_r(U) = Z \setminus \overline{B}_{|r|}(Z \setminus U)$.

Given a family $\mathcal{U}$ of subsets in a metric space Z we define $\operatorname{mesh}(\mathcal{U}) = \sup\{\operatorname{diam} U : U \in \mathcal{U}\}$. The *multiplicity* of $\mathcal{U}$, $m(\mathcal{U})$, is the maximal number of members of $\mathcal{U}$ with nonempty intersection. We say that a family $\mathcal{U}$ is *disjoint* if $m(\mathcal{U}) = 1$.

A family $\mathcal{U}$ is called a *covering* of Z if $\cup\{U : U \in \mathcal{U}\} = Z$. A covering $\mathcal{U}$ is said to be *colored* if it is the union of $m \ge 1$ disjoint families, $\mathcal{U} = \cup_{a \in A} \mathcal{U}^a$, $|A| = m$. In this case we also say that $\mathcal{U}$ is m-colored. Clearly, the multiplicity of a m-colored covering is at most m.

Let $\mathcal{U}$ be a family of open subsets in a metric space Z which cover $A \subset Z$. Given $z \in A$, we let

$$L(\mathcal{U}, z) = \sup\{\operatorname{dist}(z, Z \setminus U) : U \in \mathcal{U}\}$$

be the Lebesgue number of $\mathcal{U}$ at z, $L(\mathcal{U}) = \inf_{z \in A} L(\mathcal{U}, z)$ be the Lebesgue number of the covering $\mathcal{U}$ of A. For every $z \in A$, the open ball $B_r(z) \subset Z$ of radius $r = L(\mathcal{U})$ centered at z is contained in some member of the covering $\mathcal{U}$.

We shall use the following obvious fact (see e.g. [Bu1]).

Lemma 2.1. *Let $\mathcal{U}$ be an open covering of $A \subset Z$ with $L(\mathcal{U}) > 0$. Then for every $s \in (0, L(\mathcal{U}))$ the family $\mathcal{U}_{-s} = B_{-s}(\mathcal{U})$ is still an open covering of A.* □

2.1 Definitions of capacity and asymptotic dimensions

There are several equivalent definitions of the capacity dimension, see [Bu1]. In this paper, we use the following two.

(i) The capacity dimension of a metric space Z, $\mathrm{cdim}(Z)$, is the minimal integer $m \geq 0$ with the following property: There is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau > 0$ there exists a $(m+1)$-colored open covering $\mathcal{U}$ of Z with $\mathrm{mesh}(\mathcal{U}) \leq \tau$ and $L(\mathcal{U}) \geq \delta\tau$.

(ii) The capacity dimension of a metric space Z, $\mathrm{cdim}(Z)$, is the minimal integer $m \geq 0$ with the following property: There is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau > 0$ there exists an open covering $\mathcal{U}$ of Z with multiplicity $m(\mathcal{U}) \leq m+1$, for which $\mathrm{mesh}(\mathcal{U}) \leq \tau$ and $L(\mathcal{U}) \geq \delta\tau$.

The asymptotic dimension is a quasi-isometry invariant of a metric space introduced in [Gr]. There are also several equivalent definitions, see [Gr], [BD], and we use the following one. The asymptotic dimension of a metric space X is the minimal integer $\mathrm{asdim}\, X = n$ such that for every positive d there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \leq n+1$, $\mathrm{mesh}(\mathcal{U}) < \infty$ and $L(\mathcal{U}) \geq d$.

The following notion turns out to be useful for our purposes. This notion is called the Higson property in [DZ, sect. 4] and the asymptotic dimension of linear type in the book [Ro, Example 9.14]. We call it *asymptotic capacity dimension*. Similarly to the capacity dimension, the following two definitions, colored and covering ones, are equivalent.

(i) The asymptotic capacity dimension of a metric space X, $\mathrm{Cdim}(X)$, is the minimal integer $m \geq 0$ with the following property: There is a constant $\delta \in (0,1)$ such that for every sufficiently large $R > 1$ there exists a $(m+1)$-colored open covering $\mathcal{U}$ of X with $\mathrm{mesh}(\mathcal{U}) \leq R$ and $L(\mathcal{U}) \geq \delta R$.

(ii) The asymptotic capacity dimension of a metric space X, $\mathrm{Cdim}(X)$, is the minimal integer $m \geq 0$ with the following property: There is a constant $\delta \in (0,1)$ such that for every sufficiently large $R > 1$ there exists an open covering $\mathcal{U}$ of X with multiplicity $m(\mathcal{U}) \leq m+1$, for which $\mathrm{mesh}(\mathcal{U}) \leq R$ and $L(\mathcal{U}) \geq \delta R$.

Clearly, $\mathrm{asdim}\, X \leq \mathrm{Cdim}\, X$ for any metric space X.

3 Auxiliary facts

Here, we collect some facts needed for the proof of our main results.

The following lemma implies in particular the finite union theorem for the capacity dimension, which is similar to the appropriate theorems for the asymptotic dimension [BD] and the Assouad-Nagata dimension [LS].

Lemma 3.1. *Suppose, that Z is a metric space and A, $B \subset Z$. Let $\mathcal{U}$ be an open covering of A, $\mathcal{V}$ be an open covering of B both with multiplicity at most m. If $\mathrm{mesh}(\mathcal{V}) \leq L(\mathcal{U})/2$ then there exist an open covering $\mathcal{W}$ of $A \cup B$ with multiplicity at most m and $\mathrm{mesh}(\mathcal{W}) \leq \max\{\mathrm{mesh}(\mathcal{V}), \mathrm{mesh}(\mathcal{U})\}$, $L(\mathcal{W}) \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\}$.*

Proof. We can assume that $L(\mathcal{U}) < \infty$, i.e. no member of $\mathcal{U}$ covers Z, because otherwise we take $\mathcal{W} = \mathcal{U}$.

We put $r = L(\mathcal{U})/2$ and consider the family $\widetilde{\mathcal{U}} = \{B_{-r}(U) : U \in \mathcal{U}\}$. This family still covers A. Next, let $\widetilde{\mathcal{V}}$ be the family of all $V \in \mathcal{V}$, each of which intersects some $\widetilde{U} \in \widetilde{\mathcal{U}}$. For every $\widetilde{V} \in \widetilde{\mathcal{V}}$, we fix $\widetilde{U} \in \widetilde{\mathcal{U}}$ with $\widetilde{U} \cap \widetilde{V} \neq \emptyset$ and consider the union $W = W(\widetilde{U})$ of $\widetilde{U}$ and all $\widetilde{V} \in \widetilde{\mathcal{V}}$ assigned in this way to $\widetilde{U}$. Now, we take the family $\mathcal{W}$ consisting of all $V \in \mathcal{V}$, which do not enter $\widetilde{\mathcal{V}}$, and all $W(\widetilde{U})$, $\widetilde{U} \in \widetilde{\mathcal{U}}$.

By the remark at the beginning, the family $\mathcal{W}$ covers $A \cup B$. Since $\operatorname{diam} V \leq r$ for every $V \in \mathcal{V}$, we have $W(\widetilde{U}) \subset U$ for the corresponding $U \in \mathcal{U}$ and thus $\operatorname{diam} W(\widetilde{U}) \leq \operatorname{mesh} \mathcal{U}$. Hence, $\operatorname{mesh} \mathcal{W} \leq \max\{\operatorname{mesh}(\mathcal{U}), \operatorname{mesh} \mathcal{V}\}$.

Clearly, for the Lebesgue number of $\mathcal{W}$ we have

$$L(\mathcal{W}) \geq \min\{L(\mathcal{V}), L(\widetilde{\mathcal{U}})\} \geq \min\{L(\mathcal{V}), L(\mathcal{U})/2\}.$$

Finally, let $\mathcal{A}$ be a collection of members of $\mathcal{W}$ with nonempty intersection. If $\mathcal{A}$ contains no member of $\mathcal{V} \setminus \widetilde{\mathcal{V}}$ then every member of $\mathcal{A}$ is contained in some member of $\mathcal{U}$ and thus $|\mathcal{A}| \leq m$. Otherwise, the intersection of $\mathcal{A}$ is contained in some $V \in \mathcal{V} \setminus \widetilde{\mathcal{V}}$ and therefore, it misses the closure of any $\widetilde{U} \in \widetilde{\mathcal{U}}$. Thus, $\mathcal{A}$ contains at most as many members as the number of members of $\mathcal{V}$ contains the intersection. This shows that the multiplicity of $\mathcal{W}$ is at most m. □

We say that a metric space Z is *doubling at small scales* if there is a constant $N \in \mathbb{N}$ such that for every sufficiently small $r > 0$ every ball in Z of radius $2r$ can be covered by at most N balls of radius r.

Similarly, a metric space X is *asymptotically doubling* if there is a constant $N \in \mathbb{N}$ such that for every sufficiently large $R > 1$ every ball in X of radius $2R$ can be covered by at most N balls of radius R.

Lemma 3.2. *(1) Assume that a metric space Z is locally similar to a compact metric space Y. Then Z is doubling at small scales.*

(2) Assume that a metric space X is asymptotically similar to a compact metric space Y. Then X is asymptotically doubling.

Proof. (1) There is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\operatorname{diam} A \leq \Lambda_0/R$, $\Lambda_0 = \min\{1, \operatorname{diam} Z/\lambda\}$, there is a λ-quasi-homothetic map $f : A \to Y$ with coefficient R.

We fix a positive $\rho \leq \Lambda_0/(4\lambda)$. Because Y is compact, there is $N \in \mathbb{N}$ such that any subset $B \subset Y$ can be covered by at most N balls of radius ρ centered at points of B. Take $r > 0$ small enough so that $R = \lambda\rho/r$ satisfies the assumption above. Then for any ball $B_{2r} \subset Z$, we have

$$\operatorname{diam} B_{2r} \leq 4r \leq \Lambda_0/R,$$

and thus there is a λ-quasi-homothetic map $f : B_{2r} \to Y$ with coefficient R. The image $f(B_{2r})$ is covered by at most N balls of radius ρ centered at points of $f(B_{2r})$. The preimage under f of every such a ball is contained in a ball of radius $\leq \lambda\rho/R = r$ centered at a point in B_{2r}. Hence, B_{2r} is covered by at most N balls of radius r, and Z is doubling at small scales.

(2) Again, there are Λ_0, $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset X$ with $\operatorname{diam} A \leq R/\Lambda_0$ there is a λ-quasi-homothetic map $f : Y \to X$ with coefficient R, whose image contains an isometric copy $A' \subset X$ of A, $A' \subset f(Y)$.

We fix a positive $\rho \leq 1/(4\lambda\Lambda_0)$. Because Y is compact, there is $N \in \mathbb{N}$ such that any subset $B \subset Y$ can be covered by at most N balls of radius ρ centered at points of B. Take $R > 1$ large enough satisfying the assumption above. Then for any ball $B_{2R} \subset X$ of radius $2R$, there is a λ-quasi-homothetic map $f : Y \to X$ with coefficient $4\Lambda_0 R$, such that $f(Y)$ contains an isometric copy B'_{2R} of B_{2R}. Without loss of generality, we can assume that $B'_{2R} = B_{2R}$.

Then, the preimage $B = f^{-1}(B_{2R})$ is covered by at most N balls of radius ρ centered at points of B. The image under f of every such a ball is contained in a ball of radius $\leq 4\lambda\Lambda_0\rho R \leq R$ centered at a point in B_{2R}. Hence, B_{2R} is covered by at most N balls of radius R, and X is asymptotically doubling. $\square$

The idea of the following lemma is borrowed from [LS, Lemma 2.3] as well as its proof, which we give for convenience of the reader.

Lemma 3.3. *Assume that a metric space Z is doubling at small scales, and a metric space X is asymptotically doubling. Then* $\operatorname{cdim} Z < \infty$ *and* $\operatorname{Cdim} X < \infty$.

Proof. By the assumption, there is $n \in \mathbb{N}$ such that every ball $B_{4r} \subset Z$ of radius $4r$ is covered by at most $n+1$ balls $B_{r/2}$ for all sufficiently small $r > 0$. We fix a maximal r-separated set $Z' \subset Z$, i.e. $|zz'| > r$ for each distinct z, $z' \in Z'$. Then, the family $\mathcal{U}' = \{B_r(z) : z \in Z'\}$ is an open covering of Z.

Since every ball $B_{r/2}$ contains at most one point from Z' and $B_{4r}(z)$ is covered by at most $n+1$ balls $B_{r/2}$, the ball $B_{4r}(z)$ contains at most $n+1$ points from Z' for every $z \in Z'$. Thus, there is a coloring $\chi : Z' \to A$, $|A| = n+1$, such that $\chi(z) \neq \chi(z')$ for each distinct z, $z' \in Z$ with $|zz'| < 4r$.

For $a \in A$, we let $Z'_a = \chi^{-1}(a)$ be the set of the color a. Then $|zz'| \geq 4r$ for distinct z, $z' \in Z'_a$. Putting $\mathcal{U}_a = \{B_{2r}(z) : z \in Z'_a\}$, we obtain an open $(n+1)$-colored covering $\mathcal{U} = \cup_{a\in A}\mathcal{U}_a$ of Z with $\operatorname{mesh}(\mathcal{U}) \leq 4r$ and $L(\mathcal{U}) \geq r$. This shows that $\operatorname{cdim} Z \leq n$.

A similar argument shows that $\operatorname{Cdim} X < \infty$. We leave details to the reader as an exercise. $\square$

We shall use the following facts obviously implied by the definition of a quasi-homothetic map.

Lemma 3.4. *Let $h : Z \to Z'$ be a λ-quasi-homothetic map with coefficient R. Let $V \subset Z$, $\widetilde{\mathcal{U}}$ be an open covering of $h(V)$ and $\mathcal{U} = h^{-1}(\widetilde{\mathcal{U}})$. Then*

(1) $R\,\mathrm{mesh}(\mathcal{U})/\lambda \le \mathrm{mesh}(\widetilde{\mathcal{U}}) \le \lambda R\,\mathrm{mesh}(\mathcal{U})$;

(2) $\lambda R \cdot L(\mathcal{U}) \ge L(\widetilde{\mathcal{U}}) \ge R \cdot L(\mathcal{U})/\lambda$, *where $L(\mathcal{U})$ is the Lebesgue number of $\mathcal{U}$ as a covering of V.* □

4 Proof of Theorem 1.1

It follows from Lemmas 3.2 and 3.3, that $\mathrm{cdim}\, Z = N$ is finite. We can also assume that $\dim Y = n$ is finite. There is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau > 0$ there exists a $(N+1)$-colored open covering $\mathcal{V} = \cup_{a\in A}\mathcal{V}^a$ of Z with $\mathrm{mesh}(\mathcal{V}) \le \tau$ and $L(\mathcal{V}) \ge \delta\tau$. It is convenient to take $A = \{0,\dots,N\}$ as the color set.

There is a constant $\lambda \ge 1$, such that for every sufficiently large $R > 1$ and every $V \subset Z$ with $\mathrm{diam}\, V \le \Lambda_0/R$, $\Lambda_0 = \min\{1, \mathrm{diam}\, Y/\lambda\}$, there is a λ-quasi-homothetic map $h_V : V \to Y$ with coefficient R.

Using that Y is compact and $\dim Y = n$, we find for every $a \in A$ a finite open covering $\widetilde{\mathcal{U}}_a$ of Y with multiplicity $m(\widetilde{\mathcal{U}}_a) \le n+1$ such the following holds:

(i) $\mathrm{mesh}(\widetilde{\mathcal{U}}_0) \le \frac{\delta}{2\lambda}$;

(ii) $\mathrm{mesh}(\widetilde{\mathcal{U}}_{a+1}) \le \frac{1}{2\lambda^2}\min\{L(\widetilde{\mathcal{U}}_a), \mathrm{mesh}(\widetilde{\mathcal{U}}_a)\}$ for every $a \in A$, $a \le N-1$.

Then $l := \min\{L(\widetilde{\mathcal{U}}_N), \frac{1}{2}L(\widetilde{\mathcal{U}}_{N-1}), \dots, \frac{1}{2^N}L(\widetilde{\mathcal{U}}_0)\} > 0$ and $\mathrm{mesh}(\widetilde{\mathcal{U}}_a) \le \frac{\delta}{2\lambda}$ for every $a \in A$.

For every $V \in \mathcal{V}$, consider the slightly smaller subset $V' = B_{-\delta\tau/2}(V)$. Then, the sets $Z_a = \cup_{V\in\mathcal{V}^a} V' \subset Z$, $a \in A$, cover Z, $Z = \cup_{a\in A} Z_a$, because $L(\mathcal{V}) \ge \delta\tau$.

Given $V \in \mathcal{V}$, we fix a λ-quasi-homothetic map $h_V : V \to Z$ with coefficient $R = 1/\tau$ and put $\widetilde{V} = h_V(V')$. Now, for every $a \in A$, $V \in \mathcal{V}^a$ consider the family $\widetilde{\mathcal{U}}_{a,V} = \{\widetilde{U} \in \widetilde{\mathcal{U}}_a : \widetilde{V} \cap \widetilde{U} \ne \emptyset\}$, which is obviously a covering of $\widetilde{V}$ with multiplicity $\le n+1$. Then,

$$\mathcal{U}_{a,V} = \{h_V^{-1}(\widetilde{U}) : \widetilde{U} \in \widetilde{\mathcal{U}}_{a,V}\}$$

is an open covering of V' with multiplicity $\le n+1$.

Note that $U = h_V^{-1}(\widetilde{U})$ is contained in V for every $\widetilde{U} \in \widetilde{\mathcal{U}}_{a,V}$ because $\mathrm{dist}(v', Z \setminus V) > \delta\tau/2$ for every $v' \in V'$ and $\mathrm{diam}\, U \le \lambda\tau\,\mathrm{diam}\,\widetilde{U} \le \delta\tau/2$. Thus the family $\mathcal{U}_{a,V}$ is contained in V. Now, the family $\mathcal{U}_a = \cup_{V\in\mathcal{V}^a}\mathcal{U}_{a,V}$ covers the set Z_a of the color a, and it has the following properties

(1) for every $a \in A$, the multiplicity of $\mathcal{U}_a$ is at most $n+1$;

(2) $\operatorname{mesh}(\mathcal{U}_{a+1}) \le \frac{1}{2}\min\{L(\mathcal{U}_a), \operatorname{mesh}(\mathcal{U}_a)\}$ for every $a \in A$, $a \le N-1$ ($L(\mathcal{U}_a)$ means the Lebesgue number of $\mathcal{U}_a$ as a covering of Z_a);

(3) $\operatorname{mesh}(\mathcal{U}_a) \le \lambda\tau \operatorname{mesh}(\widetilde{\mathcal{U}}_a)$ and $L(\mathcal{U}_a) \ge \tau L(\widetilde{\mathcal{U}}_a)/\lambda$ for every $a \in A$.

Indeed, distinct V_1, $V_2 \in \mathcal{V}^a$ are disjoint and thus any $U_1 \in \mathcal{U}_{a,V_1}$, $U_2 \in \mathcal{U}_{a,V_2}$ are disjoint because $U_1 \subset V_1$, $U_2 \subset V_2$. This proves (1). Furthermore, for every $a \in A$, $a \le N-1$, and every $U \in \mathcal{U}_{a+1}$, we have

$$\begin{aligned}\operatorname{diam} U &\le \lambda\tau \operatorname{mesh}(\widetilde{\mathcal{U}}_{a+1}) \le \frac{\tau}{2\lambda}\min\{L(\widetilde{\mathcal{U}}_a), \operatorname{mesh}(\widetilde{\mathcal{U}}_a)\} \\ &\le \frac{1}{2}\min\{L(\mathcal{U}_a), \operatorname{mesh}(\mathcal{U}_a)\}\end{aligned}$$

by Lemma 3.4, hence, (2). Finally, (3) also follows from Lemma 3.4.

Now, we put $\widehat{\mathcal{U}}_{-1} = \{Z\}$, $\widehat{\mathcal{U}}_0 = \mathcal{U}_0$ and assume that for some $a \in A$, we have already constructed families $\widehat{\mathcal{U}}_0, \dots, \widehat{\mathcal{U}}_a$ so that $\widehat{\mathcal{U}}_a$ covers $Z_0 \cup \dots \cup Z_a$ with multiplicity $\le n+1$ and $\operatorname{mesh}(\mathcal{U}_a) \le \frac{1}{2}L(\widehat{\mathcal{U}}_{a-1})$, $\operatorname{mesh}(\widehat{\mathcal{U}}_a) \le \operatorname{mesh}(\mathcal{U}_0)$, $L(\widehat{\mathcal{U}}_a) \ge \min\{L(\mathcal{U}_a), \frac{1}{2}L(\widehat{\mathcal{U}}_{a-1})\}$. Then using (2), we have

$$\operatorname{mesh}(\mathcal{U}_{a+1}) \le \frac{1}{2}\min\{L(\mathcal{U}_a), \frac{1}{2}L(\widehat{\mathcal{U}}_{a-1})\} \le \frac{1}{2}L(\widehat{\mathcal{U}}_a).$$

Applying Lemma 3.1 to the pair of families $\widehat{\mathcal{U}}_a$, $\mathcal{U}_{a+1}$, we obtain an open covering $\widehat{\mathcal{U}}_{a+1}$ of $Z_0 \cup \dots \cup Z_{a+1}$ with multiplicity $\le n+1$ and with $\operatorname{mesh}(\widehat{\mathcal{U}}_{a+1}) \le \max\{\operatorname{mesh}(\widehat{\mathcal{U}}_a), \operatorname{mesh}(\mathcal{U}_{a+1})\} \le \operatorname{mesh}(\mathcal{U}_0)$ and $L(\widehat{\mathcal{U}}_{a+1}) \ge \min\{L(\mathcal{U}_{a+1}), \frac{1}{2}L(\widehat{\mathcal{U}}_a)\}$.

Proceeding by induction and using (3), we obtain an open covering $\mathcal{U} = \widehat{\mathcal{U}}_N$ of Z of multiplicity $\le n+1$ with $\operatorname{mesh}(\mathcal{U}) \le \operatorname{mesh}(\mathcal{U}_0) \le \delta\tau/2$ and $L(\mathcal{U}) \ge \min\{L(\mathcal{U}_N), \frac{1}{2}L(\mathcal{U}_{N-1}), \dots, \frac{1}{2^N}L(\mathcal{U}_0)\} \ge (l/\lambda)\tau$. Because we can choose $\tau > 0$ arbitrarily small and the constants δ, λ, l are independent of τ, this shows that $\operatorname{cdim} Z \le n$. □

Remark 4.1. A similar idea to lower the multiplicity of a colored covering via the finite union lemma (Lemma 3.3) is used in [LS, Proposition 2.8].

Proof of Corollary 1.2. We have $\dim Z \le \operatorname{cdim} Z$ for every metric space Z. By Theorem 1.1, $\operatorname{cdim} Z < \infty$ and $\operatorname{cdim} Z \le \dim Z$, hence $\operatorname{cdim} Z = \dim Z$ is finite. □

4.1 The capacity dimension versus the topological one

The following proposition allows to construct various examples of compact metric spaces with the capacity dimension arbitrarily larger than the topological dimension.

Proposition 4.2. *Let X, Y be bounded metric spaces such that for every $\varepsilon > 0$ there is $A \subset X$ and a homothety $h_\varepsilon : A \to Y$ with the ε-dense image, $\operatorname{dist}(y, h_\varepsilon(A)) < \varepsilon$ for every $y \in Y$. Then $\operatorname{cdim} X \ge \dim Y$.*

Proof. We can assume that $\dim Y > 0$, in particular, $\operatorname{diam} Y > 0$. Then, we have $\lambda(\varepsilon) \geq \lambda_0 > 0$ as $\varepsilon \to 0$ for the coefficient $\lambda(\varepsilon)$ of the homothety h_ε, because X is bounded.

Assume that $n = \operatorname{cdim} X < \dim Y$. There is $\delta > 0$ such that for every sufficiently small $\tau > 0$ there is an open covering $\mathcal{U}_\tau$ of X of multiplicity $\leq n+1$ with $\operatorname{mesh}(\mathcal{U}_\tau) \leq \tau$ and $L(\mathcal{U}_\tau) \geq \delta\tau$.

Using the estimate $\lambda(\varepsilon) \geq \lambda_0$, we can find $\tau = \tau(\varepsilon)$ such that $\lambda(\varepsilon)\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\delta\lambda(\varepsilon)\tau(\varepsilon) \geq 4\varepsilon$. Then for the covering $\mathcal{V}_\varepsilon = h_\varepsilon(\mathcal{U}_\tau)$ of $h_\varepsilon(A)$, we have $\operatorname{mesh}(\mathcal{V}_\varepsilon) \leq \lambda(\varepsilon)\tau(\varepsilon)$ and $L(\mathcal{V}_\varepsilon) \geq \delta\lambda(\varepsilon)\tau(\varepsilon)$. Furthermore, $m(\mathcal{V}_\varepsilon) \leq n+1$. Therefore, the family $\mathcal{V}'_\varepsilon = B_{-2\varepsilon}(\mathcal{V}_\varepsilon)$ still covers $h_\varepsilon(A)$. Taking the ε-neighborhood in Y of every $V \in \mathcal{V}'_\varepsilon$, we obtain an open covering $\mathcal{V}$ of Y with $\operatorname{mesh}(\mathcal{V}) \leq \operatorname{mesh}(\mathcal{V}_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let us estimate the multiplicity of $\mathcal{V}$. Assume that $y \in Y$ is a common point of members $V_j \in \mathcal{V}$, $j \in J$. By the definition of $\mathcal{V}$, for every $j \in J$, there is $a_j \in A$ such that $f(a_j) \in V'_j \in \mathcal{V}'_\varepsilon$ and $|f(a_j)y| < \varepsilon$. Then, the mutual distances of the points $f(a_j)$, $j \in J$, are $< 2\varepsilon$. Because $V'_j = B_{-2\varepsilon}(U_j)$ for $U_j \in \mathcal{V}_\varepsilon$, we see that every point $f(a_j)$, $j \in J$, is contained in every U_i, $i \in J$, and therefore $|J| \leq n+1$ because the multiplicity of $\mathcal{V}_\varepsilon$ is at most $n+1$. Hence, $m(\mathcal{V}) \leq n+1$ and $\dim Y \leq n$, a contradiction. □

As an application, we obtain the following examples. Let $Z = \{0\} \cup \{1/m : m \in \mathbb{N}\}$. Then $\operatorname{cdim} Z^n = n$ for any $n \geq 1$, while $\dim Z^n = 0$. Indeed, the spaces $X = Z^n$ and $Y = [0,1]^n$, obviously, satisfy the condition of Proposition 4.2, thus $\operatorname{cdim} Z^n \geq \dim Y = n$ (we have the equality here because $Z^n \subset Y$). For $n = 1$, this example is given in [LS] in context of the Assouad-Nagata dimension.

Combining with Corollary 1.2 and quasi-symmetry invariance of the capacity dimension, see [Bu1], we obtain: The space Z^n is not quasi-symmetric to any locally self-similar space for any $n \geq 1$.

Further examples. Take any monotone sequence of positive $\varepsilon_k \to 0$, $\varepsilon_1 = 1/3$, and repeat the construction of the standard ternary Cantor set $K \subset [0,1]$, only removing at every k-th step, $k \geq 1$, instead of the $(1/3)^k$-length intervals, the middle intervals of length $s_k = \varepsilon_k l_k$, $l_1 = 1$, where the length l_{k+1} of the segments obtained after processing the k-th step is defined recurrently by $2l_{k+1} + s_k = l_k$. The resulting compact space $K_a \subset [0,1]$ is homeomorphic to K. One easily sees that $\operatorname{cdim} K = 0$. However, $X = K_a$ and $Y = [0,1]$ satisfy the condition of Proposition 4.2, thus $\operatorname{cdim} K_a = 1$, while $\dim K_a = 0$.

Similarly, one can construct 'exotic' Sierpinski carpets, Menger curves etc with the capacity dimension strictly bigger than the topological dimension. Any of those compact metric spaces is not quasi-symmetric to any locally self-similar space, in particular, it is not quasi-symmetric to the boundary at infinity (viewed with a visual metric) of a hyperbolic group, see Theorem 1.4. To compare, it is well known that the boundary at infinity of a typical hyperbolic group is homeomorphic to the Menger curve.

5 Proof of Theorem 1.3

We actually prove that under the condition of Theorem 1.3, the following three dimensions are finite and coincide

$$\operatorname{asdim} X = \operatorname{Cdim} X = \dim Y.$$

We have $\operatorname{asdim} X \le \operatorname{Cdim} X$ for every metric space X, and by Lemmas 3.2 and 3.3, $\operatorname{Cdim} X$ is finite, because X is asymptotically similar to the compact space Y. We first show that $\dim Y \le \operatorname{asdim} X$. We let $\operatorname{asdim} X = N$. Then for every sufficiently large $\tau > 1$, there exists an open covering $\mathcal{V}$ of X with multiplicity $\le N+1$, $\operatorname{mesh}(\mathcal{V}) < \infty$ and $L(\mathcal{V}) \ge \tau$.

There are constants Λ_0, $\lambda \ge 1$, such that for every sufficiently large $R > 1$ and every $V \subset X$ with $\operatorname{diam} V \le R/\Lambda_0$, there is a λ-quasi-homothetic map $h_V : Y \to X$ with coefficient R, whose image contains an isometric copy $V' \subset X$ of V, $V' \subset h_V(Y)$.

Given $\varepsilon > 0$, we take a sufficiently large $R > 1$ with $\frac{\lambda}{R}\operatorname{mesh}(\mathcal{V}) < \varepsilon$, satisfying the condition above. Then, there is a λ-quasi-homothetic map $h : Y \to X$ with coefficient R. The family $\mathcal{U} = h^{-1}(\mathcal{V})$ is an open covering of Y with multiplicity $\le N+1$ and $\operatorname{mesh}(\mathcal{U}) \le \frac{\lambda}{R}\operatorname{mesh}(\mathcal{V}) < \varepsilon$. Hence, $\dim Y \le \operatorname{asdim} X$.

It remains to show that $\operatorname{Cdim} X \le \dim Y$. We already know that the topological dimension of Y and the asymptotic capacity dimension of X are finite, and we let $\dim Y = n$, $\operatorname{Cdim} X = N$. Starting from this point, the proof is completely parallel to that of Theorem 1.1.

According to the definition of Cdim, there is $\delta \in (0,1)$ such that for every sufficiently large R there exists a $(N+1)$-colored open covering $\mathcal{V} = \cup_{a\in A}\mathcal{V}^a$ of X with $\operatorname{mesh}(\mathcal{V}) \le R$ and $L(\mathcal{V}) \ge \delta R$. It is convenient to take $A = \{0,\dots,N\}$ as the color set.

Using that Y is compact and $\dim Y = n$, we find for every $a \in A$ a finite open covering $\widetilde{\mathcal{U}}_a$ of Y with multiplicity $m(\widetilde{\mathcal{U}}_a) \le n+1$ such the following holds:

(i) $\operatorname{mesh}(\widetilde{\mathcal{U}}_0) \le \frac{\delta}{2\lambda\Lambda_0}$;

(ii) $\operatorname{mesh}(\widetilde{\mathcal{U}}_{a+1}) \le \frac{1}{2\lambda^2}\min\{L(\widetilde{\mathcal{U}}_a), \operatorname{mesh}(\widetilde{\mathcal{U}}_a)\}$ for every $a \in A$, $a \le N-1$.

Then $l := \min\{L(\widetilde{\mathcal{U}}_N), \frac{1}{2}L(\widetilde{\mathcal{U}}_{N-1}), \dots, \frac{1}{2^N}L(\widetilde{\mathcal{U}}_0)\} > 0$ and $\operatorname{mesh}(\widetilde{\mathcal{U}}_a) \le \frac{\delta}{2\lambda\Lambda_0}$ for every $a \in A$.

We put $t = \delta R/2$, and for every $V \in \mathcal{V}$, consider the smaller subset $V_t = B_{-t}(V)$. Then, the sets $X_a = \cup_{V\in\mathcal{V}^a} V_t \subset X$, $a \in A$, cover X, $X = \cup_{a\in A} X_a$, because $L(\mathcal{V}) \ge \delta R = 2t$.

Given $V \in \mathcal{V}$, there is a λ-quasi-homothetic map $h_V : Y \to X$ with coefficient $\Lambda_0 R$ and $V' \subset h_V(Y)$ for some isometric copy $V' \subset X$ of V. Taking the inverse, we obtain a λ-quasi-isometric map $f_V : V \to Y$ with coefficient $(\Lambda_0 R)^{-1}$.

Now, for every $a \in A$, $V \in \mathcal{V}^a$ consider the family $\mathcal{U}_{a,V} = \{f_V^{-1}(\widetilde{U}) : \widetilde{U} \in \widetilde{\mathcal{U}}_a, f_V(V_t) \cap \widetilde{U} \neq \emptyset\}$, which is obviously a covering of V_t with multiplicity $\leq n+1$.

Note that every $U \in \mathcal{U}_{a,V}$, $U = f_V^{-1}(\widetilde{U})$ is contained in V because $\operatorname{dist}(v, X \setminus V) > t$ for every $v \in V_t$ and $\operatorname{diam} U \leq \lambda\Lambda_0 R \operatorname{diam} \widetilde{U} \leq t$. Thus the family $\mathcal{U}_{a,V}$ is contained in V. Now, the family $\mathcal{U}_a = \cup_{V \in \mathcal{V}^a} \mathcal{U}_{a,V}$ covers the set X_a of the color a, and it has the following properties

(1) for every $a \in A$, the multiplicity of $\mathcal{U}_a$ is at most $n+1$;

(2) $\operatorname{mesh}(\mathcal{U}_{a+1}) \leq \frac{1}{2}\min\{L(\mathcal{U}_a), \operatorname{mesh}(\mathcal{U}_a)\}$ for every $a \in A$, $a \leq N-1$ ($L(\mathcal{U}_a)$ means the Lebesgue number of $\mathcal{U}_a$ as a covering of X_a);

(3) $\operatorname{mesh}(\mathcal{U}_a) \leq \lambda\Lambda_0 R \operatorname{mesh}(\widetilde{\mathcal{U}}_a)$ and $L(\mathcal{U}_a) \geq \Lambda_0 R \cdot L(\widetilde{\mathcal{U}}_a)/\lambda$ for every $a \in A$.

For (1) and (3), the argument is literally the same as in the proof of Theorem 1.1. Furthermore, for every $a \in A$, $a \leq N-1$, and every $U \in \mathcal{U}_{a+1}$, we have

$$\begin{aligned}\operatorname{diam} U &\leq \lambda\Lambda_0 R \operatorname{mesh}(\widetilde{\mathcal{U}}_{a+1}) \leq \frac{\Lambda_0 R}{2\lambda}\min\{L(\widetilde{\mathcal{U}}_a), \operatorname{mesh}(\widetilde{\mathcal{U}}_a)\} \\ &\leq \frac{1}{2}\min\{L(\mathcal{U}_a), \operatorname{mesh}(\mathcal{U}_a)\}\end{aligned}$$

by Lemma 3.4, hence, (2).

Now, we put $\widehat{\mathcal{U}}_{-1} = \{X\}$, $\widehat{\mathcal{U}}_0 = \mathcal{U}_0$ and assume that for some $a \in A$, we have already constructed families $\widehat{\mathcal{U}}_0, \dots, \widehat{\mathcal{U}}_a$ so that $\widehat{\mathcal{U}}_a$ covers $X_0 \cup \dots \cup X_a$ with multiplicity $\leq n+1$ and $\operatorname{mesh}(\mathcal{U}_a) \leq \frac{1}{2}L(\widehat{\mathcal{U}}_{a-1})$, $\operatorname{mesh}(\widehat{\mathcal{U}}_a) \leq \operatorname{mesh}(\mathcal{U}_0)$, $L(\widehat{\mathcal{U}}_a) \geq \min\{L(\mathcal{U}_a), \frac{1}{2}L(\widehat{\mathcal{U}}_{a-1})\}$. Then using (2), we have

$$\operatorname{mesh}(\mathcal{U}_{a+1}) \leq \frac{1}{2}\min\{L(\mathcal{U}_a), \frac{1}{2}L(\widehat{\mathcal{U}}_{a-1})\} \leq \frac{1}{2}L(\widehat{\mathcal{U}}_a).$$

Applying Lemma 3.1 to the pair of families $\widehat{\mathcal{U}}_a$, $\mathcal{U}_{a+1}$, we obtain an open covering $\widehat{\mathcal{U}}_{a+1}$ of $X_0 \cup \dots \cup X_{a+1}$ with multiplicity $\leq n+1$ and with $\operatorname{mesh}(\widehat{\mathcal{U}}_{a+1}) \leq \max\{\operatorname{mesh}(\widehat{\mathcal{U}}_a), \operatorname{mesh}(\mathcal{U}_{a+1})\} \leq \operatorname{mesh}(\mathcal{U}_0)$ and $L(\widehat{\mathcal{U}}_{a+1}) \geq \min\{L(\mathcal{U}_{a+1}), \frac{1}{2}L(\widehat{\mathcal{U}}_a)\}$.

Proceeding by induction and using (3), we obtain an open covering $\mathcal{U} = \widehat{\mathcal{U}}_N$ of X of multiplicity $\leq n+1$ with $\operatorname{mesh}(\mathcal{U}) \leq \operatorname{mesh}(\mathcal{U}_0) \leq \delta R/2$ and $L(\mathcal{U}) \geq \min\{L(\mathcal{U}_N), \frac{1}{2}L(\mathcal{U}_{N-1}), \dots, \frac{1}{2^N}L(\mathcal{U}_0)\} \geq (l\Lambda_0/\lambda)R$. Because we can choose R arbitrarily large and the constants δ, λ, Λ_0, l are independent of R, this shows that $\operatorname{Cdim} X \leq n$. $\square$

6 Applications

6.1 Capacity dimension of the boundary at infinity of a hyperbolic group

Here, we describe a large class of hyperbolic spaces whose boundary at infinity is locally self-similar and prove a generalization of Theorem 1.4.

Recall necessary facts from the hyperbolic spaces theory. For more details the reader may consult e.g. [BoS]. We also assume that the reader is familiar with notions like of a geodesic metric space, a triangle, a geodesic ray etc.

Let X be a geodesic metric space. We use notation xx' for a geodesic in X between x, $x' \in X$, and $|xx'|$ for the distance between them. For $o \in X$ and for x, $x' \in X$, put $(x|x')_o = \frac{1}{2}(|xo|+|x'o|-|xx'|)$. The number $(x|x')_o$ is nonnegative by the triangle inequality, and it is called the Gromov product of x, x' w.r.t. o.

Lemma 6.1. *Let o, g, x', x'' be points of a metric space X such that $(x'|g)_o$, $(x''|g)_o \geq |og| - \sigma$ for some $\sigma \geq 0$. Then*

$$(x'|x'')_o \leq (x'|x'')_g + |og| \leq (x'|x'')_o + 2\sigma.$$

Proof. The left hand inequality immediately follows from the triangle inequality: because $|ox'| \leq |og| + |gx'|$ and $|ox''| \leq |og| + |gx''|$, we have $(x'|x'')_o \leq (x'|x'')_g + |og|$.

Next, we note that $(x'|o)_g = |og| - (x'|g)_o \leq \sigma$. This yields $|x'o| = |og|+|gx'|-2(x'|o)_g \geq |og|+|gx'|-2\sigma$ and similarly $|x''o| \geq |og|+|gx''|-2\sigma$. Now, the right hand inequality follows. □

A geodesic metric space X is called *δ-hyperbolic*, $\delta \geq 0$, if for any triangle $xyz \subset X$ the following holds: Let $y' \in xy$, $z' \in xz$ be points with $|xy'| = |xz'| \leq (y|z)_x$. Then $|y'z'| \leq \delta$. In this case, the δ-inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

holds for every base point $o \in X$ and all x, y, $z \in X$. A geodesic space is *(Gromov) hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Let X be a δ-hyperbolic space and $o \in X$ be a base point. A sequence of points $\{x_i\} \subset X$ *converges to infinity,* if

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty.$$

Two sequences $\{x_i\}$, $\{x'_i\}$ that converge to infinity are *equivalent* if

$$\lim_{i\to\infty} (x_i|x'_i)_o = \infty.$$

Using the δ-inequality, one easily sees that this defines an equivalence relation for sequences in X converging to infinity. The *boundary at infinity* $\partial_\infty X$ of X is defined as the set of equivalence classes of sequences converging to infinity. Every isometry of X canonically extends to a bijection of $\partial_\infty X$ on itself, and we use the same notation for the extension.

Every geodesic ray in X represents a point at infinity. Conversely, if a geodesic hyperbolic space X is proper (i.e. closed balls in X are compact), then every point at infinity is represented by a geodesic ray.

The Gromov product extends to $X \cup \partial_\infty X$ as follows. For points ξ, $\xi' \in \partial_\infty X$, it is defined by

$$(\xi|\xi')_o = \inf \liminf_{i\to\infty} (x_i|x'_i)_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x'_i\} \in \xi'$. Note that $(\xi|\xi')_o$ takes values in $[0,\infty]$, and that $(\xi|\xi')_o = \infty$ if and only if $\xi = \xi'$.

Similarly, the Gromov product

$$(x|\xi)_o = \inf \liminf_{i\to\infty} (x|x_i)_o$$

is defined for any $x \in X$, $\xi \in \partial_\infty X$, where the infimum is taken over all sequences $\{x_i\} \in \xi$.

Furthermore, for any ξ, $\xi' \in X \cup \partial_\infty X$ and for arbitrary sequences $\{x_i\} \in \xi$, $\{x'_i\} \in \xi'$, we have

$$(\xi|\xi')_o \le \liminf_{i\to\infty} (x_i|x'_i)_o \le \limsup_{i\to\infty} (x_i|x'_i)_o \le (\xi|\xi')_o + 2\delta.$$

Moreover, the δ-inequality holds in $X \cup \partial_\infty X$,

$$(\xi|\xi'')_o \ge \min\{(\xi|\xi')_o, (\xi'|\xi'')_o\} - \delta$$

for every ξ, ξ', $\xi'' \in X \cup \partial_\infty X$.

A metric d on the boundary at infinity $\partial_\infty X$ of X is said to be *visual*, if there are $o \in X$, $a > 1$ and positive constants c_1, c_2, such that

$$c_1 a^{-(\xi|\xi')_o} \le d(\xi,\xi') \le c_2 a^{-(\xi|\xi')_o}$$

for all ξ, $\xi' \in \partial_\infty X$. In this case, we say that d is the visual metric with respect to the base point o and the parameter a. The boundary at infinity is bounded and complete w.r.t. any visual metric, moreover, if X is proper then $\partial_\infty X$ is compact. If $a > 1$ is sufficiently close to 1, then a visual metric with respect to a does exist.

A metric space X is *cobounded* if there is a bounded subset $A \subset X$ such that the orbit of A under the isometry group of X covers X.

Proposition 6.2. *The boundary at infinity of every cobounded, hyperbolic, proper, geodesic space X is locally self-similar with respect to any visual metric.*

Proof. We can assume that the geodesic space X is δ-hyperbolic, $\delta \geq 0$, and that a visual metric d on $\partial_\infty X$ satisfies

$$c^{-1}a^{-(\xi|\xi')_o} \leq d(\xi, \xi') \leq ca^{-(\xi|\xi')_o}$$

for some base point $o \in X$, some constants $c \geq 1$, $a > 1$ and all ξ, $\xi' \in \partial_\infty X$. Note that then $\operatorname{diam} \partial_\infty X \leq c$.

There is $\rho > 0$ such that the orbit of the ball $B_\rho(o)$ under the isometry group of X covers X. Now, we put $\lambda = c^2 a^{\rho+4\delta}$. Then

$$\Lambda_0 = \min\{1, \operatorname{diam} \partial_\infty X/\lambda\} \leq 1/c.$$

Fix $R > 1$ and consider $A \subset \partial_\infty X$ with $\operatorname{diam} A \leq \Lambda_0/R$. For each ξ, $\xi' \in A$, we have

$$(\xi|\xi')_o \geq \log_a \frac{R}{c\Lambda_0} \geq \log_a R.$$

We fix $\xi \in A$. Since X is proper, there is a geodesic ray $o\xi \subset X$ representing ξ. We take $g \in o\xi$ with $a^{|og|} = R$. Then using the δ-inequality, we obtain for every $\xi' \in A$

$$(\xi'|g)_o \geq \min\{(\xi'|\xi)_o, (\xi|g)_o\} - \delta = |og| - \delta$$

because $(\xi|g)_o = |og|$.

For arbitrary ξ', $\xi'' \in A$, consider sequences $\{x'\} \in \xi'$, $\{x''\} \in \xi''$ such that $(x'|x'')_g \to (\xi'|\xi'')_g$. We can assume without loss of generality that $(x'|g)_o$, $(x''|g)_o \geq |og| - \delta$ because possible errors in these estimates disappear while taking the limit, see below.

Applying Lemma 6.1 to the points o, g, x', $x'' \in X$ and $\sigma = \delta$, we obtain

$$(x'|x'')_o - |og| \leq (x'|x'')_g \leq (x'|x'')_o - |og| + 2\delta.$$

Passing to the limit, this yields

$$(\xi'|\xi'')_o - |og| \leq (\xi'|\xi'')_g \leq (\xi'|\xi'')_o - |og| + 4\delta.$$

There is an isometry $f : X \to X$ with $|of(g)| \leq \rho$. Then

$$(\xi'|\xi'')_g - \rho \leq (f(\xi')|f(\xi''))_o \leq (\xi'|\xi'')_g + \rho$$

because the Gromov products with respect to different points differ one from each other at most by the distance between the points. The last two double inequalities give

$$(\xi'|\xi'')_o - |og| - \rho \leq (f(\xi')|f(\xi''))_o \leq (\xi'|\xi'')_o - |og| + \rho + 4\delta,$$

and therefore,

$$c^{-2}a^{-(\rho+4\delta)}Rd(\xi', \xi'') \leq d(f(\xi'), f(\xi'')) \leq c^2 a^\rho Rd(\xi', \xi'').$$

This shows that $f : A \to \partial_\infty X$ is λ-quasi-homothetic with coefficient R and hence $\partial_\infty X$ is locally self-similar. □

Now, Corollary 1.2 and Proposition 6.2 give the following

Theorem 6.3. *The capacity dimension of the boundary at infinity of every cobounded, hyperbolic, proper, geodesic space X coincides with the topological dimension,* $\operatorname{cdim}\partial_\infty X = \dim \partial_\infty X$. □

The class of spaces satisfying the condition of Theorem 6.3 is very large. It includes in particular all symmetric rank one spaces of noncompact type (i.e. the real, complex, quaternionic hyperbolic spaces and the Cayley hyperbolic plane), all cocompact Hadamard manifolds of negative sectional curvature, various hyperbolic buildings etc. The most important among them is the class of (Gromov) hyperbolic groups, and Theorem 1.4 is a particular case of Theorem 6.3.

Note that for many such spaces, the boundary at infinity is fractal in the sense that its Hausdorff dimension with respect of a natural visual metric is larger than the topological dimension. This is true e.g. for the complex, quaternionic hyperbolic spaces, the Cayley hyperbolic plane, for the Fuchsian hyperbolic buildings, see [Bou], and for the hyperbolic graph surfaces, see [Bu3].

6.2 The asymptotic dimension of a hyperbolic group

Theorem 6.3 is the decisive step in the proof of the following

Theorem 6.4. *The asymptotic dimension of every cobounded, hyperbolic, proper, geodesic space X equals topological dimension of its boundary at infinity plus 1,*

$$\operatorname{asdim} X = \dim \partial_\infty X + 1.$$

Theorem 1.5 is a particular case of Theorem 6.4. The fact that $\operatorname{asdim} X$ as well as $\dim \partial_\infty X$ are finite, $\operatorname{asdim} X$, $\dim \partial_\infty X < \infty$, is well known, it follows e.g. from [BoS] (for hyperbolic groups, there is an alternative proof [Ro, Theorem 9.25]). Our contribution is that we prove the optimal estimate, $\operatorname{asdim} X \le \dim \partial_\infty X + 1$.

The estimate from below,

$$\operatorname{asdim} X \ge \dim \partial_\infty X + 1,$$

or at least the idea of its proof is also well known, see [Gr, $1.\mathrm{E}_1'$]. More stronger estimates of different types are obtained in [Sw] and [BS2] respectively. For convenience of the reader, we give a simplified version of arguments from [BS2] adapted to the asymptotic dimension.

Let Z be a bounded metric space. Assuming that $\operatorname{diam} Z > 0$, we put $\mu = \pi / \operatorname{diam} Z$ and note that $\mu|zz'| \in [0, \pi]$ for every z, $z' \in Z$. Recall that the hyperbolic cone $\operatorname{Co}(Z)$ over Z is the space $Z \times [0, \infty)/Z \times \{0\}$ with metric defined as follows. Given $x = (z, t)$, $x' = (z', t') \in \operatorname{Co}(Z)$ we consider a triangle $\overline{o}\,\overline{x}\,\overline{x}' \subset \mathrm{H}^2$ with $|\overline{o}\,\overline{x}| = t$, $|\overline{o}\,\overline{x}'| = t'$ and the angle $\angle_{\overline{o}}(\overline{x}, \overline{x}') = \mu|zz'|$. Now, we put $|xx'| := |\overline{x}\,\overline{x}'|$. In the degenerate case $Z = \{\mathrm{pt}\}$, we define $\mathrm{Co}(Z) = \{\mathrm{pt}\} \times [0, \infty)$ as the metric product. The point $o = Z \times \{0\} \in \mathrm{Co}(Z)$ is called the *vertex* of $\mathrm{Co}(Z)$.

Proposition 6.5. *For every proper geodesic hyperbolic space X we have*

$$\operatorname{asdim} X \geq \dim \partial_\infty X + 1.$$

Proof. The same argument as in [Bu1, Proposition 6.2] shows that the hyperbolic cone $\mathrm{Co}(Z)$ over $Z = \partial_\infty X$, taken with some visual metric, can be quasi-isometrically (actually, roughly similarly) embedded in X because X is geodesic. Thus $\operatorname{asdim} X \geq \operatorname{asdim} \mathrm{Co}(Z)$, and we show that $\operatorname{asdim} \mathrm{Co}(Z) \geq \dim Z + 1$.

The *annulus* $\mathrm{An}(Z) \subset \mathrm{Co}(Z)$ consists of all $x \in \mathrm{Co}(Z)$ with $1 \leq |xo| \leq 2$. Clearly, $\mathrm{An}(Z)$ is homeomorphic to $Z \times I$, $I = [0, 1]$. Since X is proper, Z is compact. According to a well known result from the dimension theory (see [Al]), the topological dimension

$$\dim \mathrm{An}(Z) = \dim Z + 1.$$

Consider the sequence of contracting homeomorphisms $F_k : \mathrm{Co}(Z) \to \mathrm{Co}(Z)$ given by $F_k(z, t) = (z, \frac{1}{k}t)$, $(z, t) \in \mathrm{Co}(Z)$, $k \in \mathbb{N}$. Given a uniformly bounded covering $\mathcal{U}$ of $\mathrm{Co}(Z)$, the coverings $\mathcal{U}_k = F_k(\mathcal{U}) \cap \mathrm{An}(Z)$ of the annulus $\mathrm{An}(Z)$ have arbitrarily small mesh as $k \to \infty$. Therefore, $\operatorname{asdim} \mathrm{Co}(Z) \geq \dim \mathrm{An}(Z)$, and the estimate follows. □

Proof of Theorem 6.4. The estimate from below

$$\operatorname{asdim} X \geq \dim \partial_\infty X + 1$$

follows from Proposition 6.5. By the main result of [Bu1], see also [Bu2] for another proof, we have $\operatorname{asdim} X \leq \operatorname{cdim} \partial_\infty X + 1$ (the space X is certainly visual, i.e. X satisfies the condition of the cited theorems). Now, the estimate from above,

$$\operatorname{asdim} X \leq \dim \partial_\infty X + 1,$$

follows from Theorem 6.3. □

6.3 Embedding and nonembedding results

Combining Theorem 6.3 with the main result of [Bu2], we obtain

Theorem 6.6. *Every cobounded, hyperbolic, proper, geodesic space X admits a quasi-isometric embedding $X \to T_1 \times \cdots \times T_n$ into the n-fold product of simplicial metric trees $T_1, \ldots, T_n$ with $n = \dim \partial_\infty X + 1$.* □

(Every space X satisfying the condition of Theorem 6.6 is certainly visual, i.e. X also satisfies the condition of [Bu2, Theorem 1.1]). For example, the complex hyperbolic plane $\mathbb{C}\,\mathrm{H}^2$ admits a quasi-isometric embedding into the 4-fold product of simplicial metric trees etc.

Theorem 1.6 is a particular case of Theorem 6.6.

Theorem 6.3 has applications also to nonembedding results. For example, let X^n be a universal covering of a compact Riemannian n-dimensional, $n \geq 2$ manifold with nonempty geodesic boundary and constant sectional curvature -1. Then X^n satisfies the condition of Theorem 6.3, and hence $\dim \partial_\infty X^n = \operatorname{cdim} \partial_\infty X^n$. Note that X^n can be obtained from the real hyperbolic space H^n by removing a countable collection of disjoint open half-spaces, and $\partial_\infty X^n \subset S^{n-1}$ is a compact nowhere dense subset obtained by removing a countable collection of disjoint open balls. In particular, for $n = 2$, $\partial_\infty X^n \subset S^1$ is a Cantor set, for $n = 3$, $\partial_\infty X^n \subset S^2$ is a Sierpinski carpet, and for $n \geq 4$, $\partial_\infty X^n \subset S^{n-1}$ is a higher dimensional version of a Sierpinski carpet. Thus $\dim \partial_\infty X^n = n - 2$.

The space X^n contains isometrically embedded copies of H^{n-1} as the boundary components, thus by [BF], the k-fold product $X^n \times \cdots \times X^n$, $k \geq 1$ factors, contains quasi-isometrically embedded H^p for $p = k(n-2) + 1$.

Theorem 6.7. *Let X^n be the space as above, $Y_k^n = X^n \times \cdots \times X^n$ be the k-fold product, $k \geq 1$. Then there is no quasi-isometric embedding*

$$\mathrm{H}^p \to Y_k^n \times \mathbb{R}^m$$

for $p > k(n-1)$ and any $m \geq 0$.

For example, the Theorem says that there is no way to embed quasi-isometrically H^5 into $X^3 \times X^3 \times \mathbb{R}^m$ for any $m \geq 0$ though there is a quasi-isometric $\mathrm{H}^3 \to X^3 \times X^3$. In general, for an arbitrary $n \geq 2$, there is a quasi-isometric $\mathrm{H}^p \to X^n \times X^n$ for $p = 2n-3$ and there is no quasi-isometric $\mathrm{H}^p \to X^n \times X^n \times \mathbb{R}^m$ for $p = 2n - 1$ and an arbitrary $m \geq 0$.

In the case $n = 2$, the space X^2 is quasi-isometric to the binary tree T whose edges all have length 1 because X^2 covers a compact hyperbolic surface with nonempty geodesic boundary. By [DS], there is a quasi-isometric embedding $\mathrm{H}^2 \to T \times T$, and hence there is a quasi-isometric $\mathrm{H}^p \to X^n \times X^n$ in the remaining case $p = 2n - 2$ if $n = 2$. For $n \geq 3$, the question whether there is a quasi-isometric $\mathrm{H}^{2n-2} \to X^n \times X^n$ remains open. Moreover, the same question is open for quasi-isometric

$$\mathrm{H}^{k(n-1)} \to Y_k^n, \; n, k \geq 3.$$

Certainly, there is a huge range of possibilities in variation of this theme, e.g. taking as the target space the product with different dimensions of factors, considering other than X spaces, replacing H^p as source space etc. However, everything what is known here on nonembedding side is covered by ideas of the proof of Theorem 6.7.

Proof of Theorem 6.7. The key ingredient of the proof is the notion of the hyperbolic dimension of a metric space X, $\operatorname{hypdim} X$, which is introduced in [BS2] and has the following properties

(1) monotonicity: let $f : X \to X'$ be a quasi-isometric map between metric spaces X, X'. Then $\operatorname{hypdim} X \le \operatorname{hypdim} X'$;

(2) the product theorem: for any metric spaces X_1, X_2, we have

$$\operatorname{hypdim}(X_1 \times X_2) \le \operatorname{hypdim} X_1 + \operatorname{hypdim} X_2;$$

(3) $\operatorname{hypdim} X \le \operatorname{asdim} X$ for every metric space X;

(4) $\operatorname{hypdim} \mathbb{R}^m = 0$ for every $m \ge 0$.

Recall that a metric space X has *bounded growth at some scale*, if for some constants r, R with $R > r > 0$, and $N \in \mathbb{N}$ every ball of radius R in X can be covered by N balls of radius r, see [BoS].

According to the main result of [BS2], for every geodesic, Gromov hyperbolic space, which has bounded growth at some scale and whose boundary at infinity $\partial_\infty X$ is infinite, one holds

$$\operatorname{hypdim} X \ge \dim \partial_\infty X + 1.$$

In particular, $\operatorname{hypdim} \mathrm{H}^p \ge p$ (actually, the equality here is true). Now, assume that there is a quasi-isometric embedding $\mathrm{H}^p \to Y^n_k \times \mathbb{R}^m$. Then by properties (1) – (4) above, we have

$$p \le \operatorname{hypdim} \mathrm{H}^p \le k \cdot \operatorname{hypdim} X^n \le k \cdot \operatorname{asdim} X^n.$$

Using Theorem 6.6 and properties of the asymptotic dimension, or using that $\operatorname{asdim} X^n \le \operatorname{cdim} \partial_\infty X^n + 1$ ([Bu1]) and Theorem 6.3, we obtain $\operatorname{asdim} X^n \le \dim \partial_\infty X^n + 1 = n - 1$. Hence, the claim. □

6.4 Examples of strict inequality in the product theorem for the capacity dimension

An application of Corollary 1.2 is that the strict inequality in the product theorem for the capacity dimension holds for some compact metric spaces. With each $n \in \mathbb{N}$, one associates a Pontryagin surface Π_n which is a 2-dimensional compactum, $\dim \Pi_n = 2$, and for coprime m, $n \in \mathbb{N}$ one holds $\dim(\Pi_m \times \Pi_n) = 3$, that is

$$\dim(\Pi_m \times \Pi_n) < \dim \Pi_m + \dim \Pi_n.$$

According [Dr1], [Dr2, Corollary 2.3], for every prime p, there is a hyperbolic Coxeter group with a Pontryagin surface Π_p as the boundary at infinity. Taken with any visual metric, Π_p is a locally self-similar space and hence $\operatorname{cdim} \Pi_p = \dim \Pi_p = 2$ by Theorem 1.4. Obviously, the product of locally self-similar spaces is locally self-similar, and we obtain

$$\operatorname{cdim}(\Pi_p \times \Pi_q) < \operatorname{cdim} \Pi_p + \operatorname{cdim} \Pi_q$$

for prime $p \neq q$.

Unfortunately, the construction in [Dr2] of an appropriate hyperbolic Coxeter group is implicit. Here, we give explicit constructions self-similar Pontryagin surfaces.

7 Self-similar Pontryagin surfaces

For the notion of the cohomological dimension of a topological space X with respect to an abelian group G, $\dim_G X$, we refer e.g. to the survey [Dr3]. Let p be a prime number. A *Pontryagin surface* Π_p is a 2-dimensional compact space with $\dim_{\mathbb{Q}} \Pi_p = \dim_{\mathbb{Z}_q} \Pi_p = 1$ for every prime $q \neq p$ and $\dim_{\mathbb{Z}_p} \Pi_p = 2$.

Theorem 7.1. *For every prime p, there exists a Pontryagin surface Π_p with locally self-similar metric.*

Our objective is the existence of a (locally) self-similar metric space Π_p. For a simple argument, which proves the required cohomological properties of the compactum Π_p we construct, we refer to [Dr3, Example 1.9].

7.1 Construction

By a square we mean a topological space homeomorphic to $[0, 1] \times [0, 1]$. Given a natural $m \geq 2$, we consider the *m-band* B_m, which is a 2-dimensional square complex, constructed as follows. The union $\widetilde{B}_m$ of m squares with a common side can also be described as $T_m \times [0, 1]$, where T_m is the union of m copies of the segment $[0, 1]$ attached to each other along the common vertex 0. We fix a cyclic permutation τ of the segments $\sigma \subset T_m$ and define $B_m = \widetilde{B}_m / \{\sigma \times 0 \equiv \tau(\sigma) \times 1 : \sigma \subset T_m\}$. In the case $m = 2$, this gives the usual Möbius band.

The square complex B_m consists of m squares and its boundary ∂B_m as well as its singular locus sB_m corresponding to the common side of the squares both are homeomorphic to S^1.

For every a, $b > 0$, there is a well defined intrinsic metric on B_m with respect to which every square of B_m is isometric to the Euclidean rectangle whose sides have length a and b. We assume that a is the length of the common side of the squares. Then $\partial B_m \subset B_m$ is a geodesic of length ma and $sB_m \subset B_m$ is a geodesic of length a. We use notation $B_m(a, b)$ for B_m endowed with this metric. Note that $B_m(a, b)$ has nonpositive curvature (in Alexandrov sense) for every a, $b > 0$.

Lemma 7.2. *Given $a > 0$, $b \geq a\sqrt{2}/2$, let $A = B_m\left(\frac{4a}{m}, b\right)$, $B = [0,a]\times[0,a]$. Then, there is a 1-Lipschitz map*

$$q : (A, \partial A) \to (B, \partial B)$$

whose restriction to the boundary, $q|\partial A$, preserves the length of every arc.

Proof. The boundary ∂A is a closed geodesic in A of length $4a$. We subdivide ∂A into four segments of length a, called the *sides*, and map ∂A onto ∂B the sides and arc length preserving. This defines $q : \partial A \to \partial B$, which is obviously 1-Lipschitz.

Next, we extend q to the singular locus sB_m by collapsing it to the middle of the square B. This extension is still 1-Lipschitz because the distance between any points $x \in \partial A$ and $x' \in sB_m$ is at least b and $b \geq a\sqrt{2}/2$, which is the maximal distance between the middle of B and points of ∂B.

Finally, we extend already defined q to A as the affine map on every radial segment $xx' \subset A$ with $x \in \partial A$, $x' \in sB_m$ be the (unique) closest to x point. When x runs over ∂A, the segments xx' cover A so that for different $x, y \in \partial A$ the segments xx' and yy' have no common interior point. Because $|xx'| = b$, the restriction of q to xx' is 1-Lipschitz.

So defined q is smooth outside of the union of ∂A, sB_m and the four radial segments corresponding to the end points of the sides of ∂A. One easily sees that $|dq| \leq 1$ there. Since A is geodesic, it follows that q is 1-Lipschitz. □

7.1.1 First template

Fix a natural $m \geq 2$ and an odd k, $k = 2l + 1$ with $l \geq 1$. We define a 2-dimensional square complex $P = P_{m,k}$ obtained from the square $Q_k = [0,k] \times [0,k]$ by removing the open middle square

$$q_k = (l, l+1) \times (l, l+1) \subset Q_k$$

and attaching instead the m-band B_m along a homeomorphism $\partial B_m \to \partial q_k$. We consider the following square structure on P. The remainder $Q_k \setminus q_k$ consists of $k^2 - 1$ squares each of which we subdivide into m^2 subsquares. Furthermore, the m-band consists of m squares each of which we represent as the rectangle $[0,4] \times [0,m]$ with the natural square structure consisting of $4m$ squares. We assume that the side $[0,4] \times \{0\}$ corresponds to the singular locus sB_m of B_m. Assuming that the gluing homeomorphism $\partial B_m \to \partial q_k$ preserves the induced subdivisions of each circle into $4m$ segments, we obtain the desired square complex structure on P consisting of

$$s_{m,k} = (k^2 - 1)m^2 + 4m^2 = (km)^2 + 3m^2$$

squares. Speaking about squares of P we mean squares of this square complex structure.

We consider the canonical intrinsic metric on P with respect to which every square of P is isometric to the Euclidean square with the side length $1/(km)$. The space P is nonpositively curved, and the subcomplex $B_m \subset P$ is convex, isometric to $B_m(\frac{4}{km}, \frac{1}{k})$, and its boundary is geodesic in P.

The boundary ∂P consists of four sides of length 1 and one can consider P as the unit square $[0,1] \times [0,1]$ with the appropriate middle subsquare replaced by $B_m\left(\frac{4}{km}, \frac{1}{k}\right)$. We have $b > a\sqrt{2}/2$ for $a = b = 1/k$. Thus applying Lemma 7.2, we obtain a 1-Lipschitz map

$$q_0^1 : P \to [0,1] \times [0,1]$$

which is identical outside of the interior of $B_m \subset P$.

7.1.2 Constructing a sequence of polyhedra $\{P^i\}$

We construct a sequence of polyhedra $\{P^i = P^i_{m,k}\}$, $i \geq 1$, in a way that every P^i serves as a building block for P^{i+1} and as such it is called the i-th *template.* Furthermore, every P^{i+1} consists of one and the same number of building blocks P^i independent of i. Every polyhedron P^i possesses a canonical square complex structure and being endowed with an intrinsic metric, for which every square is isometric to a fixed Euclidean square, it is nonpositively curved in the Alexandrov sense. From ideological side, the construction is similar to well known constructions of self-similar compact metric spaces via a family of homotheties.

As the 0-th template we take the unit square, $P^0 = [0,1] \times [0,1]$. The first template $P^1 = P$ is already described above. It can also be described as follows. The polyhedron P^1 consists of $s_{m,k}$ blocks each of which is a $1/mk$-homothetic copy of P^0 attached along the boundary to the 1-skeleton S of P^1.

The polyhedron P^2 is obtained out of P^1 as follows. We remove every of $s_{m,k}$ open square of P^1, obtaining again the 1-dimensional complex S, and replace every removed open square by a $1/mk$-homothetic copy of P^1 attaching it along the boundary to S.

Assume that a square polyhedron P^i is already constructed for $i \geq 1$. By assumption, it is considered with the canonical intrinsic nonpositively curved metric in which every square is isometric to the Euclidean square of side length $1/(mk)^i$. The polyhedron P^i consists of $s_{m,k}$ pairwise isometric blocks each of which is mk-homothetic to the template P^{i-1}. The boundary ∂P^i is subdivided into four sides, consisting each of $(mk)^i$ segments of the square structure, and has length 4.

We construct the square polyhedron P^{i+1}, replacing every of $s_{m,k}$ open square of P^1 by a $1/mk$-homothetic copy of the template P^i, attaching it to S, so that they all together form a square complex structure of P^{i+1} and define the canonical intrinsic nonpositively curved metric in which every square is isometric to the Euclidean square of side length $1/(mk)^{i+1}$.

Our construction has the following property: for every integer j, $1 \le j \le i$, the polyhedron P^{i+1} consists of $s_{m,k}^{i+1-j}$ subblocks each of which is homothetic to the j-th template P^j with coefficient $(mk)^{i+1-j}$.

Lemma 7.3. *The diameter of P^i, $i \ge 0$, is bounded above by a constant independent of i,*

$$\operatorname{diam} P^i \le d$$

where one can take $d = d(m,k) = \frac{2m(l+1)}{mk-1} + 2$, (recall $k = 2l+1$).

Proof. Because the length of the boundary ∂P^i equals 4 for every $i \ge 1$, it suffices to estimate $\delta_i = \max\{\operatorname{dist}(x, \partial P^i) : x \in P^i\}$ from above independent of i.

For $i = 1$, the most remote points from the boundary are sitting in the singular locus of the subcomplex $B_m \subset P^1$. Moving along the 1-skeleton S of P^1, we find that $\delta_1 = m/mk + ml/mk = (l+1)/k$.

The grid S serves as a skeleton for attaching the blocks while constructing every P^i and thus it is isometrically (in the sense of the induced intrinsic metric) embedded in P^i for every $i \ge 1$. So, to estimate δ_i we can use paths in S, namely, $\operatorname{dist}_S(x, S_0) \le \delta_1$ for every $x \in S$, where $S_0 \subset S$ is identified with the boundary ∂P^i for every $i \ge 1$, and the distance is taken with respect to the intrinsic metric of S.

For $i = 2$, clearly

$$\delta_2 \le \delta_1/mk + \max_{x \in S} \operatorname{dist}_S(x, S_0) \le \delta_1/mk + \delta_1$$

because P^2 consists of blocks $1/mk$-homothetic to P^1, whose boundaries are subsets of S. Similarly, we recurrently obtain the estimate

$$\delta_{i+1} \le \delta_i/mk + \delta_1 \le \delta_1 \sum_{j=0}^{i} 1/(mk)^j,$$

hence, the claim. □

7.2 The inverse sequence $\{P^i; q_i^{i+1}\}$

The bonding map $q_0^1 : P^1 \to P^0$ is already described above. By induction, we obtain the bonding map $q_i^{i+1} : P^{i+1} \to P^i$ for every $i \ge 1$ by putting together the maps q_{i-1}^i defined on the blocks of P^{i+1}.

This map is 1-Lipschitz and it is compatible with self-similar structure of complexes, i.e. its restriction to every subblock $P^j \subset P^{i+1}$, $1 \le j \le i$, coincides with q_{j-1}^j.

The product $\mathcal{P} = \prod_{i \ge 0} P^i$ is the set of all sequences $\{x_i \in P^i : i \ge 0\}$. The limit of the inverse sequence $\{P^i; q_i^{i+1}\}$,

$$\Pi = \Pi_{m,k} = \varprojlim (P^i; q_i^{i+1}),$$

is the subset of $\mathcal{P}$ consisting of all sequences $\{x_i\}$ with $x_i = q_i^{i+1}(x_{i+1})$ for every $i \ge 0$. For every $j \ge 0$, we have the projection $q_j^\infty : \Pi \to P^j$ defined by $q_j^\infty(\{x_i\}) = x_j$ for every $\{x_i\} \in \Pi$. Clearly, $q_j^{j+1} \circ q_{j+1}^\infty = q_j^\infty$ for every $j \ge 0$.

The space $\mathcal{P}$ is compact in the product topology as the product of compact spaces, and Π is closed in $\mathcal{P}$ because all bonding maps are continuous. Thus Π is compact in the induced topology, which we call the *product topology* of Π. By the definition of the product topology, the map q_j^∞ is continuous for every $j \ge 0$ and the product topology is the roughest one among all topologies with this property.

For each $\xi = \{x_i\}$, $\xi' = \{x_i'\} \in \Pi$, the sequence of distances $|x_i x_i'|$ is bounded by Lemma 7.3 and nondecreasing because every bonding map is 1-Lipschitz. Now, we define a metric on Π by

$$|\xi\xi'| = \lim_i |x_i x_i'|$$

(this metric is of course by no means nonpositively curved despite the fact that all P^i are nonpositively curved). The corresponding metric topology on Π we call the *metric topology* of Π.

Lemma 7.4. *The metric topology of* Π *coincides with the product topology.*

Proof. If $|\xi\xi'| < r$ for some $\xi = \{x_i\}$, $\xi' = \{x_i'\} \in \Pi$, then $|x_i x_i'| \le |\xi\xi'| < r$ for every $i \ge 0$. It follows that the projection q_i^∞ is continuous in the metric topology for every $i \ge 0$ and thus every open set in the product topology is open in the metric topology.

Fix $\xi \in \Pi$, $r > 0$ and consider the (open) ball $B_r(\xi) \subset \Pi$. We show that there is an open in the product topology subset which is contained in $B_r(\xi)$ and contains ξ. There is $j \in \mathbb{N}$ such that $2d/(mk)^j < r$, where $d = d(m,k)$ is the upper bound for the diameter of P^i, see Lemma 7.3, and hence for the diameter of Π.

We consider $x_j = q_j^\infty(\xi) \in P^j$ and take the union $\overline{A}$ of all squares of P^j containing x_j. Recall that the side length of every such square is $1/(mk)^j$. Then, the point x_j is contained in the interior A of $\overline{A}$, which open in P^j. Thus $B = (q_j^\infty)^{-1}(A)$ is open in the product topology, and $\xi \in B$. On the other hand, $\operatorname{diam} B \le 2 \operatorname{diam} C$, where C is preimage under q_j^∞ of a square of P^j. We have $\operatorname{diam} C = \operatorname{diam} \Pi/(mk)^j \le d/(mk)^j$ and hence $\operatorname{diam} B \le 2d/(mk)^j < r$. It follows that $B \subset B_r(\xi)$. Therefore, every open in the metric topology subset in Π is also open in the product topology. □

7.3 Self-similarity of the space Π

Recall some notions from the theory of self-similar metric spaces, see e.g. [Fa], [Hu]. A compact metric space K is said to be *self-similar* if there is a finite collection of homotheties $f_a : K \to K$, $a \in A$, with coefficients $h_a \in (0,1)$ such that $K \subset \cup_{a\in A} f_a(K)$.

In this case, there is a unique number $\mu \geq 0$ with

$$\sum_{a\in A} h_a^\mu = 1.$$

The number μ is called the similarity dimension of K, $\mu = \dim_s K$ (more precisely, μ is the similarity dimension of the family $\{f_a\}_{a\in A}$). For example, if $h_a = h$ for all $a \in A$, then we have

$$\dim_s K = \frac{\log |A|}{\log(1/h)}.$$

One always has $\dim_H K \leq \dim_s K$ for the Hausdorff dimension $\dim_H K$ of K. The collection of the homotheties $\{f_a : a \in A\}$ satisfies the *OSC* (Open Set Condition), if there is an open set $U \subset K$ such that $f_a(U) \subset U$ for all $a \in A$ and $f_a(U) \cap f_{a'}(U) = \emptyset$ for all distinct a, $a' \in A$. In this case, the Hausdorff dimension of K coincides with the similarity dimension, $\dim_H K = \dim_s K$ (see e.g. [EG]).

Proposition 7.5. *The metric space $\Pi = \Pi_{m,k}$ is compact and self-similar for every integer $m \geq 2$ and odd $k = 2l+1$. Furthermore, the corresponding collection $\{f_a : a \in A\}$ consists of $|A| = s_{m,k}$ homotheties with coefficients $h_a = 1/(mk)$, and satisfies the OSC. In particular, the Hausdorff dimension*

$$\dim_H \Pi = 2 + \frac{\log(1+3/k^2)}{\log(mk)}.$$

Proof. It follows from Lemma 7.4 that the metric space Π is compact. Recall that the square polyhedron P^{i+1} consists of $s_{m,k} = (mk)^2 + 3m^2$ blocks with disjoint interiors, each of which is mk-homothetic to P^i for every $i \geq 0$.

We label the blocks of P^i by a finite set A, $|A| = s_{m,k}$, in a way independent of i, that is compatible with the bonding maps, and fix for each $i \geq 0$, $a \in A$ a homothety $f_a^i : P^i \to P^{i+1}$ with coefficient $h_a = 1/(mk)$ whose image is the corresponding block of P^{i+1}. We can also assume that

$$f_a^i \circ q_i^{i+1} = q_{i+1}^{i+2} \circ f_a^{i+1} \tag{$*$}$$

for each $i \geq 0$, $a \in A$. Then $P^{i+1} = \cup_{a\in A} f_a^i(P^i)$ and for the interior U^i of P^i, we have $f_a^i(U^i) \subset U^{i+1}$ while the open sets $f_a^i(U^i)$, $f_{a'}^i(U^i)$ are disjoint for different a, $a' \in A$.

The equality $(*)$ allows to pass to the limit as $i \to \infty$, which yields the collection of homotheties $f_a : \Pi \to \Pi$, $a \in A$, with coefficients $h_a = 1/(mk)$ for every $a \in A$ with the required properties. □

Proof of Theorem 7.1. We fix an odd $k = 2l+1 \geq 3$ and consider the metric space $\Pi_p = \Pi_{p,k}$, which is self-similar by Proposition 7.5. It is an easy exercise to check that every compact self-similar space is locally self-similar. We have $2 < \dim_H \Pi_p < 3$ for the Hausdorff dimension by Proposition 7.5, thus $\dim \Pi_p \leq 2$. Applying the argument from [Dr3, Example 1.9], we obtain $\dim_{\mathbb{Q}} \Pi_p = \dim_{\mathbb{Z}_q} \Pi_p = 1$ for every prime $q \neq p$ and $\dim_{\mathbb{Z}_p} \Pi_p = 2$. The last equality together with the estimate $\dim \Pi_p \leq 2$ implies that $\dim \Pi_p = 2$. □

7.4 Asymptotically self-similar Pontryagin surfaces

We fix a compact self-similar Pontryagin surface $\Pi = \Pi_{m,k}$ for some integer $m \geq 2$, odd $k = 2l+1 \geq 3$, and define a metric space $\widehat{\Pi}$ as follows. Recall Π consists of $s_{m,k} = (mk)^2 + 3m^2$ blocks, each of which is $(1/mk)$-homothetic to Π and attached along the 1-skeleton S of the polyhedron P^1, and S is isometrically (in the sense of the induced intrinsic metric) embedded in Π. For every of $4m^2$ middle blocks, which are projected into the m-band $B_m \subset P^1$, their distance to the boundary $\partial\Pi$ is at least ml/mk, because the projection $\Pi \to P^1$ is 1-Lipschitz, and the distance from B_m to the boundary ∂P^1 equals ml/mk.

We put $\lambda = mk$, $X_0 = \Pi$, and consider the λ-homothetic copy of X_0, $X_1 = \lambda X_0$. The space X_1 consists of $s_{m,k}$ blocks isometric to X_0, and we can consider X_0 as a subspace of X_1, $X_0 \subset X_1$, identifying it with some block. Moreover, we take this block in the middle of X_1 so that $\mathrm{dist}(X_0, \partial X_1) \geq ml$. Taking $X_i = \lambda^i X_0$, we obtain an increasing sequence

$$X_0 \subset X_1 \subset \ldots \subset X_i$$

of metric spaces via appropriate identifications for which $\mathrm{dist}(X_i, \partial X_{i+1}) \geq \lambda^i ml$. Now, we define $\widehat{\Pi} = \widehat{\Pi}_{m,k} := \cup_{i\in\mathbb{N}} X_i$. Given x, $x' \in \widehat{\Pi}$, there is $i \in \mathbb{N}$ with x, $x' \in X_i$. Then, the distance $|xx'|$ in $\widehat{\Pi}$ is well defined as the distance between x, x' in X_i. Therefore, $\widehat{\Pi}$ is a metric space.

Proposition 7.6. *The metric space $\widehat{\Pi}$ is asymptotically similar to the compact space Π.*

Proof. We put $\lambda = \Lambda_0 = mk$ and consider a bounded subset $A \subset \widehat{\Pi}$ with $\mathrm{diam}\, A \leq R/\Lambda_0$ for some $R > 1$. There is $i \in \mathbb{N}$ with $\lambda^i < R \leq \lambda^{i+1}$. Then, any λ^i-homothety as well as λ^{i+1}-homothety is a λ-quasi-homothety with coefficient R.

Because $\mathrm{dist}(X_j, \partial X_{j+1}) \to \infty$ as $j \to \infty$, A is contained in some X_{j+1}, and we can assume that $j \geq i$. The problem is however that j can be much larger than i, e.g. if A is sitting near the boundary of X_j. Thus we assume that $j \geq i$ is minimal with property $A' \subset X_{j+1}$ for some isometric copy $A' \subset \widehat{\Pi}$ of A. Now, we show that $j \leq i+1$.

Assume to the contrary, that $j \geq i+2$. Note that $\lambda = mk \geq 6$. Then $R/\Lambda_0 \leq \lambda^i \leq \lambda^{i+1}/6$. The space X_{j+1} is the union of $s^2_{m,k}$ subblocks each of which is isometric to X_{j-1}. Since $j-1 \geq i+1$ and the distance in X_{j+1} between any disjoint copies of X_{j-1} is at least λ^{i+1}, we obtain that A is covered by the union $Q \subset X_{j+1}$ of copies of X_{j-1} such that there is a common point of the copies. However, every such a union is isometric to a subset of X_j. Hence, X_j contains an isometric copy of A. Since this contradicts our assumption on j, we conclude that $j \leq i+1$.

It follows that the λ^{j+1}-homothety $f : \Pi \to X_{j+1} \subset \widehat{\Pi}$ is a λ-quasi-homothety with coefficient R, and its image X_{j+1} contains an isometric copy of A. Therefore, $\widehat{\Pi}$ is asymptotically similar to Π. □

Using Theorem 1.3, we obtain $\operatorname{asdim} \widehat{\Pi} = \dim \Pi = 2$. We fix an odd $k = 2l+1 \geq 3$ and put $\widehat{\Pi}_p = \widehat{\Pi}_{p,k}$.

Corollary 7.7. *For distinct prime p, q, we have*

$$\operatorname{asdim}(\widehat{\Pi}_p \times \widehat{\Pi}_q) < \operatorname{asdim} \widehat{\Pi}_p + \operatorname{asdim} \widehat{\Pi}_q.$$

Proof. If metric spaces X, X' are asymptotically similar to metric spaces Y, Y' respectively, then clearly $X \times X'$ is asymptotically similar to $Y \times Y'$. Therefore, $\operatorname{asdim}(\widehat{\Pi}_p \times \widehat{\Pi}_q) = \dim(\Pi_p \times \Pi_q) = 3$, while $\operatorname{asdim} \widehat{\Pi}_p = \dim \Pi_p = 2$ and $\operatorname{asdim} \widehat{\Pi}_q = \dim \Pi_q = 2$. □

Remark 7.8. In [Gra], examples of coarse spaces X, Y with $\operatorname{asdim}(X \times Y) < \operatorname{asdim} X + \operatorname{asdim} Y$ are given. Here, the asymptotic dimension $\operatorname{asdim} X$ is associated with the coarse structure $\mathcal{E}$ of X and it would more appropriate to use notation $\mathcal{E} \dim X$ for that dimension. Spaces X, Y are of the form $K \times [0,1)$ where K is a classical Pontryagin surface, that is a 2-dimensional compact space with $\dim_{\mathbb{Q}} K = \dim_{\mathbb{Z}_q} K = 1$ for every prime $q \neq p$ and $\dim_{\mathbb{Z}_p} K = 2$ for some prime p. The coarse structure on $X = K \times [0,1)$ is the topological coarse structure (in terms of the book [Ro]) or the continuously controlled coarse structure (in terms of [Gra]) induced by the compactification $\overline{X} = K \times [0,1]$ of X, and $\overline{X}$ is certainly metrizable.

We want to explain that these examples do not cover the case of the classical asymptotic dimension, introduced in [Gr], and for which we have constructed our asymptotically self-similar Pontryagin surfaces. The usual asymptotic dimension of a metric space (X,d) is associated with the bounded coarse structure $\mathcal{E}_d$, $\operatorname{asdim} X = \mathcal{E}_d \dim X$, where a set $E \subset X \times X$ is controlled, $E \in \mathcal{E}_d$, if and only if $\sup\{d(x,x') : (x,x') \in E\}$ is finite. However, it is known that the coarse structure induced by any metrizable compactification is nonmetrizable, see [Ro, Example 2.53 and Remark 2.54]. In particular, the coarse structure of $X = K \times [0,1)$ above is nonmetrizable, i.e. there is no metric d on X for which $\mathcal{E} = \mathcal{E}_d$, and therefore, it does not make sense to compare $\mathcal{E} \dim X$ with the classical asymptotic dimension. Rather, arguments from [Gra, Theorem 2.5.7] show that $\mathcal{E} \dim X = \dim K + 1$ coincides with the topological dimension of X, $\mathcal{E} \dim X = \dim X$.

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