Birational symplectic manifolds and their deformations

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1 Introduction

Compact complex manifolds X^{2n} with holonomy group $Sp(n)$ can algebraically be characterized as simply connected compact Kähler manifolds with a unique (up to scalars) holomorphic symplectic two-form ([2]). These manifolds, which are higher-dimensional analogues of K3 surfaces, are called irreducible symplectic.

Beauville was able to generalize the local Torelli theorem, one of the fundamental results in the theory of K3 surfaces, to all irreducible symplectic manifolds. His results show that there exists a (coarse) moduli space $\mathcal{M}$ of marked irreducible symplectic manifolds and that the period map

$$P : \mathcal{M} \to \mathbf{P}(\Gamma \otimes \mathbb{C})$$

is étale over $Q \subset \mathbf{P}(\Gamma \otimes \mathbb{C})$ – an open subset of a quadric defined by $q(x) = 0$ and $q(x+\bar{x}) > 0$. By definition, a marking is an isomorphism of lattices $\sigma : H^2(X, \mathbb{Z}) \cong \Gamma$, where $H^2(X, \mathbb{Z})$ is endowed with the quadratic form defined in [2] and Γ is a fixed lattice.

For K3 surfaces the moduli space $\mathcal{M}$ consists of two connected components which can be identified by $(X, \sigma) \mapsto (X, -\sigma)$. The global Torelli theorem for K3 surfaces asserts that the period map P restricted to either of the two components, say $\mathcal{M}_0$, is surjective and 'almost injective'. More precisely, if (X, σ) and (X', σ') are two points in $P_0^{-1}(x)$, then (X, σ), $(X', \sigma') \in \mathcal{M}_0$ are non-separated and the underlying X and X' are isomorphic K3 surfaces containing at least one (-2)-curve. Furthermore, for $x \in Q$ in the complement of the union of countably many proper closed subsets the fibre $P_0^{-1}(x)$ is a single point. In short, the failure of the injectivity of the period map P_0 is due to the non-separatedness of $\mathcal{M}_0$ and two non-separated points are given by one K3 surface equipped with two different markings related by reflections orthogonal to (-2)-curves.

In the higher-dimensional situation, the global Torelli theorem does not hold, i.e. an isomorphism of Hodge structures $H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$ compatible with the quadratic forms does not imply $X \cong X'$. In fact, for any two birational irreducible symplectic manifolds X and X' one finds markings σ and σ' such that $P(X, \sigma) = P(X', \sigma')$. Due to an example of Debarre, birational X and X' need not be isomorphic in higher dimensions.

Although, only little evidence can be provided, we cannot resist to formulate the following (cf. [17]):

Speculation *(Global Torelli theorem) The period map P_0 is almost injective, i.e. two points (X, σ) and $(X'\sigma')$ in the same fibre of P_0 are non-separated in $\mathcal{M}_0$. In particular, X and X' are birational.*

The birationality of X and X' follows from [14].
As the known counterexamples to the global Torelli theorem use birational manifolds X and X', the following conjecture can be regarded as a weaker version of this speculation:

Conjecture *Two irreducible symplectic manifolds X and X' are birational if and only if they correspond to non-separated points in the moduli space.*

This paper proves the conjecture in two fairly general cases.

Theorem 4.7 *If X and X' are projective irreducible symplectic manifolds which are birational and isomorphic in codimension two (cf. 2.2), then the corresponding points in the moduli space of symplectic manifolds are non-separated.*

Dropping the assumption on the codimension and the projectivity, but restricting to Mukai's elementary transformation, the most explicit birational correspondence, one can prove

Theorem 3.4 *If X' is the elementary transformation of an irreducible symplectic manifold X along a smooth $\mathbb{P}_N$-bundle of codimension N, then X and X' correspond to non-separated points in the moduli space.*

Both results combined will be used in Sect. 5 to deduce the conjecture for projective X and X' and birational correspondences which in codimension two are given by elementary transformations (cf. 5.5).

Unfortunately, only few examples of irreducible symplectic manifolds are knwon. Higher-dimensional examples were first described by Beauville and Fujiki. Starting with a K3 surface S, Beauville showed that the Hilbert schemes $Hilb^n(S)$ of zero-dimensional subschemes are irreducible symplectic.
As shown by Mukai [15], moduli spaces of stable sheaves on a K3 surface also admit a (holomorphic) symplectic structure. That these spaces are irreducible symplectic, provided they are compact, was shown in [9] for the rank two case and in [18] in general. The idea in both approaches is to deform the underlying K3 surface S to a special K3 surface S_0, such that the moduli space of sheaves on S_0 is birational to the Hilbert scheme $Hilb^n(S_0)$. As the moduli space of sheaves on S_0 is a deformation of the moduli space of sheaves on S, this shows that any smooth moduli space is deformation equivalent to a manifold which is birational to an irreducible symplectic manifold. This is enough to conclude that the moduli spaces of higher rank sheaves are irreducible symplectic.
Proving this result [9], we observed the following phenomenon. Let S be a K3 surface and let H and H' be two different generic polarizations. Then the moduli spaces $X := M_H$ and $X' := M_{H'}$ of H-stable, respectively H'-stable, sheaves, which in general are not isomorphic, can be realized as the special fibres of the same family, i.e. equipped with appropriate markings they correspond to non-separated points in the moduli space $\mathcal{M}$. This observation motivated the study of the general question explained above. Moreover, since the birational correspondence between M_H and $M_{H'}$ looks quite similar to the one between moduli space and Hilbert scheme on the special K3 surface S_0, we conjectured that moduli spaces of higher rank sheaves are deformation equivalent to Hilbert schemes $Hilb^n(S)$.

The general results 4.7 and 3.4 do not cover this case, since the birational correspondence of moduli space and Hilbert scheme is not an isomorphism in codimension two. But using

the result of Sect. 5 one can at least prove the rank two case.

Theorem 6.3 *If S is a K3 surface, $\mathcal{Q} \in \mathrm{Pic}(S)$ indivisible, $2n := 4c_2 - c_1^2(\mathcal{Q}) - 6 \geq 10$ and H a generic polarization, then the moduli space $M_H(\mathcal{Q}, c_2)$ of H-stable rank two sheaves E with $\det(E) \cong \mathcal{Q}$ and $c_2(E) = c_2$ is deformation equivalent to $Hilb^n(S)$.*

The assumption $2n \geq 10$ is a technical condition, whereas the assumption on the determinant and the polarization is needed to guarantee the smoothness of the moduli space. We believe that the same result can be proved for the rank > 2 moduli spaces, as well. As there is evidence that our conjecture holds in general and that the higher rank case is an immediate consequence of it, we developed the necessary modification only in the rank two case.
Due to this result it seems that all known examples of irreducible symplectic manifolds are either deformation equivalent to some Hilbert scheme $Hilb^n(S)$, where S is a K3 surface, or to a generalized Kummer variety $K^n(A)$, where A is a two-dimensional torus.

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2 Preparations

2.1. Symplectic manifolds. A complex manifold X is called *symplectic* (in this paper!) if there exists a holomorphic two-form $\omega \in H^0(X, \Omega_X^2)$ which is non-degenerate at every point. Note that the existence of ω implies that the canonical bundle K_X is trivial. If X is compact, then the symplectic structure is unique if and only if $h^0(X, \Omega_X^2) = 1$. A simply connected compact Kähler manifold with a unqiue symplectic structure is called *irreducible symplectic*. By [2] X^{2n} is irreduible symplectic if and only if its holonomy is $Sp(n)$, i.e. it is irreducible hyperkähler.
For a compact irreducible symplectic Kähler manifold Beauville introduced a quadratic form on $H^2(X, \mathbb{C})$ by

$$\alpha \mapsto \frac{n}{2} \int (\omega\bar{\omega})^{n-1}\alpha^2 + (1-n) \int \omega^{n-1}\bar{\omega}^n\alpha \cdot \int \omega^n\bar{\omega}^{n-1}\alpha$$

where $\omega \in H^0(X, \Omega_X^2) = H^{2,0}$ is the symplectic form. Using Hodge decomposition $\alpha = a\omega + \varphi + b\bar{\omega}$ with $\varphi \in H^{1,1}(X)$ and assuming $\int(\omega\bar{\omega})^n = 1$ this form can be written as $\alpha \mapsto ab + (n/2)\int(\omega\bar{\omega})^{n-1}\varphi$. It turns out that this form is non-degenerate of index $(3, b_2 - 3)$. Moreover, a positive multiple of it is integral (cf. [2], [7]). The unique positive multiple making it to a primitive integral form is called the canonical form q on $H^2(X, \mathbb{C})$. Using the weight-two Hodge structure endowed with this quadratic form Beauville's local Torelli theorem says that $\mathcal{X}_t \mapsto [H^{2,0}(\mathcal{X}_t)] \in \mathbb{P}(H^2(X, \mathbb{C}))$ induces a local isomorphism of the Kuranishi space $Def(X)$ with the quadric in $\mathbb{P}(H^2(X, \mathbb{C}))$ defined by $q(\alpha) = 0$.

2.2. Birational symplectic manifolds. Let $f : X \to X'$ be a birational map between two compact symplectic manifolds and assume that the symplectic structure on X is unique. Then the largest open subset $U \subset X$ where f is regular satisfies $\mathrm{codim}(X \setminus U) \geq 2$. Moreover, one shows $f|_U$ is an embedding: Since ω_X is unique and $\mathbb{C} = H^0(X, \Omega_X^2) = H^0(U, \Omega_U^2)$, the pull-back $f^*\omega_{X'}$ is a non-trivial multiple of ω_X. Thus f is quasi-finite on U. Since it is generically one-to-one, it is an embedding. Note that, as a consequence, the symplectic structure on X' is unique, too. Thus, if $U \subset X$ and $U' \subset X'$ denote the maximal open subsets where f and f^{-1}, respectively, are regular, then $U \cong U'$ and $\mathrm{codim}(X \setminus U)$, $\mathrm{codim}(X' \setminus U') \geq 2$. A birational correspondence is by definition an *isomorphism in codimension two* if and only if $\mathrm{codim}(X \setminus U)$, $\mathrm{codim}(X' \setminus U') \geq 3$. Recall, that a birational map between two K3 surfaces can always be extended to an isomorphism.

If X is a projective manifold and $U \subset X$ is an open subset with $\mathrm{codim}(X \setminus U) \geq 2$, then the restriction defines an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}(U)$. In particular, for two birational projective manifolds X and X' with unique symplectic structures one has $\mathrm{Pic}(X) \cong \mathrm{Pic}(U) \cong \mathrm{Pic}(U') \cong \mathrm{Pic}(X')$. The corresponding line bundles on X and X' will usually be denoted by L and L', or M and M'. In particular, the Picard numbers $\rho(X)$ and $\rho(X')$ are equal. Using the exponential sequence one gets the same result for non-projective X and X'.

Frequently, we will use the following result due to Scheja [S]. If E is a locally free sheaf on X and $U \subset X$ is an open subset, then the restriction map $H^i(X, E) \to H^i(U, E|_U)$ is injective for $i \leq \mathrm{codim}(X \setminus U) - 1$ and bijective for $i \leq \mathrm{codim}(X \setminus U) - 2$. In particular, this can be applied to the line bundles L and L'. Thus, $H^0(X, L) = H^0(U, L|_U) = H^0(U', L'|_{U'}) = H^0(X', L')$ and if $\mathrm{codim}(X' \setminus U') \geq 3$ we get $H^1(X, L) \subset H^1(X', L')$.

If X and X' are birational irreducible symplectic manifolds, then there exists an isomorphism between their weight-two Hodge structures compatible with the canonical forms q_X and $q_{X'}$ ([17], [18]).

2.3. Deformations. Any compact Kähler manifold X with trivial canonical bundle K_X has unobstructed deformations, i.e. the base space of the Kuranishi family $Def(X)$ is smooth. This is originally due to Bogomolov, Tian and Todorov ([5], [22], [23]). For an algebraic proof see [19] and [11].

If L is a line bundle on X, such that the cup-product $c_1(L) : H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O}_X)$ is surjective, then the deformations of the pair (X, L) are unobstructed as well. This follows from the fact that the infinitesimal deformations of (X, L) are parametrized by $H^1(X, \mathcal{D}(L))$ and the obstructions are contained in $H^2(X, \mathcal{D}(L))$. Here $\mathcal{D}(L)$ is the sheaf of differential operators of order ≤ 1 on L. The symbol map induces an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}(L) \longrightarrow \mathcal{T}_X \longrightarrow 0$$

whose boundary map $H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O}_X)$ is the cup-product with $c_1(L)$. In particular, $H^2(X, \mathcal{D}(L)) \to H^2(X, \mathcal{T}_X)$ is injective. Since X is unobstructed, all obstructions of (X, L) vanish.

All this can be applied to irreducible symplectic manifolds. Using $H^1(X, \mathcal{T}_X) \cong H^1(X, \Omega_X)$ one finds that $Def(X)$ is smooth of positive dimension. Any small deformation of X is again Kähler (cf. [12]) and irreducible symplectic. In fact, any Kähler deformation of X is irreducible symplectic [2]. Under the isomorphism $H^1(X, \mathcal{T}_X) \cong H^1(X, \Omega_X)$ the kernel

of $c_1(L) : H^1(X,\mathcal{T}_X) \to H^2(X,\mathcal{O}_X) = \mathbb{C}$ is identified with the kernel of $q(c_1(L), \ ) : H^1(X,\Omega_X) \to \mathbb{C}$ (cf. [2]). In particular, if L is non-trivial, then $c_1(L) : H^1(X,\mathcal{T}_X) \to H^2(X,\mathcal{O}_X)$ is surjective and thus $Def(X,L)$ is a smooth hypersurface of $Def(X)$. For the tangent space of $Def(X,L)$ we have $T_0Def(X,L) \cong H^1(X,\mathcal{D}(L)) \cong \ker(H^1(X,\mathcal{T}_X) \stackrel{c_1(L)}{\longrightarrow} H^2(X,\mathcal{O}_X)) \cong \ker(H^1(X,\mathcal{T}_X) \cong H^1(X,\Omega_X) \stackrel{q(c_1(L),\ )}{\longrightarrow} \mathbb{C})$. If $c_1(L)$ and $c_1(M)$ are linearly independent, then the deformation spaces $Def(X,L)$ and $Def(X,M)$ intersect transversely.

2.4. Moduli spaces. Due to Beauville's local Torelli theorem one can easily construct a moduli space $\mathcal{M}$ of marked irreducible symplectic manifolds. Here a marking consists of an isomorphism of $H^2(X,\mathbb{Z})$ with a fixed lattice compatible with the quadratic form q. As for K3 surfaces the space of marked irreducible symplectic Kähler manifolds is smooth but non-separated. In contrast to the K3 surface case, the moduli space $\mathcal{M}$ is in general not fine. This is due to the fact that higher-dimensional irreducible symplectic manifolds permit automorphisms inducing the identity on $H^2(X,\mathbb{Z})$ (cf. [3]).
The quotient of $\mathcal{M}$ by the orthogonal group of (H^2,q) is the moduli space of unmarked manifolds, but this space is not expected to have any reasonable analytic structure. The theme of this paper is to prove statements like: X and X' correspond to non-separated points in the moduli space. Here, we usually refer to the moduli space of marked manifolds, though this distinction does not really matter for our purposes. Explicitly, this means that there are two one-dimensional deformations $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ (S is smooth), which are isomorphic over $S \setminus \{0\}$ and the special fibres are $\mathcal{X}_0 \cong X$ and $\mathcal{X}'_0 \cong X'$.

3 Elementary transformations

An explicit birational correspondence between two symplectic manifolds was introduced by Mukai [15]. We briefly want to recall the construction.
Let X be a complex manifold of dimension $2n$ which admits a holomorphic everywhere non-degenerate two-form $\omega \in H^0(X,\Omega_X^2)$. Furthermore, let $P \subset X$ be a closed submanifold which itself is a projective bundle $P = \mathbf{P}(F) \stackrel{\phi}{\longrightarrow} Y$. Here, F is a rank-$(N+1)$ vector bundle on the manifold Y. Using the symplectic structure one can define the elementary transformation X' of X along P as follows.
Since a projective space $\mathbb{P}_N$ does not admit any regular two-form, the restriction of ω to any fibre of ϕ is trivial. More is true, the relative tangent bundle $\mathcal{T}_\phi$ of ϕ is orthogonal to $\mathcal{T}_P$ with respect to the restriction of ω, i.e. $\omega|_P : \mathcal{T}_\phi \times \mathcal{T}_P \to \mathcal{O}_P$ vanishes. Indeed, this follows from the isomorphism $H^0(Y,\Omega_Y^2) \cong H^0(P,\Omega_P^2)$, i.e. $\omega|_P$ is the pull-back of a two-form on Y. Thus the composition of $\mathcal{T}_P \subset \mathcal{T}_X|_P$ with the isomorphism $\mathcal{T}_X|_P \cong \Omega_X|_P$ and the projection $\Omega_X|_P \longrightarrow\!\!\!\!\!\rightarrow \Omega_P \longrightarrow\!\!\!\!\!\rightarrow \Omega_\phi$ vanishes. Hence ω induces a vector bundle homomorphism $\mathcal{N}_{P/X} \cong \mathcal{T}_X|_P/\mathcal{T}_P \to \Omega_\phi$.
Now let $\operatorname{codim} P = N$. Then both vector bundles $\mathcal{N}_{P/X}$ and Ω_ϕ are of rank N and, since ω is non-degenerate, the homomorphism $\mathcal{N}_{P/X} \to \Omega_\phi$ is an isomorphism.
Let $\tilde{X} \to X$ denote the blow-up of X in $P \subset X$ and let $D \subset \tilde{X}$ be the exceptional divisor. The projection $D \to P$ is isomorphic to the projective bundle $\mathbf{P}(\mathcal{N}_{P/X}) \cong \mathbf{P}(\Omega_\phi) \to P$.
The natural isomorphism of the incidence variety $\{(x,H)|x \in H\} \subset \mathbb{P}_N \times \mathbb{P}_N^*$ as a projective

bundle over $\mathbf{P}_N$ with the projective bundle $\mathbf{P}(\Omega_{\mathbf{P}_N}) \to \mathbf{P}_N$ can be generalized to the relative situation, i.e. there is a canonical embedding $D = \mathbb{P}(\Omega_\phi) \subset \mathbb{P}(F) \times_Y \mathbb{P}(F^*)$ compatible with the projection to $\mathbb{P}(F)$. The other projection $D \to \mathbf{P}(F^*)$ is a projective bundle as well. If $\mathcal{O}_{\tilde{X}}(D)$ restricts to $\mathcal{O}(-1)$ on every fibre of $D \to \mathbf{P}(F^*)$ then there exists a blow-down $\tilde{X} \to X'$ to a smooth manifold X' such that $D \subset \tilde{X}$ is the exceptional divisor and $D \to X'$ is the projection $D \to \mathbf{P}(F^*) \subset X'$ (cf. [8]). Adjunction formula shows that $\mathcal{O}_{\tilde{X}}(D)$ indeed satisfies this condition.

Definition 3.1 *$X' = elm_P X$ is called the elementary transformation of the symplectic manifold X along the projective bundle P.*

Mukai also shows that an elementary transformation $elm_P X$ of a symplectic manifold X is again symplectic.

Example 3.2 In the case of a K3 surface S, which is a two-dimensional symplectic manifold, and a (-2)-curve $P = \mathbb{P}_1 \subset S$ one obviously has $elm_P S \cong S$. The Hilbert scheme $X := Hilb^n(S)$, which is irreducible symplectic, then contains the projective space $\mathbf{P}_n \cong S^n(P) = Hilb^n(P)$. The elementary transformation of $Hilb^n(S)$ along this projective space is in general not isomorphic to $Hilb^n(S)$. This is due to an example of Debarre [6]. Though in his example the K3 surface S, and hence $X = Hilb^n(S)$, is only Kähler, it is expected that one can also find examples $X \ncong elm_P X$, where X is projective. Also note that there are examples where an elementary transformation of $Hilb^n(S)$ is isomorphic to $Hilb^n(S)$ ([3]).

The following question was raised in [17].

Question 3.3 *Are the symplectic manifolds X and $X' = elm_P X$ deformation equivalent?*

We want to give an affirmative answer to this question in the case of compact Kähler manifolds.

Theorem 3.4 *Let X be a compact symplectic Kähler manifold and let $P \subset X$ be a smooth $\mathbf{P}_N$-bundle of codimension N. Then there exist two smooth proper families $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ over a smooth and one-dimensional base S, such that $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic as families over $S \setminus \{0\}$ and the fibres over $0 \in S$ satisfy $\mathcal{X}_0 \cong X$ and $\mathcal{X}'_0 \cong X' \cong elm_P X$.*

Note that the theorem is in fact stronger than what the original question suggests. The theorem shows that X and X' correspond to non-separated points in the moduli space of symplectic manifolds. In particular, one has

Corollary 3.5 *The higher-weight Hodge structures of X and $elm_P X$ are isomorphic.* □

The following lemma is needed for the proof of the theorem. Consider a deformation $\mathcal{X} \to S$ of X and assume that S is smooth and one-dimensional. Let $v \in H^1(X, \mathcal{T}_X)$ be its Kodaira-Spencer class, i.e. $\mathbb{C} \cdot v$ is the image of the Kodaira-Spencer map $T_0 S \to H^1(X, \mathcal{T}_X)$. Furthermore, denote by $\bar{v} \in H^1(X, \Omega_X)$ the image of v under the isomorphism $H^1(X, \mathcal{T}_X) \cong H^1(X, \Omega_X)$ induced by the symplectic structure.

Lemma 3.6 *Assume that $\bar{v} \in H^1(X, \Omega_X)$ is a Kähler class. Then the normal bundle $\mathcal{N}_{P/\mathcal{X}}$ is isomorphic to $\phi^* F^* \otimes \mathcal{O}_\phi(-1)$.*

Proof: We certainly can assume that Y is connected and hence $H^1(P, \mathcal{N}_{P/X}) \cong H^1(P, \Omega_\phi) \cong H^0(Y, \mathcal{O}_Y) \cong \mathbb{C}$.
By construction, the isomorphism $\mathcal{N}_{P/X} \cong \Omega_\phi$ commutes with the projections $\mathcal{T}_X \to \mathcal{N}_{P/X}$, $\Omega_X \to \Omega_\phi$ and the symplectic structure $\mathcal{T}_X \cong \Omega_X$. In particular, the image ξ of v under $H^1(X, \mathcal{T}_X) \to H^1(P, \mathcal{N}_{P/X})$ is non-zero if and only if $\bar{v}$ maps to a non-zero class under $H^1(X, \Omega_X) \to H^1(P, \Omega_\phi)$. Since $\bar{v}$ is Kähler and thus its restriction to the fibres of ϕ non-trivial, one concludes that ξ is the extension class of the unique (up to scalars) non-trivial extension of $\mathcal{O}_P$ by $\mathcal{N}_{P/X} \cong \Omega_\phi$. Thus it is isomorphic to the relative Euler sequence

$$0 \longrightarrow \Omega_\phi \longrightarrow \phi^* F^* \otimes \mathcal{O}_\phi(-1) \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

Therefore, it suffices to show that ξ is also the extension class of the canonical sequence

$$0 \longrightarrow \mathcal{N}_{P/X} \longrightarrow \mathcal{N}_{P/\mathcal{X}} \longrightarrow \mathcal{N}_{X/\mathcal{X}}|_P \longrightarrow 0,$$

where we use $\mathcal{N}_{X/\mathcal{X}} \cong \mathcal{O}_X$. This follows easily from the definition of the Kodaira-Spencer class v as the extension class of

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathcal{X}}|_X \longrightarrow \mathcal{N}_{X/\mathcal{X}} \longrightarrow 0.$$

□

Proof of 3.4: By 2.3 a one-dimensional deformation $\mathcal{X} \to S$ of X such that $\bar{v}$ is Kähler always exists. Denote the blow-up of $\mathcal{X}$ in P by $\tilde{\mathcal{X}} \to \mathcal{X}$. By lemma 3.6 the exceptional divisor $\mathcal{D} \to P$ is isomorphic to the projective bundle $\mathbf{P}(\phi^* F^*) \to P$. Obviously, $\mathbb{P}(\phi^* F^*) \cong \mathbb{P}(F) \times_Y \mathbf{P}(F^*)$. Now consider the second projection $\mathcal{D} \cong \mathbf{P}(\phi^* F^*) \to \mathbf{P}(F^*)$. As before one checks that $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ restricts to $\mathcal{O}(-1)$ on every fibre of this projection, i.e. the condition of the Nakano-Fujiki criterion is satisfied. Thus $\tilde{\mathcal{X}}$ can be blown-down to a smooth manifold $\mathcal{X}'$ such that the exceptional divisor $\mathcal{D}$ is contracted to $\mathbb{P}(F^*)$. By the very construction $\mathcal{X}' \leftarrow \tilde{\mathcal{X}} \to \mathcal{X}$ is compatible with $X' \leftarrow \tilde{X} \to X$, i.e. $\mathcal{X}' \to S$ is a smooth proper family, isomorphic to $\mathcal{X}$ over $S \setminus \{0\}$, and its special fibre $\mathcal{X}'_0$ is isomorphic to X'. □

Note that the two families $\mathcal{X}$ and $\mathcal{X}'$ are not isomorphic. In particular, one gets the well-known

Corollary 3.7 *If X is a K3 surface with a (-2)-curve $P \subset X$, then there exist non-isomorphic families $\mathcal{X}, \mathcal{X}' \to S$ which are isomorphic over $S \setminus \{0\}$ and $\mathcal{X}_0 \cong \mathcal{X}'_0 \cong X$.* □

4 Non-separated points in the moduli space

In this section we discuss other situations where birational symplectic manifolds present non-separated points in their moduli space.
Elementary transformations, dealt with previously, define very explicit birational correspondences between symplectic manifolds. But birational correspondences encountered in the examples are usually more complicated. This section is devoted to general birational correspondences. The result is analogous to 3.4, though we restrict to projective manifolds and

birational correspondences which are isomorphisms in codimension two. Later (cf. Sect. 5) the result will be generalized to the case where in codimension two the birational correspondence is given by an elementary transformation.

Let us fix the following notations: X and X' denote irreducible symplectic manifolds which are isomorphic on the open sets $U \subset X$ and $U' \subset X'$ (cf. 2.2). If v is a class in $H^1(X, \mathcal{T}_X)$, then the symplectic structure $\mathcal{T}_X \cong \Omega_X$ induces a class $\bar{v} \in H^1(X, \Omega_X)$. The following proposition does not make any assumptions either on the projectivity of X or on the codimension of $X \setminus U$. It is not needed for the proof of the main theorem, but shows how and to what extent the idea of Sect. 3 works in the general context.

Proposition 4.1 *Let S be smooth and one-dimensional and let $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ be deformations of $\mathcal{X}_0 = X$ and $\mathcal{X}'_0 = X'$, respectively. If $\mathcal{X}$ and $\mathcal{X}'$ are S-birational and the Kodaira-Spencer class v of $\mathcal{X} \to S$ induces a class $\bar{v} \in H^1(X, \Omega_X)$ which is non-trivial on all rational curves in $X \setminus U$, then $\mathcal{X}|_{S\setminus\{0\}} \cong_S \mathcal{X}'|_{S\setminus\{0\}}$ (possibly after shrinking S to an open neighbourhood of 0).*

Remarks 4.2 *i)* $\bar{v}$ non-trivial on a rational curve means that the pull-back of $\bar{v} \in H^2(X, \mathbb{C})$ evaluated on the fundamental class of such a curve is non-trivial.
ii) The condition on v is satisfied if $\bar{v}$ is contained in the cone spanned (over $\mathbf{R}$) by classes which are ample on $X \setminus U$, e.g. if $\bar{v}$ is ample. Note that the rational curves could be singular and reducible.
iii) Whenever X is projective there are deformations with Kodaira-Spencer class v such that $\bar{v}$ is ample. The problem is to construct $\mathcal{X}' \to S$ simultaneously. If the codimensions of $X \setminus U$ and $X' \setminus U'$ are at least three, then the isomorphisms $H^1(X, \mathcal{T}_X) \cong H^1(U, \mathcal{T}_U) \cong H^1(U', \mathcal{T}_{U'}) \cong H^1(X', \mathcal{T}_{X'})$ suggest that deformations of X can be related to deformations of X' via the big open subsets U and U'. I don't know how to make this rigorous. In particular, it is not clear to me what deformations of U should really mean.
iv) In the proof of 3.4 the family $\mathcal{X}' \to S$ was constructed explicitly from $\mathcal{X} \to S$ as a blow-up followed by a blow-down. For the general situation this approach seems to fail.

Proof of 4.1: If the S-birational map $\mathcal{X} \to \mathcal{X}'$ does not extend to an isomorphism $\mathcal{X}_t \cong \mathcal{X}'_t$ for generic t, then there exists a surface $\mathcal{C}$ together with a flat morphism $\mathcal{C} \to S$ such that:
i) $\mathcal{C}$ is smooth and irreducible.
ii) For generic t the fibre $\mathcal{C}_t$ is a disjoint union of smooth rational curves.
iii) There exists a finite S-morphism $\alpha : \mathcal{C} \to \mathcal{X}$ that maps $\mathcal{C}_0$ to $X \setminus U$.
This follows from resolution of singularities: By shrinking S we can assume that there is a sequence of monoidal transformations $\mathcal{Z}_n \to \mathcal{Z}_{n-1} \to \ldots \to \mathcal{Z}_1 \to \mathcal{X}'$ with smooth centers, which either dominate S or are contained in the fibre over $0 \in S$, and such that there exists a morphism $\mathcal{Z}_n \to \mathcal{X}$ which resolves the birational map $\mathcal{X} \to \mathcal{X}'$. If $\mathcal{X}_t \to \mathcal{X}'_t$ does not extend to an isomorphism for generic t, then at least one monoidal transformation $\mathcal{Z}_i \to \mathcal{Z}_{i-1}$ with smooth center T_i dominating S occurs. Let i be maximal with this property. Next one finds a morphism $S' \to T_i$ from a smooth, irreducible curve S' such that the composition $S' \to T_i \to S$ is finite and smooth over $S \setminus \{0\}$. Then $\mathcal{Z}_i \times_{\mathcal{Z}_{i-1}} S' \to S'$ is a projective bundle. Since i is maximal, we have $(\mathcal{Z}_i \times_{\mathcal{Z}_{i-1}} S') \times_S S \setminus \{0\} \subset \mathcal{Z}_n \times_S S'$, Now pick a $\mathbf{P}_1$-bundle

contained in $\mathcal{Z}_i \times_{\mathcal{Z}_{i-1}} S' \to S'$ such that its restriction to $S' \times_S S \setminus \{0\}$ maps generically finite to $\mathcal{X}$ under $\mathcal{Z}_n \to \mathcal{X}$. The resolution of the closure of it in $\mathcal{Z}_n$ gives the surface $\mathcal{C}$.

Now we want to show how one can use the existence of $\mathcal{C}$ to derive a contradiction. First, we claim that the composition

$$T_t S \to H^1(\mathcal{X}_t, \mathcal{T}_{\mathcal{X}_t}) \cong H^1(\mathcal{X}_t, \Omega_{\mathcal{X}_t}) \xrightarrow{\alpha_t^*} H^1(\mathcal{C}_t, \Omega_{\mathcal{C}_t})$$

vanishes for generic t (Here, the first map is the Kodaira-Spencer map and the isomorphism is induced by the symplectic structure on $\mathcal{X}_t$). This is a generalization of an argument explained in the proof of 3.6. One can either use deformation theory to show that the existence of $\mathcal{C} \to S$ implies the vanishing of the obstruction to deform $\mathcal{C}_{t\neq 0} \to \mathcal{X}_{t\neq 0}$, which in turn gives the desired vanishing, or one makes this explicit by the following argument: Note that we can assume $\mathcal{C}_{t\neq 0} \cong \mathbb{P}_1$. Then, let $\mathcal{N}_t$ be the generalized normal sheaf of α_t, i.e. the cokernel of the injection $\mathcal{T}_{\mathcal{C}_t} \to \alpha_t^* \mathcal{T}_{\mathcal{X}_t}$. Since for $t \neq 0$ we know $\mathcal{T}_{\mathcal{C}_t} \cong \mathcal{T}_{\mathbf{P}_1}$ and $\mathrm{Hom}(\mathcal{T}_{\mathbf{P}_1}, \Omega_{\mathbf{P}_1}) = 0$, the pull-back of the symplectic structure on $\mathcal{X}_t$ to $\mathcal{C}_t$ induces for $t \neq 0$ a commutative diagram

$$\begin{array}{ccc} \alpha_t^* \mathcal{T}_{\mathcal{X}_t} & \twoheadrightarrow & \mathcal{N}_t \\ \downarrow & & \downarrow \\ \alpha_t^* \Omega_{\mathcal{X}_t} & \to & \Omega_{\mathcal{C}_t} \end{array}$$

Thus, in order to show that $T_t S \to H^1(\mathcal{C}_t, \Omega_{\mathcal{C}_t})$ is trivial, it is enough to prove that $T_t S \to H^1(\mathcal{X}_t, \mathcal{T}_{\mathcal{X}_t}) \to H^1(\mathbf{P}_1, \alpha_t^* \mathcal{T}_{\mathcal{X}_t}) \to H^1(\mathbf{P}_1, \mathcal{N}_t)$ vanishes. The image of this map is spanned by the extension class of

$$0 \longrightarrow \mathcal{N}_t \longrightarrow \alpha_t^*(\mathcal{T}_{\mathcal{X}}|_{\mathcal{X}_t})/\mathcal{T}_{\mathcal{C}_t} \longrightarrow \alpha_t^*(\mathcal{T}_{\mathcal{X}}|_{\mathcal{X}_t}/\mathcal{T}_{\mathcal{X}_t}) \longrightarrow 0$$

(cf. proof of 3.6). Since $\mathcal{N}_{\mathcal{C}_t/\mathcal{C}}$ together with the natural inclusion $\mathcal{N}_{\mathcal{C}_t/\mathcal{C}} \subset \alpha_t^*(\mathcal{T}_{\mathcal{X}}|_{\mathcal{X}_t})/\mathcal{T}_{\mathcal{C}_t}$ induced by $\mathcal{T}_{\mathcal{C}} \to \alpha^* \mathcal{T}_{\mathcal{X}}$ splits this sequence, we conclude that $T_t S \to H^1(\mathbf{P}_1, \mathcal{N}_t)$ is trivial. Hence $T_t S \to H^1(\mathcal{C}_t, \Omega_{\mathcal{C}_t})$ is trivial as well.
The Kodaira-Spencer map $\mathcal{T}_S \to R^1\pi_* \mathcal{T}_{\mathcal{X}/S}$ composed with the isomorphism $R^1\pi_* \mathcal{T}_{\mathcal{X}/S} \cong R^1\pi_* \Omega_{\mathcal{X}/S}$ provides a global section of $R^1\pi_* \Omega_{\mathcal{X}/S} \otimes \Omega_S$. Trivializing $\mathcal{T}_S$ we can think of it as an element in $H^0(S, R^1\pi_* \Omega_{\mathcal{X}/S})$ or, using Hodge decomposition, as a C^∞-section of $R^2\pi_* \mathbb{C}_{\mathcal{X}} \otimes \mathcal{O}_S$. Moreover, making S small enough we have $R^2\pi_* \mathbb{C}_{\mathcal{X}} \cong H^2(\mathcal{X}, \mathbb{C})$. Thus $\mathcal{T}_S \to R^1\pi_* \mathcal{T}_{\mathcal{X}/S}$ induces a C^∞-map $t \mapsto \bar{v}_t \in H^2(\mathcal{X}, \mathbb{C})$. Restricting it to $\mathcal{C}$ we get $\bar{w}_t \in H^2(\mathcal{C}, \mathbb{C})$. The vanishing we just proved implies $\langle \bar{w}_t, [\mathcal{C}_t] \rangle = 0$ for $t \neq 0$. Since also $t \mapsto [\mathcal{C}_t] \in H^2(\mathcal{C}, \mathbb{C})$ is continous, we can conclude $\langle \bar{w}_0, [\mathcal{C}_0] \rangle = 0$. This contradicts the assumption on $\bar{v}|_{X \setminus U}$, since $\mathcal{C}_0$ as a degeneration of rational curves is still rational, though singular, reducible or even non-reduced [20]. □

If we in addition assume that X' is projective, then birational deformations of X and X' can be produced using the following proposition.

Proposition 4.3 *Suppose $L' \in Pic(X')$ is very ample and the corresponding line bundle $L \in \mathrm{Pic}(X)$ satisfies $H^1(X, L^n) = 0$ for $n > 0$. Let $\mathcal{X} \to S$ be a deformation of $X = \mathcal{X}_0$ over a smooth and one-dimensional base S and assume that there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}_0 := \mathcal{L}|_{\mathcal{X}_0} \cong L$. Then, replacing S by an open neighbourhood of $0 \in S$ if necessary, there exists a deformation $\mathcal{X}' \to S$ of $\mathcal{X}'_0 = X'$ which is S-birational to $\mathcal{X}$.*

Proof: First, shrink S to the open subset of points $t \in S$ such that $H^1(\mathcal{X}_t, \mathcal{L}_t) = 0$. Since $H^1(X, L) = 0$, this is an open neighbourhood of $t = 0$. By base change theorem (cf. [10], III. 12.11) $h^0(\mathcal{X}_t, \mathcal{L}_t)$ is constant on S, since it can only jump at a point t if $H^1(\mathcal{X}_t, \mathcal{L}_t) \neq 0$. Hence $\pi_*\mathcal{L}$ is locally free on S with fibre $(\pi_*\mathcal{L})(t) = H^0(\mathcal{X}_t, \mathcal{L}_t)$.
By the very ampleness of L' the base locus $Bs(L)$ of L is contained in $X \setminus U$ and therefore of codimension at least 2. The set $\cup_{t\in S} Bs(\mathcal{L}_t)$ is a closed subset of $\mathcal{X}$ and hence $\operatorname{codim}_{\mathcal{X}_t} Bs(\mathcal{L}_t) \geq 2$ for t in an open neighbourhood of $t = 0$ (semicontinuity of the fibre dimension). Since $Bs(\mathcal{L}_t^n) \subset Bs(\mathcal{L}_t)$ we can assume that $\operatorname{codim}_{\mathcal{X}_t} Bs(\mathcal{L}_t^n) \geq 2$ for all $n > 0$ and $t \in S$.
The rational maps $\phi_{|\mathcal{L}_t|} : \mathcal{X}_t - - \to \mathbf{P}(H^0(\mathcal{X}_t, \mathcal{L}_t)^*)$, defined by the complete linear system $|\mathcal{L}_t|$, glue to a rational S-map $\phi : \mathcal{X} - - \to \mathbb{P}((\pi_*\mathcal{L})^*)$. Then ϕ is regular at all points of $\mathcal{X}_t \setminus Bs(\mathcal{L}_t)$ $(t \in S)$. Let $\mathcal{Z}$ be the scheme-theoretic closure of the graph Γ_ϕ of ϕ in $\mathcal{X} \times_S \mathbf{P}((\pi_*\mathcal{L})^*)$, i.e. the closure of Γ_ϕ with the reduced induced structure.
The projection $\varphi : \mathcal{Z} \to \mathcal{X}$ is isomorphic over every point of $\mathcal{X}_t \setminus Bs(\mathcal{L}_t)$, $t \in S$. Note that a fibre $\mathcal{Z}_t$ of $\mathcal{Z}$ over $t \in S$ does not necessarily coincide with the closure of the graph of $\phi_{|\mathcal{L}_t|}$. However, since $\mathcal{X}$ has irreducible fibres and hence Γ_ϕ, the generic fibre of $\mathcal{Z} \to S$ is irreducible as well. Thus, shrinking to an open neighbourhood of $t = 0$, we can assume that $\mathcal{Z}_t$ is irreducible for $t \neq 0$. In particular, $\mathcal{Z}_{t\neq 0}$ equals the closure of the graph of $\phi_{|\mathcal{L}_t|}$ in $\mathcal{X}_t \times \mathbf{P}(H^0(\mathcal{X}_t, \mathcal{L}_t)^*)$ at least set-theoretically. Since $\mathcal{Z}$ is integral, i.e. irreducible and reduced, and S is smooth and one-dimensional, the dominant morphism $\mathcal{Z} \to S$ is flat ([10], III. 9.7.).
Now consider the other projection $\psi : \mathcal{Z} \to \mathbb{P}((\pi_*\mathcal{L})^*)$ and denote its image by $\mathcal{X}' \subset \mathbf{P}((\pi_*\mathcal{L})^*)$. Strictly speaking, $\mathcal{X}'$ is the scheme-theoretic image of ψ and since $\mathcal{Z}$ is reduced, this is the image with the reduced induced structure. Since $\mathcal{X}'$ then is integral and $\mathcal{X}' \to S$ is dominant, $\mathcal{X}'$ is flat over S.
Obviously, X' is contained in $\mathcal{X}'_0$. To conclude that $X' = \mathcal{X}'_0$ it is enough to show that $h^0(X', \mathcal{O}(n)|_{X'}) \geq h^0(\mathcal{X}'_0, \mathcal{O}(n)|_{\mathcal{X}'_0})$ for $n \gg 0$, where $\mathcal{O}(1)$ is the tautological ample line bundle on $\mathbf{P}(H^0(\mathcal{X}_0, L)^*)$. Since $\mathcal{O}(1)|_{X'} \cong L'$ and $h^0(X', L'^n) = h^0(X, L^n)$, this is equivalent to $h^0(X, L^n) \geq h^0(\mathcal{X}'_0, \mathcal{O}(n)|_{\mathcal{X}'_0})$ for $n \gg 0$. For any n there exists an open neighbourhood $S_n \subset S$ of $0 \in S$, such that $H^1(\mathcal{X}_t, \mathcal{L}_t^n) = 0$ for $t \in S_n$. This follows from the vanishing of $H^1(X, L^n)$ for all n. On the intersection $\cap S_n \subset S$, which is the complement of countably many points, all the cohomology groups $H^1(\mathcal{X}_t, \mathcal{L}_t^n)$ vanish and therefore $h^0(\mathcal{X}_t, \mathcal{L}_t^n) = h^0(X, L^n)$. Using this and the flatness of $\mathcal{X}' \to S$, the inequality $h^0(X, L^n) \geq h^0(\mathcal{X}'_0, \mathcal{O}(n)|_{\mathcal{X}'_0})$ is equivalent to $h^0(\mathcal{X}_t, \mathcal{L}_t^n) \geq h^0(\mathcal{X}'_t, \mathcal{O}(n)|_{\mathcal{X}'_t})$ for $n \gg 0$. But the latter can be derived using the composition

$$H^0(\mathcal{X}'_t, \mathcal{O}(n)|_{\mathcal{X}'_t}) \xrightarrow{\psi^*} H^0(\mathcal{Z}_t, \psi^*\mathcal{O}(n)) \xrightarrow{i^*} H^0(\mathcal{X}_t \setminus Bs(\mathcal{L}_t), \mathcal{L}_t^n) \cong H^0(\mathcal{X}_t, \mathcal{L}_t^n).$$

Indeed, ψ^* is injective since $\mathcal{Z}_t \to \mathcal{X}'_t$ is surjective, and i^* is injective, since it is induced by the dense open embedding $\mathcal{X}_t \setminus Bs(\mathcal{L}_t) \subset \mathcal{Z}_t$ $(t \neq 0)$. The last isomorphism is a consequence of $\operatorname{codim}(\mathcal{X}_t \setminus Bs(\mathcal{L}_t)) \geq 2$. This shows that $\mathcal{X}'_0 = X'$. Shrinking S further we can also assume that $\mathcal{X}' \to S$ is smooth ([10],III. Ex. 10.2).
It remains to show the assertion on the birationality. Let $\mathcal{Z}^*$ and $\mathcal{X}'^*$ denote the fibre products $\mathcal{Z} \times_S (S \setminus \{0\})$ and $\mathcal{X}' \times_S (S \setminus \{0\})$, respectively. Stein factorization decomposes $\psi : \mathcal{Z}^* \to \mathcal{X}'^*$ into a finite morphism $f : \mathcal{Y} \to \mathcal{X}'^*$ and a morphism $\mathcal{Z}^* \to \mathcal{Y}$ with connected fibres. One first shows that $f : \mathcal{Y} \to \mathcal{X}'^*$ is in fact an isomorphism. Since $f_t : \mathcal{Y}_t \to \mathcal{X}'_t$ is finite, the line

bundle $f_t^*\mathcal{O}(1)$ is ample. Thus $f_t^*\mathcal{O}(n)$ is very ample for $n \gg 0$. In order to prove that f is an isomorphism, it is therefore enough to show that $f_t^* : H^0(\mathcal{X}'_t, \mathcal{O}(n)) \to H^0(\mathcal{Y}_t, f_t^*\mathcal{O}(n))$ is surjective. We argue as above: Consider

$$H^0(\mathcal{X}'_t, \mathcal{O}(n)|_{\mathcal{X}'_t}) \xrightarrow{f_t^*} H^0(\mathcal{Y}_t, f_t^*\mathcal{O}(n)) \hookrightarrow H^0(\mathcal{Z}_t, \psi^*\mathcal{O}(n)) \hookrightarrow H^0(\mathcal{X}_t, \mathcal{L}_t^n)$$

and use $h^0(\mathcal{X}'_t, \mathcal{O}(n)|_{\mathcal{X}'_t}) = h^0(\mathcal{X}'_0, \mathcal{O}(n)|_{\mathcal{X}'_0}) = h^0(X', L'^n) = h^0(X, L^n) = h^0(\mathcal{X}_t, \mathcal{L}_t^n)$ for all $t \in \cap S_n$. Hence f_t^* is bijective for t in the complement of countably many points and therefore $\mathcal{Y} \cong \mathcal{X}'^*$ after shrinking S. Thus $\mathcal{Z}^* \to \mathcal{X}'^*$ has connected fibres. On the other hand $\dim \mathcal{Z}_t = \dim \mathcal{X}_t = \dim \mathcal{X}'_t$. Hence $\mathcal{Z}_t \to \mathcal{X}'_t$ is birational for $t \neq 0$. □

Note that the condition $H^1(X, L^n) = 0$ is automatically satisfied if $\operatorname{codim}(X' \setminus U') \geq 3$, i.e. if X' and X are isomorphic in codimension two. Indeed, $H^1(X, L^n) \subset H^1(U, L^n|_U) = H^1(U', L'^n_{U'}) = H^1(X', L'^n) = 0$ by Kodaira vanishing and [21]. It is at this point that the assumption on the codimension of $X \setminus U$ enters.
Also note that the existence of $\mathcal{L}$ implies $q(c_1(L), \bar{v}) = 0$, where $\bar{v} \in H^1(X, \Omega_X)$ is induced by the Kodaira-Spencer class $v \in H^1(X, \mathcal{T}_X)$ of $\mathcal{X} \to S$ (cf. 2.3).
Next, combining 4.1 and 4.3 we get

Corollary 4.4 *Let X and X' be birational projective irreducible symplectic manifolds isomorphic in codimension two. Assume there exists a line bundle $L \in \operatorname{Pic}(X)$ and a class $\bar{v} \in H^1(X, \Omega_X)$ such that:*
- The induced line bundle $L' \in \operatorname{Pic}(X')$ is ample.
- The restriction of $\bar{v}$ to any rational curve in $X \setminus U$ is non-trivial.
- $q(c_1(L), \bar{v}) = 0$.
Then X and X' correspond to non-separated points in the moduli space.

Proof: By taking a high power of L we can assume that L' is very ample. Furthermore, $H^1(X, L^n) = H^1(X', L'^n) = 0$ for $n > 0$. The deformation space $Def(X, L)$ of the pair (X, L) is a smooth hypersurface of $Def(X)$. Since $q(c_1(L), \bar{v}) = 0$ and $T_0 Def(X, L) \cong \ker(H^1(X, \mathcal{T}_X) \cong H^1(X, \Omega_X) \xrightarrow{q(c_1(L),\ )} \mathbb{C})$ (cf. 2.3), the class $v \in H^1(X, \mathcal{T}_X)$ is tangent to $Def(X, L)$. Therefore, there exist a deformation $\mathcal{X} \to S$ over a smooth and one-dimensional base S with Kodaira-Spencer class v and a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}_0 \cong L$. Then Proposition 4.3 shows that there exists a deformation $\mathcal{X}' \to S$ of X' which is S-birational to $\mathcal{X}$ and we conclude by Proposition 4.1. □

Remarks 4.5 *i)* If $\mathbf{P}_n \cong P \subset X$ is of codimension n, then X and $X' := elm_P X$ satisfy the assumptions of the corollary provided they are projective. Indeed, if $L' \in \operatorname{Pic}(X')$ is ample, then either there exists an element $\bar{v} \in H^1(X, \Omega_X)$ orthogonal to $c_1(L)$ or X and X' are isomorphic. The restriction $\pm\bar{v}|_P$ is either ample, hence non-trivial on any rational curve in P, or zero. In the latter case, change $\bar{v}$ and L by a small rational multiple of an ample divisor H on X. Thus we get $\bar{v}_1 := \bar{v} + \beta c_1(H)$ and $L_1 := L + \gamma H$. By adjusting β and γ we can assume $q(c_1(L'_1), \bar{v}_1) = 0$ and L'_1 ample for small γ. Obviously, $\bar{v}_1|_P \neq 0$ and therefore $\bar{v}_1$ and L_1 satisfy the conditions of the corollary. Thus 3.4 for elementary transformations along a projective space can be seen as a corollary of 4.4 if X and X' are projective. Does 4.4 work

for general elementary transformations?
ii) It is sometimes hard to check if $\bar{v}$ and L satisfying the conditions of 4.4 can be found. I don't know the answer for the examples discussed in Sect. 6.

Using 4.3 one can in fact prove corollary 4.4 without the assumptions on v. The proof relies on the fact that a compact Moishezon Kähler manifold is projective. It can be used to prove the following

Lemma 4.6 *If X and X' are birational compact irreducible symplectic Kähler manifolds with $\rho(X) = \rho(X') = 1$ and X' is projective, then $X \cong X'$.*

Proof: X is Kähler and Moishezon, hence projective. Thus, if L' is the ample generator of $\mathrm{Pic}(X')$, then $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$ and either L or L^* is ample. Since $H^0(X, L^n) = H^0(X', L'^n) \neq 0$ for $n \gg 0$, one concludes that L is ample and hence $X \cong X'$. □

Note that the isomorphism can be chosen such that it extends the birational map.

Here now is the main theorem of this paper.

Theorem 4.7 *Let X and X' be projective irreducible symplectic manifolds which are birational and isomorphic in codimension two. Then X and X' correspond to non-separated points in their moduli space.*

Proof: Assume X and X' are not isomorphic. Then $\rho(X) \geq 2$. Let L' be very ample on X' and let L be the associated line bundle on X. Then $Def(X, L) \subset Def(X)$ is a smooth hypersurface of positive dimension $h^1(X, \Omega) - 1$. Since $\mathrm{Pic}(X)$ is countable and any line bundle $M \in \mathrm{Pic}(X)$ defines a smooth hypersurface $Def(X, M)$ intersecting $Def(X, L)$ transversely if $M^n \ncong L^m$ $(n \cdot m \neq 0)$ ([2] and 2.3), there exists a generic smooth and one-dimensional $S \subset Def(X, L)$ such that $S \cap Def(X, M) = \{0\}$ for all line bundles M linearly independent of L. Let $(\mathcal{X}, \mathcal{L}) \to S$ be the associated deformation of $(\mathcal{X}_0, \mathcal{L}_0) \cong (X, L)$. Then $\rho(\mathcal{X}_t) = 1$ for general $t \in S$, i.e. t in the complement of countably many points. Now apply Proposition 4.3. We get a deformation $\mathcal{X}' \to S$ of $\mathcal{X}'_0 \cong X'$ which is S-birational to $\mathcal{X}$. Moreover, the proof of 4.3 shows that there is a line bundle $\mathcal{L}'$ on $\mathcal{X}'$ such that $\mathcal{L}'_0 \cong L'$. For small t the fibre $\mathcal{X}_t$ is still Kähler and $\mathcal{L}'_t$ is still ample on $\mathcal{X}'_t$. Thus the lemma applies and shows $\mathcal{X}_t \cong \mathcal{X}'_t$ for general t extending the S-birational map $\mathcal{X} - - \to \mathcal{X}'$. Since the set of points $t \in S$, where $\mathcal{X}_t - - \to \mathcal{X}'_t$ cannot be extended to an isomorphism is closed, we can shrink S such that $\mathcal{X} - - \to \mathcal{X}'$ is an isomorphism over $S \setminus \{0\}$. □

We want to emphasize that the condition on the codimension of $X \setminus U$ is only needed in order to apply 4.3. If for the deformation $\mathcal{X} \to S$ considered in the proof the dimension $h^0(\mathcal{X}_t, \mathcal{L}_t^n)$ does not jump in $t = 0$, then the argument goes through. This will be discussed in length in the next section.
As an immediate consequence of the theorem we have the

Corollary 4.8 *If X and X' are as in theorem 4.7, then they are diffeomorphic and their weight-n Hodge structures are isomorphic for all n.* □

5 The codimension two case

As before, let X and X' be birational projective irreducible symplectic manifolds. Let $L' \in \mathrm{Pic}(X')$ be an ample line bundle and denote by $L \in \mathrm{Pic}(X)$ the corresponding line bundle on X. The assumption on the codimension of $X \setminus U$ in theorem 4.7 was only needed to ensure $H^1(X, L^n) = 0$ for $n > 0$. If $\mathrm{cod}(X \setminus U) = 2$, then $H^1(X, L^n)$ is not necessarily zero. Indeed, consider an elementary transformation of a four-dimensional manifold X along a projective plane $\mathbb{P}_2 \subset X$. Then a standard calculation shows $H^1(X, L^n) \neq 0$ if $n \geq 2$. The vanishing $H^1(X, L^n) = 0$ was only needed at one point in the line of arguments. Namely, we used it in proposition 4.3 to conclude that $h^0(\mathcal{X}_t, \mathcal{L}_t^n) \equiv const$ for a family $\mathcal{X} \to S$. One might hope that this holds for another reason. Indeed, if $X' = elm_P X$ is an elementary transformation in codimension two and $\mathcal{X} \to S$ is as in 3.4, then $h^0(\mathcal{X}_t, \mathcal{L}_t^n) \equiv const$. To prove this use the family $\mathcal{X}' \to S$ constructed explicitly in the proof of 3.4 and the equality $h^0(\mathcal{X}_t, \mathcal{L}_t^n) = h^0(\mathcal{X}_t', \mathcal{L}_t'^n) \equiv const$, since $H^1(X', L'^n) = 0$. For a general birational correspondence the situation is more complicated, since we need $h^0(\mathcal{X}_t, \mathcal{L}_t) \equiv const$ in the first place in order to construct $\mathcal{X}' \to S$ (cf. 4.3).

First, we will show that under the above assumption (L' ample) the condition $h^0(\mathcal{X}_t, \mathcal{L}_t^n) \equiv const$ holds true infinitesimally, i.e. for any deformation $\pi : (\mathcal{X}, \mathcal{L}) \to S = \mathrm{Spec}(k[\varepsilon])$ of (X, L) the direct image $\pi_* \mathcal{L}$ is locally free. This is not quite enough to prove 4.7 in complete generality, but makes it highly plausible.
Under an additional assumption (cf. 5.2) one can in fact prove $h^0(\mathcal{X}_t, \mathcal{L}_t^n) \equiv const$. This is the second goal of the section and the result 5.5 will be applied in Sect. 6 to moduli space and Hilbert scheme on a K3 surface.

Consider (X, L) as above and let s be a global section of L. Then there exists a Kuranishi space $Def(X, L, s)$ of deformations of the triple (X, L, s) together with the forgetful maps $Def(X, L, s) \to Def(X, L) \to Def(X)$. The induced map between the tangent spaces of $Def(X, L, s)$ and $Def(X, L)$ is surjective for all s if and only if for any deformation $\pi : (\mathcal{X}, \mathcal{L}) \to \mathrm{Spec}(k[\varepsilon])$ the direct image $\pi_* \mathcal{L}$ is locally free. Therefore, in order to prove that $\pi_* \mathcal{L}$ is locally free we have to describe the tangent spaces of $Def(X, L, s)$ and $Def(X, L)$ and the homomorphism between them. Note, if one could prove that $Def(X, L, s)$ is smooth, the infinitesimal result would immediately imply that $h^0(\mathcal{X}_t, \mathcal{L}_t) \equiv const$.
We already know $T_0 Def(X, L) \cong H^1(X, \mathcal{D}(L))$ (cf. 2.3).

Proposition 5.1 *i) The Zariski tangent space of $Def(X, L, s)$ is naturally isomorphic to the first hypercohomology of the complex*

$$\begin{array}{rccc} \mathcal{D}(L, s): & \mathcal{D}(L) & \xrightarrow{s} & L \\ & D & \mapsto & D(s) \end{array}$$

ii) The map between the Zariski tangent spaces $\mathbb{H}^1(X, \mathcal{D}(L, s))$ and $H^1(X, \mathcal{D}(L))$ is given by the E_1-spectral sequence relating hypercohomology and cohomology.
iii) If (X, L) and (X', L') are as above, then $\mathbf{H}^1(X, \mathcal{D}(L, s)) \to H^1(X, \mathcal{D}(L))$ is surjective.

Proof: i) and *ii)* are well-known ([24]). For *iii)* we write down the beginning of the spectral

sequence:

$$0 \to \mathbb{C} \to H^0(X,L) \to \mathbb{H}^1(X,\mathcal{D}(L,s)) \to H^1(X,\mathcal{D}(L)) \xrightarrow{s} H^1(X,L).$$

Therefore, $\mathbb{H}^1(X,\mathcal{D}(L,s)) \to H^1(X,\mathcal{D}(L))$ is surjective if and only if $H^1(X,\mathcal{D}(L)) \xrightarrow{s} H^1(X,L)$ vanishes. Hence, we have to show that the pairing

$$\begin{array}{ccccc} H^1(X,\mathcal{D}(L)) & \otimes & H^0(X,L) & \to & H^1(X,L) \\ (D & , & s) & \mapsto & D(s) \end{array}$$

is trivial. Consider the injections $H^1(X,\mathcal{D}(L)) \hookrightarrow H^1(U,\mathcal{D}(L_U))$ and $H^1(X,\mathcal{D}(L')) \hookrightarrow H^1(U',\mathcal{D}(L'_{U'}))$. It suffices to show that under the natural isomorphism $H^1(U,\mathcal{D}(L_U)) \cong H^1(U',\mathcal{D}(L'_{U'}))$, given by $L'_{U'} \cong L_U$, the two spaces are identified. Indeed, if so then the commutative diagram

$$\begin{array}{ccccccc} H^1(X,\mathcal{D}(L)) & \otimes & H^0(X,L) & \to & H^1(X,L) & \hookrightarrow & H^1(U,L_U) \\ \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong \\ H^1(X',\mathcal{D}(L')) & \otimes & H^0(X',L') & \to & H^1(X',L') & \hookrightarrow & H^1(U',L'_{U'}) \end{array}$$

and the vanishing $H^1(X',L') = 0$ prove the assertion. In order to compare $H^1(X,\mathcal{D}(L))$ and $H^1(X',\mathcal{D}(L'))$ as subspaces of $H^1(U,\mathcal{D}(L_U)) \cong H^1(U',\mathcal{D}(L'_{U'}))$ we make use of the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}(L) \longrightarrow \mathcal{T}_X \longrightarrow 0.$$

Its cohomology sequence provides the short exact sequence

$$0 \longrightarrow H^1(X,\mathcal{D}(L)) \longrightarrow H^1(X,\mathcal{T}_X) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow 0.$$

We first show that the two subspaces $H^1(X,\mathcal{T}_X) \hookrightarrow H^1(U,\mathcal{T}_U)$ and $H^1(X',\mathcal{T}_{X'}) \hookrightarrow H^1(U',\mathcal{T}_{U'})$ are identified under $H^1(U,\mathcal{T}_U) \cong H^1(U',\mathcal{T}_{U'})$. Using the symplectic structures this is equivalent to the analogous statement for Ω_X and $\Omega_{X'}$. Let $X \leftarrow Z \to X'$ be a smooth resolution of the birational correspondence $U \cong U'$. Then $H^{1,1}(X) \oplus \bigoplus_i \mathbb{C} \cdot D_i \cong H^{1,1}(Z) \cong H^{1,1}(X') \oplus \bigoplus_i \mathbb{C} \cdot D_i$, where the D_i's are the exceptional divisors. Since the D_i's are trivial on $U \cong U' \subset Z$, the induced isomorphism $H^{1,1}(X) \cong H^{1,1}(X')$ is compatible with restriction.
To conclude the proof we have to show that under the identification of $H^1(X,\mathcal{T}_X)$ and $H^1(X',\mathcal{T}_{X'})$ as subspaces of $H^1(U,\mathcal{T}_U)$ the homomorphisms $c_1(L) : H^1(X,\mathcal{T}_X) \to H^2(X,\mathcal{O}_X)$ and $c_1(L') : H^1(X',\mathcal{T}_{X'}) \to H^2(X',\mathcal{O}_{X'})$ have the same kernel. Since $\ker(c_1(L)\cdot\ )$ is identified with $\ker(q(c_1(L),\ ))$ under the isomorphism $H^1(X,\mathcal{T}_X) \cong H^1(X,\Omega_X)$, this follows immediately from the fact that $H^2(X,\mathbb{C}) \cong H^2(X',\mathbb{C})$ respects q_X and $q_{X'}$ (cf. [17],[18]). □

The proposition gives evidence that $h^0(\mathcal{X}_t,\mathcal{L}_t) \equiv const$ holds in general. In fact, I believe that the same technique should show the vanishing of the higher obstructions to deform (X,L,s), but I don't know how to prove this.
In the examples it seems as if a birational correspondence between irreducible symplectic manifolds might be non-isomorphic in codimension two, but that in such a case the birational correspondence is in codimension two given by an elementary transformation. Thus, it is not completely unlikely, that the following assumption is always satisfied. For the birational

correspondence between the moduli space of rank two sheaves and the Hilbert scheme this is established in Sect. 6.

Assumption 5.2 There exist open subsets $U \subset V \subset X$ and $U' \subset V' \subset X'$ such that $\operatorname{codim}(X \setminus V), \operatorname{codim}(X' \setminus V') \geq 3$ and $V' := elm_{V \setminus U} V$. In particular, we assume that $P := V \setminus U$ is a $\mathbb{P}_2$-bundle $\mathbb{P}(F) \to Y$ over a smooth not necessarily compact manifold Y. If X and X' are isomorphic in codimension two we set $U = V$ and $U' = V'$.

We are going to prove 4.7 under this additional assumption (without using 5.1).

First note that a modification of the proof of 3.4 immediately yields

Corollary 5.3 *Assume X and X' satisfy 5.2. If $\mathcal{X} \to S$ is a deformation as in the proof of 3.4 (i.e. $\bar{v}$ is non-trivial on the fibres of $P \to Y$), then there exists a smooth morphism $\mathcal{V}' \to S$ such that $\mathcal{V}'|_{S \setminus 0} \cong \mathcal{X}_{S \setminus \{0\}}$ and $\mathcal{V}_0 \cong V'$.* □

It can be used to prove

Proposition 5.4 *Let X and X' be as before, in particular L' ample, and assume that 5.2 is satisfied. If $(\mathcal{X}, \mathcal{L}) \to S$ is a deformation over a smooth and one-dimensional base S such that the class $\bar{v} \in H^1(X, \Omega_X)$ associated to the Kodaira-Spencer class is non-trivial on the fibres of $P \to Y$, then $h^0(\mathcal{X}_t, \mathcal{L}_t) \equiv const$ in a neighbourhood of $t = 0$.*

Since replacing L' by another ample line bundle (if necessary) ensures that the generic deformation $\mathcal{X} \to S$ in $Def(X, L)$ has a Kodaira-Spencer class v such that $\bar{v}$ is non-trivial on the fibres of $P \to Y$ (cf. 4.5), the proposition immediately shows

Corollary 5.5 *If X and X' are projective irreducible symplectic manifolds such that X' is an elementary transformation of X in codimension two, i.e. 5.2 holds, then X and X' present non-separated points in the moduli space.* □

Proof of 5.4: Let s be the local parameter of S at $0 \in S$ and let S_n denote the closed subspace $\operatorname{Spec}(k[s]/s^{n+1}) \subset S$. Furthermore, let $\mathcal{X}_n := \mathcal{X} \times_S S_n$ and $\mathcal{L}_n : \mathcal{L}|_{\mathcal{X}_n}$. In order to show that $h^0(\mathcal{X}_t, \mathcal{L}_t) \equiv const$, it suffices to prove that for all n the restriction $H^0(\mathcal{X}_n, \mathcal{L}_n) \to H^0(\mathcal{X}_{n-1}, \mathcal{L}_{n-1})$ is surjective. This will be achieved by comparing it with the analogous restriction maps for the family $\mathcal{V}' \to S$. For this purpose we introduce the following notations. $\mathcal{U}_n$ denotes the space $(U, \mathcal{O}_{\mathcal{X}_n}|_U)$ and is considered as a deformation of U over S_n. Analogously, let $\mathcal{V}'_n := \mathcal{V}' \times_S S_n$ and $\mathcal{U}'_n := (U', \mathcal{O}_{\mathcal{V}'_n}|_{U'})$, which is isomorphic to $\mathcal{U}_n$. The line bundle $\mathcal{L}$ induces a line bundle $\mathcal{L}'$ on $\mathcal{V}'$. Its restrictions to $\mathcal{V}'_n$ are denoted by $\mathcal{L}'_n$. In particular $\mathcal{L}'_0$ is isomorphic to $L'|_{V'}$.

First, $H^0(\mathcal{V}'_n, \mathcal{L}'_n) \to H^0(\mathcal{V}'_{n-1}, \mathcal{L}'_{n-1})$ is surjective for all n. Indeed, using the exact sequence

$$0 \longrightarrow L'|_{V'} \longrightarrow \mathcal{L}'_n \longrightarrow \mathcal{L}'_{n-1} \longrightarrow 0$$

this follows from $H^1(V', L'|_{V'}) = H^1(X', L') = 0$. Next, $H^0(\mathcal{U}'_n, \mathcal{L}'_n|_{\mathcal{U}'_n}) \to H^0(\mathcal{U}'_{n-1}, \mathcal{L}'_{n-1}|_{\mathcal{U}'_{n-1}})$ is surjective and $H^0(\mathcal{V}'_n, \mathcal{L}'_n) \to H^0(\mathcal{U}'_n, \mathcal{L}'_n|_{\mathcal{U}'_n})$ is an isomorphism. This is proved by induction

starting with $H^0(V', L'|_{V'}) = H^0(U', L'|_{U'})$ and the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^0(V', L'|_{V'}) & \to & H^0(\mathcal{V}'_n, \mathcal{L}'_n) & \to & H^0(\mathcal{V}'_{n-1}, \mathcal{L}'_{n-1}) & \to & 0 \\ & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\ 0 \to & H^0(U', L'|_{U'}) & \to & H^0(\mathcal{U}'_n, \mathcal{L}'_n|_{\mathcal{U}'_n}) & \to & H^0(\mathcal{U}'_{n-1}, \mathcal{L}'_{n-1}|_{\mathcal{U}'_{n-1}}) & \to & \end{array}$$

The isomorphism $H^0(\mathcal{U}'_n, \mathcal{L}'_n|_{\mathcal{U}'_n}) \cong H^0(\mathcal{U}_n, \mathcal{L}_n|_{\mathcal{U}_n})$ and a similar induction argument prove $H^0(\mathcal{X}_n, \mathcal{L}_n) \cong H^0(\mathcal{U}_n, \mathcal{L}_n)$ and $H^0(\mathcal{X}_n, \mathcal{L}_n) \twoheadrightarrow H^0(\mathcal{X}_{n-1}, \mathcal{L}_{n-1})$. In the analogous diagram one in addition has to use $H^1(X, L) \hookrightarrow H^1(U, L|_U)$. □

6 Application to moduli spaces of bundles on K3 surfaces

We briefly recall some facts from [9] that are necessary for our purposes.
Let S be a K3 surface, let $\mathcal{Q} \in \mathrm{Pic}(S)$ be an indivisible line bundle and let $c_2 \in \mathbb{Z}$ such that $2n := 4c_2 - c_1^2(\mathcal{Q}) - 6 \geq 10$. Assume that H is a generic polarization, i.e. an ample line bundle such that a rank two sheaf E with $\det(E) \cong \mathcal{Q}$ and $c_2(E) = c_2$ is H-semi-stable if and only if it is H-stable. Then the moduli space $M_H(\mathcal{Q}, c_2)$ of H-stable rank-two sheaves with determinant $\mathcal{Q}$ and second Chern number c_2 is smooth and projective. By [15] the moduli space $M_H(\mathcal{Q}, c_2)$ admits a symplectic structure.
Next, one finds a K3 surface S_0 such that $\mathrm{Pic}(S_0) \cong \mathbb{Z} \cdot H_0$, where H_0 is ample, and $H_0^2/2+3 = n$. In [9] we showed that under all these assumptions the moduli space $M_H(\mathcal{Q}, c_2)$ is deformation equivalent to the moduli space $M_{H_0}(H_0, n)$ of sheaves on S_0. In particular, $M_H(\mathcal{Q}, c_2)$ is irreducible symplectic if and only if $M_{H_0}(H_0, n)$ is irreducible symplectic. Moreover, both spaces have the same Hodge numbers.
In order to prove that $M_{H_0}(H_0, n)$ is irreducible symplectic we used Serre correspondence to relate this space to the Hilbert scheme $Hilb^n(S_0)$. Roughly, the generic sheaf $[E] \in M_{H_0}(H_0, n)$ admits exactly one global section and the zero set of this section defines a point in $Hilb^n(S_0)$. To make this more explicit we consider the moduli space N of H_0-stable pairs $(E, s \in H^0(S_0, E))$, such that $\det(E) \cong H_0$ and $c_2(E) = n$. The parameter in the stability condition for such pairs is chosen very small and constant. As explained in [9] the maps $(E, s) \mapsto Z(s)$ and $(E, s) \mapsto E$ define morphisms $\varphi : N \to Hilb^n(S_0)$ and $\psi : N \to M_{H_0}(H_0, n)$, respectively. For the fibers we have

$$\varphi^{-1}(Z) \cong \mathbb{P}(\mathrm{Ext}^1(I_Z \otimes H_0, \mathcal{O}_{S_0}))$$

and

$$\psi^{-1}(E) \cong \mathbb{P}(H^0(S_0, E)).$$

Generically, $h^1(S_0, I_Z \otimes H_0) = 1$ and $h^0(S_0, E) = 1$. Thus

$$X := Hilb^n(S_0) \xleftarrow{\varphi} N \xrightarrow{\psi} M_{H_0}(H_0, n) =: X'$$

defines a birational correspondence between irreducible symplectic manifolds.
Next, we want to show that $X \xleftarrow{\varphi} N \xrightarrow{\psi} X'$ satisfies the assumption 5.2.
Using the exact sequence

$$0 \longrightarrow I_Z \otimes H_0 \longrightarrow H_0 \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

the vanishing $H^1(S_0, H_0) = 0$ and $H_0^2/2 + 3 = n$, one shows $h^1(S_0, I_Z \otimes H_0) = 1 + h^0(S, I_Z \otimes H_0)$. Therefore, $f := \varphi \circ \psi^{-1}$ is regular at points Z which are not contained in any divisor $D \in |H_0|$.

Let $\mathcal{D} \to |H_0|$ denote the family of divisors parametrized by the complete linear system $|H_0|$ and let $Hilb^n(\mathcal{D}) \to |H_0|$ be the relative Hilbert scheme. Then f is regular on the complement U of the image of the natural map $g : Hilb^n(\mathcal{D}) \to Hilb^n(S_0) = X$. Since $\dim Hilb^n(\mathcal{D}) = n + h^0(S_0, H_0) - 1 = 2n - 2 = \dim Hilb^n(S_0) - 2$, the birational correspondence f is not isomorphic in codimension two.

Let $\mathcal{C} \to B \subset |H_0|$ denote the family of smooth curves. The relative Hilbert scheme over B is just the relative symmetric product $S^n(\mathcal{C}/B) \to B$, which factorizes naturally through the relative Picard $\mathrm{Pic}^n(\mathcal{C}/B) \to B$.

The fibre of the factorization $\phi : S^n(\mathcal{C}/B) \to \mathrm{Pic}^n(\mathcal{C}/B)$ over a point $L \in \mathrm{Pic}^n(\mathcal{C}_t)$ is naturally isomorphic to $\mathbb{P}(H^0(\mathcal{C}_t, L))$. Note that by Riemann-Roch $\chi(\mathcal{C}_t, L) = n - H_0^2/2 = 3$. Hence $h^0(\mathcal{C}_t, L) \geq 3$. Let $Y \subset \mathrm{Pic}^n(\mathcal{C}/B)$ denote the open set of line bundle $L \in \mathrm{Pic}^n(\mathcal{C}_t)$ such that $h^0(\mathcal{C}_t, L) = 3$ and let $\phi : P := \phi^{-1}(Y) \to Y$ be the induced $\mathbb{P}_2$-bundle.

Proposition 6.1 *i) The morphism $g : Hilb^n(\mathcal{D}) \to X$ restricted to P is an embedding.*
ii) The union V of U and $g(P)$ is open and $\mathrm{codim}(X \setminus V) \geq 3$.
iii) $V \xleftarrow{\varphi} \varphi^{-1}(V) \xrightarrow{\psi} \psi(\varphi^{-1}(V))$ is an elementary transformation along the $\mathbb{P}_2$-bundle P.

Proof: There is a number of little things to check.

First, by our assumption $n \geq 5$ we have $H_0^2 \geq 4$. Thus we can apply a result of Saint-Donat (cf. [16]) to conclude that H_0 is very ample. Hence $B \subset |H_0|$ is dense. Moreover, $Hilb^n(\mathcal{C})$ is dense in $Hilb^n(\mathcal{D})$ (cf. [1], Thm. 5).

Next, we show that $Y \subset \mathrm{Pic}^n(\mathcal{C}/B)$ is non-empty and, therefore, dense in $\mathrm{Pic}^n(\mathcal{C}/B)$. Indeed, if $x_1, ..., x_{n-2}$ are generic points in S_0, then there is exactly one smooth curve $C \in |H_0|$ containing them all. Let x_{n-1} and x_n be two more generic points on C and let $Z := \{x_1, ..., x_n\}$. Then $h^0(S_0, I_Z \otimes H_0) = 1$ and hence $h^1(S_0, I_Z \otimes H_0) = 2$. Using the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_0} \longrightarrow I_Z \otimes H_0 \longrightarrow \mathcal{O}_C(-Z) \otimes H_0 \longrightarrow 0$$

we get $h^1(C, \mathcal{O}_C(-Z) \otimes H_0) = h^1(S_0, I_Z \otimes H_0) + h^2(S_0, \mathcal{O}_{S_0}) = 3$ and therefore $h^0(C, \mathcal{O}_C(Z)) = 3$. Thus the line bundle $L := \mathcal{O}_C(Z)$ defines a point in Y. Note that one could invoke a result by Lazarsfeld [13] to prove $Y \neq \emptyset$. His result also shows that for the generic curve $\mathcal{C}_t$ the complemet of $Y \cap \mathrm{Pic}^n(\mathcal{C}_t) \subset \mathrm{Pic}^n(\mathcal{C}_t)$ has at least codimension four.

Since P is obviously smooth and any $Z \in \mathrm{Im}(g)$ satisfies $h^0(S_0, I_Z \otimes H_0) = 1$, the morphism g is an embedding on P.

By definition V is the intersection of the open set $\{Z | h^0(S_0, I_Z \otimes H_0) \leq 1\}$ and the complement of $g(Hilb^n(\mathcal{D}) \setminus Hilb^n(\mathcal{C}))$. Hence V is open. The assertion on the codimension follows from $Y \neq \emptyset$ and the irreducibility of $Hilb^n(\mathcal{D})$ (cf. [1]).

It remains to prove *iii)*. Here we make essential use of the moduli space N.

Let N_P denote $\varphi^{-1}(P)$. We first show that $\psi : N_P \to X'$ respects the projective bundle $\phi : P \to Y$, i.e. a fibre of ψ maps to a fibre of ϕ. Indeed, if $[E] \in X'$ and $s_1, s_2 \in H^0(S_0, E)$

are two linearly independent global sections, then we have a diagram

$$\begin{array}{ccccc} & & \mathcal{O}_{S_0} & & \mathcal{O}_{S_0} \\ & & s_1 \downarrow & & \tilde{s}_1 \downarrow \\ \mathcal{O}_{S_0} & \xrightarrow{s_2} & E & \longrightarrow & I_{Z(s_2)} \otimes H_0 \\ =\downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{S_0} & \xrightarrow{\tilde{s}_2} & I_{Z(s_1)} \otimes H_0 & \longrightarrow & \mathcal{H} \end{array}$$

Thus $\tilde{s}_1$ and $\tilde{s}_2$ vanish along the same curve $C \in |H_0|$ and $\mathcal{H} \cong \mathcal{O}_C(-Z(s_i)) \otimes H_0$ for $i = 1, 2$. Hence $\mathcal{O}_C(Z(s_1)) \cong \mathcal{O}_C(Z(s_2))$, i.e. $\phi(\varphi(E, s_1)) = \phi(\varphi(E, s_2))$.
This reduces assertion *iii)* to the following problem. Let $L \in \mathrm{Pic}^n(\mathcal{C}_t) \cap Y$, let $P_L := \mathbf{P}(H^0(\mathcal{C}_t, L)) \cong \mathbf{P}_2$ and let $N_{P_L} := \varphi^{-1}(P_L)$, which is a $\mathbf{P}_1$-bundle over $\mathbf{P}_2$. Identify $P_L \leftarrow N_{P_L} \rightarrow \psi(N_{P_L})$ with $\mathbf{P}_2 \leftarrow \mathbf{P}(\Omega_{\mathbf{P}_2}) \rightarrow \mathbf{P}_2^*$!!
The argument goes as follows. Any point $Z \in P_L$ gives an exact sequence

$$0 \longrightarrow \mathcal{O}_{S_0} \longrightarrow I_Z \otimes H_0 \longrightarrow \underset{\cong L^* \otimes K_{\mathcal{C}_t}}{\mathcal{O}_{\mathcal{C}_t}(-Z) \otimes H_0} \longrightarrow 0$$

Now use the canonical isomorphisms

$$\mathbf{P}(H^0(\mathcal{C}_t, L)) \cong \mathbf{P}(H^1(\mathcal{C}_t, L^* \otimes K_{\mathcal{C}_t})^*) \cong \mathbf{P}(\mathrm{Ext}^1(L^* \otimes K_{\mathcal{C}_t}, \mathcal{O}_{S_0}))$$

to obtain the exact sequence

$$0 \longrightarrow q^*\mathcal{O}_{S_0} \otimes p^*\mathcal{O}_{P_L}(1) \longrightarrow I_{\mathcal{Z}} \otimes q^*H_0 \otimes p^*\mathcal{O}_{P_L}(a) \longrightarrow q^*(L^* \otimes K_{\mathcal{C}_t}) \longrightarrow 0,$$

where q and p are the two projections from $S_0 \times P_L$ and $I_{\mathcal{Z}}$ is the ideal sheaf of the universal subscheme $\mathcal{Z} \subset S_0 \times P_L$. By restricting to $\{x\} \times P_L$, where $x \in S_0 \setminus \mathcal{C}_t$, we deduce $a = 1$. The push-forward under p induces the exact sequence

$$0 \longrightarrow R^1p_*(I_{\mathcal{Z}} \otimes q^*H_0) \otimes \mathcal{O}_{P_L}(1) \longrightarrow H^1(\mathcal{C}_t, L^* \otimes K_{\mathcal{C}_t}) \otimes \mathcal{O}_{P_L} \longrightarrow \mathcal{O}_{P_L}(1) \longrightarrow 0.$$

Hence $R^1p_*(I_{\mathcal{Z}} \otimes q^*H_0) \cong \Omega_{P_L}$. It is straightforward to identify $N_{P_L} \rightarrow P_L$ with $\mathbf{P}(R^1p_*(I_{\mathcal{Z}} \otimes q^*H_0)^*)$. Thus $(N_{P_L} \rightarrow P_L) \cong (\mathbf{P}(\mathcal{T}_{\mathbf{P}_2}) \rightarrow \mathbf{P}_2) \cong (\mathbf{P}(\Omega_{\mathbf{P}_2}) \rightarrow \mathbf{P}_2)$.
It remains to show that $\varphi : P_E := \mathbf{P}(H^0(S_0, E)) = \psi^{-1}(E) \rightarrow P_L$ is a linear embedding. On P_E we have

$$0 \longrightarrow \mathcal{O} \longrightarrow q^*E \otimes p^*\mathcal{O}_{P_E}(1) \longrightarrow (1 \times \varphi)^*(I_{\mathcal{Z}} \otimes q^*H_0) \otimes p^*\mathcal{O}_{P_E}(a) \longrightarrow 0,$$

where by abuse of notation q and p are again the projections from $S_0 \times P_E$. As above one finds $a = 2$. Taking direct images we obtain

$$0 \longrightarrow H^1(S_0, E) \otimes \mathcal{O}_{P_E}(1) \longrightarrow \varphi^*(R^1p_*(I_{\mathcal{Z}} \otimes q^*H_0)) \otimes \mathcal{O}_{P_E}(2) \longrightarrow \mathcal{O}_{P_E} \longrightarrow 0,$$

i.e. $0 \longrightarrow \mathcal{O}_{P_E}(1) \longrightarrow \varphi^*\Omega_{P_L} \otimes \mathcal{O}_{P_E}(2) \longrightarrow \mathcal{O}_{P_E} \longrightarrow 0$. Thus $\varphi^*\mathcal{O}_{P_L}(1) \cong \mathcal{O}_{P_E}(1)$. □

Remark: The identification $N_{P_L} \cong \mathbf{P}(\mathcal{T}_{P_L})$ seems to show that the birational correspondence described by N is not some kind of "nested elementary transformation": It is only in the codimension two case where one has $\mathbf{P}(\Omega_{P_L}) \cong \mathbf{P}(\mathcal{T}_{P_L})$.

Corollary 5.5 now immediately implies

Corollary 6.2 *If S_0 is a K3 surface with $\mathrm{Pic}(S_0) = \mathbb{Z} \cdot H_0$ and $H_0^2 \geq 4$, then $M_{H_0}(H_0, n)$ and $Hilb^n(S_0)$ correspond to non-separated points in the moduli space of symplectic manifolds ($n = H_0^2/2 + 3$).* □

Thus we can conclude

Theorem 6.3 *If S is a K3 surface, $\mathcal{Q} \in \mathrm{Pic}(S)$ indivisible, $2n := 4c_2 - c_1^2(\mathcal{Q}) - 6 \geq 10$ and H a generic polarization, then the moduli space $M_H(\mathcal{Q}, c_2)$ of H-stable rank two sheaves E with $\det(E) \cong \mathcal{Q}$ and $c_2(E) = c_2$ is deformation equivalent to $Hilb^n(S)$.* □

Note that in particular moduli space and Hilbert scheme are just different complex structures on the same differentiable manifold.

Remark: O'Grady works instead of S_0 with an elliptic surface and shows that every moduli space is deformation equivalent to a moduli space on an elliptic surface [18]. The birational correpondence between moduli space and Hilbert scheme on the elliptic surface is again given by Serre correspondence. The picture there is slightly more complicated than what we have encountered above. Nevertheless I believe, that also in his situation the assumptions 5.2 are satisfied and that moduli space and Hilbert scheme are deformation equivalent rank> 2 as well.

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