A converse of Bézout's theorem

by

Wolfgang Vogel

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

Martin–Luther–University
Dep. of Mathematics
P.O. Box
4010 Halle
German Democratic Republic

MPI/90-53

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1. Introduction and main result

The theorem of Bézout can be seen as a landmark in algebraic geometry and it has been extensively applied. It has taken more than three centuries of hard work by some of the foremost mathematicians to master Bézout's theorem. This theorem has been extended to intersections with arbitrary excess (see, e.g., [4], [12,15] [6], [9]). Another possible direction might be to state a converse of Bézout's theorem. It is not immediately clear how such a statement should look. Using the new multiplicity of [3] for improper intersections we will prove a converse of Bézout's theorem (see corollary 1). The aim of this paper is to prove the following statement.

<u>Theorem:</u> Let k be an algebraically closed field of arbitrary characteristic. Let X,Y be pure dimensional projective subschemes of P^n_k, say $X=\bigcup_{i=1}^{r} X_i$ and $Y=\bigcup_{j=1}^{s} Y_j$ where $\dim X = \dim X_i$, $\dim Y = \dim Y_j$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. Let $r_{ij} \geq 0$ be the integer such that $(X_i \cup Y_j)_{red}$ is contained in an $(n-r_{ij})$-plane but not lying in an $(n-r_{ij}-1)$-plane. Consider the following conditions:

(i) $\deg X \cdot \deg Y = \sum_{\substack{C \subseteq X \cap Y \\ \dim C = \dim X \cap Y}} \tilde{j}(X,Y;C) \cdot \deg C$

(ii) $\deg X \cdot \deg Y \leqslant \sum_{\substack{C \subseteq X \cap Y \\ \dim C = \dim X \cap Y}} \tilde{j}(X,Y;C) \cdot \deg C$

(iii) r_{ij} is independent of i and j, and the excess dimension

$e := \dim X \cap Y - (\dim X + \dim Y - n) = r_{ij}$

(iv) $\deg X \cdot \deg Y = \sum_{\substack{C \subseteq X \cap Y \\ \dim C = \dim X \cap Y}} j(X,Y;C) \cdot \deg C$

Then we have

(a) (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv)

(b) The implications (iii) $\Longrightarrow$ (ii), and (iv) $\Longrightarrow$ (iii) are not true in general even in the case that X and Y are arithmetically Cohen-Macaulay and $X \cap Y$ is pure dimensional.

This statement yields a converse of Bézout's theorem.

Corollary 1 (A converse of Bézout's theorem): With the same notations as in the theorem, assume that there is an integer $r_{ij} = 0$. Then the following conditions are equivalent:

(i) $\deg X \cdot \deg Y = \sum_{\substack{C \subseteq X \cap Y \\ \dim C = \dim X \cap Y}} \tilde{j}(X,Y;C) \cdot \deg C$

(ii) $e = 0$.

Remark: 1. Our example 2 of § 5 shows that we cannot replace the intersection number $\tilde{j}(X,Y,;C)$ in (i) of corollary 1 by the intersection multiplicity as presented in [4] which is the same multiplicity defined in [12]. This coincidence of both intersection multiplicities follows from the magnificent thesis of L. van Gastel [7]

Of course, if one condition of the corollary holds then $\tilde{j}(X,Y;C)=i(X,Y;C)$, where $i(X,Y;C)$ is given by Weil's i-symbol [17] defined for proper intersections.

2. An application of corollary 1 is the following: We assume (i) of corollary 1 for an improper intersection, that is, $e > o$. Then the reduced scheme of every pair of components of X and Y is contained in a hyperplane (see example 1 of § 5). Suppose that X and Y are arithmetically Cohen-Macaulay then we get stronger results of this application (see, e.g., proposition 3 of § 4). We therefore study in § 4 the particular situation when X and Y are locally or arithmetically Cohen-Macaulay schemes.

2. Notations and preliminary results

Before proving the theorem we must prove several preliminary results. First we want to recall the definition of the intersection numbers $\tilde{j}(X,Y;C)$ and $j(X,Y;C)$. Let X,Y be pure dimensional subschemes of $\mathbb{P}^n_k$ with defining ideals $I(X)$ and $I(Y)$ in $k[x_o,\ldots,x_n]=:R_x$. We introduce a second copy $k[y_o,\ldots,y_n]=:R_y$ of R_x and denote by $I(Y)'$ the ideal in R_y corresponding to $I(Y)$. We consider the polynomial ring $R:=k[x_o,\ldots,x_n,y_o,\ldots,y_n]$ and the ideal $\underline{c} = (x_o-y_o,\ldots,x_n-y_n)\cdot R$. Furthermore, we introduce new independent variables $u_{i,m}$ over k where $i,m=o,1,\ldots,n$. Let $\bar{k}$ be the algebraic closure of $k(u_{oo},\ldots,u_{nn})$. Put

$$\bar{R} := \bar{k}[x_o,\ldots,x_n,y_o,\ldots,y_n] \quad \text{and} \quad l_i := \sum_{m=o}^{n} u_{i,m}(x_m-y_m) \quad \text{in } \bar{R}$$

for $i=o,\ldots,n$.

Let C be an isolated component of $X \cap Y$ with defining prime ideal $I(C)$ of Krull-dimension $j \leq$ Krull-dim $X \cap Y$. We want to construct two well-defined primary ideals belonging to $(I(C)+\underline{c})\cdot\bar{R}$ such that

$j(X,Y;C)$ and $\tilde{j}(X,Y;C)$ are given by the length of these primary ideals, resp. We set $\delta := $ Krull-dim of $I(X)+I(Y)'$ in $\bar{R}$, and $d :=$ Krull-dim of $I(X)+I(Y)$ in R_x. Take the linear forms $l_o, \ldots, l_{\delta-d-1}$ and put

$(I(X)\cdot\bar{R} + I(Y)'\cdot\bar{R})_{-1} := I(X)\cdot\bar{R} + I(Y)'\cdot\bar{R}$, and

$(I(X)\cdot\bar{R} + I(Y)'\cdot\bar{R})_r := l_r\cdot\bar{R} + U((I(X)\cdot\bar{R} + I(Y)'\cdot\bar{R})_{r-1})$

for any $r=o,\ldots,\delta-d-1$, where $U(\ldots)$ is the intersection of all highest dimensional primary ideals belonging to the ideal $(\ldots)$. Furthermore, we put

$\underline{a}_o := U((I(X)\cdot\bar{R} + I(Y)'\cdot\bar{R})_{\delta-d-2})$, and if $j < d$

$\underline{a}_s :=$ intersection of all primary ideals belonging to $U(\underline{a}_{s-1} + l_{\delta-d+s-2}\cdot\bar{R})$ such that $\underline{c}\cdot\bar{R}$ is not contained in their associated primes for all $s=1,\ldots,d-j$. Following [12,15] we define

$j(X,Y;C) :=$ length of $(\underline{a}_{d-j} + l_{\delta-j-1}\cdot\bar{R})_{(I(C)+\underline{c})\cdot\bar{R}}$.

Following [3] we define

$\tilde{j}(X,Y;C) :=$ length of $(\underline{a}_{d-j} + \underline{c}\cdot\bar{R})_{(I(C)+\underline{c})\cdot\bar{R}}$.

In particular, if $j=d$, then we have

$j(X,Y;C) =$ length of $((I(X)\cdot\bar{R} + I(Y)'\cdot\bar{R})_{n-e})_{(I(C)+\underline{c})\cdot\bar{R}}$, and

$\tilde{j}(X,Y;C) =$ length of $((I(X)\cdot\bar{R} + I(Y)'\cdot\bar{R})_{n-e} + \underline{c}\cdot\bar{R})_{(I(C)+\underline{c})\cdot\bar{R}}$,

where e is the excess dimension of X and Y, that is, $e := \dim X \cap Y - (\dim X + \dim Y - n)$. Moreover, for every isolated component C of $X \cap Y$ we put

$l(X,Y;C) :=$ length of $(R_x/I(X)+I(Y))_{I(C)}$.

Let us collect some properties of these intersection numbers (see [3], proposition 3.8).

Lemma 1: Let C be an isolated component of $X \cap Y$. Then we have

(i) $\tilde{j}(X,Y;C) \leq \min\{j(X,Y;C), l(X,Y;C)\}$.

(ii) If $\underline{O}_{X,C}$ and $\underline{O}_{Y,C}$ are Cohen-Macaulay then

$\tilde{j}(X,Y;C) = l(X,Y;C)$.

(iii) If $e(C) := \dim C + n - \dim X - \dim Y = o$ then

$\tilde{j}(X,Y;C) = j(X,Y;C) = i(X,Y;C)$,where $i(X,Y;C)$ is defined by Weil's i-symbol.

Proof: Assertion (i) follows from the definition of the intersection numbers.(ii) follows from [1],lemma 3,and we get (iii) from [15],q.e.d.

Moreover,we need the main result of [3] which is a key result in proving our theorem.

Lemma 2: Let X,Y be irreducible and reduced subschemes of $\underline{P}^n_k$. Assume that $X \cup Y$ is not contained in a hyperplane then we have

$$\deg X \cdot \deg Y \geq \sum \tilde{j}(X,Y;C) \cdot \deg C + e,$$

where the sum is taken over all irreducible components C of $X \cap Y$ and e is the excess dimension, that is, $e = \dim X \cap Y - \dim X - \dim Y + n$.

We will also apply a certain bilinear property of the intersection algorithm in the join construction as developed in Sect.2. This property was first stated by R.Achilles and L.van Gastel.The work of L.van Gastel (see,e.g.,[6,7])shows the usefulness of this bilinearity.Before stating this result we need some notations.

Let X,Y be pure dimensional subschemes of $\underline{P}^n_k$.The intersection algorithm as presented in [12,15] gives also a collection of subvarieties $C \subset X \cap Y$ in $\underline{P}^n_k$ counted with multiplicities $j(X,Y;C)$ such that

$$\deg X \cdot \deg Y = \sum_C j(X,Y;C) \cdot \deg C \qquad (1)$$

We shall denote here this collection of subvarieties by $\mathcal{C}(X,Y)$. When there is no possibility of confusion we will denote $\mathcal{C}(X,Y)$ simply by $\mathcal{C}$. We note that every irreducible component C of $X \cap Y$ belongs to $\mathcal{C}$. Let $\mathcal{C}_{irr}$ be the set of all these irreducible components, and let $\mathcal{C}_h$ be the set of all components C of $X \cap Y$ with $\dim C = \dim X \cap Y$. Hence we have

$$\mathcal{C}_h \subseteq \mathcal{C}_{irr} \subseteq \mathcal{C}.$$

We now state a theorem of additivity and a reduction theorem needed for the proof of the theorem.

Lemma 3: Let X,Y be pure dimensional subschemes of $\mathbb{P}^n_k$ with defining ideals $I(X)$ and $I(Y)$ in $k[x_0,\dots,x_n]$. We consider primary decompositions of $I(X)$ and $I(Y)$, say $I(X)=\underline{q}_1 \cap \dots \cap \underline{q}_r$ where $\underline{q}_i$ is $\underline{p}_i$-primary, and $I(Y)=\underline{q}'_1 \cap \dots \cap \underline{q}'_s$ where $\underline{q}'_j$ is $\underline{p}'_j$-primary. We set $X = \bigcup_{i=1}^{r} X_i$ and $Y = \bigcup_{j=1}^{s} Y_j$ where X_i is defined by $\underline{q}_i$, and Y_j is given by $\underline{q}'_j$ for $i=1,\dots,r$, and $j=1,\dots,s$, resp. We define reduced and irreducible subschemes V_i and W_j defined by the prime ideals $\underline{p}_i, 1 \le i \le r$, and $\underline{p}'_j, 1 \le j \le s$, resp., that is, $(X_i)_{red} := V_i$ and $(Y_j)_{red} := W_j$. We set

$l_i :=$ length of $\underline{q}_i$ for $i=1,\dots,r$, and

$m_j :=$ length of $\underline{q}'_j$ for $j=1,\dots,s$.

Then we have

(i) $\mathcal{C}(X,Y) = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} \mathcal{C}(V_i,W_j)$.

(ii) For every $C \in \mathcal{C}(X,Y)$ we get

$$j(X,Y;C) = \sum_{i=1}^{r} \sum_{j=1}^{s} l_i \cdot m_j \cdot j(V_i,W_j;C) ,$$

where we set $j(V_i,W_j;C) = 0$ if $C \notin \mathcal{C}(V_i,W_j)$.

<u>Proof:</u> (i) follows from the intersection algorithm in the join construction by considering the radical of the corresponding ideals. There are different arguments in proving (ii). For example, L. van Gastel [6,7] described geometrically and partly generalized the intersection theory of [12,15]. Hence the assertion (ii) is clear from the definition of the algorithm (see [6], remark 4.4), q.e.d.

3. Proof of the theorem

Before embarking on the proof of the theorem we need a suitable deepening of investigations on the new intersection number $\tilde{j}(X,Y;C)$. This gives us the second key result in proving the theorem.

<u>Lemma 4:</u> Let $X=\bigcup_{i=1}^{r} X_i$ and $Y=\bigcup_{j=1}^{s} Y_j$ be pure dimensional subschemes of $\mathbb{P}^n_k$. Let C be an element of the collection $\mathcal{C}(X,Y)$. If $j(X,Y;C) = \tilde{j}(X,Y;C)$ then we have

(a) $\tilde{j}(X_{red},Y_{red};C) = j(X_{red},Y_{red};C)$

(b) $\tilde{j}((X_i)_{red},(Y_j)_{red};C) = j((X_i)_{red},(Y_j)_{red};C)$ for all $i=1,\dots,r$ and $j=1,\dots,s$.

<u>Proof:</u> There is not loss of generality in assuming that $C \in \mathcal{C}(X_i,Y_j)$. Consider the above intersection algorithm in the join construction. Let Q,Q' and Q_{ij} be the primary ideals defining the intersection numbers $j(X,Y;C)$, $j(X_{red},Y_{red};C)$ and $j((X_i)_{red},(Y_j)_{red};C)$, resp. Since

$I(X) + I(Y)' \subseteq I(X)_{red} + I(Y)'_{red} \subseteq I(X_i)_{red} + I(Y_j)'_{red}$

the intersection algorithm provides $Q \subseteq Q' \subseteq Q_{ij}$. The assumption $j(X,Y;C) = \tilde{j}(X\ \ ,Y\ \ ;C)$ gives $\underline{c} \subseteq Q$ with respect the localization at the prime ideal P belonging to the primary ideals Q,Q' and Q_{ij}.

Therefore $\underline{c} \subseteq Q' \subseteq Q_{ij}$ at this localization. This property shows the assertions of lemma 4 , q.e.d.

Proving the implications (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii) of the theorem we need the following lemma.

Lemma 5: With the same notations as in the theorem, assume the condition (ii). Then we obtain the condition (i) of the theorem. Moreover, we get the following properties:

(a) $\underline{\mathscr{C}}(X,Y) = \underline{\mathscr{C}}_h(X,Y)$, and $j(X,Y;C) = \tilde{j}(X,Y;C)$ for all elements C of $\underline{\mathscr{C}}(X,Y)$.

(b) $\dim X \cap Y = \dim X_i \cap Y_j$ for all $i=1,\ldots,r$ and $j=1,\ldots,s$.

(c) $\underline{\mathscr{C}}(X_i,Y_j) = \underline{\mathscr{C}}_h(X_i,Y_j)$ for all i and j.

Moreover, let V_i and W_j be the embedding of $(X_i)_{red}$ and $(Y_j)_{red}$, resp. in $P^{n-r_{ij}}$ then we have

(d) $\underline{\mathscr{C}}(V_i,W_j) = \underline{\mathscr{C}}_h(V_i,W_j)$, and $\tilde{j}(V_i,W_j;C) = j(V_i,W_j;C)$ for all $C \in \underline{\mathscr{C}}(V_i,W_j)$ and for all $i=1,\ldots,r$ and $j=1,\ldots,s$.

Proof: Consider

$$\sum_{C \in \underline{\mathscr{C}}_h} j(X,Y;C)\cdot\deg C \geqslant \sum_{C \in \underline{\mathscr{C}}_h} \tilde{j}(X,Y;C)\cdot\deg C, \text{ by lemma 1,(i)}$$

$$\geqslant \deg X\cdot\deg Y \text{ , by assumption of lemma 5}$$

$$= \sum_{C \in \underline{\mathscr{C}}} j(X,Y;C)\cdot\deg C \text{ , by (1) of § 3}$$

$$\geqslant \sum_{C \in \underline{\mathscr{C}}_h} j(X,Y;C)\cdot\deg C \text{ , since } \underline{\mathscr{C}}_h \subseteq \underline{\mathscr{C}}.$$

This shows condition (i) and (a). Applying lemma 3,(i) we get (b) and (c). There is a 1-1 correspondence between the elements of $\mathscr{C}(X_i,Y_j) = \mathscr{C}((X_i)_{red},(Y_j)_{red})$ and of $\underline{\mathscr{C}}(V_i,W_j)$. Hence we also have the first assertion of (d). The second one follows from lemma 4,(b) and the intersection algorithm, q.e.d.

<u>Proof of the theorem:</u> Lemma 5 shows (i)$\Longleftrightarrow$(ii).

(ii) $\Longrightarrow$ (iii): Consider the following excess dimensions:

$e(X,Y) := \dim X \cap Y - \dim X - \dim Y + n,$

$e(X_i,Y_j) := \dim X_i \cap Y_j - \dim X_i - \dim Y_j + n.$

Lemma 5,(b) shows that $e(X,Y) = e(X_i,Y_j)$ for all $i=1,\dots,r$ and $j=1,\dots,s$. Hence we get $e(X,Y)=e(X_i,Y_j)=e(V_i,W_j)+r_{ij}$, where V_i and W_j are the embeddings of $(X_i)_{red}$ and $(Y_j)_{red}$, resp. in $\underline{P}^{n-r_{ij}}$, and $e(V_i,W_j)=\dim V_i \cap W_j - \dim V_i - \dim W_j + (n-r_{ij})$.

<u>Claim:</u> $e(V_i,W_j)=0$ for all $i=1,\dots,r$ and $j=1,\dots,s$.

<u>Proof of the claim:</u> We fix V_i and W_j for arbitrary $1 \leq i \leq r$ and $1 \leq j \leq s$. We then obtain:

$$\sum_{C \in \underline{\mathcal{C}}_h(V_i,W_j)} j(V_i,W_j;C)\cdot \deg C = \deg V_i \cdot \deg W_j \quad \text{,by lemma 5,(d)}$$

and the statement (1) of § 3,

$$\geq \sum_{C \in \underline{\mathcal{C}}_{irr}(V_i,W_j)} \tilde{j}(V_i,W_j;C)\cdot \deg C + e(V_i,W_j) \quad \text{,by lemma 2}$$

$$= \sum_{C \in \underline{\mathcal{C}}_h(V_i,W_j)} j(V_i,W_j;C)\cdot \deg C + e(V_i,W_j) \quad \text{,by lemma 5,(d).}$$

Hence we get $e(V_i,W_j)=0$. This shows our claim.

The claim now provides $r_{ij}=e(X,Y)$ for all $i=1,\dots,r$ and $j=1,\dots,s$. This completes the proof of (ii)$\Longrightarrow$(iii).

(iii) $\Longrightarrow$(iv): Since $e(X,Y)=r_{ij}$ for all i,j we get:

$$e(X,Y) \geq e(X_i,Y_j) = e(V_i,W_j) + e(X,Y),$$

that is, $e(V_i,W_j)=0$. Therefore $V_i \cap W_j$ is a proper intersection in $\underline{P}^{n-e}$ for all $i=1,\dots,r$ and $j=1,\dots,s$. Hence we have Weil's intersection multiplicities $i(V_i,W_j;C)$ for all $C \in \underline{\mathcal{C}}_h(V_i,W_j)=\underline{\mathcal{C}}(V_i,W_j)$. Using some notations of lemma 3 we therefore obtain:

$$\deg X \cdot \deg Y = \left(\sum_{i=1}^{r} \deg X_i\right)\cdot\left(\sum_{j=1}^{s} \deg Y_j\right)$$

$$= \sum \sum l_i \cdot m_j \cdot \deg(X_i)_{red} \cdot \deg(Y_j)_{red}$$

$$= \sum \sum l_i \cdot m_j \cdot \deg V_i \cdot \deg W_j$$

$$= \sum \sum l_i \cdot m_j \Big[\sum_{C \in \mathcal{C}_h(V_i, W_j)} i(V_i, W_j; C) \cdot \deg C \Big] \text{, by Bézout's}$$

theorem in $\underline{P}^{n-e}$ and the above properties,

$$= \sum \sum l_i \cdot m_j \Big[\sum_{C \in \mathcal{C}((X_i)_{red}, (Y_j)_{red})} j((X_i)_{red}, (Y_j)_{red}; C) \cdot \deg C \Big],$$

by the intersection algorithm with the schemes of $\underline{P}^n$,

$$= \sum_{C \in \mathcal{C}(X,Y)} j(X, Y; C) \cdot \deg C \text{ , by lemma 3.}$$

This shows the implication (iii) $\Longrightarrow$ (iv).

The examples 2 and 3 of Sect.5 show that the implications (iv) $\Longrightarrow$ (iii) and (iii) $\Rightarrow$ (ii) ,resp. are not true in general.This completes the proof of the theorem,q.e.d.

4. The Cohen-Macaulay case

In this section we study the particular situation when X and Y are locally or arithmetically Cohen-Macaulay schemes.

We first recall that

$$\deg X \cap Y = \sum_{\substack{C \subseteq X \cap Y \\ \dim C = \dim X \cap Y}} \operatorname{length}(\underline{O}_{X \cap Y, C}) \cdot \deg C.$$

Therefore the following proposition 1 follows immediately from the theorem and lemma 1,(ii).

Proposition 1: In addition to the hypothesis of corollary 1, assume that the local rings $\underline{O}_{X,C}$ and $\underline{O}_{Y,C}$ are Cohen-Macaulay for all irreducible components C of $X \cap Y$ with $\dim C = \dim X \cap Y \geq o$. Then the following conditions are equivalent:

(i) $\deg X \cdot \deg Y \leq \deg X \cap Y$

(ii) $\deg X \cdot \deg Y = \deg X \cap Y$

(iii) $e = o$

This proposition yields an extended version of the corollary stated in the introduction of [3].

We now assume that X and Y are arithmetically Cohen-Macaulay, that is, the defining ideals I(X) and I(Y) are perfect. In this case we will get another description of the excess dimension $e = \dim X \cap Y - (\dim X + \dim Y - n)$. We will reach this desired result by a suitable deepening of the approach as presented in [11]. In this connection I am grateful for insightful comments made by students and coworkers attending my lectures on intersection theory given in 1988/89 at the University of Halle. Before stating our results, we give some general observations and collect some known results that we need. We will consider the intersection algorithm in the join construction for $r \geq 2$ subschemes $X_1, \ldots, X_r$ of $\underline{P}_k^n$ (see also [10], and for a wealth of background material see F. Gaeta [5]).

Let $X_1, \ldots, X_r$ $(r \geq 2)$ be pure dimensional subschemes of $\underline{P}_k^n$ defined by homogeneous ideals $I(X_1), \ldots, I(X_r)$ in $k[x_o, \ldots, x_n] =: R_o$. We introduce r copies $R_i := k[x_{io}, \ldots, x_{in}]$, $1 \leq i \leq r$, of R_o and denote by I_i' the ideal in R_i corresponding to $I(X_i)$, $1 \leq i \leq r$. We put $N := r(n+1)-1$, $R := k[x_{ij}$ with $1 \leq i \leq r, o \leq j \leq n]$ and $\underline{c}$:= the diagonal ideal in R generated by $\{x_{1j}-x_{ij}$ with $2\leq i\leq r, o\leq j\leq n\}$. Let $J(X_1,\dots,X_r)$ be the join-variety defined by $(I_1'+\dots+I_r')\cdot R$ in $\mathbb{P}_k^N$. Moreover, we introduce new independent variables u_{kij} over the ground field, $o\leq k\leq m:=(r-1)\cdot(n+1)-1, 1\leq i\leq r$ and $o\leq j\leq n$. Let $\bar{k}$ be the algebraic closure of $k(u_{kij}$ with $o\leq k\leq m, 1\leq i\leq r, o\leq j\leq n)$. We put

$$\bar{R} := \bar{k}[x_{ij} \text{ with } 1\leq i\leq r, o\leq j\leq n].$$

We then introduce so called generic linear forms $l_o,\dots,l_m$:

$$l_k := \sum_{\substack{2\leq i\leq r\\ o\leq j\leq n}} u_{kij}(x_{1j}-x_{ij}) \text{ for } k=o,\dots,m \text{ in } \bar{R}.$$

Note that $\underline{c}\cdot\bar{R} = (l_o,\dots,l_m)\cdot\bar{R}$, and we recall that $m=(r-1)\cdot(n+1)-1$.

Let $[I]_i$ denote the i-th graded part of a homogeneous ideal I. We put $\dim_k[I]_i =: V(i,I)$. Then we will prove the following result.

<u>Proposition 2:</u> Let $X_1,\dots,X_r, r\geq 2$, be arithmetically Cohen-Macaulay schemes of $\mathbb{P}_k^n$. Assume that

$\deg(X_1\cap\dots\cap X_r)\geq \prod_{i=1}^{r}\deg(X_i)$ then we get for the excess dimension

$$e := \dim(X_1\cap\dots\cap X_r) - \sum_{i=1}^{r}\dim(X_i) + (r-1)\cdot n :$$

$$e = \dim_k[(I_1' + \dots + I_r')\cdot R\cap \underline{c}\cdot R]_1 .$$

<u>Proof:</u> We set $J:=I_1'+\dots+I_r'$. We have $V(1,(J,l_o,\dots,l_{m-e})\cdot\bar{R}) = V(1,(J,l_o,\dots,l_{m-e-1})\cdot\bar{R}) + V(1,(l_{m-e})\cdot\bar{R}) - V(1,(J,l_o,\dots,l_{m-e-1})\cdot\bar{R} \cap (l_{m-e})\cdot\bar{R})$. Since $V(1,(J,l_o,\dots,l_{m-e-1})\cdot\bar{R}\cap(l_{m-e})\cdot\bar{R}) =$

$V(o,(J,l_o,\dots,l_{m-e-1})\cdot\bar{R} : (l_{m-e})\bar{R}) = o$ we obtain

$$V(1,(J,l_o,\dots,l_{m-e})\cdot\bar{R}) = V(1,J\cdot\bar{R}) + m - e + 1.$$

On the other hand, the assumptions of Proposition 2 yields:

$V(1,(J,l_o,\dots,l_{m-e})\cdot\bar{R}) = V(1,(J+\underline{c})\cdot\bar{R})$.

This important fact follows by analyzing the intersection algorithm in the join construction (see Hilfssatz 3 and Folgerung 4 of [11]). Hence we get

$V(1,(J,l_o,\dots,l_{m-e})\cdot\bar{R}) = V(1,J\cdot\bar{R}) + V(1,\underline{c}\cdot\bar{R}) - V(1,J\cdot\bar{R}\cap\underline{c}\cdot\bar{R}) =$

$V(1,J\cdot\bar{R}) + (m+1) - V(1,J\cdot\bar{R}\cap\underline{c}\cdot\bar{R})$. Therefore we have

$e = V(1,J\cdot\bar{R}\cap\underline{c}\cdot\bar{R}) = V(1,J\cdot R\cap\underline{c}\cdot R)$.

This shows Proposition 2 , q, e,d.

Proposition 2 does provide interesting applications. For example, the following corollary describes a certain converse of the main result of [11].

Corollary 2: Let $X_1,\dots,X_r, r\geq 2$, be arithmetically Cohen-Macaulay schemes of $\mathbb{P}^n_k$ such that

$$\deg(X_1\cap\dots\cap X_r)\geq\prod_{i=1}^{r}\deg(X_i).$$

Then the following conditions are equivalent:

(i) $e = o$

(ii) $\dim_k[J\cdot R\cap\underline{c}\cdot R]_1 = o$

(iii) $X_i\cup(X_1\cap\dots\cap X_{i-1}\cap X_{i+1}\cap\dots\cap X_r)$ is not lying in a hyperplane for all $i=1,\dots,r$.

Proof: The equivalence of (i) and (ii) follows from Proposition 2. We note that the equivalence of (ii) and (iii) is always true for pure dimensional subschemes $X_1,\dots,X_r$ of $\mathbb{P}^n_k$. This follows by analyzing the proof of Hilfssatz 4 of [11], q.e.d.

Moreover, we get the following result in case that the excess dimension $e(X,Y)\neq o$:

Proposition 3: Let X and Y be arithmetically Cohen-Macaulay schemes of $\underline{P}^n_k$. Let e be the excess dimension of X and Y, that is, $e = \dim X \cap Y - \dim X - \dim Y + n \geq o$. Then the following conditions are equivalent:

(i) $\deg X \cdot \deg Y = \deg X \cap Y$

(ii) $X \cup Y$ is contained in an (n-e)-plane but not lying in an (n-e-1)-plane.

Proof: (i) $\Rightarrow$ (ii): Proposition 2 shows that $e = \dim_k [J \cdot R \cap \underline{c} \cdot R]_1$. It is not difficult to see that for two arbitrary subschemes X and Y we always have:

$$\dim_k [J \cdot R \cap \underline{c}\, R]_1 = \dim_k [I(X) \cap I(Y)]_1 \ .$$

This assertion follows again by analyzing the proof of Hilfssatz 4 of [11] in case of two subschemes. Hence we obtain (ii).

(ii) $\Rightarrow$ (i): We thus have $e = \dim_k [I(X) \cap I(Y)]_1$. Let V and W be the embedding of X and Y, resp. in $\underline{P}^{n-e}$. Then V and W are again arithmetically Cohen-Macaulay schemes of $\underline{P}^{n-e}$. Moreover, we obtain for the excess dimension of V and W, say e(V,W):

$e(V,W) = \dim V \cap W - (\dim V + \dim W - (n-e)) = \dim X \cap Y - (\dim X + \dim Y - (n-e)) = e - e = o$, that is, $V \cap W$ is a proper intersection in $\underline{P}^{n-e}$. Therefore we have $\deg V \cdot \deg W = \deg V \cap W$ since V and W are arithmetically Cohen-Macaulay. This provides our assertion (i), q.e.d.

This Proposition 3 improves an important result of [11] (see Corollary 3 of [11]). However, it is not clear (to me) what the analogue of Proposition 3 should be when we consider the intersection of $r \geq 2$ subschemes of P^n. Perhaps we should then replace condition (ii) of Proposition 3 by the statement

$e = \dim_k [J \cdot R \cap \underline{c}\, R]_1$.

5. Examples and open problems

We discuss in conclusion some examples and open questions.

The first example shows that the hypothesis in the theorem and proposition 1 that $(X_i \cup Y_j)_{red}$ is not contained in a hyperplane cannot be replaced by the weaker one that $(X_i \cup Y_j)$ is not lying in a hyperplane.

Example 1: Let X,Y be curves in $\mathbb{P}^3_k$ with defining ideals

$I(X) = (x_0^2, x_1^2, x_0x_1, x_0x_3 - x_1x_2)$ and $I(Y) = (x_1, x_3)$, resp. Then X and Y are irreducible, and $X \cap Y$ is given by the intersection point, say C: $x_0 = x_1 = x_3 = o$ counted with multiplicity $\tilde{j}(X,Y;C) = 2$ by lemma 1,(ii). Hence we get

$2 = \deg X \cdot \deg Y = \tilde{j}(X,Y;C) \cdot \deg C = \deg X \cap Y$ but $e = 1$.

Note that $(X \cup Y)_{red}$ is lying in the hyperplane $x_1 = o$.

Moreover, this example also shows that the statement of corollary 2 replaced "Cohen-Macaulay" by "Buchsbaum" is wrong (see [13], Theorem III.3.2,(iii) for the Buchsbaum property of X).

Analyzing the proof of the theorem we want to state the following question asked first by L. van Gastel [8].

Problem 1: Let X,Y be pure dimensional subschemes of $\mathbb{P}^n_k$. Under which circumstances is the following implication true:

$$\deg X \cdot \deg Y = \sum_{C \in \mathcal{C}_{irr}(X,Y)} j(X,Y;C) \cdot \deg C \implies$$

$\dim X \cap Y = \dim X + \dim Y - n$?

The following example shows that this is not true in general. Moreover, example 2 also shows that the implication (iv) $\implies$ (iii) of the theorem is not true in general.

Example 2 (see also [7], remark 4.7,(3) on p.65): Let X be an arithmetically Cohen-Macaulay reduced and irreducible curve of degree $d > 1$ and Y a disjoint line in $\mathbb{P}^4_k$. By coning we obtain in $\mathbb{P}^5$

an intersection with excess dimension $e=1$ of X' and Y', supported by the vertex, say C. It follows from the intersection algorithm that $j(X',Y';C) = \deg X' = d$, that is, $\deg X' \cdot \deg Y' = j(X',Y';C) \cdot \deg C$ but $e=1$.

This example and the theorem with corollary 1 yield the importance of the new intersection multiplicity $\tilde{j}(X,Y;C)$. It also proves the assertion in the remark stated after corollary 1.

The following example 3 now shows that the implication (iii) $\Longrightarrow$ (iv) of the theorem is not true in general even in the case that X and Y are arithmetically Cohen-Macaulay and $X \cap Y$ is pure dimensional.

Example 3: Let X and Y be the curves of $\mathbb{P}^3_k$ with the following defining ideals in $k[x_0,\ldots,x_3]$: $I(X) = (x_0,x_1) \cap (x_1,x_2) \cap (x_2,x_3)$ and $I(Y) = (x_1,x_3)$. Then we have: X and Y are arithmetically Cohen-Macaulay such that $r_{ij}=1$ for all $i=1,2,3$ and $j=1$.

$I(X) + I(Y) = (x_0,x_1,x_3) \cap (x_1,x_2,x_3)$, that is, $X \cap Y$ has precisely two intersection points, say C_1 given by $x_0=x_1=x_3=0$ and C_2 given by $x_1=x_2=x_3=0$. Hence the excess dimension $e=1$, that is, the condition (iii) holds. Lemma 1, (ii) gives for the multiplicities $\tilde{j}(X,Y;C_1)=\tilde{j}(X,Y;C_2)=1$. Hence condition (ii) is not true. We note that condition (iv) of the theorem is indeed true since $j(X,Y;C_1)=1$ and $j(X,Y;C_2)=2$. This follows from the criterion for intersection multiplicity one of [1].

Remark: It is not difficult to see that lemma 3, (ii) is not true in general for the new intersection multiplicity $\tilde{j}(X,Y;C)$. This is the deeplying reason that condition (iii) does not imply condition (ii) of the theorem.

Moreover, we want to pose the following problem.

Problem 2: Would Kirby's arguments of [9] yield similar results to those in this paper ?

Of course, there are further interesting problems concerning Bézout's theorem, see, e.g., [7], [14], [15], [16].
Finally, we note that a quite different inverting of Bézout's theorem in the plane is discussed by E.D.Davis (see, e.g., [2]).

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Martin-Luther-University
Dep. of Mathematics
P.O.Box
DDR-4010 Halle
German Democratic Republic