TOPOLOGY OF UNITARY DUAL OF

NON-ARCHIMEDEAN GL(n)

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Introduction

Let G be a locally compact group. The set of all equivalence classes of irreducible unitary representations of G is denoted by $\hat{G}$. The set $\hat{G}$ is called the unitary dual of G and it carries a natural topology (see [5]).

Let F be a local non-archimedean field. In this paper we consider properties of the representation theory of $GL(n,F)$-groups related to the topology of the unitary dual of $GL(n,F)$. The main results of this paper are: classification of all isolated points modulo unramified characters in $GL(n,F)^\wedge$, description of composition factor of ends of complementary series representations and description of $GL(n,F)^\wedge$ as (abstract) topological space.

For reductive groups over local fields the unitary dual as topological space has been determined as this author knows for the following groups: $SL(2,\mathbb{C})$ by J.M.G. Fell in [6] (1961), $SL(2,\mathbb{R})$ by D. Miličić in [11] (1971), universal covering group of $SL(2,\mathbb{R})$ by H. Kraljević and D. Miličić in [10] (1972), universal covering group of $SU(n,1)$ by H. Kraljević in [9] (1973) and $SL(2,k)$ where k is non-archimedean by this author in [17] (1982).

The unitary dual $GL(n,F)^\wedge$ as the set is parametrized in [19]. The results we need in our study of the topology of the unitary dual are contained in [6],[12] and [18].

Now we shall describe more detailed the content of this paper.

In the first section we collect the basic definitions and results related to the topology of the unitary duals of reducitve groups over local non-archimedean fields. In this section we introduce the notion of isolated point in $\hat{G}$ modulo unramified characters. Let $U^u(G)$ be the group of all unitary unramified characters of G. For $\pi \in \hat{G}$ the set $U^u(G)\pi \subseteq \hat{G}$ is always closed and connected. Therefore each open and closed subset of G containing π contains also $U^u(G)\pi$. Therefore we define $\pi \in \hat{G}$ to be isolated modulo unramified characters if $U^u(G)\pi$ is an open subset of $\hat{G}$. If G has no non-trivial split toruses in the center (for example if G is semi-simple), then this notion is equal to the standard notion of isolated point (or isolated representation) in $\hat{G}$.

The second section deals with the basic topological properties of $GL(n,F)^\wedge$.

In the third section we introduce the notation related to the non-unitary dual of $GL(n,F)$ and recall of the main results of Bernstein and Zelevinsky from [3] and [22]. We recall of Langlands and Zelevinsky classifications.

In the fourth section we recall of the parametrisation of $GL(n,F)^\wedge$ obtained in [19]. Set

$$\mathrm{Irr}^u = \bigcup_{n\geq 0} GL(n,F)^\wedge \quad .$$

We consider Irr^u as topological space with the topology of the disjoint union. Let D^u be the set of all classes of square integrable representations in Irr^u. We attached in [19] to each $\delta \in D^u$ and each natural number n an irreducible unitary representation

$$u(\delta,n)$$

and to $0 < \alpha < 1/2$ we attached a complementary series representation

$$\pi(u(\delta,n),\alpha).$$

Now each $\pi \in \mathrm{Irr}^u$ is in a unique way parabolically induced by a tensor product of representations $u(\delta,n)$'s and $\pi(u(\delta,n),\alpha)$'s (see [19] or the fourth section of this paper).

In the fifth section we classify all isolated points modulo unramified characters in $GL(n,F)^\wedge$. Let C^u be the set of all classes of cuspidal representations in D^u. Then to each $\rho \in C^u$ and each natural number m by the J.N. Bernstein description of D^u we can attach a square integrable representation $\delta(\rho,m)$ (see the fourth section). Now we can describe the isolated points modulo unramified characters: $\pi \in \mathrm{Irr}^u$ is isolated modulo the unramified characters if and only if $\pi = u(\delta(\rho,m),n)$ for some ρ,m,n with $m \neq 2$ and $n \neq 2$.

The composition series of ends of complementary series $\pi(u(\delta,n),\alpha)$ are described in the sixth section. We prove that the end of complementary series $\pi(u(\delta(\rho,m),\alpha)$ has exactly two different irreducible subquotients, each one has multiplicity one and one of them is parabolically induced by

$$u(\delta(\rho,m),n+1) \otimes u(\delta(\rho,m),n-1)$$

while the other one is parabolically induced by

$$u(\delta(\rho,m+1),n) \otimes u(\delta(\rho,m-1),n).$$

We expect that we have the similar situation in the case of archimedean fields (see [20] and also [16]).

In the seventh section we give a description of $GL(n,F)^\wedge$ (or Irr^u) as abstract topological space. J.N. Bernstein proved that the parabolically induced representation of $GL(n,F)$ by an irreducible unitary representation of Levi subgroups of a parabolic subgroup of $GL(n,F)$, is irreducible. Roughly speaking, by the Bernstein result the topology of $GL(n,F)^\wedge$ is locally described by the obvious topology on the Langlands parameters except at the ends of complementary series where we know what is happening by the previous section. From this realization of $GL(n,F)^\wedge$ we can conclude that $GL(n,F)^\wedge$ are all homeomorphic, for fixed n and different F's.

In this paper the field of complex numbers is denoted by $\mathbb{C}$ and the subring of real ones is denoted by $\mathbb{R}$. The subring of the rational integers is denoted by $\mathbb{R}$ and the subset of the non-negative ones is denoted by $\mathbb{Z}_+$. The subset of positive ones is denoted by $\mathbb{N}$.

The basic ideas of the topology of unitary dual I learned from D. Miličić. Discussions with H. Kraljević helped a lot to improve my understanding of this topological spaces. I use this ocassion to express my thankfulness to the both of them.

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1. Basic facts about topology of the dual space of a reductive group over local non-archimedean field

Let G be a locally compact group. The set of all equivalence classes of irreducible unitary representations of G is denoted by $\hat{G}$.

We fix a left Haar measure on G. Let $C_0(G)$ be the convolution algebra of all compactly supported continuous complex-valued functions on G. For each unitary representation (π, H_π) of G and $f \in C_0(G)$ set

$$\pi(f) = \int_G f(g)\pi(g)\,dg.$$

Let $\|f\|_\pi$ be the operator norm of $\pi(f)$, and let $\|f\|$ be the suppremum of all $\|f\|_\pi$ when π goes over all unitary representations of G. The completion of $C_0(G)$ with respect to norm $\|\ \|$ is denoted by $C^*(G)$ and called the C^*-algebra of G. Clearly $C^*(G)$ is C^*-algebra. If π is a unitary representation of G then π lifts in a natural way to the representation of the C^*-algebra $C^*(G)$.

Let $\mathrm{Prim}\, C^*(G)$ be the set of kernels of irreducible non-zero representations of the algebra $C^*(G)$ on vector spaces. For $T \subseteq \mathrm{Prim}\, C^*(G)$ set

$$C\ell(T) = \{I \in \mathrm{Prim}\, C^*(G);\ \bigcap_{J \in T} J \subseteq I\}.$$

Now $C\ell$ is the closure operator for so-called Jacobson topology on Prim $C^*(G)$.

There is a canonical surjection

$$\hat{G} \longrightarrow \text{Prim } C^*(G),$$

$$\pi \longmapsto \ker \pi .$$

We supply $\hat{G}$ with the weakest topology so that the above surjection is continuous.

The set $\hat{G}$ supplied with the above topology is called the dual space of G.

In the rest of this paragraph, G will denote the set of all rational points of a connected reducitve group defined over a non-archimedean local field.

The group G is totally disconnected. There exist a countable basis of neighborhoods of identity, consisting of open and compact subgroups of G.

The subalgebra of all locally constant functions in $C_c(G)$ is denoted by $H(G)$ or $C_c^\infty(G)$. Let K be an open compact subgroup of G. The subalgebra of all functions in $H(G)$ constant on double K-classes in G is denoted by $H(G,K)$.

For $(\pi, H_\pi) \in \hat{G}$ set $H_\pi^\infty = \pi(H(G))H_\pi$. Now H_π^∞ is G-invariant and possesses a G-invariant inner product. For each $v \in H_\pi^\infty$ the stabilizer of v in G is open, and for each open compact subgroup K in G K-invariants $(H_{\pi}^{\infty})^{K}$ in H_{π}^{∞} are finite-dimensional. Thus the representation $(\pi^{\infty}, H_{\pi}^{\infty})$ is smooth and admissible (here $\pi^{\infty}(g) = \pi(g)|H_{\pi}^{\infty})$. This representation of G is algebraically irreducible and possesses a G-invariant inner product (which is unique up to a constant). The mapping $(\pi, H_{\pi}) \longmapsto (\pi^{\infty}, H_{\pi}^{\infty})$ is a bijection from $\hat{G}$ onto the set of all equivalence classes of irreducible smooth admissible representations of G which possesses G-invariant inner product. When there is no possibility of confusion we shall not make difference between an irreducible unitary representation (π, H) of G, its class in $\hat{G}$ and its smooth part $(\pi^{\infty}, H_{\pi}^{\infty})$.

The set of all equivalence classes of irreducible smooth admissible representations will be denoted by $\tilde{G}$. An irreducible smooth admissible representation (π, V) will be said unitarizable if V possesses an G-invariant inner product. Now $\hat{G}$ can be identified with the set of all unitarizable classes in $\tilde{G}$.

By [1] G is a liminal group, or in other terminology, G is CCR group. It means that $\pi(C^*(G))$ consists of compact operators for $\pi \in \hat{G}$.

Since G is liminal, the canonical mapping

$$\hat{G} \longrightarrow \mathrm{Prim}\, C^*(G)$$

is bijection and thus it is homeomorphism ([5], 4.4.1.). Points in $\hat{G}$ are closed ([5], 4.4.1.). The topology of $\hat{G}$ has a countable basis of open sets since $C^*(G)$ is separable ([5] , 3.3.4.).

Let $T \subseteq \hat{G}$. Since G has the topology with countable basis, $C\ell\ T$ is just the set of all limits of convergent sequences in $\hat{G}$.

Let X,Y be two topological spaces with countable basis of open sets and $f: X \to Y$ mapping. One can see directly that f is continuous if and only if for each convergent sequence (x_n) in X and each limit x of (x_n), $(f(x_n))$ converges to $f(x)$. Similarly, continuity of f is equivalent to the following property: if (x_n) is a convergent squence in X and x a limit of (x_n), then there exist a subsequence (x_{n_k}) of (x_n) so that $(f(x_{n_k}))$ converges to $f(x)$.

For a fixed open compact subgroup K of G , the function

$$(\pi, H_\pi) \longmapsto \dim_{\mathbb{C}} \pi(H(G,K)H_\pi) = \dim_{\mathbb{C}} H_\pi^K$$

on $\hat{G}$ is bounded ([1] , Theorem 1).

For an admissible smooth representation (π,V) of G we shall denote by Θ_π its character. For fixed $f \in H(G)$, the function

$$(\pi, H_\pi) \longmapsto \Theta_\pi(f)$$

on $\hat{G}$ is bounded. It means that $C^*(G)$ is by D. Miličić

C^*-algebra with bounded trace ([12]). Therefore, we have D. Miličić description of the topology of $\hat{G}$.

Let $H(G)^*$ be the space of all linear forms on $H(G)$ supplied with the weak topology. Let $X \subseteq \hat{G}$. Set $X^* = \{\Theta_\pi ; (\pi, H_\pi) \in X\}$. We denote by $C\ell\ X$ (resp $C\ell\ X^*$) the closure of X (resp X^*) in $\hat{G}$ (resp. $H(G)^*$). By D. Miličić results in [12], $(\pi, H_\pi) \in C\ell(X)$ if and only if there exist $\varphi \in C\ell(X^*)$, a discrete closed subset S of $\hat{G}$ and positive integers $n_\sigma, \sigma \in S$ so that

$$\Theta_\pi + \sum_{\sigma \in S} n_\sigma\ \Theta_\pi = \varphi.$$

Note that in $H(G)^*$, a sequence (φ_n) converges to φ if and only if $(\varphi_n(f))$ converges to $\varphi(f)$ in $\mathbb{C}$ for any $f \in H(G)$. For $Y \subseteq H(G)^*$, $C\ell(Y)$ consists of all limits of all convergent sequences contained in Y.

Let ${}^0G$ be the group of all $g \in G$ so that $|\chi(g)|_F = 1$ for all F-rational characters of G. Here $|\ |_F$ denotes the normalized absolute value on F.

A continuous homomorphism χ from G into $\mathbb{C}^x$ will be called character of G. If the image of χ is contained in $\mathbf{T} = \{z \in \mathbb{C};\ |z| = 1\}$, then χ will be called unitary character of G. If χ is trivial on ${}^0G$ then χ will be called unramified character of G.

The group of all unramified characters of G will be denoted by $U(G)$ and $U(G)$ is endowed with the topology of uniform convergence over compacts. The subgroup of all unitary characters in $U(G)$ is denoted by $U^u(G)$. The group $G/{}^0G$ is free abelian of finite rank. If $n = \text{rank}\,(G/{}^0G)$ then $U(G)$ is isomorphic to $(\mathbb{C}^\times)^n$ and $U^u(G)$ to $(\mathbb{T})^n$.

The groups $U(G)$ and $U^u(G)$ act on $\tilde{G}$ in the natural way. Also $U^u(G)$ acts on $\hat{G}$.

Let P be a parabolic subgroup of G, and let $P = MN$ be Levi decomposition of P. If σ is a smooth representation of M, then $\text{Ind}\,(\sigma|P,G)$ will denote the induced representation of G from P by σ. The induction we consider is normalized.

Let (π_n) be a convergent sequence in $\hat{G}$. Then by Theorem 5.6. of [18] there exist a parabolic subgroup $P = MN$ of G, an irreducible cuspidal representation σ of M, a convergent sequence (χ_n) of unramified characters of M, $n_0 \geq 1$ so that π_n is a subquotient of $\text{Ind}(\chi_n\sigma|P,G)$ for $n \geq n_0$ and each limit of (π_n) is a composition factor of $\text{Ind}\,((\lim_n \chi_n)\sigma|P,G)$. Therefore, the set of limits is finite and its cardinal number is bounded by the order of the Weyl group of G.

The above discussion on convergent sequences implies that the set S in the D. Miličić description of the topology of $\hat{G}$ must be finite.

We shall note now few facts about convergence of sequences.

Let (π_n) be a convergent sequence in $\hat{G}$. The D. Miličić description of the topology implies that there exist a subsequence (π_{n_k}) of (π_n) such that $\Theta_{\pi_{n_k}}$ converges in $H(G)^*$ and

$$\lim_k \Theta_{\pi_{n_k}} = \sum_{\sigma \in \hat{G}} n_\sigma \Theta_\sigma$$

where n_σ are non-negative integers, different from zero only for finitely many $\sigma \in \hat{G}$.

Let (π_n) be a sequence in $\hat{G}$ and suppose that there exist a subsequence (π_{n_k}) so that $\lim \Theta_{\pi_{n_k}} = 0$ in $H(G)^*$. Then (π_n) is not convergent sequence.

Let (π_n) be a sequence in $\hat{G}$. Suppose that (Θ_{π_n}) converges in $H(G)^*$ to a non-zero element. Suppose that there exist a finite set $S \subseteq \tilde{G}$ and positive integers $n_\sigma, \sigma \in S$ so that

$$\lim_n \Theta_{\pi_n} = \sum_{\sigma \in S} n_\sigma \Theta_\sigma .$$

Then (π_n) is convergent, $S \subseteq \hat{G}$ and S is the set of all limits of (π_n) ([12]).

For a parabolic subgroup $P = MN$ of G and irreducible smooth cuspidal representation σ of M we shall denote by $\tilde{G}(\sigma)$ the set of all irreducible

subquotients of $\operatorname{Ind}(\chi\sigma|P,G)$ when χ runs over $U(M)$.
Set

$$\hat{G}(\sigma) = \tilde{G}(\sigma) \cap \hat{G} .$$

By [18], sets $\hat{G}(\sigma)$ define a partition of $\hat{G}$ and they are open and closed subsets of $\hat{G}$. Also sets $\tilde{G}(\sigma)$ define a partition of $\tilde{G}$.

Suppose that $P_i = M_i N_i$, $i = 1,2$, are two associated parabolics and σ_i , $i = 1,2$, associated representations. Then

$$\tilde{G}(\sigma_1) = \tilde{G}(\sigma_2) .$$

Let $P = MN$ be a parabolic subgroup in G and let σ_1, σ_2 be irreducible cuspidal representations of M.
If $\tilde{G}(\sigma_1) = \tilde{G}(\sigma_2)$ then there exist an element w from the Weyl group of G normalizing M so that

$$w\sigma_1 \in U(M)\sigma_2$$

where $w\sigma_1(m) = \sigma_1(w m w^{-1})$ (see [18]) .

On the set of all irreducible cuspidal representations we define an equivalence relation by $\tau \sim \sigma$ if and only if $U(G)\tau = U(G)\sigma$. The equivalence class of τ will be denoted by $[\tau]$.

Let $P = MN$ be a parabolic subgroup of G and σ a smooth admissible representation of M. The formula for the character of the induced representation in [4] implies that

$$\begin{aligned} \chi &\longmapsto \Theta_{\mathrm{Ind}(\chi\sigma|PG)} \\ U(M) &\longrightarrow H(G)^* \end{aligned}$$

is continuous (see also Lemma 2.1. of [17]).

Let $P = MN$ be a parabolic subgroup of G. Suppose that (σ_n) is a sequence of admissible representations of M and σ an admissible representation of M such that $\lim_n \Theta_{\sigma_n} = \Theta_\sigma$ in $H(M)^*$. By [4] there exist a linear mapping

$$\Lambda : H(G) \longrightarrow H(M)$$

so that

$$\Theta_{\mathrm{Ind}(\sigma_n|P,G)} = \Theta_{\sigma_n} \circ \Lambda ,$$

$$\Theta_{\mathrm{Ind}(\sigma|P,G)} = \Theta_\sigma \circ \Lambda .$$

Therefore

$$\lim_n \Theta_{\mathrm{Ind}(\sigma_n|P,G)} = \Theta_{\mathrm{Ind}(\sigma|P,G)} .$$

Let σ be an irreducible cuspidal representation of M where $P = MN$ is a parabolic subgroup of G. By Theorem 3.1 of [18] the set of all $\chi \in U(M)$ so that $\mathrm{Ind}\,(\chi\sigma|P,G)$ contains a unitarizable subquotient, is a compact subset of $U(M)$.

Suppose that (π_n) is a sequence in $\hat{G}(\sigma)$. Then π_n is a subquotient of some $\mathrm{Ind}(\chi_n\sigma|P,G)$. Now we can take a subsequence (π_{n_k}) so that (χ_{n_k}) converges. Using Lemma 3.5 and previous considerations we obtain that (π_n) contains a convergent subsequence.

For an admissible representation π of G, $\tilde{\pi}$ will denote the contragradient representation of π, and $\bar{\pi}$ will denote the Hermitian conjugate of π. Set $\pi^+ = \tilde{\bar{\pi}}$. Then π^+ is called the Hermitian conjugate of π. Let σ be an irreducible cuspidal representation of M where $P = MN$ is a parabolic subgroup of G. Since each irreducible cuspidal representation can be twisted by an unramified character to a unitarizable representation, we have that

$$(\tilde{G}(\sigma))^+ = \tilde{G}(\sigma).$$

Let $\sigma \in \hat{G}$. By the description of the topology of $\hat{G}$ we have directly that

$$\begin{array}{ccc} \chi & \longmapsto & \chi\sigma \\ U^u(G) & \longrightarrow & \hat{G} \end{array}$$

is continuous. Since $U^u(G)$ is connected set, $U^u(G)\sigma$ is also connected set. We shall now show that $U^u(G)\sigma$ is also a closed subset of $\hat{G}$. Let $\pi \in C\ell(U^u(G)\sigma)$. Then there exist a sequence (π_n) in $U^u(G)\sigma$ converging to π. Set $\pi_n = \chi_n\sigma$ with $\chi_n \in U^u(G)$. Since $U^u(G)$ is compact, we can find a convergent subsequence (χ_{n_k}) of (χ_n).

Set $\chi = \lim_k \chi_{n_k}$. Now $(\chi_{n_k}\sigma)$ converges to π , and also $\chi\sigma$ is the only limit of $(\chi_{n_k}\sigma)$. Thus $\chi\sigma = \pi$ and $\pi \in U^u(G)\sigma$. This proves that $U^u(G)\sigma$ is closed.

If $\sigma \in \hat{G}$ and $U \subseteq \hat{G}$ is an open and closed subset containing σ then it must contain $U^u(G)\sigma$.

1.1. Definition: Let $\sigma \in \hat{G}$. We say that σ is isolated in $\hat{G}$ modulo the unramified characters if $U^u(G)\sigma$ is open set.

2. Basic facts about the topology of the dual space of GL(n,F).

Let F be a non-archimedean local field. We put $G_n = GL(n,F)$. Let $\text{Alg } G_n$ be the category of all smooth representations of G_n of finite length. Let $\sigma_i \in G_{n_i}$, $i = 1,2$ and let P be the standard maximal compact subgroup whose Levi factor is naturally isomorphic to $G_{n_1} \times G_{n_2}$. The representation of $G_{n_1+n_2}$ induced from P by $\sigma_1 \otimes \sigma_2$ is denoted by $\sigma_1 \times \sigma_2$. The induction is normalized.

Let R_n be the Grothendieck group of $\text{Alg } G_n$. Set

$$R = \bigoplus_{n\geq 0} R_n .$$

The induction functor

$$(\tau,\sigma) \mapsto \tau\times\sigma \, , \ \text{Alg } G_n \times \text{Alg } G_m \to \text{Alg } G_{m+n}$$

induces the structure of a graded associative commutative algebra on R. The graduation will be denoted by $gr: R \to \mathbb{Z}_+$. Set

$$\text{Irr} = \bigcup_{n\geq 0} \tilde{G}_n \, ,$$

$$\text{Irr}^u = \bigcup_{n\geq 0} \hat{G}_n \, .$$

We supply Irr^u with the topology of disjoint union. We consider Irr as the subset of R. In fact, Irr is a

$\mathbb{Z}$-basis of R .

Let $C(G_n)$ be the subset of all classes of cuspidal representations in Irr. Set

$$C = \bigcup_{n\geq 1} C(G_n) ,$$

$$C^u = C \cap \mathrm{Irr}^u ,$$

$$C_{\otimes} = C \cup (C \otimes C) \cup (C \otimes C \otimes C) \cup \ldots$$

where

$$\underbrace{C \otimes \ldots \otimes C}_{n\text{-times}} = \{\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n \; ; \; \rho_i \in C\} .$$

If $\rho_i \in \tilde{G}_{m_i}$, then $\rho_1 \otimes \ldots \otimes \rho_n$ is considered to be in $G_{m_1} \times \ldots \times G_{m_n}$.

For a set $X, M(X)$ will denote the set of all finite multisets in X . It is the union of all X^n , $n \geq 0$, derived by the action of symmetric group S_n acting by permutation of coordinates. For $x_i \in X^{n_i}$, $i = 1,2$, we consider $(x_1,x_2) \in X^{n_1+n_2}$. This induces the structure of semigroup on $M(X)$. This semigroup will be denoted additively.

We have a natural projection

$$\kappa : C_{\otimes} \longrightarrow M(C)$$

given by

$$\rho_1 \otimes \ldots \otimes \rho_n \longmapsto (\rho_1, \ldots, \rho_n).$$

Let P be a standard parabolic subgroup of G_n . Then Levi factor M is ismorphic to some product

$$G_{n_i} \times \ldots \times G_{n_k}$$

where $n_1 + \ldots + n_k = n$. Now an irreducible cuspidal representation of M is isomorphic to

$$\rho_1 \otimes \ldots \otimes \rho_{n_k}$$

where $\rho_i \in C(G_{n_i})$.

We have also

$$U(M) \cong U(G_{n_1}) \times \ldots \times U(G_{n_k}) .$$

Let $\sigma = \rho_1 \otimes \ldots \otimes \rho_k \in C_{\otimes}$. Then σ is a cuspidal representation of a parabolic subgroup of suitable G_n . Since n is completely determined by σ , we put

$$\tilde{G}(\sigma) = \tilde{G}_n(\sigma) ,$$

$$\hat{G}(\sigma) = \hat{G}_n(\sigma) .$$

Let $c \in M(C)$, $c \neq \emptyset$. Choose $\sigma \in C_{\otimes}$ so that $c(\sigma) = c$. Since $\tilde{G}(\sigma)$ and $\hat{G}(\sigma)$ do not depend on such choice of σ we set

$$\tilde{G}(c) = \tilde{G}(\sigma) ,$$

$$\hat{G}(c) = \hat{G}(\sigma) .$$

Let $C|_{\sim}$ be the set of all equivalence classes of the eigenvalence relation $\sim$ in C. There is the projection map

$$\varphi : C \to C|_\sim \ .$$

This map lifts to the map

$$\Phi : M(C) \to M(C|_\sim) \ .$$

Let $\gamma \in C|_\sim$. Choose $c \in M(C)$ so that $\Phi(c) = \gamma$. Now $\tilde{G}(c)$ does not depend on c, but only on γ . Thus we define

$$\tilde{G}(\gamma) = \tilde{G}(c) \ .$$

Let $\gamma \in M(C|_\sim)$. Choose $(\rho_1,\ldots,\rho_m) \in M(C)$ so that $\Phi((\rho_1,\ldots,\rho_m)) = \gamma$. We define $\mathrm{gr}\,\gamma$ to be $\mathrm{gr}\,\rho_1 + \ldots + \mathrm{gr}\,\rho_m$.

From the first section one obtains directly

2.1. Proposition: (i) The disjoint union of all

$$\tilde{G}(\gamma),\ \gamma \in M(C|_\sim)$$

equals to Irr.

(ii) The disjoint union of all

$$\hat{G}(\gamma)\ ,\ \gamma \in M(C|_\sim)$$

equals to Irr^u .

(iii) Let $n \geq 1$. The disjoint union of all

$$\hat{G}(\gamma),\ \gamma \in M(C|_\sim)$$

with $\mathrm{gr}\,\gamma = n$ equals to $\hat{G}_n$.

Note that the inclusion $C^u \hookrightarrow C$ induces the bijection

$$(C^u|_\sim) \hookrightarrow (C|_\sim)$$

and thus the bijection

$$M(C^u|_\sim) \hookrightarrow M(C|_\sim) .$$

Let $\gamma = (\gamma_1,\ldots,\gamma_n), \delta = (\delta_1,\ldots,\delta_m) \in M(C|_\sim)$. We say that γ and δ are disjoint if $\gamma_i \neq \delta_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

For $c,d \in M(C)$ we say that are disjoint if $\Phi(c)$ and $\Phi(d)$ are disjoint. For $\rho,\sigma \in C_\otimes$ we say that are disjoint if $\kappa(\rho)$ and $\kappa(\sigma)$ are disjoint.

By Corollary 8.2. of [2] , Irr^u is multiplicatively closed subset of R.

2.2 <u>Proposition</u> (i) The multiplication of R defines the structure of topological semigroup on Irr^u

(ii) If $c_1,c_2 \in M(C)$ then

$$\hat{G}(c_1) \times \hat{G}(c_2) \subseteq \hat{G}(c_1 + c_2) .$$

(iii) If $c_1,c_2 \in M(C)$ are not connected then

$$(\tau,\sigma) \mapsto \tau \times \sigma$$

is a homeomorphism of $\hat{G}(c_1) \times \hat{G}(c_2)$ onto $\hat{G}(c_1 + c_2)$.

<u>Proof:</u> The statement (ii) is a direct consequence of the exactness of the induction functor. Since Irr^u is a disjoint union of $\hat{G}_n$'s it is enough to show that

$$(\tau,\sigma) \longmapsto \tau \times \sigma$$

$$\hat{G}_m \times \hat{G}_n \longrightarrow \hat{G}_{m+n}$$

is continuous. Suppose that $((\tau_i^1,\tau_i^2))_i$ converges to (τ^1,τ^2). Then $(\tau_i^j)_i$ converges to τ^j for $j = 1,2$. Now we can pass to a subsequence to that

$$\lim_i \Theta_{\tau_i^j} = \sum_\rho n_\rho^j \Theta_\rho$$

where $n^j_{\tau^j} \geq 1$ for $j = 1,2$. Now

$$\lim_i \Theta_{\tau_i^1 \times \tau_i^2} = \sum_{\rho_1,\rho_2} n^1_{\rho_1} n^2_{\rho_2} \Theta_{\rho_1 \times \rho_2} .$$

Since $n^1_{\tau^1} n^2_{\tau^2} \geq 1$, $(\tau_i^1 \times \tau_i^2)_i$ converges to $\tau^1 \times \tau^2$. This proves (i).

Let $c_1, c_2 \in M(C)$. Suppose that c_1 and c_2 are not connected. Then analogically as in Proposition 1.5.2. of [23] we obtain that

$$(\tau,\sigma) \longrightarrow \tau \times \sigma ,$$

$$\tilde{G}(c_1) \times \tilde{G}(c_2) \rightarrow \tilde{G}(c_1 + c_2)$$

is a bijection. Denote this bijection by Ψ . Thus the restriction

$$\Psi:\hat{G}(c_1)\times\hat{G}(c_2) \to \hat{G}(c_1+c_2)$$

is an injection. Now we shall show that it is also a surjection. Let $\rho \in \hat{G}(c_1+c_2)$. There exists $\rho_i \in \tilde{G}(c_i)$ so that $\rho = \rho_1 \times \rho_2$. Since ρ is unitarizable we have

$$\rho = \rho_1 \times \rho_2 = \rho_1^+ \times \rho_2^+ \quad .$$

We noticed that $\rho_i^+ \in \tilde{G}(c_i)$. Thus $\rho_i^+ = \rho_i$, $i = 1,2$. Now Corollary 8.2. of [2] implies that $\rho_i \in \hat{G}(c_i)$. Thus

$$\Psi:\hat{G}(c_1)\times\hat{G}(c_2) \to \hat{G}(c_1+c_2)$$

is a bijection. We shall show now that Ψ^{-1} is continuous.

Suppose that $\tau_i^1\times\tau_i^2$ converges to $\tau^1\times\tau^2$ in $\hat{G}(c_1+c_2)$, where $\tau_1^j, \tau^j \in \hat{G}(c_j)$. Passing to a subsequence we can suppose that $\Theta_{\tau_i^j}$, $j = 1,2$, are convergent. Now $\Theta_{\tau_i^1\times\tau_i^2}$ converges and we put

$$\lim_i \Theta_{\tau_i^1\times\tau_i^2} = \sum n_\sigma \Theta_\sigma \quad .$$

We know that $n_{\tau^1\times\tau^2} \neq 0$. Set

$$\lim_i \Theta_{\tau_i^j} = \sum n_\sigma^i \Theta_\sigma \quad .$$

Then

$$\sum n_\sigma\Theta_\sigma = \sum n_\tau^1 n_\rho^2 \Theta_{\tau\times\rho} \quad .$$

Now the fact that Ψ is a bijection and the linear independence of the characters implies $n^1_{\tau^1}\, n^2_{\tau^2} \neq 0$. Thus (τ^j_1) converges to τ^j, $j = 1,2$. This proves that Ψ^{-1} is continuous.

Let $c \in C|_\sim$ and $n \in \mathbb{Z}_+$. We define

$$\hat{G}(c,m) = \hat{G}(\underbrace{(c,c,\ldots,c)}_{n\text{-times}}).$$

For $n = 0$ we set $\hat{G}(C,0) = \hat{G}_0$.

The set of all functions from $C|_\sim$ to $\mathbb{Z}_+$ with finite support will be denoted by $M^*(C|_\sim)$. Note that $M^*(C|_\sim)$ is in natural bijection with $M(C|_\sim)$.

Now the previous proposition implies:

2.3. <u>Proposition:</u> (i) The disjoint union of all

$$\coprod_{f \in M^*(C|_\sim)} \hat{G}(c,f(c))$$

is homeomorphic to Irr^u .

(ii) Let $n \geq 1$. The disjoint union of all

$$\coprod_{f \in M^*(C|_\sim)} \hat{G}(c,f(c))$$

with

$$\sum_{c \in C|_\sim} f(c)\,\mathrm{gr}\, c = n \quad .$$

is homeomorphic to $\hat{G}_n$.

2.4. Proposition: Let $n \geq 1$. The set

$$C(G_n)|_\sim$$

is countable infinite.

Proof: Recall that

$$C(G_n)|_\sim = (C(G_n) \cap \hat{G}_n)|_\sim =$$

$$= \{U^u(G_n)\rho;\ \rho \in C(G_n) \cap \hat{G}_n\} .$$

Since $U^n(G_n)\rho$,$\rho \in C(G_n) \cap \hat{G}_n$ are just $\hat{G}_n(\rho)$ and these are open and closed subsets of $\hat{G}_n$, we have that $C(G_n)|_\sim$ is countable since $\hat{G}_n$ has a countable basis of open sets.

It remains to show that $C(G_n)|_\sim$ is infinite.

Note first that $C(G_n) \neq \emptyset$ (see for example [7]).

Let $\rho \in C(G_n)$ and let ω_ρ be the central character of ρ . Let T be the maximal compact subgroup in F^x . Recall that the center $Z(G_n)$ of G_n is isomorphic to F^x and thus we identify $Z(G_n)$ with F^x .

Suppose that $\sigma \in U(G_n)\rho$. Then obviously

$$\omega_\sigma|_T = \omega_\rho|_T \quad .$$

Now the family

$$[\omega_{(\chi \,\circ\, \det)\sigma}]|_T ; x \in \hat{F}$$

is infinite since

$$\chi^n, \chi \in \hat{T}$$

is infinite. Otherwise there exist an exponent m so that $\chi^m = 1$ for all $\chi \in \hat{T}$ what implies that $x^m = 1$ for all $x \in \hat{\hat{T}} \cong \hat{T}$. This is impossible since F is a field and T is infinite.

The above considerations imply that $C(G_n)|_{\sim}$ is infinite.

For $\chi \in U(G_1) = U(F^x)$ and $\pi \in \tilde{G}_n$ we shall denote $(\chi \circ \det)\pi$ also $\chi\pi$.

Very often we shall identify $U(G_1) = U(F^x)$ with $U(G_n)$ by the identification $\chi \longmapsto \chi \circ \det$.

3. Parametrizations of the non-unitary dual of $GL(n,F)$

Let $m \in \mathbb{N}$. Set

$$\nu(g) = |\det(g)|_F$$

where $|\ |_F$ is the normalized absolute value on F.

For $\rho \in C$ and $n \in \mathbb{N}$ set

$$\Delta[n]^{(\rho)} = \{\nu^{-(n-1)/2+i}\rho\ ;\quad 0 \leq i \leq n-1\}.$$

The set $\Delta[n]^{(\rho)}$ is called a segment in C. The set of all segments in C is denoted by $S(C)$.

If $\Delta = \Delta[n]^{(\rho)} \in S(C)$, then we denote

$$\Delta^- = \{\nu^{-(n-1)/2+i}\rho\ ;\quad 0 \leq i \leq n-2\},$$

$$^-\Delta = \{\nu^{-(n-1)/2+i}\rho\ ;\quad 1 \leq i \leq n-1\}.$$

Let $a = (\Delta_1,\ldots,\Delta_m) \in M(S(C))$. Set

$$a^- = (\Delta_1^-,\ldots,\Delta_m^-).$$

Consider $\Delta_i \subseteq C$ as multisets in C in a natural way. Then we define

$$\operatorname{supp} a = \Delta_1 + \ldots + \Delta_m \in M(C).$$

The number m is called the cardinal number of a.

For $\Delta = \{\rho,\nu\rho,\ldots,\nu^m\rho\} \in S(C)$ set

$$t(\Delta) = (\{\rho\},\{\nu\rho\},\ldots,\{\nu^m\rho\}) \in M(S(C))$$

Two segments $\Delta_i = \Delta[n_i]^{(\rho_i)}$ $i = 1,2$, are said to be linked if $\Delta_1 \cup \Delta_2$ is again a segment and

$$\Delta_1 \cup \Delta_2 \notin \{\Delta_1, \Delta_2\} .$$

If Δ_1 and Δ_2 are linked, then there exists $\alpha \in \mathbb{R}$ so that $\rho_2 = \nu^\alpha \rho_1$. If $\alpha > 0$ then we say that Δ_1 precedes Δ_2 and we write

$$\Delta_1 \longrightarrow \Delta_2 .$$

Let $a = (\Delta_1, \ldots, \Delta_m) \in M(S(C))$. Suppose that Δ_i and Δ_j are linked $(1 \leq i < j \leq m)$. If we set

$$b = (\Delta_1, \ldots, \Delta_{i-1}, \Delta_i \cup \Delta_j, \Delta_{i+1}, \ldots, \Delta_{j-1}, \Delta_i \cap \Delta_j, \Delta_{j+1}, \ldots, \Delta_m)$$

then we write $b \dashv a$. For $a_1, a_2 \in M(S(C))$ we write $a_1 < a_2$ if there exist $b_1, b_2, \ldots, b_k \in M(S(C))$, $k \geq 2$, so that

$$a_1 = b_1 \dashv b_2 \dashv \ldots \dashv b_k = a_2 .$$

For $\alpha \in \mathbb{C}$ and $\Delta = \Delta[n]^{(\rho)} \in S(C)$ set

$$\nu^\alpha \Delta = \Delta[n]^{(\nu^\alpha \rho)} .$$

For $a = (\Delta_1, \ldots, \Delta_m) \in M(S(C))$ set

$$\nu^\alpha a = (\nu^\alpha \Delta_1, \ldots, \nu^\alpha \Delta_m) .$$

Let $\Delta = \{\rho, \nu\rho, \ldots, \nu^m \rho\} \in S(C)$. Then the representation

$$\rho \times \nu\rho \times \ldots \times \nu^m \rho$$

has the unique irreducible subrepresentation which is denoted by $Z(\Delta)$, and the unique irreducible quotient which is denoted by $L(\Delta)$.

Let $a = (\Delta_1,\dots,\Delta_n) \in M(S(C))$. We can choose an order on the segments satisfying

$$\Delta_i \to \Delta_j \implies i > j \quad .$$

Up to an isomorphism, representations

$$\zeta(a) = Z(\Delta_1) \times\dots\times Z(\Delta_n) \quad ,$$

$$\lambda(a) = L(\Delta_1) \times\dots\times L(\Delta_n)$$

are uniquely determined by a. The representation $\zeta(a)$ has a unique irreducible subrepresentation which is denoted by $Z(a)$, and $\lambda(a)$ has a unique irreducible quotient which is denoted by $L(a)$.

Now $a \longmapsto Z(a)$, $M(S(C)) \mapsto \mathrm{Irr}$ is a bijection and this is Zelevinsky classification of Irr ([22]). The mapping is a version of Langlands classification of Irr ([14]).

The mapping

$$Z(a) \longmapsto L(a), \; a \in M(S(C))$$

extends uniquely to an additive mapping of R which will be denoted by

$$t : R \longrightarrow R \; .$$

Let $\pi \in \mathrm{Irr}$. Choose $a, b \in M(S(C))$ so that

$$\pi = L(a) = Z(b).$$

Then $\operatorname{supp} a = \operatorname{supp} b$ and we define

$$\operatorname{supp} \pi = \operatorname{supp} a = \operatorname{supp} b.$$

Let D be the set of all essentially square integrable representations in Irr. Set $D^u = D \cap \mathrm{Irr}^u$.

Let $d = (\delta_1, \ldots, \delta_n) \in M(D)$. Now we can write $\delta_i = \nu^{\alpha_i} \delta_i^u$ where $\alpha_i \in \mathbb{R}$ and $\delta_i^u \in D^u$. After a suitable permutation we can suppose that

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n .$$

The representation

$$\lambda(d) = \delta_1 \times \ldots \times \delta_n$$

has the unique irreducible quotient which will be denoted by $L(d)$. The mapping

$$d \mapsto L(d), \quad M(D) \to \mathrm{Irr}$$

is a bijection and this is a version of Langlands classification.

The following lemma is obvious.

3.1. <u>Lemma</u>: If $\alpha \in \mathbb{C}$ and $\alpha \in M(S(C))$ then

$$\nu^\alpha Z(a) = Z(\nu^\alpha a) ,$$

$$\nu^\alpha L(a) = L(\nu^\alpha a) .$$

The above lemma implies

$$t(\nu^\alpha(\pi)) = \nu^\alpha(t(\pi))$$

for $\pi \in$ Irr.

Let R_+ be the set of all possible $n_1\pi_1 + \ldots + n_k\pi_k$ with $k \in \mathbf{N}$, $n_i \in \mathbf{Z}_+$ and $\pi_i \in$ Irr. For an additive mapping φ on R we say that it is positive if

$$x \in R_+ \Longrightarrow \varphi(x) \in R_+ .$$

Let $x \in R_+$. We say that x contains $\pi \in \mathrm{Irr}$ if $x - \pi \in R_+$. We say that x contains π with multiplcity one if

$$x - \pi \in R_+ \quad \text{and} \quad x - 2\pi \notin R_+ .$$

If x contains π we shall also say that π is a composition factor of x.

Now we shall recall of the derivatives of representations in a way that is convenient for our purposes.

Since R is a polynomial ring over all $Z(\Delta)$, $\Delta \in S(C)$, there exist a unique ring morphism $\mathcal{D}$ of R so that

$$\mathcal{D}(Z(\Delta)) = Z(\Delta) + Z(\Delta^-) .$$

For $x \in R$, $\mathcal{D}(x)$ will be called the derivative of X. Let

$$\mathcal{D}(x) = \sum_{i \geq 0} y_i$$

where $y_i \in R_i$. If $\mathcal{D}(x) = 0$ then 0 is called the highest derivative of x. If $\mathcal{D}(x) \neq 0$ then y_{i_0} is called the highest derivative of x if $y_{i_0} \neq 0$ and $y_j = 0$ for $j < i_0$. The highest derivative of x will be denoted by $\mathcal{D}_h(x)$. Now we list the most important properties of $\mathcal{D}$ (see [3] and [22]) .

(i) $\mathcal{D}$ is a ring morphism of R determined by

$$\mathcal{D}(Z(\Delta)) = Z(\Delta) + Z(\Delta^-), \; \Delta \in S(C).$$

(ii) For $\pi \in \mathrm{Irr}$ $\mathcal{D}(\pi) = \pi + x$ where

$$\mathrm{gr}\, x < \mathrm{gr}\, \pi.$$

This implies that $\mathcal{D}$ is an automorphism.

(iii) The operator $\mathcal{D}$ is positive.

(iv) For $\Delta \in S(C)$ we have

$$\mathcal{D}(L(\Delta)) = L(\Delta) + L({}^-\Delta) + L({}^{--}\Delta) + \ldots + L(\emptyset) \; .$$

(v) The highest derivative of an irreducible representation is an irreducible representation. Moreover

$$\mathcal{D}_h(Z(a)) = Z(a^-)$$

for $a \in M(S(C))$.

Note that $\mathcal{D}_h : R \to R$ is a multiplicative mapping.

3.2. <u>Lemma</u>: Let $\Delta_1, \Delta_2 \in S(C)$.

(i) If Δ_1 and Δ_2 are not linked, then

$$L(\Delta_1) \times L(\Delta_2) = L((\Delta_1, \Delta_2)).$$

(ii) If Δ_1 and Δ_2 are linked, then

$$L(\Delta_1) \times L(\Delta_2) = L((\Delta_1, \Delta_2)) + L((\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2)).$$

<u>Proof:</u> By Proposition A.4 of [19] $L(\Delta_1) \times L(\Delta_2)$ contains $L((\Delta_1, \Delta_2))$ with multiplicity one. If Δ_1 and Δ_2 are not linked then $L(\Delta_1) \times L(\Delta_2)$ is irreducible by Theorem 9.7. of [22]. This implies (i).

Suppose that Δ_1 and Δ_2 are linked. Then $L(\Delta_1) \times L(\Delta_2) = Z(t(\Delta_1)) \times Z(t(\Delta_2))$ contains $Z(t(\Delta_1) + t(\Delta_2))$ with multiplicity one by Proposition 8.4. of [22]. Now

$$Z(t(\Delta_1) + t(\Delta_2)) = Z(t(\Delta_1 \cup \Delta_2) + t(\Delta_1 \cap \Delta_2)) =$$

$$= Z(t(\Delta_1 \cup \Delta_2)) \times Z(t(\Delta_1 \cap \Delta_2)) =$$

$$= L(\Delta_1 \cup \Delta_2) \times L(\Delta_1 \cap \Delta_2) =$$

$$= L((\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2)) .$$

If we show that $L(\Delta_1) \times L(\Delta_2)$ can have only one composition factor except $L(\Delta_1 \cup \Delta_2) \times L(\Delta_1 \cap \Delta_2)$, then (ii) will be proved.

We can suppose that

$$\Delta_1 = \{\rho, \nu\rho, \ldots, \nu^p\rho\} \quad ,$$

$$\Delta_2 = \{\nu^q\rho, \nu^{q+1}\rho, \ldots, \nu^r\rho\}$$

where $p,q,r \in \mathbf{N}$, $q \leq p+1$ and $p < r$. Let π be a composition factor of $L(\Delta_1) \times L(\Delta_2)$ different from

$$L(\Delta_1 \cup \Delta_2) \times L(\Delta_1 \cap \Delta_2) .$$

Then $\mathcal{D}_h(\pi)$ is a composition factor of

$$\mathcal{D}(L(\Delta_1) \times L(\Delta_2)) - \mathcal{D}(L(\Delta_1 \cup \Delta_2) \times L(\Delta_1 \cap \Delta_2)) =$$

$$= L(\Delta_1) + L(^-\Delta_1) + \ldots + L(\emptyset)) \times (L(\Delta_2) + L(^-\Delta_2) + \ldots + L(\emptyset))$$

$$-(L(\Delta_1 \cup \Delta_2) + L(^-(\Delta_1 \cup \Delta_2)) + \ldots + L(\emptyset))(L(\Delta_1 \cap \Delta_2) + \ldots + L(\emptyset)) .$$

The property (v) of the operator $\mathcal{D}$ implies that $\nu^r\rho$ can not be in the support of $\mathcal{D}_h(\pi)$. Therefore $\mathcal{D}_h(\pi)$ is a composition factor of

$$(L(\Delta_1) + (L(^-\Delta_1) + \ldots + L(\emptyset)) \times L(\emptyset) -$$

$$- (L(\Delta_1 \cup \Delta_2) + \ldots + L(\emptyset)) \times L(\emptyset) =$$

$$= L(\{\rho, \nu\rho, \ldots, \nu^p\rho\}) + L(\{\nu\rho, \ldots, \nu^p\rho\}) + \ldots + L(\{\nu^{q-1}\rho, \ldots, \nu^r\rho\}) =$$

$$= Z((\{\rho\}, \{\nu\rho\}, \ldots, \{\nu^p\rho\})) + \ldots + Z((\{\nu^{q-1}\rho\}, \ldots, \{\nu^r\rho\})) .$$

Now $Z((\{\nu^i\rho\}, \ldots, \{\nu^r\rho\}))$, $0 \leq i < q-1$, can not be the highest derivative because the multiplicity of $\nu^{q-1}\rho$ in $\operatorname{supp} \pi$ should be 2 in that case. Thus

$$Z((\{\nu^{q-1}\rho\},\ldots,\{\nu^{r}\rho\}))$$

equals $\mathcal{D}_h(\pi)$. Recall $\operatorname{supp}\pi = \operatorname{supp}(\Delta_1,\Delta_2)$.
Since the support and the highest derivative completely determine the representation, we have proved (ii).

4. Unitary dual of $GL(n,F)$

For $\sigma \in \mathrm{Irr}$ and $\alpha \in \mathbb{R}$ set

$$\pi(\sigma,\alpha) = (\nu^{\alpha}\sigma) \times (\nu^{\alpha}\sigma)^{+} = (\nu^{\alpha}\sigma) \times (\nu^{-\alpha}\sigma^{+}) .$$

Put

$$a(n,d)^{(\rho)} = (\nu^{(n-1)/2}\Delta[n]^{(\rho)}, \nu^{(n-1)/2-1}\Delta[n]^{(\rho)}, \ldots$$
$$\ldots, \nu^{-(n-1)/2}\Delta[n]^{(\rho)})$$

for $n,d \in \mathbf{N}$, $\rho \in C$;

$$u(\delta,n) = L(\nu^{(n-1)/2}\delta, \nu^{(n-1)/2-1}\delta, \ldots, \nu^{-(n-1)/2}\delta)$$

for $\delta \in D$, $n \in \mathbf{N}$;

$$\delta(\rho,m) = L(\Delta[m]^{(\rho)})$$

for $\rho \in C$, $m \in \mathbf{N}$.

In [19] we have proved the following:

4.1. Theorem: (i) Set

$$B = \{u(\delta,n); \pi(u(\delta,n),\alpha);\ \delta \in D^{u} ,\ n \in \mathbf{N} ,\ 0 < \alpha < 1/2\} .$$

If $\sigma_1,\ldots,\sigma_m \in B$, then $\sigma_1 \times \ldots \times \sigma_m \in \mathrm{Irr}^u$. If $\pi \in \mathrm{Irr}^u$, then there exist $\tau_1,\ldots,\tau_k \in B$ unique up to a permutation so that

$$\pi = \tau_1 \times \ldots \times \tau_k$$

(ii) $$L(a(n,d)^{(\rho)}) = Z(a(d,n)^{(\rho)}) = u(\delta(\rho,d),n) .$$

5. Isolated representations

In this section we shall classify isolated representations modulo unramified characters. We prove:

5.1. Theorem: The set

$$I = \{u(\delta(\rho,n),m) \ ; \rho \in C^u \ , \ n,m \in \mathbf{N} \ , \ n \neq 2 \ , \ m \neq 2\} =$$

$$= \{Z(a(n,m)^{(\rho)}) \ ; \rho \in C^u \ , \ n,m \in \mathbf{N}, \ n \neq 2, \ m \neq 2\}$$

$$= \ L(a(n,m)^{(\rho)}); \rho \in C^u \ , n,m \in \mathbf{N} \ , \ n \neq 2 \ , \ m \neq 2\}$$

is the set of all isolated points modulo unramified characters in Irr^u .

Proof: Let π be an isolated point modulo unramified characters. First we shall show that π is not unitarily induced i.e. that there do not exists $\tau, \sigma \in Irr^u$, $gr\ \tau$, $gr\, \sigma \geq 1$, so that

$$\pi = \tau \times \sigma .$$

To prove that suppose $\pi = \tau \times \sigma$ with $\tau, \sigma \in Irr^u$ $gr\,\tau$, $gr\,\sigma \geq 1$. Let $\sigma \in \hat{G}_m$, $\tau \in \hat{G}_n$. Now the mapping

$$\chi \mapsto \tau \times (\chi\sigma)$$

$$U^u(F^x) \cong U^u(G_n) \to Irr^u .$$

is continuous. Since $U^u(G_n)$ is connected and $\pi = \tau \times \sigma$ is isolated modulo unramified characters , we have

$$(*) \qquad \{\tau \times (\chi\sigma); \chi \in U^u(F^x)\} \subseteq \{\mu(\tau \times \sigma);\ u \in U^u(F^x)\} \quad .$$

Let $\operatorname{supp} \tau = (\tau_1,\dots,\tau_t)$ and $\operatorname{supp} \sigma = (\sigma_1,\dots,\sigma_s)$. Then

$$\operatorname{supp}\ (\tau \times (\chi\sigma)) = (\tau_1,\dots,\tau_t\ ,\chi\sigma_1,\dots,\chi\sigma_s) \ ,$$

$$\operatorname{supp}\ (\mu\tau \times \mu\sigma) = (\mu\tau_1,\dots,\mu\tau_t,\mu\sigma_1,\dots,\mu\sigma_s) \ .$$

Now (*) implies that for each $\chi \in U^u(F^x)$ there exist $\mu(\chi) \in U^u(F^x)$ so that

$$\operatorname{supp}\ (\tau \times (\chi\sigma)) = \operatorname{supp}\ (\mu\tau \times \mu\sigma) \ .$$

Thus there exist $1 \leq i \leq t$ or $1 \leq j \leq s$ so that

$$\tau_1 \cong \mu(\chi)\tau_i \quad \text{or} \quad \tau_1 \cong \mu(\chi)\sigma_j$$

for χ in some infinite family $X \subseteq U^u(F^x)$. Considering the central characters one sees directly that

$$\{\mu(\chi);\ \chi \in X\}$$

is a finite set. This implies that

$$\{\chi\sigma;\ \chi \in X\}$$

is a finite set. This is a contradiction.

We can conclude now from Theorem 4.1. that either

$$\pi = u(\delta,n)$$

or

$$\pi = \pi(u,\delta),\alpha_0) \quad , \quad 0 < \alpha_0 < 1/2 \ .$$

Now we shall show that the second possibility can not happen. Similarly as above we can see that

$$\{\pi(u(\delta,\infty),\alpha) \ ; \quad 0 < \alpha < 1/2\}$$

is a connected set. Thus

$$\{\pi(u(\delta,n),\alpha) \ ; \ 0 < \alpha < 1/2\} \subseteq$$

$$\subseteq \{\pi(\chi u(\delta,n),\alpha_0) ; \quad \chi \in U^u(F^x)\} \ .$$

Considering the support we can obtain in the same way as above that this is not possible.

Thus $\qquad \pi = Z(a(2,d)^{(\rho)})$.

Now we shall see that $n \neq 2$. Note that $Z(a(z,d)^{(\rho)})$ is in the closure of the complementary series

$$\{\pi(Z(a(1,d)^{(\rho)}),\alpha) \ ; \quad 0 < \alpha < 1/2\}$$

which is a connected set. In the same way as before we get

$$\{\pi(Z(a(1,d)^{(\rho)}),\alpha) ; \ 0 < \alpha < 1/2\} \nsubseteq$$

$$\nsubseteq \{\chi Z(a(2,d)^{(\rho)}) \ ; \quad \chi \in U^u(F^x)\} \ .$$

Thus $Z(a(2,d)^{(\rho)})$ can not be isolated modulo the unramified characters.

In the same way we obtain that $L(a(2,d)^{(\rho)}) = Z(a(d,2)^{(\rho)})$ can not be isolated modulo the unramified characters.

We have proved that each isolated point modulo the unramified characters is contained in I. In the rest of the proof we shall show that each $\pi \in I$ is isolated. This will prove the theorem.

Let $\pi \in I$ i.e. $\pi = Z(a(n,d)^{(\rho)})$, $n \neq 2$ and $d \neq 2$. Suppose that π is not isolated modulo the unramified characters. Then there exists a sequence (π_m) in Irr^u converging to π such that

$$\pi_m \notin \{\chi\pi ; \chi \in U^u(F^{\times})\} ,$$

for all m . Passing to a subsequence, we can suppose that (ch_{π_m}) is convergent sequence. Let

$$\pi_m = \left[\prod_{i=1}^{p_{im}} Z(a(k_i^m,d_i^m)^{(\rho_i^m)}) \right] \times \left[\prod_{i=1}^{q_{im}} \pi(Z(a(\tilde{k}_i^m,\tilde{d}_i^m)^{(\sigma_1^m)}) ; \alpha_i^m) \right] .$$

Since the decomposition of Irr^u into $\hat{G}(c)$ $, c \in M(C|_{\sim})$, passing to a subsequence we can suppose that

$$\rho_i^m \sim \rho \quad \text{and} \quad \sigma_i^m \sim \rho$$

for all $m \in \mathbf{N}$ and $1 \leq i \leq p_m$, $1 \leq j \leq q_m$.

Passing to a subsequence we can suppose that

$$\begin{aligned} p_m &= p \\ k_i^m &= k_i \\ d_i^m &= d_i \\ \tilde{k}_i^m &= \tilde{k}_i \\ \tilde{d}_i^m &= \tilde{d}_i \end{aligned}$$

for all m and that

$$\{\rho_i^m\}_m,\ \{\sigma_i^m\}_m \quad \text{and} \quad \{\alpha_i^m\}_m$$

are convergent sequences. Let

$$\lim_m \rho_i^m = \rho_i$$

$$\lim_m \sigma_i^m = \sigma_i$$

$$\lim_m \alpha_i^m = \alpha_i \quad .$$

By the first section, the set of all limits of (π_m) equals to

$$\prod_{i=1}^{p} Z(a(k_i,d_i)^{(\rho_i)}) \times \prod_{i=1}^{q} \sigma_i$$

where σ_i is a composition factor of $\pi(Z(a(\tilde{k}_i,\tilde{d}_i)^{(\sigma_i)}),\alpha_i)$. Note that σ_i are unitarizable. If

$$p + q > 1$$

then Theorem 4.1. together with the above considerations implies that (π_m) is not converging to π since π is not unitarily induced.

Now we have two possibilities. First suppose that

$$\pi_m = Z(a(k_1^m,d_1^m)^{(\rho_1^m)}) \quad .$$

This sequence has only one limit which is

$$Z(a(k_1,d_1)^{(\rho_1)})$$

Thus $k_1 = n$, $d_1 = d$, $\rho_1 = \rho$. Since $\rho_1^m \sim \rho$ we have that

$$\pi_m \in U^u(F^x)\ Z(a(n,d)^{(\rho)})$$

what is a contradiction. Thus

$$\pi_m = \pi(Z(a(\tilde{k}_1,\tilde{d}_1)^{(\sigma_1^m)}),\alpha_1^m) \ .$$

Now the set of all limits of (π_m) equals to the set of all composition factors of

$$\pi(Z(a(\tilde{k}_1,\tilde{d}_1)^{(\sigma_1)},\alpha_1) \ .$$

If $0 \leq \alpha_1 < 1/2$, then it is clear that

$$\pi \neq \pi(Z(a(\tilde{k}_1,\tilde{d}_1)^{(\sigma_1)},\alpha_1)$$

from Theorem 4.1. Thus $\alpha_1 = 1/2$ i.e. π is a composition factor of

$$\pi(Z(a(\tilde{k}_1,\tilde{d}_1)^{(\sigma_1)}),1/2) \ .$$

First we have that $\sigma_1 = \rho$ i.e. π is a composition factor of

$$\pi(Z(a(\tilde{k}_1,\tilde{d}_1)^{(\rho)}),1/2).$$

This implies that

$$\operatorname{supp} a(n,d)^{(\rho)} = \operatorname{supp} (\nu^{-1/2}a(\tilde{k}_1,\tilde{d}_1)^{(\rho)} + \nu^{1/2}a(\tilde{k}_1,\tilde{d}_1)^{(\rho)}) \ .$$

We shall show that the last relation is not possible.

Since

$$\operatorname{supp} a(r,s)^{(\rho)} = \operatorname{supp} a(s,r)^{(\rho)} ,$$

we can suppose that $n \leq d$ and $\tilde{k}_1 \leq \tilde{d}_1$. Consider the multiset $\operatorname{supp} a(n,d)^{(\rho)}$ as the function from C to $\mathbf{Z}_+$. If

$$(\operatorname{supp}\ a(n,d)^{(\rho)})(\sigma) \neq 0$$

then $\sigma \in \{\nu^{\alpha}\rho \,;\, \alpha \in [(n+d)/2 + \mathbf{Z}]\}$.

Let $\sigma = \nu^{\alpha}\rho,\ \alpha \in [(n+d)/2 + \mathbf{Z}]$. Direct computation gives:

$$(\operatorname{supp}\ a(n,d)^{(\rho)})(\nu^{\alpha}\rho) = \begin{cases} 0 & \alpha \leq -(n+d)/2 \\ \alpha + (n+d)/2 & 1-(n+d)/2 \leq 2 \leq (n-d)/2 \\ n & (n-d)/2 \leq \alpha \leq (d-n)/2 \\ -\alpha + (n+d)/2 & (d-n)/2 \leq \alpha \leq -1+(n+d)/2 \\ 0 & (n+d)/2 \leq \alpha \end{cases} .$$

Set $a = \nu^{-1/2} a(\tilde{k}_1,\tilde{d}_1)^{(\rho)} + \nu^{1/2} a(\tilde{k}_1,\tilde{d}_1)^{(\rho)}$. Now we have

$$(\operatorname{supp}\ a)(\sigma) \neq 0$$

only when $\sigma \in \{\nu^{\alpha}\rho \,;\, \alpha \in [(\tilde{k}_1 + \tilde{d}_1 + 1)/2 + \mathbf{Z}]\}$.

Let $\sigma = \nu^{\alpha}\rho,\ \alpha \in [(\tilde{k}_1 + \tilde{d}_1 + 1)/2 + \mathbf{Z}]$. We have in the case $k_1 > 1$

$$(\text{supp } a)(\nu^{\alpha}\rho) = \begin{cases} 0 & \alpha \leq -(\tilde{k}_1 + \tilde{d}_1)/2 - 1/2 \\ 1 & \alpha = -(\tilde{k}_1 + \tilde{d}_1)/2 + 1/2 \\ 3 & \alpha = -(\tilde{k}_1 + \tilde{d}_1)/2 + 3/2 \\ - & ---------------- \\ - & ---------------- \end{cases}$$

Direct comparison with $\text{supp } a(n,d)^{(\rho)}$ gives

$$-(n+d)/2 = -(\tilde{k}_1 + \tilde{d}_1)/2 - 1/2 \Longrightarrow n + d = \tilde{k}_1 + \tilde{d}_1 + 1 \ .$$

Since

$$(\text{supp } a(n,d)^{(\rho)})(\nu^{-(\tilde{k}_1+\tilde{d}_1)/2+3/2}\rho) =$$

$$= (\text{supp } a(n,d)^{(\rho)})(\nu^{-(n+d)/2+2}\rho) = 2 \neq$$

$$\neq 3 = (\text{supp } a)(\nu^{-(\tilde{k}_1+\tilde{d}_1)/2+3/2}) \ .$$

It remains to show that the case $k_1 = 1$ is not possible. We have

$$\text{supp } (\nu^{-1/2}a(1,\tilde{d}_1)^{(\rho)} + \nu^{1/2}a(1,\tilde{d}_1)^{(\rho)}) =$$

$$= \text{supp } a(2,\tilde{d}_1)^{(\rho)} \ .$$

The formula for $\text{supp } a(n,d)^{(\rho)}$ implies that $\text{supp } a(n,d)^{(\rho)}$ determines uniquely $a(n,d)^{(\rho)}$ with $n \leq d$. Thus $a(2,\tilde{d}_1)^{(\rho)} = a(n,d)^{(\rho)}$ implies $n = 2$. This is impossible since

$$Z(a(n,d)^{(\rho)}) \in I \quad .$$

This finishes the proof of the theorem.

6. Ends of complementary series

The purpose of this section is to prove:

6.1. Theorem: Let $n,d \in \mathbf{N}$, $\rho \in C^u$. Then we have in R:

(i) $$\nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)}) =$$

$$= Z(a(n+1,d)^{(\rho)}) \times Z(a(n-1,d)^{(\rho)}) +$$

$$+ Z(a(n,d+1)^{(\rho)}) \times Z(a(n,d-1)^{(\rho)})$$

(ii) $$\nu^{1/2}L(a(n,d)^{(\rho)}) \times \nu^{-1/2}L(a(n,d)^{(\rho)}) =$$

$$= L(a(n+1,d)^{(\rho)}) \times L(a(n-1,d)^{(\rho)}) +$$

$$+ L(a(n,d+1)^{(\rho)}) \times L(a(n,d-1)^{(\rho)})$$

(iii) $$\nu^{1/2}u(\delta(\rho,d),n) \times \nu^{-1/2}u(\delta(\rho,d),n) =$$

$$= u(\delta(\rho,d),n+1) \times u(\delta(\rho,d),n-1) +$$

$$+ u(\delta(\rho,d+1),n) \times u(\delta(\rho,d-1),n) \quad .$$

In the above theorem we take for $p,q \in \mathbb{Z}_+$

$$Z(a(p,q)^{(\rho)}) = L(a(p,q)^{(\rho)}) = u(\delta(\rho,p),q) = Z(\emptyset) = L(\emptyset)$$

if $p = 0$ or $q = 0$.

By Theorem 4.1. it is enough to prove only the first statement.

First we have the following simple

6.2. Lemma: Let

$$a_i = (\Delta_1^i, \ldots, \Delta_{k_i}^i) \in M(S(C)),$$

$i = 1,2,3,4$. Suppose that $a_4 \leq a_1 + a_2$ and that there do not exist $b \in M(S(C))$ so that $b < a_4$.

(i) If $Z(a_3)$ is a composition factor of $Z(a_1) \times Z(a_2)$ then

$$k_4 \leq k_3 \leq k_1 + k_2 .$$

(ii) If $L(a_3)$ is a composition factor of $L(a_1) \times L(a_2)$ then

$$k_4 \leq k_3 \leq k_1 + k_2 .$$

Proof: If $Z(a_3)$ is a composition factor of $Z(a_1) \times Z(a_2)$, then $Z(a_3)$ is a composition factor of $\zeta(a_1) \times \zeta(a_2)$ what is equal to $\zeta(a_1 + a_2)$ in R. Thus

$$a_3 \leq a_1 + a_2$$

by Proposition 7.7 of [22]. The definition of the order on $M(S(C))$ implies that

$$a_4 \leq a_3 \leq a_1 + a_2 .$$

Again the definition of $\leq$ implies

$$k_4 \leq k_3 \leq k_1 + k_2 .$$

This proves (i).

Using Remark in 5.3. of [14] and Theorem 5.1. of [14] we prove (ii) in the same way as (i).

6.3. <u>Lemma:</u> Let

$$a_1 = (\Delta_1,\dots,\Delta_m) \in M(S(C)) \ ,$$

$$a_2 = (\Delta_{m+1},\dots,\Delta_n) \in M(S(C)) \quad .$$

Suppose that

$$\Delta_i \cap \Delta_j \neq \emptyset \ , \quad 1 \leq i,j \leq n \ .$$

Then the number of all composition factors of $Z(a_1) \times Z(a_2)$ is equal to the number of all composition factors of $Z(a_1^-) \times Z(a_2^-)$.

<u>Proof</u>: First we observe that there exists $\rho \in C$ so that

$$\Delta_i \subseteq \{\nu^\alpha\rho; \alpha \in \mathbb{Z}\}$$

for all $1 \leq i \leq n$. If $a \leq a_1 + a_2$ then the cardinal number of a is n. Therefore the highest derivatives of all composition factors of $Z(\Delta_1) \times Z(\Delta_2)$ have the same graduation and they are composition factors of

$$\mathcal{D}_h(Z(\Delta_1) \times Z(\Delta_2)) = Z(\Delta_1^-) \times Z(\Delta_2^-) \ .$$

Since the highest derivative of an irreducible representation is irreducible, we have the lemma.

Now we go to

<u>Proof of Theorem 6.1.</u> By Proposition 8.4. of [22] we have that

$$Z(\nu^{1/2}a(n,d)^{(\rho)} + \nu^{-1/2}a(n,d)^{(\rho)}) =$$

$$= Z(a(n+1,d)^{(\rho)} + a(n-1,d)^{(\rho)}) =$$

$$= Z(a(n+1,d)^{(\rho)}) \times Z(a(n-1,d)^{(\rho)})$$

is a composition factor of

$$\nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)})$$

with the multiplicity one.

Since

$$\nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)}) =$$

$$= \nu^{1/2}L(a(d,n)^{(\rho)}) \times \nu^{-1/2}L(a(d,n)^{(\rho)}) ,$$

Proposition A.4. of [19] implies in the same way that

$$L(a(d+1,n)^{(\rho)}) \times L(a(d-1,n)^{(\rho)})$$

is a composition factor with the multiplicity one. Since

$$L(a(d+1,n)^{(\rho)}) \times L(a(d-1,n)^{(\rho)}) =$$

$$= Z(a(n,d+1)^{(\rho)}) \times Z(a(n,d-1)^{(\rho)}) ,$$

for the proof of Theorem 6.1. it is enough to show that the representation

$$\nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)})$$

has at most two different composition factors.

Suppose that we have shown it in the case $n = d$. Now we shall show how this case implies the general case. Note that

$$(*) \qquad \nu^{1/2}\mathcal{D}_h(\nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)}) =$$

$$= \nu^{1/2}Z(a(n,d-1)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d-1)^{(\rho)}) \; .$$

Since we suppose that

$$\nu^{1/2}Z(a(n,n)^{(\rho)}) \times \nu^{-1/2}Z(a(n,n)^{(\rho)})$$

has at most two different composition factors, the fact that the highest derivative of $\pi \in \mathrm{Irr}$ is irreducible and (*) implies that

$$\nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)})$$

has at most two different composition factors when $d \leq n$. The case of $d > n$ is a consequence of the formula (*) and Lemma 6.3.

It remains to consider the case of $d = n$. Let π be a composition factor of

$$\nu^{1/2}Z(a(n,n)^{(\rho)}) \times \nu^{-1/2}Z(a(n,n)^{(\rho)}) \; .$$

We shall show that π must be either

$$Z(a(n+1,n)^{(\rho)}) \times Z(a(n-1,n)^{(\rho)})$$

or

$$Z(a(n,n+1)^{(\rho)}) \times Z(a(n,n-1)^{(\rho)})$$

and this will prove the theorem.

Since π is a composition factor of the end of complementary series, we know that π is unitarizable. By Theorem 4.1. we can write

$$\pi = \left(\prod_{i=1}^{p} Z(a(n_i,d_i)^{(\rho_i)}) \right) \times \left(\prod_{j=1}^{q} \pi \; (Z(a(n_j',d_j')^{(\rho_j')}),\alpha_j) \right)$$

where $n_i, d_i, n_j', d_j' \in \mathbf{N}$, $\rho_i, \rho_j' \in C^u$ and $0 < \alpha_j < 1/2$. The fact that

$$\text{supp } \pi = \text{supp } (\nu^{1/2} a(n,n)^{(\rho)} + \nu^{-1/2} a(n,n)^{(\rho)})$$

implies that

$$\pi = \prod_{i=1}^{p} Z(a(n_i,d_i)^{(\rho_i)})$$

and again the equality of supports implies $\rho_i = \rho$ i.e.

$$\pi = \prod_{i=1}^{p} Z(a(n_i,d_i)^{(\rho)}) \; .$$

Let us consider more carefully the equality of supports:

$$\text{supp } (\nu^{1/2} a(n,n)^{(\rho)} + \nu^{-1/2} a(n,n)^{(\rho)}) =$$

$$= \sum_{i=1}^{p} \text{supp } (a(n_i,d_i)^{(\rho)}) \; .$$

Note that the left hand side is supported in $\{\nu^{\alpha}\rho; \alpha \in (1/2 + \mathbf{Z})\}$. Thus all $n_i + d_i$ are odd numbers. Now the formula for the support of $a(n,d)^{(\rho)}$ in the proof of Theorem 5.1. gives

$$(\operatorname{supp}(a(n_i,d_i)^{(\rho)}))(\nu^{1/2}\rho) =$$

$$= \min\{n_i,d_i\}$$

and also

$$(\operatorname{supp}(\nu^{1/2}a(n,n)^{(\rho)} + \nu^{-1/2}a(n,n)^{(\rho)}))(\nu^{1/2}\rho) =$$

$$= 2n - 1 \quad .$$

Thus

$$\text{(i)} \qquad \sum_{i=1}^{p} \min\{n_i,d_i\} = 2n - 1 \ .$$

By (i) of Lemma 6.2. we have

$$\text{(ii)} \qquad 2n - 1 \le \sum_{i=1}^{p} n_i \le 2n \quad .$$

Note that

$$\pi = \prod_{i=1}^{p} Z(a(n_i,d_i)^{(\rho)}) =$$

$$= \prod_{i=1}^{p} L(a(d_i,n_i)^{(\rho)})$$

is a composition factor of

$$\nu^{1/2} Z(a(n,n)^{(\rho)}) \times \nu^{-1/2} Z(a(n,n)^{(\rho)}) =$$

$$= \nu^{1/2} L(a(n,n)^{(\rho)}) \times \nu^{-1/2} L(a(n,n)^{(\rho)})$$

Now (ii) of Lemma 6.2. implies

(iii) $$2n - 1 \leq \sum_{i=1}^{p} d_i \leq 2n.$$

Since $a(r,r)^{(\rho)}$ is supported in

$$\{\nu^{\alpha}\rho \ ; \ \alpha \in \mathbb{Z}\}$$

we have that $n_i \neq d_i$ for all i . We shall show that

(iv) $$|n_1 - d_i| = 1 \quad \text{for} \quad 1 \leq i \leq p \ .$$

To show this suppose

$$|n_i - d_i| > 1 \quad .$$

Let us suppose that $n_i > d_i$. Then $n_i - d_i > 1$ and

$$\sum_{j=1}^{p} n_j \geq n_i + \sum_{j \neq i} \min \{n_j, d_j\} =$$

$$= n_i - d_i + \sum_{j=1}^{p} \min \ \{n_j, d_j\} > 1 + 2n - 1 = 2n$$

This contradicts to (ii).

The case of $d_i > n_i$ can be checked analogously.

The representation $\nu^{n-1/2}\rho$ is in the support of

$$\nu^{1/2}a(n,n)^{(\rho)} + \nu^{-1/2}a(n,n)^{(\rho)}$$

and $\nu^{\alpha}\rho$ with $\alpha > n - 1/2$ is not in the support. Let $\nu^{n-1/2}\rho$ be in the support of $a(n_1,d_1)^{(\rho)}$. Now we know that

$$\max\{n_1,d_1\} = n + 1 .$$

By (iv) we have

$$\{n_1,d_1\} = \{n,n + 1\} .$$

Repeating the previous step we obtain that

$$\{n_2,d_2\} = \{n,n - 1\} ,$$

and also that $p = 2$.

Now we have four possibilities for π :

$$Z(a(n,n+1)^{(\rho)}) \times Z(a(n-1,n)^{(\rho)}),$$

$$Z(a(n+1,n)^{(\rho)}) \times Z(a(n,n-1)^{(\rho)}),$$

$$Z(a(n,n+1)^{(\rho)}) \times Z(a(n,n-1)^{(\rho)}),$$

$$Z(a(n+1,n)^{(\rho)}) \times Z(a(n-1,n)^{(\rho)}).$$

Since the first two possibilities do not satisfy either (ii) or (iii) we have that π must be either

$$Z(a(n,n+1)^{(\rho)}) \times Z(a(n,n-1)^{(\rho)})$$

or

$$Z(a(n+1,n)^{(\rho)}) \times Z(a(n-1,n)^{(\rho)}) \quad .$$

This proves the theorem.

6.4. <u>Remark:</u> In the proof of (ii) Lemma 6.2. we have used one result which is equivalent to the fact that $t : R \to R$ is multiplicative, i.e. a ring morphism. It was announced that J.N. Bernstein have proved it. The written proof of that fact known to this author does not exist.

Let us denote by R_ρ, for $\rho \in C$, the subgroup of R generated by all $\pi \in \mathrm{Irr}$ with

$$\operatorname{supp} \pi \in M(\{\nu^\alpha \rho \,,\, \alpha \in \mathbb{Z}\}).$$

Then $t(R_\rho) \subseteq R_\rho$. Let t_ρ be the restriction of t

$$t_\rho : R_\rho \to R_\rho .$$

For the proof of the multiplicativity of t it is enough to prove the multiplicativity of t_ρ for all ρ.

J.L. Waldspurger proved in [21] the result equivalent to the multiplicativity of t_ρ, for great number of representations $\rho \in C$.

The proof of (ii) of Lemma 6.2. which is not using the multiplicativity of t, one can obtain using Lemma 3.2.

and the factorisation of intertwining operators obtained by F. Shahidi in [15].

At the rest of this section

$$(\tau,\sigma) \longmapsto \tau\times\sigma, \ \mathrm{Alg}\ G_n \times \mathrm{Alg}\ G_m \longrightarrow \mathrm{Alg}\ G_{m+n}$$

will denote the induction functor.

6.5 <u>Proposition:</u> Let $n,d \in \mathbf{N}, \rho \in C^u$. There exists an exact sequence of representations

$$\begin{aligned} 0 &\longrightarrow Z(a(n+1,d)^{(\rho)} \times Z(a(n-1,d)^{(\rho)}) \longrightarrow \\ &\longrightarrow \nu^{1/2} Z(a(n,d)^{(\rho)}) \times \nu^{-1/2} Z(a(n,d)^{(\rho)}) \longrightarrow \\ &\longrightarrow Z(a(n,d+1)^{(\rho)}) \times Z(a(n,d-1)^{(\rho)}) \longrightarrow 0 . \end{aligned}$$

This sequence does not split.

<u>Proof:</u> We prove the proposition by induction with respect to d. For $d = 1$ we know that

$$\nu^{1/2} Z(a(n,1)^{(\rho)}) \times \nu^{-1/2} Z(a(n,1)^{(\rho)}) = \nu^{1/2} L(\Delta[n]^{(\rho)}) \times \nu^{-1/2} L(\Delta[n]^{(\rho)})$$

has a unique irreducible quotient (properties of the Langlands classification). This quotient is

$$L(a(2,d)^{(\rho)}) = Z(a(d,2)^{(\rho)}).$$

Now Theorem 6.1 implies the proposition for $d = 1$. Suppose that we have proved the statement of the proposition for some d. Consider the case of $d + 1$.

For $\tau \in \mathrm{Alg}\, G_m$, J.N. Bernstein and A.V. Zelevinsky defined in 4.3 of [3] derivatives $\tau^{(k)} \in \mathrm{Alg}\, G_{m-k}$, $k = 0,1,\dots,m$. Recall that $\tau^{(k)}$ is called the highest derivative if $\tau^{(k)} \neq 0$ and $\tau^{(i)} = 0$ for $k < i \leq m$.

Let $\tau_i \in \mathrm{Alg}\, G_{m_i}$ and let σ_i be the highest derivatives of τ_i $(i = 1,2)$. By Corollary 4.14.,(c) of [3], the highest derivative of $\tau_1 \times \tau_2$ is isomorphic to $\sigma_1 \times \sigma_2$.

We know that $\pi = \nu^{1/2}Z(a(n,d+1)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d+1)^{(\rho)})$ has two factors with multiplcity one (Theorem 6.1.). Suppose that π is a completely reducible representation. Since the highest derivative of an irreducible representation is irreducible, we get that the highest derivative $Z(a(n,d)^{(\rho)}) \times \nu^{-1}Z(a(n,d)^{(\rho)})$ of π has composition series of maximal length 2. Since $\nu^{1/2}(Z(a(n,d)^{(\rho)}) \times \nu^{-1}Z(a(n,d)^{(\rho)}))$ $\cong \nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)})$, the inductive assumption implies that the highest derivative of π is not completely reducible. From the other side, the complete reducibility of π implies the complete reducibility of the highest derivative of π. This is a contradiction. This proves that π is not completely reducible.

Now the fact that $Z(a(n,d+2)^{(\rho)}) \times Z(a(n,d)^{(\rho)})$ is a quotient of π we obtain in a similar way from the inductive assumption using the exactness of the functors which enter in the definition of derivatives ([B], Proposition 3.2.).

Considering contragradients one obtains

6.6. Corollary: There exists an exact sequence

$$0 \longrightarrow Z(a(n,d+1)^{(\rho)}) \times Z(a(n,d-1)^{(\rho)}) \longrightarrow$$

$$\longrightarrow \nu^{-1/2} Z(a(n,d)^{(\rho)}) \times \nu^{1/2} Z(a(n,d)^{(\rho)}) \longrightarrow$$

$$\longrightarrow Z(a(n+1,d)^{(\rho)}) \times Z(a(n-1,d)^{(\rho)}) \longrightarrow 0.$$

This sequence does not split.

7. The topology of the unitary dual of $GL(n,F)$.

Let X be a Hausdorff topological space possessing a countable basis of open sets. For $n \in \mathbf{N}$ let S_n be the permutation group of n-elements which acts on X^n by the permutation of coordinates. On X^n we have the natural product topology. We supply the quotient space X^n/S_n with the finest topology so that the canonical projection

$$\Lambda_n : X^n \rightarrow X^n/S_n$$

is continuous. Then Λ_n is also open mapping. One can directly obtain that $(\Lambda_n(x_i))_i$, $x_i \in X^n$, converges to $\Lambda_n(X)$, $x \in X^n$, if and only if there exists a sequence $(\sigma_i) \in S_n$ so that $(\sigma_i x_i)_i$ converges to x in X^n.

We supply

$$M(X) = \bigcup_{n \geq 0} X^n/S_n$$

with the topology of the disjoint union.

Let

$$B = \{Z(a(n,d)^{(\rho)}), \pi(Z(a(n,d)^{(\rho)}), \alpha) ;$$
$$n,d \in \mathbf{N},\ \rho \in C^u,\ 0 < \alpha < 1/2\}$$

and

$$B^* = \{Z(a(n,d)^{(\rho)}), \pi(Z(a(n,d)^{(\rho)}), \alpha);$$
$$n,d \in \mathbf{N}, \rho \in C^u,\ 0 \leq \alpha \leq 1/2\} .$$

We consider $B \subseteq B^* \subseteq R$.

The set C^u is the disjoint union of

$$U^u(F^x)\rho \;,\quad \rho \in C^u \;.$$

On $U^u(F^x)\rho$ we consider the unique topology so that

$$\chi \mapsto \chi\rho \;,\; U^u(F^x) \to U^u(F^x)\rho$$

is open and continuous mapping. Note that there is a finite subgroup X of $U^u(F^x)$ so that we have bijection

$$U^u(F^x)/X \to U^u(F^x)\rho$$

which is also homeomorphism. We supply C^u with the topology of the disjoint union of the sets $U^u(F^x)\rho$. From the first section we obtain that (ρ_n) converges to ρ if and only if ch_{ρ_n} converges to ch_ρ.

We consider

$$X = (\mathbf{N} \times \mathbf{N} \times C^u) \cup (\mathbf{N} \times \mathbf{N} \times C^u \times [0,1/2])$$

with the topology of disjoint union. The mapping

$$\varphi : X \to B^* \;,$$

$$(n,d,\rho) \mapsto Z(a(n,d)^{(\rho)}) \;,$$

$$(n,d,\rho,\alpha) \mapsto \pi(Z(a(n,d)^{(\rho)}),\alpha),$$

is a bijection. We supply B^* with the unique topology so that φ is a homeomorphism.

For $\sigma \in B^*$ let cont σ be the set of all composition factors of σ. Note that

$$\text{cont } (\pi(Z(a(n,d)^{\rho}),1/2) =$$

$$= \{Z(a(n+1,d)^{(\rho)}) \times Z(a(n-1,d)^{(\rho)}) ,$$

$$Z(a(n,d+1)^{(\rho)}) \times Z(a(n,d-1)^{(\rho)})\}.$$

and in all other cases

$$\text{cont } (\sigma) = \{\sigma\} .$$

For $m = (\sigma_1,\dots,\sigma_n) \in M(B^*)$ set

$$\text{cont } m = \{\tau_1 \times \dots \times \tau_n \ ; \ \tau_i \in \text{cont } \sigma_i\} .$$

By Theorem 4.1. the mapping

$$\Theta : M(B) \to \text{Irr}^u , \ (\sigma_1,\dots,\sigma_n) \mapsto \sigma_1 \times \dots \times \sigma_n$$

is a bijection.

7.1. <u>Theorem:</u> Let $X \subseteq \text{Irr}^u$. Then the closure of X in Irr^u is the union of all

$$\text{cont } m$$

when m goes through the closure of $\Theta^{-1}X$ in $M(B^*)$.

<u>Proof:</u> Let $C\ell X$ be the closure of X and

$$Y = \bigcup_{m \in C\ell\, \Theta^{-1}X} \text{cont } m$$

Now the first section implies

$$Y \subseteq C\ell\ X .$$

Suppose $\pi \in C\ell\ X$. Then π is a limit of some sequence (π_n) in X. In the same way as in the proof of Theorem 5.1. we can pass to a subsequence so that $\Theta^{-1}(\pi_n)$ converges in $M(B^*)$. Let $m = \lim_n \Theta^{-1}(\pi_n)$. Now cont m equals to the set of all limits of $(\Theta^{-1}(\pi_n))$. Thus $\pi \in Y$. Therefore,

$$Y = C\ell\ X ,$$

and this finishes the proof.

We have one direct consequence of Theorem 7.1. and Proposition 2.4.

7.2. Corollary: If E and F are local non-archimedean fields, then $GL(n,E)^\wedge$ and $GL(n,F)^\wedge$ are homeomorphic.

7.3. Remark: By Proposition 2.3. Irr^u is homeomorphic to

$$\prod_{f \in M^*(C|\sim)} \hat{G}(c,f(c)) .$$

Thus for the topology of $\hat{G}_n$ it is sufficient to know

$$\hat{G}(\rho,n) , \rho \in C|_\sim , n \in \mathbf{N} .$$

First we note that $\hat{G}(\rho,n)$ are homeomorphic for all $\rho \in C|_\sim$. Thus we define

$$\hat{G}(n)$$

to be $\hat{G}(\rho,n)$.

Now it could be interesting to find geometric realisation of topological spaces $\hat{G}(n)$.

One sees directly that $\hat{G}(1)$ is homeomorphic to the circle.

Let X be the Möbious strip. Take two copies of X and identify corresponding interior points of these two copies of X. Let $\tilde{X}$ be the topological space obtained in this way. Then $\hat{G}(2)$ is homeomorphic to $\tilde{X}$.

We expect also that $\hat{G}(n)$ for $n > 2$ have some reasonable geometric description.

REFERENCES

[1] J.N. Bernstein, All reductive p-adic groups are tame, Functional Anal. Appl. 8(1974), 91 - 94.

[2] J.N. Bernstein, P-invariant distributions on GL(N) and the classification of unitary representations of GL(N) (non-archimedean case), in Lie Group Representations III, Proceedings, University of Maryland 1982-1983, Lecture Notes in Math. vol. 1041, Springer-Verlag, Berlin, 1983, 50 - 102.

[3] J.N. Bernstein, A.V. Zelevinsky, Induced representations of reductive p-adic groups, I, Ann. Scient. Ecole Norm. Sup. 10 (1977), 441 - 472.

[4] G. van Dijk, Computation of certain induced characters of P-adic groups, Math. Ann. 199(1972), 229 - 240.

[5] J. Dixmier, Les C*-algebres et leurs représentations, Gauthiers-Villars, Paris, 1969.

[6] J.M.G. Fell, The structure of algebras of operator fields, Acta Math. 106 (1961), 233 - 280.

[7] P. Gerardin, Cuspidal unramified series for central simple algebras over local fields, in Proc. Sympos. Pure Math. vol. XXXIII, part I, Amer. Math. Soc., Providence, R.I., 1973, 157 - 169.

[8] D.A. Kazhdan, Connection of the dual space of a group with the structure of its closed subgroups, Functional Anal. Appl. 1(1967), 63 - 65.

[9] H. Kraljević, Representations of the universal covering group of the group SU(n,1), Glasnik Mat. 8(28) (1973), 23 - 72.

[10] H. Kraljević, D. Miličić, The C*-algbra of the universal covering group of SL(2,ℝ), Glasnik Mat. 7(27)(1972), 35 - 48.

[11] D. Miličić, Topological representation of the group C*-algebra of SL(2,ℝ), Glasnik Mat. 6(26) (1971), 231 - 246.

[12] D. Miličić, On C*-algebras with bounded trace, Glasnik Mat. 8(28) (1973), 7 - 22.

[13] D. Miličić, On structure of dual of semisimple groups, Ph. D. Thesis, University of Zagreb, 1973.

[14] F. Rodier, Représentations de GL(n,k) où k est un corps p-adique Séminaire Bourbaki n°587 (1982), Astérisque 92 - 93 (1982), 201 - 218.

[15] F. Shahidi, On certian L-functions, Amer. J. Math. 103(1981), 297 - 355.

[16] B.E.M. Speh, Some results on principal series for GL(n,$\mathbb{R}$), Ph. D. Thesis, M.I.T., 1977.

[17] M. Tadić, The C*-algebra of SL(2,k), Glasnik Mat. 17(37) (1982), 249 - 263.

[18] M. Tadić, The topology of the dual space of a reductive group over a local field, Glasnik Mat. 18(38)(1983), 259 - 279.

[19] M. Tadić, Classification of unitary representations in irreducible representations of general linear group (non-archimedean case), to appear in Ann. Scient. Ecole Norm. Sup.

[20] M. Tadić, Unitary representation of general linear group over real and complex field, preprint, Bonn, 1985.

[21] J.-L. Waldspurger, Algebres de Hecke et induites de representations cuspidales, pour GL(N), preprint, Paris, 1985.

[22] A.V. Zelevinsky, Induced representations of reductive p-adic groups II, Ann. Scient. Ecole Norm. Sup. 13(1980), 165 - 210.

[23] A.V. Zelevinsky, p-adic analogue of the Kazhdan-Lusztig conjecture, Functional Anal. Appl. 15(1981), 83 - 92.