Smoothing Fano 3-folds

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Introduction

Let X be a Moishezon 3-fold with only Gorenstein terminal singularities. Then we call X a generalized Fano 3-fold if there is a *small* birational proper morphism π from X to a Fano 3-fold Y with Gorenstein terminal singularities (i.e. $-K_Y$ is ample, and the exceptional locus of π is a curve.).

One of main results proved in the paper is the following.

Theorem 11. *Let X be a generalized Fano 3-fold with Gorenstein terminal singularities. Then X can be deformed to a smooth generalized Fano 3-fold.*

In particular, any Fano 3-fold with Gorenstein terminal singularities is smoothable by a flat deformation, and as a consequence, such a 3-fold is a degeneration of the 3-folds classified by Iskovskih [Is 1, 2] and Mori-Mukai [M-M]. To obtain the result, we do not use the classification of Fano 3-folds, but use the deformation theory developed in [Na 2]. We note that, as a corollary of the classification, similar results to Theorem 11 are obtained by Mukai [Mu] and Sano [Sa]. The present work is also motivated by those of Burns-Wahl [B-W] and Friedman [Fr], in which deformations of surfaces with rational double points, and of 3-folds with ordinary double points are studied respectively. For example, it is an advantage of our method that we can estimate the number of singular points on X in the following sense:

Theorem 13. *Assume that a smooth Fano 3-fold Z is degenerated to a Fano 3-fold X with Gorenstein terminal singularities by a flat deformation. Then we have*

$$21 - (1/2)e(Z) \geq \Sigma_{p \in Sing(X)} \mu(X, p) + \#\{ \text{ ordinary double points on } X \}.$$

Here the invariant μ is defined for an isolated rational singularity V as follows. Let $\nu : \hat{V} \longrightarrow V$ be a resolution of V. Then the dimension of the $\mathbf{C}$-vector space $Coker[(1/2\pi i)dlog : H^1(\hat{V}, \mathcal{O}^*_{\hat{V}}) \otimes_{\mathbf{Z}} \mathbf{C} \longrightarrow H^1(\hat{V}, \Omega^1_{\hat{V}})]$ is independent of the choice of the resolution by [Na 1, §5]. We denote this number by $\mu(V)$. For an ordinary double point of dimension 3, this invariant vanishes. However, it is positive for other Gorenstein terminal singularities by [Na-St, Theorem(2.2)]. Thus, Theorem 13 gives an estimate of the number of singular points on X. Note also that, since smooth Fano 3-folds form a bounded family, their Euler numbers are bounded, and hence, Theorem 13 gives a universal bound of the number of singular points for all Fano 3-folds.

We shall briefly sketch the proofs of the results. Theorem 11 is obtained as follows. Let $\alpha : Ext^1(\Omega^1_X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X))$ be the natural homomorphism from the space of 1-st order global deformations of a generalized Fano 3-fold X to the space of 1-st order local deformations of X. When X has only ordinary double points, α is proved to be surjective (cf. Proposition 4), which is a slight generalization of (4.1) from [Fr]. In particular, X is smoothable by a flat deformation in such a case because any 1-st order deformation of X is unobstructed (cf. Proposition 3). However, in general, α is not surjective for a gereralized Fano 3-fold with Gorenstein terminal singularities (Example 5). Thus, in a general case, we consider the deformation of a pair (X, D) of a generalized Fano 3-fold X with Gorenstein terminal singularities and an anti-canonical divisor D instead of X itself. By Shokurov [Sh] and Reid [Re], a general anti-canonical divisor of a Fano 3-fold with Gorenstein terminal singularities admits only rational double points. Thus, the same thing holds for (X, D). Moreover, by Alexeev [Al], (X, D) has only log canonical singularities. By using these, we shall investigate the map $\alpha_{log} : Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{E}xt^1(\Omega^1_X(log\ D), \mathcal{O}_X))$. This map is not surjective. But the deformation theory of (X, D) has a deep connection with Hodge theory in the case where X is a Fano 3-fold. This point of view origintc from our previous paper [Na-St]. Here the invariant μ defined above has an important role. We finally obtained Proposition 10; it says, in some sense, that there is a good direction $\eta \in Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X)$ which makes the singularity of (X, D) mild. By a succession of small deformations along these η, (X, D) eventually becomes a pair of a smooth generalized Fano 3-fold and a smooth K3 surface, or otherwise, X becomes a generalized Fano 3-fold with only ordinary double points. In the first case, we have finished, and in the second case, we only have to apply Proposition 4.

Next consider Theorem 13. By Riemann-Roch theorem, the left hand side of the inequality is $dim_{\mathbf{C}} H^1(Z; \Theta_Z(-\log D))$. Note that $dim_{\mathbf{C}} Ext^1(\Omega^1_X(\log D), \mathcal{O}_X)) =$

$dim_{\mathbf{C}} H^1(Z, \Theta_Z(-\log D))$ (cf.Lemma 12). Thus, we have to show that $dim_{\mathbf{C}} Def(X, D) \geq \Sigma_{p \in Sing(X)} \mu(X, p) + \#\{$ ordinary double points on X $\}$, where $Def(X, D)$ denotes the Kuranishi space of the pair (X, D), and it is a smooth analytic space by virtue of Proposition 3. For simplicity of the argument, we consider the following two cases:

(1) X has only ordinary double points;

(2) all singularities of X differ from an ordinary double point.

In the case (1), the inequality follows from Proposition 4, which states that the homomorphism α must be surjective, and from the surjectivity of the natural homomorphism $Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X) \longrightarrow Ext^1(\Omega^1_X, \mathcal{O}_X)$. In the case (2), we can show that $dim_{\mathbf{C}} Im(\alpha_{log}) \geq \Sigma_{p \in Sing(X)} \mu(X, p)$ by looking at Lemma 8 carefully, which implies the inequality above.

When X has both ordinary double points and non-ordinary double points, the argument will be a little bit difficult, and we must use Theorem 11. We shall discuss it in the final part of the paper.

§1.

Let (X, D) be a pair of a normal complex space X with $dim\ X \geq 3$ and its Cartier divisor D. Assume that both X and D have only isolated complete intersection singularities. Set $Sing(X) \bigcup Sing(D) = \{p_1, ..., p_n\}$ and let U_i be an Stein open neighborhood of p_i in X. Let $\Omega^1_X(log\ D)$ be the sheaf of logarithmic differential forms and let $\Theta_X(-log\ D)$ be its dual sheaf. Note that $\Omega^1_X(log\ D)$ is, in general, not a locally free sheaf at p_i.

Lemma 1.
(1) $H^0(U_i, \mathcal{E}xt^1(\Omega^1_X(log\ D)) \cong H^1(U_i - p_i, \Theta_X(-log\ D))$;
(2) $Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X) \cong H^1(X - \{p_1, ..., p_n\}, \Theta_X(-log\ D))$.

Proof. (1): Let $D_i = D \bigcap U_i$ and let $\bar{U}_i = U_i - p_i$. If $D_i = \emptyset$, then the result follows from [Sch]. If $D_i \neq \emptyset$, then we consider the exact sequence:

$$0 \longrightarrow \Omega^1_{U_i} \longrightarrow \Omega^1_{U_i}(log\ D_i) \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0.$$

Since $Ext^1(\mathcal{O}_{D_i}, \mathcal{O}_X) \cong H^0(D_i, \mathcal{O}_{D_i}(D_i))$, we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} H^0(U_i, \Theta_{U_i}) & \to & H^0(D_i, \mathcal{O}_{D_i}(D_i)) & \to & Ext^1(\Omega^1_{U_i}(log\ D)), \mathcal{O}_{U_i}) & \to & Ext^1(\Omega^1_{U_i}, \mathcal{O}_{U_i}) & \to & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \\ H^0(\bar{U}_i, \Theta_{U_i}) & \to & H^0(\bar{D}_i, \mathcal{O}_{D_i}(D_i)) & \to & Ext^1(\Omega^1_{\bar{U}_i}(log\ D)), \mathcal{O}_{\bar{U}_i}) & \to & Ext^1(\Omega^1_{\bar{U}_i}, \mathcal{O}_{\bar{U}_i}) & \to & 0. \end{array}$$

Since $\Theta_{U_i} = \mathcal{H}om(\Omega^1_{U_i}, \mathcal{O}_{U_i})$ is a reflexive sheaf, α_1 is an isomorphism. α_2 is also an isomorphism by the same reason. α_4 is an isomorphism by [Sch]. Hence α_3 is an isomorphism, which implies (1).

(2): By (1) we have $H^0(X, \mathcal{E}xt^1(\Omega^1_X(log\ D), \mathcal{O}_X) \cong \bigoplus H^1(U_i - p_i, \Theta_X(-log\ D)) \cong \bigoplus H^2_{p_i}(X, \Theta_X(-log\ D))$. Consider the commutative diagram of the "local to global spectral sequence" and the "exact sequence of local cohomology":

$$\begin{array}{ccccccc} 0 \to H^1(\Theta_X(-log\ D)) & \to & Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X) & \to & H^0(\mathcal{E}xt^1(\Omega^1_X(log\ D), \mathcal{O}_X)) & \to & H^2(\Theta_X(-log\ D)) \\ \| & & \downarrow & & \downarrow & & \| \\ 0 \to H^1(\Theta_X(-log\ D)) & \to & H^1(X - \{p_1, ..., p_n\}; \Theta_X(\log D)) & \to & \bigoplus H^2_{p_i}(\Theta_X(-log\ D)) & \to & H^2(\Theta_X(-log\ D)). \end{array}$$

By this diagram, we obtain (2).

Let (X, D) be a generalized Fano 3-fold with Gorenstein terminal singularities and its general anti-canonical divisor. By definition, there is a small birational morphism $\pi : X \longrightarrow Y$, where Y is a Fano-3-fold with Gorenstein terminal singularities. Since D is the pull-back of an anti-canonical divisor $\bar{D}$ of Y by π, and since $\bar{D}$ admits only rational double points by Shokurov [Sh] or Reid [Re], D also admits only rational double points. Indeed, let $\nu : \hat{D} \longrightarrow D$ be the normalization of D, and let $\phi : \tilde{D} \longrightarrow \hat{D}$ be a resolution. Define h to be the birational morphism from $\tilde{D}$ to $\bar{D}$. Assume that ν is not an isomorphism. $\hat{D}$ is a Cohen-Macauley surface because $\hat{D}$ is a normal surface. Let $\omega_{\hat{D}}$ be its dualizing sheaf. Then $\omega_{\hat{D}} \cong \nu^*(\omega_D) \bigotimes I_D$, where I_D is the conductor ideal, which does not coincide with $\mathcal{O}_{\hat{D}}$ by the assumption. It can be checked that $\hat{D}$ has only rational singularities. Hence, we have $\phi_*\omega_{\tilde{D}} = \omega_{\hat{D}}$. This implies that $h_*\omega_{\tilde{D}} = \omega_{\bar{D}} \bigotimes I$ for some ideal $I \neq \mathcal{O}_{\bar{D}}$. On the other hand, $h_*\omega_{\tilde{D}} = \omega_{\bar{D}}$ because $\bar{D}$ has only rational singularities. Thus, D is a normal surface with trivial dualizing sheaf. Now, it is easily checked that D admits only rational double points. Moreover, by Alexeev [Al], $(Y, \bar{D})$ has only log canonical singularities (cf. [KMM]), and hence (X, D) also has only canonical singularities; there is a resolution of singularities $f : \tilde{X} \longrightarrow X$ such that the union of the exceptional locus and $f^{-1}(D)$ is a divisor with normal crossings, and that $K_{\tilde{X}} + \tilde{D} = f^*(K_X + D) + \Sigma b_j F_j$ with the coefficients $b_j \geq -1$, where F_j are f-exceptional divisors and $\tilde{D}$ is the proper transform of D by f.

In the remainings, (X, D) is assumed to have the above properties.

Proposition 2 *In the above situation, the natural homomorphism* $Pic(X) \longrightarrow Pic(D)$ *is an injection.*

Proof. We shall first prove that (X, D) satisfies the Lefschetz condition $Lef(X, D)$ (cf. [Gro], [Ha]). By the argument of [Ha, Proposition(1.1), p.165] it suffices to show that for any coherent sheaf F on $X - D$, $H^i(X - D, F) = 0$ for $i > 1$. This can be checked by using Leray spectral sequence because $Y - \bar{D}$ is an affine variety by definition, and π is a small partial resolution of Y with only isolated sinularities. We next note that any Zariski open set U of X containing D intersects every effective divisor of X. Thus, the complement $X - U$ is of codimension 2 in X. Let L be a line bundle on X. Denote by $\hat{X}$ the formal completion of X along D and denote by $\hat{L}$ the restriction of L to $\hat{X}$. Since $Lef(X, D)$ holds, there is a Zariski open subset U of W containing D such that $H^0(U, L) \cong H^0(\hat{X}, \hat{L})$. Assume that $\hat{L}$ is trivial line bundle. If we take the U sufficiently small, then there is a nowhere vanishing section $s \in H^0(U, L)$. This implies that $L|_U \cong \mathcal{O}_U$. Since $Pic(X) \cong Pic(U)$, we have shown that $Pic(X) \longrightarrow Pic(\hat{X})$ is an injection. Since $\mathcal{O}_D(D)$ is a nef and big line bundle, we have $H^1(D, \mathcal{O}_D(-nD)) = 0$ for all $n > 0$ by Kawamata-Viehweg vanishing theorem. By using this, it is easily checked that the natural homomorphism $Pic(\hat{X}) \longrightarrow Pic(D)$ is an injection. Q.E.D.

Proposition 3.

$$Ext^2(\Omega^1_X, \mathcal{O}_X) = Ext^2(\Omega^1_X(log\ D), \mathcal{O}_X) = 0$$

In particular, the deformations of X and (X, D) are unobstructed.

Proof. Since $Ext^2(\mathcal{O}_D, \mathcal{O}_X) = 0$, there is an injection from $Ext^2(\Omega^1_X(log\ D), \mathcal{O}_X)$ to $Ext^2(\Omega^1_X, \mathcal{O}_X)$. Hence, it is enough to show that $Ext^2(\Omega^1_X, \mathcal{O}_X) = 0$. By the Serre duality, $Ext^2(\Omega^1_X, \mathcal{O}_X)^* = H^1(X, \Omega^1_X \otimes K_X)$. By the exact sequence

$$0 \longrightarrow \Omega^1_X \otimes K_X \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X|_D \longrightarrow 0$$

we have an exact sequence

$$H^0(\Omega^1_X|_D) \to H^1(\Omega^1_X \otimes K_X) \to H^1(\Omega^1_X) \xrightarrow{\beta} H^1(\Omega^1_X|_D).$$

It is easily checked that $H^0(\Omega^1_X|_D) = 0$. We shall claim that β is an injection. Since D has only rational double points, and since $H^1(D, \mathcal{O}_D) = 0$, the map $(1/2\pi i)dlog$: $H^1(D, \mathcal{O}^*_D) \otimes_{\mathbf{Z}} \mathbf{C} \longrightarrow H^1(D, \hat{\Omega}^1_D)$ is an injection by the Hodge theory of V-manifolds, where $\hat{\Omega}^1_D$ is a double dual of Ω^1_D. Thus, the homomorphism $H^1(D, \mathcal{O}^*_D) \otimes_{\mathbf{Z}} \mathbf{C} \longrightarrow H^1(D, \Omega^1_D)$ is also an injection. Consider the commutative diagram

$$H^1(X,\Omega^1_X) \xrightarrow{j} H^1(D,\Omega^1_D)$$

$$(1/2\pi i)dlog \uparrow \qquad\qquad \uparrow (1/2\pi i)dlog$$

$$H^1(X,\mathcal{O}^*_X)\otimes\mathbf{C} \xrightarrow{k} H^1(D,\mathcal{O}^*_D)\otimes\mathbf{C}$$

By Proposition 2, k is an injection. By the same argument as [Na1, (2.2)], the vertical map on the left-hand side is an isomorphism. Therefore, j is an injection by the above diagram. The map j is factored through $H^1(D,\Omega^1_X|_D)$. Hence β is an injection. Q.E.D.

Consider the "local to global" spectral sequence of Ext:

$$0 \to H^1(X,\Theta_X) \to Ext^1(\Omega^1_X,\mathcal{O}_X) \xrightarrow{\alpha} H^0(X,T^1_X) \to H^2(X,\Theta_X),$$

where $\Theta_X := \mathcal{H}om(\Omega^1_X,\mathcal{O}_X)$, and $T^1_X := \mathcal{E}xt^1(\Omega^1_X,\mathcal{O}_X)$. Since α may be regarded as the homomorphism from the space of 1-st order global deformations of X to the space of 1-st order local deformations of X, X is smoothable by a flat deformation by Proposition 3 if α is a surjection.

Proposition 4. *Let X be a generalized Fano 3-fold with only ordinary double points. Then we have $H^2(X,\Theta_X)=0$. In particular, X is smoothable by a flat deformation.*

Proof. Let $f: Y \longrightarrow X$ be a small resolution of X. By definition, the exceptional locus of f over a singular point is a smooth rational curve C with $N_{C/Z} \cong \mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$. Then we have an exact sequence:

$$H^1(Z,\Theta_Z) \to H^0(X,R^1f_*\Theta_Z) \to H^2(X,\Theta_X) \to H^2(Z,\Theta_Z)$$

It is easily checked that $H^0(X,R^1f_*\Theta_Z)=0$ (cf. [Fr2, §2.]). We shall show that $H^2(Z,\Theta_Z)=0$. By the Serre duality $H^2(Z,\Theta_Z)^* = H^1(Z,\Omega^1_Z\otimes K_Z)$. Let D' be the proper transform of D by f. Consider the exact sequence

$$0 \longrightarrow \Omega^1_Z\otimes K_Z \longrightarrow \Omega^1_Z(log\ D')\otimes K_Z \longrightarrow K_Z|_{D'} \longrightarrow 0.$$

Since $H^0(D',K_Z|_{D'})=0$, it is sufficient to show that $H^1(\Omega^1_Z(log\ D')(-D'))=0$. In order to do that, we consider the exact sequence

$$0 \longrightarrow \Omega^1_Z(log\ D')(-D') \longrightarrow \Omega^1_Z \longrightarrow \Omega^1_{D'} \longrightarrow 0.$$

Since there is an injection $\Omega^1_{D'} \longrightarrow \hat{\Omega}^1_{D'}$, and since $H^0(D', \hat{\Omega}^1_{D'}) = 0$, we have $H^0(D', \Omega^1_{D'}) = 0$. The homomorphism $H^1(Z, \Omega^1_Z) \longrightarrow H^1(D, \Omega^1_D)$ is an injection. In fact, by Proposition 2 and Hodge theory, we have an injection $H^1(Z, \Omega^1_Z) \longrightarrow H^1(D, \hat{\Omega}^1_D)$. Since this homomorphism factors through $H^1(D, \Omega^1_D)$, the injectivity is proved. Q.E.D.

Example 5. This is an example where the map $\alpha : Ext^1(\Omega^1_X, \mathcal{O}_X) \longrightarrow H^0(X; T^1_X)$ is not a surjection for a Fano 3-fold with Gorenstein terminal singularities. In particular, $H^2(X; \Theta_X) \neq 0$.

Let $S = \mathbf{P}^1 \times \mathbf{P}^1$ and $L = p_1{}^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes p_2{}^*\mathcal{O}_{\mathbf{P}^1}(-1)$. We denote by x, y, and z the natural inclusions

$$L^{\otimes 2} \longrightarrow \mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3}$$
$$L^{\otimes 3} \longrightarrow \mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3}$$
$$\mathcal{O}_S \longrightarrow \mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3},$$

respectively. Take sections $a \in H^0(S; L^{\otimes -4})$ and $b \in H^0(S; L^{\otimes -6})$. Then we have an elliptic 3-fold $W \subset \mathbf{P}_S(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3})$ defined by the equation $y^2 z = x^3 + axz^2 + bz^3$. The elliptic 3-fold $\pi : W \longrightarrow S$ has a section $\Sigma = \{x = y = 0\}$. W is smooth along Σ. One has $N_{\Sigma/W} \cong p_1{}^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes p_2{}^*\mathcal{O}_{\mathbf{P}^1}(-1)$. We obtain a 3-fold X by contracting $\Sigma \subset W$ to a point. It can be checked that the deformation of W is unobstructed and that any deformation of W is realized as an elliptic 3-fold in $\mathbf{P}_S(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3})$ described above.

We now take $a = 0$ and $b = p_1^*\varphi \otimes p_2^*\psi$, where φ and ψ are general elements of $H^0(\mathbf{P}^1; \mathcal{O}_{\mathbf{P}^1}(6))$. The divisor $div(b)$ defined by $b = 0$ has 36 singular points. Then W has just 36 singular points $p_i (1 \leq i \leq 36)$ which are all analytically isomorphic to the singularity $A_{1,2} = \{(x, y, t, s) \in \mathbf{C}^4; y^2 = x^3 + t^2 + s^2\}$. Hence X has 36 singular points $\bar{p}_i$ as the image of p_i and one ordinary double point q as the image of Σ. There is an injection $\mu_* : Ext^1(\Omega^1_W, \mathcal{O}_W) \longrightarrow Ext^1(\Omega^1_X, \mathcal{O}_X)$ corresponding to the birational contraction $\mu : W \longrightarrow X$, and $dim_{\mathbf{C}} Coker(\mu_*) \leq 1$. In fact, we have an exact commutative diagram

$$\begin{array}{ccccccc} 0 \to H^1(X - \{\bar{p}_1, ..., \bar{p}_{36}\}, \Theta_X) & \to & Ext^1(\Omega^1_W, \mathcal{O}_W) & \to & H^0(X, R^1\mu_*\Theta_W) = 0 \\ \| & & \downarrow & & \downarrow \\ 0 \to H^1(X - \{\bar{p}_1, ..., \bar{p}_{36}\}, \Theta_X) & \to & Ext^1(\Omega^1_X, \mathcal{O}_X) & \to & H^0_q(X, T^1_X)(= \mathbf{C}) \end{array}$$

In order to investigate the homomorphism α, we consider the following commutative diagram.

$$\begin{array}{ccc} Ext^1(\Omega^1_W, \mathcal{O}_W) & \stackrel{\tilde{\alpha}}{\longrightarrow} & H^0(W, T^1_W) \\ \mu_* \downarrow & & \downarrow \mu_* \\ Ext^1(\Omega^1_X, \mathcal{O}_X) & \stackrel{\alpha}{\longrightarrow} & H^0(X, T^1_X) \end{array}$$

Note that the vertical map on the right-hand side is an injection. We shall show that $dim_{\mathbf{C}} Coker(\tilde{\alpha}) \geq 11$. If this is proved, then one can see that $dim_{\mathbf{C}} Coker(\alpha) \geq 10$ by the diagram above. Let (T_0, T_1, S_0, S_1) be a homogenous coordinates of $\mathbf{P}^1 \times \mathbf{P}^1$. Let $\mathbf{C}^1 \times \mathbf{C}^1$ be the affine space defined by $T_0 \neq 0$ and $S_0 \neq 0$. Set $t = T_1/T_0$ and $s = S_1/S_0$. Wemay assume that $p_i \in \pi^{-1}(\mathbf{C}^1 \times \mathbf{C}^1) - \Sigma$ for all i. $W_0 := \pi^{-1}(\mathbf{C}^1 \times \mathbf{C}^1) - \Sigma$ is the hypersurface of $\mathbf{C}^2 \times \mathbf{C}^1 \times \mathbf{C}^1$ defined by the equation $y^2 = x^3 + b(t, s)$. By definition we have $b(t, s) = \prod_{1 \leq i \leq 6}(t - \alpha_i) \prod_{1 \leq j \leq 6}(s - \beta_j)$ with some $\alpha_i, \beta_j \in \mathbf{C}$. Let $p_{i,j} = (0, 0, \alpha_i, \beta_j)$. These are nothing but the 36 singular points on W. The semi-universal deformation of W at $p_{i,j}$ is described as follows by using 2 parameters $\sigma_{i,j}$ and $\tau_{i,j}$:

$$y^2 = x^3 + \sigma_{i,j} x + b(t, s) + \tau_{i,j}.$$

We then have an identification of $H^0(W; T^1_W)$ with 72 dimensional $\mathbf{C}$-vector space $\bigoplus_{i,j}(\mathbf{C}_{\sigma_{i,j}} \bigoplus \mathbf{C}_{\tau_{i,j}})$. There is a surjection $\psi : H^0(S, L^{\otimes -4}) \bigoplus H^0(S, L^{\otimes -6}) \longrightarrow Ext^1(\Omega^1_W, \mathcal{O}_W)$ which sends $(c, d) \in H^0(S; L^{\otimes -4}) \bigoplus H^0(S; L^{\otimes -6})$ to the element of $Ext^1(\Omega^1_W, \mathcal{O}_W)$ corresponding to the infinitesimal deformation of W given by

$$y^2 z = x^3 + (\epsilon c) x z^2 + (b + \epsilon d) z^3$$

Then the following diagram commutes:

$$\begin{array}{ccc} H^0(S, L^{\otimes -4}) \bigoplus H^0(S, L^{\otimes -6}) & \stackrel{\beta}{\longrightarrow} & Ext^1(\Omega^1_W, \mathcal{O}_W) \\ \downarrow & & \downarrow \tilde{\alpha} \\ \bigoplus_{i,j}(\mathbf{C}_{\sigma_{i,j}} \bigoplus \mathbf{C}_{\tau_{i,j}}) & \cong & H^0(W, T^1_W) \end{array}$$

Since $dim_{\mathbf{C}} H^0(S; L^{\otimes -4}) = 25$, we infer that $dim_{\mathbf{C}} Coker(\tilde{\alpha}) \geq 11$.

As the example shows in the above, we need a different method from Proposition 4. to make a generalized Fano 3-fold smooth. Hence, we shall begin with recalling some invariant of an isolated rational singularity. Let V be the germ of an isolated rational singularity and let $\tilde{V}$ be the resolution. We then define

$$\mu(V) = dim_{\mathbf{C}} Coker[(1/2\pi i) dlog : H^1(\tilde{V}, \mathcal{O}^*_{\tilde{V}}) \otimes_{\mathbf{Z}} \mathbf{C} \to H^1(\tilde{V}; \Omega^1_{\tilde{V}})]$$

$\mu(V)$ is independent of the choice of resolutions by [Na 1, §5]. Moreover, we have the following proposition.

Proposition 6.[Na-St, Theorem(2.2)] *Let V be a Gorenstein terminal singularity of dimension 3. Then $\mu(V) = 0$ if and only if V is a smooth point or an ordinary double point.*

Let (X, D) be a pair of a generalized Fano 3-fold X with Gorenstein terminal singularities and its general anti-canonical divisor D. We shall use the same notation as above. Set $U := X - \{p_1, ..., p_n\}$ and let U_i be a contractible Stein open neighborhood of p_i. Let $f : \tilde{X} \longrightarrow X$ be a resolution such that the union F of the f-exceptional divisors and $f^{-1}(D)$ is a divisor with normal crossing and such that f is an isomorphism over U. Let $E_i = f^{-1}(p_i)$ and let $\tilde{D}$ be the proper transform of D by f.

Lemma 7 *There is an injection*

$$\Theta_{\tilde{X}}(-log\ F) \longrightarrow \Omega^2_{\tilde{X}}(log\ F).$$

Proof. It suffices to show that $K_{\tilde{X}} + F$ is an effective divisor, which follows from the fact that (X, D) has only log canonical singularities. Q.E.D.

We shall investigate the homomorphism

$$\alpha_{log} : Ext^1(\Omega^1_X(logD), \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{E}xt^1(\Omega^1_X(logD), \mathcal{O}_X)$$

in order to study the deformation of (X, D). Since $H^1(U_i - p_i, \Theta_X(-log\ D)) \cong H^2_{p_i}(X; \Theta_X(-log\ D))$, α_{log} is identified with the coboundary map of local cohomology:

$$H^1(U, \Theta_X(-log\ D)) \longrightarrow \bigoplus H^2_{p_i}(X, \Theta_X(-log\ D))$$

Since $\Theta_X(-log\ D)|_U \cong \Omega^2_X(log\ D)|_U$ and $H^2_{p_i}(X, \Theta_X(-log\ D)) \cong H^2_{p_i}(X, f_*\Omega^2_{\tilde{X}}(log\ D))$, we have a commutative diagram

(**)

$$\begin{array}{ccccc} H^1(f^{-1}(U), \Omega^1_{\tilde{X}}(log\ F)) & \longrightarrow & \bigoplus H^2_{E_i}(\tilde{X}, \Omega^2_{\tilde{X}}(log\ F)) & \stackrel{\bigoplus \gamma_i}{\longrightarrow} & H^2(\tilde{X}, \Omega^2_{\tilde{X}}(log\ F)) \\ \| & & \uparrow \bigoplus \tau_i & & \\ H^1(U, \Theta_X(-log\ D)) & \longrightarrow & \bigoplus H^2_{p_i}(X, \Theta_X(-log\ D)) & & \end{array}$$

The top horizontal sequence of the diagram is an exact sequence of local cohomology and the vertical homomorphisms are edge homomorphism of the Leray spectral sequence. We note that the horizontal homomorphism at the bottom can be identified with α_{log}.

Lemma 8. *Assume that $p_i \in X$ is neither a smooth point nor an ordinary double point. Then*

$$\gamma_i : H^2_{E_i}(\tilde{X}, \Omega^2_{\tilde{X}}(log\ F)) \longrightarrow H^2(\tilde{X}, \Omega^2_{\tilde{X}}(log\ F))$$

is not an injection. Moreover, $dim_{\mathbf{C}} Ker(\gamma_i) \geq \mu(U_i)$.

Proof. Let U_i be a contractible Stein neighborhood of $p_i \in X$. Let $V_i = f^{-1}(U_i)$. By taking the dual of γ_i, we have

$$\gamma_i^* : H^1(\tilde{X}, \Omega^1_{\tilde{X}}(log\ F) \otimes \mathcal{O}_{\tilde{X}}(-F)) \longrightarrow H^1(V_i, \Omega^1_{V_i}(log\ F) \otimes \mathcal{O}_{V_i}(-F)).$$

We shall show that γ_i^* is not a surjection. Since there is an exact sequence

$$0 \longrightarrow \Omega^1_{\tilde{X}}(log\ F)(-F) \longrightarrow \Omega^1_{\tilde{X}} \longrightarrow \hat{\Omega}^1_F \longrightarrow 0$$

where $\hat{\Omega}^1_F$ is the quotient of Ω^1_F by its torsion part, one has an exact commutative diagram

$$\begin{array}{ccc}
H^1(\tilde{X}; \Omega^1_{\tilde{X}}(log\ F)(-F)) & \xrightarrow{\gamma_i^*} & H^1(V_i, \Omega^1_{V_i}(-log\ F)(-F)) \\
\downarrow & & \downarrow \\
H^1(\tilde{X}, \Omega^1_{\tilde{X}}) & \xrightarrow{\beta_1} & H^1(V_i, \Omega^1_{V_i}) \\
\downarrow & & \sigma \downarrow \\
H^1(F; \hat{\Omega}^1_{V_i}) & \xrightarrow{\beta_2} & H^1(F_i, \hat{\Omega}^1_{F_i}),
\end{array}$$

where $F_i = F \cap V_i$. Assume that γ_i^* is a surjection. Then we have an inequality

$$(\#) \qquad dim_{\mathbf{C}} Coker(\beta_1) \leq dim_{\mathbf{C}} Coker(\sigma \circ \beta_1)$$

On the other hand, consider the following commutative diagrams

$$\begin{array}{ccc}
H^1(\tilde{X}; \Omega^1_{\tilde{X}}) & \xrightarrow{\beta_1} & H^1(V_i, \Omega^1_{V_i}) \\
\uparrow & & \uparrow \\
H^1(\tilde{X}, \mathcal{O}^*_{\tilde{X}}) \otimes_{\mathbf{Z}} \mathbf{C} & \xrightarrow{\beta_1'} & H^1(V_i, \mathcal{O}^*_{V_i}) \otimes_{\mathbf{Z}} \mathbf{C}
\end{array}$$

$$\begin{array}{ccc}
H^1(V_i, \Omega^1_{V_i}) & \xrightarrow{\sigma} & H^1(F_i, \hat{\Omega}^1_{F_i}) \\
\uparrow \varphi & & \uparrow \phi (= (1/2\pi i) dolg) \\
H^1(V_i, \mathcal{O}^*_{V_i}) \otimes_{\mathbf{Z}} \mathbf{C} & \xrightarrow{\sigma'} & H^1(F_i, \mathcal{O}^*_{F_i}) \otimes_{\mathbf{Z}} \mathbf{C}
\end{array}$$

We claim that σ' is an isomorphism, and that ϕ is a surjection. If this claim is verified, then we see that $dim_{\mathbf{C}}Coker(\sigma \circ \beta_1) \leq dim_{\mathbf{C}}Coker(\beta_1')$ by the diagram. The homomorphism φ is an injection by [Na 1, (2.2), CLAIM] because $p_i \in X$ is a rational singularity. It is, however, not a surjection because $\mu(U_i) > 0$ by Proposition 6. Hence, we have $dim_{\mathbf{C}}Coker(\beta_1') < dim_{\mathbf{C}}Coker(\beta_1)$ by the diagram, which implies that

$$(\#\#) \qquad dim_{\mathbf{C}}Coker(\sigma \circ \beta_1) < dim_{\mathbf{C}}Coker(\beta_1).$$

This contradicts (#), and therefore, γ_i^* is not a surjection. This argument also implies that $dim_{\mathbf{C}}Coker(\gamma_i^*) \geq \mu(U_i)$.

From now on, we shall prove the claim. First, for simplicity, we introduce some notation. Put $\tilde{D}_i = \tilde{D} \cap V_i$, and $C_i = \tilde{D}_i \cap E_i$. Note that $F_i = E_i \cup \tilde{D}_i$. E_i is a variety with normal crossing, and it can be embedded into some projective space by definition. Moreover, $\tilde{D}_i$ is a resolution of $D_i = D \cap U_i$ which is a sufficiently small open neighborhood of a rational double point of a surface.

Sublemma 9.

$$\text{(a) } H^1(F_i, \mathcal{O}^*_{F_i}) \cong H^1(E_i, \mathcal{O}^*_{E_i});$$
$$\text{(b) } H^1(F_i, \hat{\Omega}^1_{F_i}) \cong H^1(E_i, \hat{\Omega}^1_{E_i})$$

Proof. (a): We shall show that $H^1(F_i, \mathcal{O}^*_{F_i}) \cong H^2(F_i, \mathbf{Z})$ and that $H^1(E_i, \mathcal{O}^*_{E_i}) \cong H^2(E_i, \mathbf{Z})$. If these are proved, then the result follows since there is a deformation retract of F_i to E_i. Consider the commutative diagram

$$\begin{array}{ccc} E_1^{p,q} = H^q(V_i, \Omega^p_{V_i}) & \Rightarrow & H^{p+q}(V_i, \mathbf{C}) \\ \downarrow & & \| \\ F_1^{p,q} = H^q(E_i, \hat{\Omega}^p_{E_i}) & \Rightarrow & H^{p+q}(E_i, \mathbf{C}) \end{array}$$

The second spectral sequence degenerates at F_1-terms (cf.[Fr 1]). Since $H^j(V_i, \mathcal{O}_{V_i}) = 0$ for $j > 0$, we have $H^j(E_i, \mathcal{O}_{E_i}) = 0$ for $j > 0$ by the diagram. Thus, we have $H^1(E_i, \mathcal{O}^*_{E_i}) \cong H^2(E_i, \mathbf{Z})$ by the exponential sequence. Let $\nu : F_i^{[0]} := \tilde{D}_i \coprod E_i \longrightarrow F_i$ be the natural projection. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{F_i} \longrightarrow \nu_* \mathcal{O}_{F_i^{[0]}} \longrightarrow \mathcal{O}_{C_i} \longrightarrow 0.$$

Since D_i has a rational double point, $H^j(\mathcal{O}_{\tilde{D}_i}) = H^j(\mathcal{O}_{C_i}) = 0$ for $j > 0$. Hence, we have $H^j(\mathcal{O}_{F_j}) = 0$ for $j > 0$ by the above exact sequence, which implies that $H^1(F_i, \mathcal{O}^*_{F_i}) \cong H^2(F_i, \mathbf{Z})$.

(b): One has an exact sequence

$$0 \longrightarrow \hat{\Omega}^1_{F_i} \longrightarrow \nu_*\hat{\Omega}^1_{F_i[0]} \longrightarrow \hat{\Omega}^1_{C_i} \longrightarrow 0.$$

Since C_i is a tree of smooth rational curves, the following sequence is exact:

$$0 \longrightarrow H^1(F_i, \hat{\Omega}^1_{F_i}) \longrightarrow H^1(F_i, \nu_*\hat{\Omega}^1_{F_i[0]}) \longrightarrow H^1(C_i, \hat{\Omega}^1_{C_i}).$$

Now the result would follow if the homomorphism $H^1(\tilde{D}_i, \Omega^1_{\tilde{D}_i}) \longrightarrow H^1(C_i, \hat{\Omega}^1_{C_i})$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} H^1(\tilde{D}_i, \mathcal{O}^*_{\tilde{D}_i}) \otimes_{\mathbf{Z}} \mathbf{C} & \longrightarrow & H^1(\tilde{D}_i, \hat{\Omega}^1_{\tilde{D}_i}) \\ \downarrow & & \downarrow \\ H^1(C_i, \mathcal{O}^*_{C_i}) \otimes_{\mathbf{Z}} \mathbf{C} & \longrightarrow & H^1(C_i, \hat{\Omega}^1_{C_i}) \end{array}$$

The horizontal map on the top is an isomorphism because D_i is a V-manifold and hence, $\mu(D_i) = 0$. The horizontal map on the bottom is also an isomorphism by the Hodge theory on C_i (cf. [Fr 1]) because $H^0(C_i, \hat{\Omega}^1_{C_i}) = H^1(C_i, \mathcal{O}_{C_i}) = H^2(C_i, \mathcal{O}_{C_i}) = 0$. Since $H^1(\tilde{D}_i, \mathcal{O}^*_{\tilde{D}_i}) \cong H^2(\tilde{D}_i, \mathbf{Z})$ and since $H^1(C_i, \mathcal{O}^*_{C_i}) \cong H^2(C_i, \mathbf{Z})$, the vertical map on the left-hand side is an isomorphism. Thus, we have the result. Q.E.D.

We shall return to the proof of the claim. First we have $H^1(V_i, \mathcal{O}^*_{V_i}) \cong H^1(E_i, \mathcal{O}^*_{E_i}) \cong H^2(E_i, \mathbf{Z})$. Next, by (a) of sublemma 9. we have $H^1(F_i, \mathcal{O}^*_{F_i}) \cong H^2(E_i, \mathbf{Z})$. Thus, σ' is an isomorphism.

The surjectivity of ϕ follows from the commutative diagram

$$\begin{array}{ccc} H^1(F_i, \mathcal{O}^*_{F_i}) \otimes_{\mathbf{Z}} \mathbf{C} & \longrightarrow & H^1(F_i, \hat{\Omega}^1_{F_i}) \\ \downarrow & & \downarrow \\ H^1(E_i, \mathcal{O}^*_{E_i}) \otimes_{\mathbf{Z}} \mathbf{C} & \longrightarrow & H^1(E_i, \hat{\Omega}^1_{E_i}) \end{array}$$

Here the vertical maps are both isomorphisms by sublemma 9, and the horizontal map at the bottom is a surjection. Q.E.D.

There is a natural homomorphism β_i from the space of 1-st order deformations of (V_i, F_i) to the space of 1-st order deformations of (U_i, D_i). This homomorphism can be expressed as the composition of the following homomorphisms:

$H^1(V_i, \Theta_{V_i}(-log\ F)) \to H^1(V_i - E_i, \Theta_{V_i}(-log\ F)) = H^1(U_i - p_i, \Theta_{U_i}(-log\ D)) \cong H^0(U_i, \mathcal{E}xt^1(\Omega^1_X(log\ D), \mathcal{O}_X))$

Proposition 10. *Consider the following diagram*

$$Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X) \xrightarrow{\alpha_{log}} \oplus_i H^0(U_i, \mathcal{E}xt^1(\Omega^1_X(log\ D), \mathcal{O}_X)) \xleftarrow{\bigoplus \beta_i} \oplus_i H^1(V_i, \Theta_{V_i}(-log\ F))$$

Then there is an element $\eta \in Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X)$ such that $\alpha_{log}(\eta)_i \notin Image(\beta_i)$ for all i such that $p_i \in X$ is neither a smooth point nor an ordinary double point.

Proof. Consider the exact commutative diagram (**) above Lemma 8. If $p_i \in X$ is neither a smooth point nor an ordinary double point, then γ_i is not an injection by Lemma 8. Hence there is an element $\eta \in Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X)$ such that $\tau_i(\alpha_{log}(\eta)_i) \neq 0$ by the diagram. By Lemma 7, τ_i is factored as follows:

$$H^2_{p_i}(X; \Theta_X(-log\ D)) \xrightarrow{\tau_i'} H^2_{E_i}(\tilde{X}, \Theta_{\tilde{X}}(-log\ F)) \longrightarrow H^2_{E_i}(\tilde{X}, \Omega^2_{\tilde{X}}(log\ F)).$$

Hence we have $\tau_i'(\alpha_{log}(\eta)_i) \neq 0$. Consider the exact sequence obtained by the spectral sequence of local cohomology

$$H^0(X; R^1 f_* \Theta_{\tilde{X}}(-log\ F)) \xrightarrow{\beta_i} H^2_{p_i}(X, \Theta_X(-log\ D)) \xrightarrow{\tau_i'} H^2_{E_i}(\tilde{X}, \Theta_{\tilde{X}}(-log\ F)).$$

We then have $\alpha_{log}(\eta)_i \notin Image(\beta_i)$. Q.E.D.

Let $Def(U_i; D_i)$ be the semi-universal space of the deformations of the pair $(U_i; D_i)$, which is a complex manifold since $Ext^2(\Omega^1_{U_i}(log\ D_i), \mathcal{O}_{U_i}) = 0$. Let $g_i : (\mathcal{U}_i, \mathcal{D}_i) \longrightarrow Def(U_i, D_i)$ be the semi-universal family over $Def(U_i, D_i)$. We shall construct a stratification of $Def(U_i, D_i)$ into locally closed (in the Zariski topology) smooth subsets. We must define a terminology before constructing the stratification. Let $g : \mathcal{X} \longrightarrow S$ be a smooth morphism of complex manifolds and let $\mathcal{D} = \Sigma_j \mathcal{D}_j$ be a divisor of $\mathcal{X}$ with simple normal crossing. Then $g : (\mathcal{X}, \mathcal{D}) \longrightarrow S$ is called log smooth if (1) $D_t = \Sigma_j D_{j,t}$ is a divisor of X_t with simple normal crossing for every point $t \in S$, and (2) g is locally a trivial deformation of (X_t, D_t) around any point $p \in \mathcal{X}$, where $t = g(p)$.

We note that $g_i : (\mathcal{U}_i, \mathcal{D}_i) \longrightarrow Def(U_i, D_i)$ is log smooth over a non-empty Zariski open subset S_i^0 of $Def(U_i, D_i)$. By an inductive process, we can construct a stratification $Def(U_i, D_i) = \coprod_{k \geq 0} S_i^k$ with the following properties:

(1) S_i^0 is a Zariski open subet of $Def(U_i, D_i)$, and $(\mathcal{U}_i, \mathcal{D}_i)$ is log smooth over S_i^0;

(2) S_i^k is a locally closed smooth subset of pure codimension, and $codim(S_i^k, Def(U_i, D_i)) < codim(S_i^{k+1}, Def(U_i, D_i))$ ($k \geq 0$);

(3) If $k > l$, then $\bar{S}_i^k \cap S_i^l = \emptyset$, and

(4) $(\mathcal{U}_i, \mathcal{D}_i)$ has a simultaneous resolution over each S_i^k:

Let $g_i^k : (\mathcal{U}_i^k, \mathcal{D}_i^k) \to S_i^k$ be the base change of g_i by $S_i^k \to Def(U_i, D_i)$. Then there is a resolution $\nu_i^k : (\mathcal{V}_i^k, \mathcal{F}_i^k) \longrightarrow (\mathcal{U}_i^k, \mathcal{D}_i^k)$ such that $(\nu_i^k)^{-1}(\mathcal{D}_i^k) = \mathcal{F}_i^k$ and that $g_i^k \circ \nu_i^k : (\mathcal{V}_i^k, \mathcal{F}_i^k) \longrightarrow U_i^k$ is a log smooth.

Theorem 11 *Let X be a generalized Fano 3-fold with Gorenstein terminal singularities. Then X can be deformed to a smooth generalized Fano 3-fold.*

Proof. Fix such a stratification of $Def(U_i, D_i)$ as above for each $p_i \in X$. Let $q \in Def(U_i, D_i)$ be the origin corresponding to (U_i, D_i). Let S_i^k be the stratum which contains q_i. Then the ν_i^k induces a resolution $\nu_i : V_i \longrightarrow U_i$ of U_i. Since ν_i is an isomorphism over $U_i - p_i$, we have a global resolution $\nu : \tilde{X} \longrightarrow X$. Let F be the union of ν-exceptional locus and $\nu^{-1}(D)$, and apply Proposition 10. Let $g : (\mathcal{X}, \mathcal{D}) \longrightarrow \Delta_\epsilon^1$ be the small deformation of (X, D) determined by $\eta \in Ext^1(\Omega_X^1(log\ D), \mathcal{O}_X)$. There is a holomorphic map $\phi_i : \Delta_\epsilon^1 \longrightarrow Def(\acute{U}_i, D_i)$ with $\phi_i(0) = q_i$ for each i. If p_i is neither a smooth point nor an ordinary double point, then the image of ϕ_i is not contained in the stratum S_i^k by Proposition 10. Hence, if we take a suitable point $t \in \Delta_\epsilon^1 - \{0\}$, then $\phi_i(t) \in S_i^{k'}$ for some $k' < k$. We shall consider (X_t, D_t). Then the $\nu_i^{k'}$ induces a resolution $\nu_t : \tilde{X}_t \longrightarrow X_t$ of X_t. Let F_t be the union of the ν_t-exceptional locus and $\nu_t^{-1}(D_t)$. We apply Proposition 10 to $(\tilde{X}_t, F_t)$. Since $Def(U_i, D_i)$ is versal at every point near the origin q_i, we can continue these process until X becomes a generalized Fano 3-fold with only ordinary double points. By Proposition 4, a generalized Fano 3-fold with ordinary double points is smoothable by a flat deformation. Thus, we have proved the result.

§2.

In this section we shall estimate the number of singular points on a generalized Fano 3-fold with Gorenstein terminal singularities.

Lemma 12. *Let (X, D) be a pair of a generalized Fano 3-fold X and its general anti-canonical divisor D. Then we have*

$$H^0(X, \Theta_X(-log\ D)) = 0.$$

Proof. By the Serre duality, it is enough to prove that $H^3(X, \Omega^1_X(log\ D)(-D)) = 0$. Consider the exact sequence

$$0 \longrightarrow \Omega^1_X(log\ D)(-D) \longrightarrow \Omega^1_X \longrightarrow \Omega^1_D \longrightarrow 0$$

We have $H^2(D; \Omega^1_D) = 0$ as follows. By the depth argument, Ω^1_D is a torsion free sheaf. Since there is an injection $\Omega^1_D \longrightarrow \hat{\Omega}^1_D$ and since its cokernel has a support only on the singular points, we have $H^2(D, \Omega^1_D) = H^2(D, \hat{\Omega}^1_D)$. By the Hodge symmetry for a V-manifold, $h^2(D, \hat{\Omega}^1_D) = h^1(D, K_D)$. Since D is a K3 surface with rational double points, we have $h^1(D, K_D) = 0$. Thus, $H^2(D, \Omega^1_D) = 0$ follows. We only have to show that $H^3(X, \Omega^1_X) = 0$ by the above exact sequence. By the Serre duality, it is enough to prove that $H^0(X, \Theta_X(-log\ D)) = 0$. Let $f : \tilde{X} \longrightarrow X$ be an equivariant resolution of X. Since X has only terminal singularities, we have $K_{\tilde{X}} = f^*K_X + \Sigma a_i E_i$ with $a_i > 0$ for all i. From this fact it follows that $f_*(\Theta_{\tilde{X}} \otimes K_{\tilde{X}}) \cong \Theta_X \otimes K_X$. Therefore, it is sufficient to prove that $H^0(\tilde{X}, \Theta_{\tilde{X}} \otimes K_{\tilde{X}}) = 0$. But this proved as follows. By Serre duality, we only have to show that $H^3(\tilde{X}, \Omega^1_{\tilde{X}}) = 0$. By the Hodge symmetry, $h^3(\tilde{X}, \Omega^1_{\tilde{X}}) = h^1(\tilde{X}, K_{\tilde{X}})$, which vanishes because X is a generalized Fano 3-fold with only terminal singularities. Q.E.D.

Theorem 13. *Assume that a smooth Fano 3-fold Z is degenerated to a Fano 3-fold X with Gorenstein terminal singularities by a flat deformation. Then we have*

$$21 - (1/2)e(Z) \geq \Sigma_{p \in Sing(X)} \mu(X, p) + \#\{\text{ ordinary double points on } X\ \}.$$

Proof. Let $f : \tilde{X} \longrightarrow X$ be a small partial resolution of X such that f is an isomorphism over all points of X except ordinary double points, and that f is a small resolution of the singularity around each ordinary double point. By definition, $\tilde{X}$ is a generalized Fano 3-fold with Gorenstein terminal singularities. Let $\tilde{D}$ be the pull back of D by f. By Theorem 11, $\tilde{X}$ has a flat deformation to a smooth generalized Fano 3-fold $\tilde{X}_t$. Since there is a closed immersion $Def(\tilde{X})$ into $Def(X)$ (cf. [Na 2, (1.6),(1) or (2.3),(1)]); the proof there can be applied to our situation.), there is a small birational contraction $f_t : \tilde{X}_t \longrightarrow X_t$ such that X_t becomes a small deformation of X. $\tilde{D}$ moves sideway in the flat family because $H^1(\tilde{D}; N_{\tilde{D}/\tilde{X}}) = 0$. Thus, we have an anti-canonical divisor $\tilde{D}_t$ of $\tilde{X}_t$ which is a small deformation of $\tilde{D}$. Define D_t to be the image of $\tilde{D}_t$ by f_t. D_t is an anti-canonical divisor of X_t and it is a small deformation of D. By Proposition 3 and Lemma 12, the Kuranishi space $Def(X, D)$ is universal. Thus, for $[(X_t, D_t)] \in Def(X, D)$, $Def(X, D)$ itself becomes the Kuranishi space of (X_t, D_t) at $[(X_t, D_t)]$. The similar things also hold for $Def(\tilde{X}, \tilde{D})$. We shall prove the following two results:

(1) $dim\ Def(\tilde{X}, \tilde{D}) \geq \Sigma_{p\in\tilde{X}}\mu(\tilde{X}, p)$;
(2) $dim\ Def(X_t, D_t) - dim\ Def(\tilde{X}_t, \tilde{D}_t) \geq \#\{$ ordinary double points on X $\}$.

Note that $\Sigma_{p\in\tilde{X}}\mu(\tilde{X}, p) = \Sigma_{p\in X}\mu(X, p)$ because $\mu = 0$ for an ordinary double point. Thus, if the above results are proved, then we have $dim\ Def(X_t, D_t) \geq \Sigma_{p\in X}\mu(X, p) + \#\{$ ordinary double points on X $\}$. Let S be a general anti-canonical divisor of Z. By the observation above, $dim\ Def(Z, S) = dim\ Def(X, D) = dim\ Def(X_t, D_t)$. To calculate $dim\ Def(Z, S) = dim_{\mathbf{C}} H^1(Z, \Theta_Z(-log\ S))$, we consider the exact sequence

$$0 \longrightarrow \Theta_Z(-log\ S) \longrightarrow \Theta_Z \longrightarrow N_{S/Z} \longrightarrow 0$$

By using this sequence and Riemann Roch theorem, we have $dim_{\mathbf{C}} H^1(Z, \Theta_Z(-log\ S)) = 21 - (1/2)e(Z)$. This implies the statement of Theorem.

Proof of (1) Consider the exact commutative diagram (**) above Lemma 8. By Lemma 8, $dim_{\mathbf{C}} Ker(\gamma_i) \geq \mu(X, p_i)$. Hence by (**) we easily see that $dim_{\mathbf{C}} Image(\alpha_{log}) \geq \Sigma_{p\in X}\mu(X, p)$, which implies that $dim_{\mathbf{C}} Ext^1(\Omega^1_X(log\ D), \mathcal{O}_X) \geq \Sigma_{p\in X}\mu(X, p)$. Q.E.D.

Proof of (2) Since a $(-1, -1)$-curve is stable under a flat deformation, X_t has the same number of ordinary double points as X. Since $\tilde{X}_t$ is smooth, X_t has no other singular points. By applying Proposition 4 to X_t, we have that $\alpha_t : Ext^1(\Omega^1_{X_t}, \mathcal{O}_{X_t}) \longrightarrow H^0(X, T^1_{X_t})$ is surjective. We have a homomorphism $f_* : H^1(\tilde{X}_t, \Theta_{\tilde{X}_t}(log\ \tilde{D}_t) \longrightarrow Ext^1(\Omega^1_{X_t}(log\ D_t), \mathcal{O}_{X_t})$ corresponding to $f : (\tilde{X}_t, \tilde{D}_t) \longrightarrow (X_t, D_t)$. Since f is a small birational morphism, f is an injection (cf. [Na, 2, (1.6),(1) or (2.3)(1)]). We shall show that cokernel of f_* has at least n dimension as a $\mathbf{C}$-vector space, where $n = \#\{$ ordinary double points on X_t $\}$. Let $\eta_i \in H^0(X_t, T^1_{X_t})$ be a non-zero section such that η_i has the support only on the ordinary double point $p_i \in X_t$. Consider the commutative diagram

$$\begin{array}{ccc} Ext^1(\Omega^1_{X_t}(log\ D_t), \mathcal{O}_{X_t}) & \longrightarrow & H^0(X_t, \mathcal{E}xt^1(\Omega^1_{X_t}(log\ D_t), \mathcal{O}_{X_t})) \\ \downarrow & & \downarrow \\ Ext^1(\Omega^1_{X_t}, \mathcal{O}_{X_t}) & \longrightarrow & H^0(X_t, T^1_{X_t}) \end{array}$$

The vertical maps are both surjective since $Ext^2(\mathcal{O}_{D_t}, \mathcal{O}_{X_t}) = 0$ and $\mathcal{E}xt^2(\mathcal{O}_{D_t}, \mathcal{O}_{X_t}) = 0$. Thus, by the diagram, we have a lifting $\zeta_i \in Ext^1(\Omega^1_{X_t}(log\ D_t), \mathcal{O}_{X_t})$ of η_i. Assume that $\zeta_i = f_*(\zeta_i')$ for some $\zeta_i' \in H^1(\tilde{X}_t, \Theta_{\tilde{X}_t}(-log\ \tilde{D}_t))$. Note that the composition of the maps $H^1(\tilde{X}_t, \Theta_{\tilde{X}_t}(-log\ \tilde{D}_t)) \longrightarrow Ext^1(\Omega^1_{X_t}(log\ D_t), \mathcal{O}_{X_t}) \longrightarrow H^0(X_t, \mathcal{E}xt^1(\Omega^1_{X_t}(log\ D_t), \mathcal{O}_{X_t})) \longrightarrow H^0(X_t, T^1_{X_t})$ and the composition of the maps $H^1(\tilde{X}_t, \Theta_{\tilde{X}_t}(-log\ \tilde{D}_t)) \longrightarrow H^1(\tilde{X}_t, \Theta_{\tilde{X}_t}) \longrightarrow Ext^1(\Omega^1_{X_t}, \mathcal{O}_{X_t}) \longrightarrow H^0(X_t, T^1_{X_t})$ coincide. Moreover, the sequence

$$0 \longrightarrow H^1(\tilde{X}_t, \Theta_{\tilde{X}_t}) \longrightarrow Ext^1(\Omega^1_{X_t}, \mathcal{O}_{X_t}) \longrightarrow H^0(X_t, T^1_{X_t})$$

is exact. In fact, $H^1(X_t, \Theta_{X_t}) \cong H^1(\tilde{X}_t, \Theta_{\tilde{X}_t})$ because $H^0(X_t, R^1 f_{t*}\Theta_{\tilde{X}_t}) = 0$. Thus, by using the second composition of the maps above, we conclude that $\eta_i = 0$. But, this is a contradiction. Hence ζ_i is not contained in the image of f_*. Since $\{\zeta_i\}_{1 \leq i \leq n}$ are C-linear independent sections of the cokernel of f_*, we have proved that $dim_{\mathbf{C}} Coker(f_*) \geq n$.

References

[Al] Alexeev, V.A.: Theorems about good divisors on log Fano varieties, Lecture Note in Math. **1479**, 1-9

[B-W] Burns, D.M.Jr, Wahl, J.M.: Local contributions to global deformations of surfaces, Invent. Math. **26**, 67-88 (1974)

[Fr 1] Friedman, R.: Global smoothings of varieties with normal crossings, Ann of Math. **118**, 75-114 (1983)

[Fr 2] Friedman, R.: Simultaneous resolution of threefold double points, Math. Ann. **274**, 671-689 (1986)

[Gro] Grothendieck, A.: Cohomologie Locale des Faisceaux Coherents et Theoremes de Lefschetz Locale et Globaux (SGA 2) North-Holland Publ. Co., Amsterdam (1962)

[Ha] Hartshorne, R.: Ample subvarieties of Algebraic Varieties, Lecture Notes in Math. **156**, Springer-Verlag (1970)

[Is 1, 2] Iskovskih, V.A.: Fano 3-folds I, II, Math. USSR-Izv, **11** 485-527 (1977), **12** 469-506 (1978)

[K-M-M] Kawamata, Y. Matsuda, K. Matsuki, K.: Introduction to minimal model problems, Advanced Studies in Math. **10**, 283-360 (1987), Kinokuniya, North-Holland

[M-M] Mori, S. Mukai, S.: On Fano 3-folds with $B_2 \geq 2$, Advanced Studies in Math. **1**, 101-129 (1983), Kinokuniya, North-Holland

[Mu] Mukai, S.: Gorenstein Fano threefolds, Proceedings of Algebraic Geometry Symposium, Saitama (1993)

[Na 1] Namikawa, Y.: On deformations of Calabi-Yau 3-folds with terminal singularities, Topology **33**, (3) 429-446 (1994)

[Na 2] Namikawa, Y.: Stratified local moduli of Calabi-Yau 3-folds, Preprint (1995)

[Na-St] Namikawa, Y., Steenbrink, H.H.M.: Global smoothing of Calabi-Yau threefolds, preprint

[Re] Reid, M.: Projective morphism according to Kawamata, Preprint (Warwick University) (1983)

[Sa] Sano, T.: On classification of **Q**-Fano 3-folds whose Fano indices are one, Preprint

[Sch] Schlessinger, M.: Rigidity of quotient singularities, Invent. Math. **14**, 17-26 (1971)

[Sh] Shokurov, V.V.: Smoothness of the general anti-canonical divisor of a Fano 3-fold, Math. USSR-Izv, **14**, 395-405 (1980)

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