Theta function formulae

for classical tops

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§5 Tracks and cylindrical homotopy groups

We consider homotopy classes of homotopies which we call tracks. They are 2-morphisms in any I-category which canonically has the structure of a groupoid enriched category. Using track addition we obtain the 'cylindrical homotopy groups' in I-categories.

Let $\underline{\underline{C}}$ be an I-category and let $Y \rightarrowtail X$ be a cofibration in $\underline{\underline{C}}$. For maps $x, y : X \to U$ with $x|Y = y|Y$ we consider homotopies $G : I_Y X \to U$ which extend $(x,y) : X \cup_Y X \to U$, that is $G : x \simeq y$ rel Y. Let

$$(5.1) \qquad H_Y(x,y) = [I_Y X, U]^{(x,y)}$$

be the set of homotopy classes relative (x,y) of such homotopies. The elements $\{G\}$ of this set are called <u>tracks</u> from x to y rel Y, we also write $\{G\} : x \simeq y$ rel Y.
A map $g : U \to V$ and a map $f : (A,B) \to (X,Y)$ between cofibrant pairs induce functions

(1) $$f^* = (If)^* : H_Y(x,y) \to H_B(xf, yf),$$

(2) $$g_* \quad : H_Y(x,y) \to H_Y(gx, gy)$$

compare (4.9). Here the map f^* is induced by $If : I_B A \to I_Y X$. Moreover we have the following structure $(+, -, 0)$ of a groupoid for the sets of tracks in (5.1). Let $p : I_Y X \to X$ be the projection of the cylinder, then

(3) $$0 = \{xp\} \in H_Y(x,x)$$

is the trivial track. Now we obtain

Introduction.

The fact that some problems of classical mechanics are solved in terms of theta functions was discovered in the 18 th century by Euler, who solved in this way the equations of motion of the rigid body around the center of mass in Jacobi functions. The multidimensional theta functions were first applied by C. Neumann in 1859 [38] to the problem of motion of the particle constrained on the sphere under the action of quadratic potential. The most famous mechanics system of this type is undoubtly the Kowaleski top [18], which was in the focus of interest in the 19 th centure.

Despite the dicovery of numerous examples of finite–dimensional systems integrable in terms of multidimensional theta functions there was at that time no general approach to solving the equation of motion of these systems. Each time the success integration was based on finding a rather non–trivial change of variables leading to a Jacobi inversion problem. After Kowalewski the most important results in this direction were obtained by Kötter [42], [44], [47].

We solve the equations of motion of classical tops with the help of the finite–gap integration theory. Such an application of the modern theory to the classical problems proves its universality. From the other side modern theory gives a possibility of obtaining important new results even for the classical tops (particularly for the Kowalewski top [2]). The formulae for the solutions obtained here are considerably simpler than the classical ones.

The theory of the finite–gap integration for the finite–dimensional systems is based on the representation of the equation of motion in the Lax form:

$$\frac{d}{dt} L(\lambda) + [L(\lambda),A(\lambda)] = 0 \ .$$

When the Lax representation is found, all the machinery of the finite–gap integration theory may be used. At the same time the construction of the Lax representation for the concrete system is a transcendental problem, solved however, now for all famous tops studied in classical papers. The recent results in this direction are the various Lax pairs for the Kowalewski top [48], [49], [40]. One of them found by Reymann and Semenov–Tian–Shansky is used by us for the integration of the equation of motion.

It should be mentioned that there now exists a direct approach to solving integrable systems with two degrees of freedom. This approach was devised by Adler and van Moerbere et al. [21]. It is based on the study of singularities of the solutions and goes back to fundamental ideas *) of the Kowalewski paper [18]. With the help of the direct approach, the geometry of algebraic curves and Abelian tori arising in various problems of classical mechanics, was investigated [21], [22], [23], [31], [49], [50], where important isomorphisms were found. The advantage of the direct approach is that it starts directly from the equations of motion without a priori knowledge of the Lax representation. However, the theta functional formulae were not derived in this way.

The main instrument used in constructing the general solution of the problem is the Baker–Akhiezer function. It is interesting that in some cases the Baker–Akhiezer function itself appears to be very useful for the description of dynamics. The motion in the laboratory frame is described in this way (Sect. 9).

It is worth mentioning also that the Lax representations for all classical systems considered here has nontrivial reduction groups. This in turn leads to the fact that the associated spectral curves represent complicated coverings and that the Baker–Akhiezer functions have specific analytic properties. And as a final result we have a more complicated integration procedure.

We restrict ourselves to the tops investigated in the 19 th century and do not

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*) The idea of finding of the integrable cases by analyzing the singularities of the solutions apparently was first proposed by Weierstrass [19].

discuss the numerous examples of integrable systems in classical mechanics found recently with the help of the inverse transform method. A good survey of these modern results can be found in [28], [29], [52].

The present paper is a translation of the improved version of author's preprint [1] and is supposed to be a chapter of the joint book Belokoles, Bobenko, Enolskii, Its, Matveev "Algebraic Geometrical Approach to Nonlinear Integrable Equations".

1. The Lax Equation and Analytic Properties of the Baker–Akhiezer Function

The analytic properties of the Baker–Akhiezer functions are deducible from the corresponding Lax representations

$$\frac{d}{dt} L(\lambda) + [L(\lambda), A(\lambda)] = 0 \ . \tag{1.1}$$

Let

$$L = \Psi \hat{\mu} \Psi^{-1}$$

be the diagonal form of the matrix L, where $\hat{\mu}$ is the eigenvalue matrix. It satisfies the equation

$$\hat{\mu}_t = [\hat{\mu}, \Psi^{-1} A \Psi + \Psi^{-1} \Psi_t] \tag{1.2}$$

and, as a corollary, it does not depend on t since the RHS of (1.) has a zero diagonal part.

The Baker–Akhiezer function is an eigenfunction of the operator L

$$L\psi = \mu\psi \ . \tag{1.3}$$

Here L is an $(N\times N)$ matrix and ψ is an N–dimensional vector. The eigenvalues of L do not depend on t. Therefore, the characteristic polynomial

$$\det(L(\lambda)-\mu) = 0 \tag{1.4}$$

is also independent of t. It defines a spectral curve X. The suitably normalized ψ–function is an analytic function on the Riemann surface of the spectral curve (1.4).

Let us consider the simplest, but simultaneously the most general example where L and A are the rational functions of λ, and there are no reductions. It is evident that X is the N–sheeted covering of the λ–plane. So we have N values $\mu^1,\ldots,\mu^N$ (every value is counted according to its multiplicity), corresponding to every λ. Respectively, we have N eigenvectors $\psi^i = \psi(\lambda,\mu^i)$ of the matrix L.

Let us consider the function

$$(\det \Psi)^2 = (\det(\psi^1,\ldots,\psi^N))^2 \ . \tag{1.5}$$

It is a single–valued function of λ with the divisor of the poles of degree 2K, where K is the degree of the divisor of the poles of ψ on X. The degree of the divisor of zeroes of function (1.5) is equal to the sum of all branch numbers $\Sigma\, \nu_i$. By equating these degrees and taking into account the Riemann–Hurwitz formula we get

$$2K = \Sigma\, \nu_i = 2g-2+2N \ ,$$

where g is the genus of X. Finally we see that ψ has the divisor of the poles of the degree

$$K = g+N-1 \ .$$

Differentiating (1.3) by t, we see that the Baker–Akhiezer function satisfies the equation

$$(L-\mu)(\psi_t - A\psi) = 0 \ ,$$

which in turn gives

$$\psi_t = A\psi + a(\lambda,t)\psi \ .$$

Here $a(\lambda,t)$ is some scalar function, which can be eliminated by the suitable renormalization of ψ. We see that ψ has essential singularities at the poles of A.

Finally we see that the Baker–Akhiezer function is a solution of the system

$$L\psi = \mu\psi \ , \quad \psi_t = A\psi \ .$$

It is an analytic function on X, having the divisor of poles independent of t and the essential singularities at the poles of A.

Let us mention that the various possible reductions of L–A pairs lead to symmetry of the spectral curves and to the specificity of properties of ψ–functions. Below we consider L–A pairs with reductions.

2. Integrable Systems

The Kirchhoff equations are important in classical mechanics and hydrodynamics:

$$\dot{p} = [p,\omega]\,, \quad \dot{M} = [M,\omega] + [p,\omega]\,, \quad \omega^i = \frac{\partial H}{\partial M_i}, \quad u^i = \frac{\partial H}{\partial p_i}\,. \tag{2.1}$$

Here, $[\ ,\ ]$ denotes the vector product in R^3. As was remarked in [6] equations (2.1) are Hamilton's equations of motion

$$\dot{f} = \{H,f\}\,, \quad \dot{f} = f_t \tag{2.2}$$

with respect to the following Poisson brackets:

$$\{M_i,M_j\} = \epsilon_{ijk}M_k\,, \quad \{M_i,p_j\} = \epsilon_{ijk}p_k\,, \quad \{p_i,p_j\} = 0\,, \tag{2.3}$$

$$i,j,k = 1,2,3\,, \quad \epsilon_{123} = 1\,.$$

The Poisson brackets (2.3) are, in fact, the Lie–Poisson bracket for the Lie algebra $e(3)$ of the motion group $E(3)$ of the Euclidean space. Notice that

$$f_1 = p^2 = \sum_i p_i^2\,, \qquad f_2 = pM = \sum_i p_i M_i \tag{2.4}$$

are trivial integrals of motion (Casimir functions) for the Poisson bracket (2.3). Thus we have a four–dimensional phase space. For the corresponding system to be completely integrable, it is sufficient to possess one additional (besides the Hamiltonian) integral of motion K.

The rotation of a rigid body about a fixed point is described by (2.1). In this case the orthogonal frame is attached to the body and coincides with the axes of the inertia ellipsoid. The origin is chosen to be at the fixed point. The Hamiltonian is

$$H = \tfrac{1}{2}(I_1 M_1^2 + I_2 M_2^2 + I_3 M_3^2) + \Gamma_1 p_1 + \Gamma_2 p_2 + \Gamma_3 p_3 \ . \tag{2.5}$$

Here M is the angular momentum of the body, p is the unit gravitational field vector, the constant vector $(\Gamma_1, \Gamma_2, \Gamma_3)$ indicates the center of mass and $I_1^{-1}, I_2^{-1}, I_3^{-1}$ are the main moments of inertia of the body.

The following cases are integrable:

(1.) The Euler case: $\Gamma = 0$, $K = M^2$.

(2.) The Lagrange case: $I_1 = I_2$, $\Gamma_1 = \Gamma_2 = 0$, $K = M_3$.

(3.) The Kowalewski case: $I_1 = I_2 = I_3/2$, $\Gamma_3 = 0$.

(4.) The Goryachev–Chaplygin case: $I_1 = I_2 = I_3/4$, $\Gamma_3 = 0$

and the constant f_2 vanishes, i.e. $pM = 0$. In the last case we have the integrable Hamiltonian system on only one integral level. The formulae for the additional integrals K for the Kowalewski and the Goryachev–Chaplygin cases are presented in the Sects. 3, 4.

We remark that in all integrable cases presented above K is a polynomial of M and p. In the paper [7] it was shown that there are no additional cases of integrability if K is a meromorphic function of M and p.

For the quadratic Hamiltonians

$$H = \tfrac{1}{2} \sum (a_{ij} M_i M_j + 2 b_{ij} M_i p_j + c_{ij} p_i p_j)$$

equations (2.1) coincide with the Kirchhoff equations of the motion of a rigid body in an ideal incompressible liquid being at rest at infinity. The orthogonal frame is attached to the body and is chosen so that the inertia tensor is diagonal. In this case M and p are respectively the complete angular momentum and the complete impulse of the body–liquid system. The non–trivial integrable cases of Clebsch [8] and Steklov [9] are known and are the only cases with the quadratic additional integral K, the corresponding expressions for which are presented in Sects. 7, 8.

We also consider the Euler equations on the Lie algebra $SO(4)$, which have interesting applications in hydrodynamics. The Lie algebra $SO(4)$ is isomorphic to the direct sum of two copies of $SO(3)$. In the following we shall always use the isomorphism $SO(4) = SO(3) + SO(3)$. The Euler equations with the Hamiltonian

$$H = \frac{1}{2}\sum_{ij} (a_{ij}S_iS_j + 2b_{ij}S_jT_i + c_{ij}T_iT_j)\ ,\quad a_{ij} = a_{ji},\ c_{ij} = c_{ji} \tag{2.6}$$

and the Lie–Poisson bracket

$$\{S_i,S_j\} = \epsilon_{ijk}S_k\ ,\quad \{S_i,T_j\} = 0\ ,\quad \{T_i,T_j\} = \epsilon_{ijk}T_k\ ,\quad i,j,k = 1,2,3 \tag{2.7}$$

are as follows

$$\dot{S} = [S, AS + B^T T]\ ,\quad \dot{T} = [T, BS + CT]\ .$$

Here A, B, C denote the matrices of the coefficients of the Hamiltonian (2.6).

Two trivial integrals

$$g_1 = S^2 = \sum_i S_i^2\ ,\quad g_2 = T^2 = \sum T_i^2 \tag{2.8}$$

show that as in the $e(3)$ case, we have four–dimensional orbits. The additional integral of motion K exists in the integrable cases of Manakov [10] and Steklov [11] [1]. These are the only cases with quadratic K. Recently another case of integrability with quadratic H and quartic K was found [16, 17].

Another classical problem, integrable in terms of two–dimensional theta functions, is the Neumann system. The equations of motion are

$$[S_{tt}+IS,S] = 0\ ,\ \ I = \mathrm{diag}(I_1,I_2,I_3)\ ,\ \ S^2 = 1\ . \tag{2.9}$$

It describes the motion of a particle restricted to the unit sphere under the quadratic potential

$$U(S) = \frac{1}{2}\sum_i I_i\, S_i^2\ .$$

Below we construct theta functional formulae for all systems mentioned above except the Lagrange top (the Euler and the Lagrange tops are easily solved in elliptic functions and are investigated in detail).

3. Kowalewski Top

In here celebrated paper [18] published in 1889, Kowalewski found a new and highly nontrivial integrable case of the motion of a heavy rigid body abour a fixed point,

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[1] The Euler equations on SO(4) of special type (special A, B, C) describe the motion of a rigid body with an ellipsoidal cavity filled with liquid. A family of integrable cases of such systems depending on 3 parameters was found by Steklov [12]. When the problem of finding integrable Euler equations on SO(4) was investigated later, these same integrable cases were found [13, 14, 15]. The number of arbitrary parameters increased to 6. We shall call this case a second Steklov case.

completing the list of integrable tops. Two previous known integrable cases are Euler's top in which the stationary point coincides with the center of mass, and Lagrange's top which is axially symmetric. The third case discovered by Kowalewski is rather bizarre: the moments of inertia have a fixed ratio $2:2:1$, ans the center of mass lies in the equatorial plane of the top.

In this section we follow the paper [2], where calculations omitted here are presented.

3.1 *Kowalewski's Paper*

The starting point of Kowalewski's work was her observation that Euler's and Lagrange's tops are solved in terms of Jacobi functions. Therefore, here initial idea was to try to solve the equations of motion of a general heavy rigid body about a fixed point in terms of Abelian functions. However, Weierstrass pointed out that a general solution of this form does not exist in the general case and may be possible only for some particular geometries of the top [19]. Thus Kowalewski started her search of the tops of this type.

She considered the equations of motion for the general top (2.5)

$$\dot{M} = [M, IM] + [p, \Gamma] \ , \ \ \dot{p} = [p, IM] \ ,$$
$$IM = (I_1 M_1, I_2 M_2, I_3, M_3) \ , \ \ \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) \tag{3.1}$$

and substituted the series

$$M_i = \frac{m_i}{t-t_0} + \dots \ , \ \ p_i = \frac{n_i}{(t-t_0)^2} + \dots \tag{3.2}$$

in the neighbourhood of the singularity point $(t_0 \in \mathbb{C})$ into the equations (3.1). The question was: for what tops series (3.2) is there a general solution of equations (3.1) (i.e. which tops series have a sufficient number of independent constants).

In her paper Kowalewski obtained three remarkable results. First she proved (with some gaps which were filled in later, see comments in [20]) that the only tops with the property that the general solution is given by meromorphic functions of the complex time variable are Euler's and Lagrange's tops and a new top with Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) - p_1 \ . \tag{3.3}$$

She also found the additional integral

$$K = (M_1^2 - M_2^2 + 2p_1)^2 + 4(M_1M_2 + p_2)^2 \tag{3.4}$$

for the top (3.3) which now carries her name. Finally, using a non–trivial change of variables, Kowalewski reduced the equations of motion to the Jacobi inversion problem for the hyperelliptic curve (Kowalewski curve) of genus 2 [2]

$$\mu^2 = ((\lambda - H)^2 - \tfrac{K}{4})(\lambda((\lambda - H)^2 + (1 - \tfrac{K}{4})) - (pM)^2) \ . \tag{3.5}$$

Kowalewski's paper became very popular, especially the first part which attracted attention and was widely discussed (see comments in [20]). At the same time the extremely technical part, devoted to explicit integration of the top, remained, for a long time, only a sequence of well–guessed substitutions and calculations. The relationship

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[2] Here and below, with no loss of generality, we shall assume in the sequel that $p^2 = 1$.

among the three problems considered by Kowalewski was also unclear. This was recently clarified in [21, 22, 23].

3.2 *The Lax Pair for the Kowalewski Top*

Let us consider the Kowalewski gyrostat (KG). This is a system with the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2 + 2\gamma M_3) - p_1 \ . \tag{3.4}$$

Theorem 1. *The Kowalewski gyrostat, defined by the Hamiltonian* (3.4) *is completely integrable and admits a Lax representation* $dL/dt + [L,A] = 0$ *given by*

$$L = \frac{i}{\lambda}\begin{bmatrix} 0 & p_- & 0 & -p_3 \\ -p_+ & 0 & p_3 & 0 \\ 0 & -p_3 & 0 & -p_+ \\ p_3 & 0 & p_- & 0 \end{bmatrix} + i\begin{bmatrix} -\gamma & 0 & M_- & 0 \\ 0 & \gamma & 0 & -M_+ \\ M_+ & 0 & -2M_3-\gamma & -2\lambda \\ 0 & -M_- & 2\lambda & 2M_3+\gamma \end{bmatrix},$$

$$A = \frac{i}{2}\begin{bmatrix} 2M_3+\gamma & 0 & M_- & 0 \\ 0 & -2M_3-\gamma & 0 & -M_+ \\ M_+ & 0 & -2M_3-\gamma & -2\lambda \\ 0 & -M_- & 2\lambda & 2M_3+\gamma \end{bmatrix}, \tag{3.7}$$

$$p_\pm = p_1 \pm ip_2 \, , \ M_\pm = M_1 \pm iM_2 \ .$$

These matrices obey the symmetry relations

$$L(-\lambda) = \eta L(\lambda)\eta \ , \quad \eta = \mathrm{diag}(1,-1,1,-1) = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} \ , \tag{3.8}$$

$$L(\lambda)^T = -\begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix} L(\lambda) \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix} \ . \tag{3.9}$$

We recall the definition of the Pauli matrices σ_i :

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ .$$

The invariants of the matrix $L(\lambda)$ are integrals of the motion in involution H , $f_1 = p^2 = 1$, $f_2^2 = (pM)^2$ and

$$K = (M_1^2 - M_2^2 + 2p_1)^2 + 4(M_1M_2 + p_2)^2 - 4\gamma((M_3+\gamma)(M_1^2+M_2^2) + 2M_1p_3) \ .$$

This integral is an extension of the integral (3.4) found by Kowalewski and was discussed in [24, 25] . The Lax pair (3.7) as well as broader generalizations of the Kowalewski top and the corresponding Lax pairs are obtained by Reyman and Semenov–Tian–Shanski [2] .

3.3 *The Spectral Curve*

Let us now turn to the original Kowalewski top where $\gamma = 0$. We shall consider complex equations of motion. Lax equations are linearizable on the Jacobian of the spectral curve $\hat{X}$ defined by the equation

$$\det(L(\lambda)-\mu) = 0 \ . \tag{3.10}$$

The characteristic equation (3.10) for the Lax matrix of the KT takes the form

$$\begin{aligned}
&\mu^4 - 2d_1(\lambda^2)\mu^2 + d_2(\lambda^2) = 0 \ , \\
&d_1(z) = z^{-1} - 2H + 2z \ , \\
&d_2(z) = z^{-2} + 4[(Mp)^2 - H)]\, z^{-1} + K \ .
\end{aligned}$$

The symmetries (3.8, 9) give rose to two commuting involutions τ_1, τ_2 on $\hat{X}$

$$\tau_1(\lambda,\mu) = (-\lambda,\mu) \ , \ \ \tau_2(\lambda,\mu) = (\lambda,-\mu) \ , \tag{3.11}$$

which in turn induce the coverings $\hat{X} \longrightarrow X$ and $X \longrightarrow E$, given by a change of variables $z = \lambda^2$ and $y = \mu^2$. So the curves X and E are defined by the equations

$$\mu^4 - 2d_1(z)\mu^2 + d_2(z) = 0 \tag{3.12}$$

and

$$y^2 - 2d_1(z)y + d_2(z) = 0 \ .$$

The covering $\hat{X} \longrightarrow X$ is unramified and thus is determined by a cycle $\mathscr{L}$ (mod 2) on X : a loop γ on X lifts to a closed loop on $\hat{X}$ if and only if $\langle \gamma, \mathscr{L} \rangle = 0$ (mod 2) , where $\langle \gamma, \mathscr{L} \rangle$ is the intersection number. To put it another way, the function $\lambda = \sqrt{z}$ acquires a factor $(-1)^{\langle \gamma, \mathscr{L} \rangle}$ upon a circuit of γ.

For later use, we must have a closer look at the covering $X \longrightarrow E$. The elliptic curve E is a two–sheated cover of the Z–plane. There are two points $\infty_\pm$ at "infinity" where z has simple poles, and one point 0 where z has a double zero. The function y has a simple pole at ∞_+ , a simple zero at ∞_- , a double pole at 0 , and hence two other simple zeros at some points P_1, P_2 . The branch points of the function $\mu = \sqrt{y}$ on E are therefore ∞_+ , ∞_- , P_1, P_2 . Thus X is obtained by glueing together two copies of E along suitable cuts $[\infty_+,\infty_-]$ and $[P_1,P_2]$.

We choose the cut $[\infty_+,\infty_-]$ such that the function $\lambda = \sqrt{z}$ becomes unramified on $E\setminus[\infty_+,\infty_-]$ (notice that $\infty_\pm$ are the only branch points of $\sqrt{z}$).

The curve $\hat{X}$ may be thought of as the Riemann surface of the function $\lambda = \sqrt{z}$ on X . One can always choose a canonical basis in $H_1(X,\mathbb{Z})$ so that

$$\pi a_1 = -a_3 \ , \ \ \pi b_1 = -b_3 \ , \ \ \pi a_2 = -a_2 \ , \ \ \pi b_2 = -b_2 \ ,$$

where $\pi : (\mu,z) \longrightarrow (-\mu,z)$, and also $a_2 = \mathscr{L}$ (mod 2) (see Fig. 1).

We may now identify $\hat{X}$ with two copies of X glued together along $\mathscr{L}$: $\hat{X} = X^{(1)} \cup_{\mathscr{L}} X^{(2)}$. It is natural to choose the contour $\mathscr{L}$ such that $\pi\,\mathscr{L} = -\mathscr{L}$ (the minus sign denotes reversed orientation). The involution τ_1 acts on $\hat{X}$ by permuting the sheets $X^{(i)}$.

The final thing we need is the behaviour of μ near the points of $\hat{X}$ where $\lambda = \infty$ or $\lambda = 0$. These are the points $\infty_\pm^{(i)}$ and $0_\pm^{(i)}$ on the sheets $C^{(i)}$. If we arrange the points $\infty_\pm^{(i)}$ onto a 4–tuple $(\infty_-^{(1)}, \infty_-^{(2)}, \infty_+^{(1)}, \infty_+^{(2)})$, the 4 barnches of μ near $\lambda = \infty$ can be combined into a row–vector

$$\mu(\lambda) \sim (0,0,2\lambda,-2\lambda) + o(1) \ . \tag{3.13}$$

In a similar way, with respect to the ordering $(0_-^{(1)}, 0_+^{(2)}, 0_+^{(1)}, 0_-^{(2)})$ (this particular ordering is convenient for the calculations in Sect. 3.7), we have

$$\mu(\lambda) \sim -\epsilon\lambda^{-1}(1,-1,1,-1) \tag{3.14}$$

near $\lambda = 0$, with $\epsilon = \pm 1$ depending on the location of $\mathscr{L}$. It is always possible to choose $\mathscr{L}$ such that $\epsilon = 1$.

3.4 *Analyticity Properties of the Baker–Akhiezer Function*

The Baker–Akhiezer function, defined as a solution of the linear system

$$L(\lambda(P))\psi(P) = \mu(P)\psi(P)\ ,\ \frac{\partial}{\partial t}\,\psi(P) = A(\lambda(P))\psi(P) \tag{3.15}$$

has certain analyticity properties as a vector–valued function on $\hat{X}$. We can also require ψ to be symmetric with respect to the first of the involutions (3.11)

$$\psi(\tau_1 P) = \eta\psi(P)\ . \tag{3.16}$$

This enables us to regard ψ as a double–valued function on the curve $X = \hat{X}/\tau_1$, which makes all calculations much simpler.

Let us now state the analyticity properties of ψ .

1. *ψ is meromorphic on $\hat{X}$ except at $\lambda = \infty$ and $\psi\exp(-t\mu/2)$ is meromorphic on $\hat{X}$ except at $\lambda = 0$.*
2. *The divisor of poles of ψ , denoted by $\hat{\mathscr{D}}$, has degree 8 and is time independent.*
3. *ψ satisfies the symmetry condition* (3.16).

The divisor $\hat{\mathscr{D}}$ is not, however, completely determined by these conditions. If f is a meromorphic function on X and $(f) \leq \hat{\mathscr{D}}$ on $\hat{X}$, then ψ can be replaced by $f\psi$. Using this freedom, we can fix two points of $\hat{\mathscr{D}}$ to be $\infty_+^{(1)}$ and $\infty_+^{(2)}$. Then $\hat{\mathscr{D}}$ is the pull–back to $\hat{X}$ of a divisor $\mathscr{D} \cup \infty_+$ on X , and $\deg \mathscr{D} = 3$.

The behaviour of ψ near $\lambda = \infty$ can be reformulated in a more convenient matrix form. Let $\Psi(\lambda)$ be the 4×4 matrix whose j^{th} column is the value of ψ on the j^{th} sheet of $\hat{X} \longrightarrow \{\lambda\}$ near $\lambda = \infty$ (the ordering of sheets corresponds to the ordering of points over $\lambda = \infty$, described in Sect. 3.3). We can then write $\Psi(\lambda)$ as

$$\Psi(\lambda,t) = (\Phi+S\lambda^{-1} + \ldots)\mathrm{diag}(1,1,\lambda e^{\lambda t},-i\lambda e^{-\lambda t}) \tag{3.17}$$

(the factor $-i$ in the last entry of (3.17) is taken for notational convenience). Denoting

$$L(\lambda) = L_{-1}\lambda^{-1}+L_0+L_1\lambda\ ,\ \ A(\lambda) = A_0+A_1\lambda\ ,$$

we have from (3.15, 17, 13)

$$L_1 = 2\Phi\begin{bmatrix}0 & 0\\ 0 & \sigma_3\end{bmatrix}\Phi^{-1}\ ,\ \ L_0 = [S\Phi^{-1},L_1]\ , \tag{3.18}$$

$$A_1 = \Phi\begin{bmatrix}0 & 0\\ 0 & \sigma_3\end{bmatrix}\Phi^{-1}\ ,\ \ A_0 = \Phi_t\Phi^{-1}-[S\Phi^{-1},A_1]\ . \tag{3.19}$$

The symmetry condition (3.16) takes the form (notice that τ_1 permutes the sheets):

$$\Psi(-\lambda) = \begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}\Psi(\lambda)\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{bmatrix}\ ,$$

which gives the symmetry relations for Φ and S :

$$\begin{bmatrix}\sigma_3 & \\ & \sigma_3\end{bmatrix}\Phi\begin{bmatrix}\sigma_1 & \\ & \sigma_2\end{bmatrix} = \Phi\ ,\ \ \begin{bmatrix}\sigma_3 & \\ & \sigma_3\end{bmatrix}S\begin{bmatrix}\sigma_1 & \\ & \sigma_2\end{bmatrix} = -S\ .$$

Combined with (3.18) this implies that $\Phi(t)$ has the form

$$\Phi(t) = c\,\mathrm{diag}(q_1(t),q_2(t),1,1)\begin{bmatrix}1 & 1 & & \\ 1 & -1 & & \\ & & 1 & -i \\ & & -i & 1\end{bmatrix} .$$

We can set $c = 1$. The relation (3.19) yields differential equations for $q_i(t)$:

$$\frac{dq_1}{dt} = iM_3q_1 , \quad \frac{dq_2}{dt} = -iM_3q_2 ,$$

so that

$$q_1(t) = \alpha \exp(i\int^t M_3dt) , \quad q_2(t) = \beta \exp(-i\int^t M_3dt) . \tag{3.20}$$

In a similar way, arranging the 4 eigenvectors $\psi(0_\pm^{(i)})$ into a 4×4 matrix $\Psi(0)$ according to the ordering of the points $0_\pm^{(i)}$ described in Sect. 3.3, we have

$$\begin{bmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{bmatrix}\Psi(0)\begin{bmatrix}0 & \sigma_1 \\ \sigma_1 & 0\end{bmatrix} = \Psi(0) \tag{3.21}$$

and using (3.14, 15), we find

$$L_{-1} = -\epsilon\,\Psi(0)\begin{bmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{bmatrix}\Psi^{-1}(0) . \tag{3.22}$$

The strategy of our further computations will be as follows. Using the symmetry property (3.16) of ψ , we reformulate the problem entirely in terms of the curve X : the functions ψ_1, ψ_3, $\lambda\psi_2$, $\lambda\psi_4$ are single–valued functions on X . The properties of ψ

stated above allow us to write explicit formulae for ψ_3, ψ_4 , which in turn serve to compute the coefficients S_{ij} of S for $i,j = 3,4$. From (3.18) we have the relation

$$M_3 = -i\, S_{33} - S_{43} , \tag{3.23}$$

which by (3.20) yileds expressions for q_1, q_2 . After that we can write down the remaining components ψ_1, ψ_2 . To determine the constant factors that occur in these formulae, we must use the second symmetry relation (3.9). Combining it with (3.18), we come down to the Prym condition for the divisor $\mathscr{D}$ and determine the Baker–Akhiezer function completely. Finally to derive the evolution of $p(t)$ we use (3.22).

3.5 *Explicit Formulae for the Baker–Akhiezer Function*

We now begin to implement the programme outlined above. First of all we have to introduce certain Abelian differentials.

Let $d\Omega$ be a normalized Abelian differential of the second kind on X with a pole at ∞_+ such that

$$\Omega(P) = \int^{P} d\Omega = \lambda + 0(\lambda^{-1}) \text{ as } P \longrightarrow \infty_+$$

(recall that there is a well–defined branch of λ on $X \backslash \mathscr{S}$; its sign is specified by requiring that $\mu \sim 2\lambda$ at ∞_+). Let us denote by

$$V = (V_1, V_2, V_3) , \; V_j = \int_{b_j} d\Omega$$

the b–period vector of $d\Omega$.

Let $d\Omega_3$ be a normalized Abelian differential of the third kind which has simple poles at ∞_+ and ∞_- with residues 1 and –1, respectively. We choose a path ℓ from ∞_+ to ∞_- and normalize $d\Omega_3$ by the condition $\int_{a_j} d\Omega_3 = 0$ where the cycles a_j are supposed not to intersect ℓ. It is easily checked that $d\Omega_3$ is the pullback to X of a differential on E given by

$$d\Omega_3 = \frac{(1+qz^{-1})dz}{y-d_1(z)}$$

with some constant q, so that $\pi^* d\Omega_3 = d\Omega_3$ $(\pi(z,\mu) = (z,-\mu))$. We put

$$\Omega_3(P) = \int^P d\Omega_3$$

and fix the constant of integration by the condition

$$e^{\Omega_3(P)} = \lambda + 0(1) \text{ as } P \longrightarrow \infty_+ .$$

We shall need the values of the multivalued functions $\Omega(P)$, $e^{\Omega_3(P)}$, $\int^P \omega$ at the points $\infty_\pm$ and $0_\pm$. For that purpose we shall specify the choice of the path ℓ joining ∞_+ and ∞_- coinciding with the cut $[\infty_+,\infty_-]$ (see Fig. 1), i.e., (a) its projection to E passes through 0 and is symmetric with respect to $0 \in E$, (b) the cycle $\ell - \pi\ell$ is homologous to a_2, (c) ℓ does not intersect the ramification contour $\mathscr{L}$.

Since the periods of $d\Omega$, $d\Omega_3$, ω_1 and ω_3 over a_2 are all zero, the multi–valued analytic functions $\Omega(P)$, $e^{\Omega_3(P)}$, $\int^P \omega_{1,3}$ have single–valued branches in a neighbourhood of the contour $\ell \cup \pi\ell$. Also we set $\int^{\infty_+} \omega_j = 0$.

Standard calculations [2] show that

$$\int_{\infty_+}^{\infty_-} \omega = \int_{\ell} \omega = (r, \pi i, -r), \quad \int_{b_2} d\Omega_3 = \pi i \ ,$$

$$\Omega(\infty_-) = 0, \quad e^{\Omega_3(P)} = -\frac{a^2}{\lambda} + 0(1) \ \text{ as } \ P \longrightarrow \infty_- \ ,$$

where r and a are defined as follows:

$$r = \int_{b_1} d\Omega_3 = -\int_{b_3} d\Omega_3 \ , \quad a = e^{\Omega_3(0_+)} = -e^{\Omega_3(0_-)} \ .$$

In particular we see that $e^{\Omega_3(P)}$ changes sign when analytically continued along b_2.

Let D be a vector in $\mathbb{C}^3$ such that the divisor of $\theta(\int_{\infty_+}^{P} \omega + D)$ on X coincides with the divisor $\mathscr{D}$ introduced above.

Theorem 2. *The Bakher–Akhiezer function* $\psi(P,t)$ *is given by*

$$\psi_1 = q_1 \frac{\theta(\int^P \omega + Vt + D)\, \theta[\epsilon]\,(D+R)}{\theta(\int^P \omega + D)\, \theta[\epsilon]\,(Vt+D+R)} e^{\Omega(P)t}$$

$$\psi_2 = q_2 \frac{\theta[\epsilon]\,(\int^P \omega + Vt + D)\, \theta[\epsilon](D+R)}{\theta(\int^P \omega + D)\, \Theta\,(Vt+D+R)} e^{\Omega(P)t}$$

$$\psi_3 = \frac{\theta[\epsilon]\,(\int^P \omega + Vt + D + R)\,\theta(D)}{\theta(\int^P \omega + D)\,\theta[\epsilon]\,(Vt + D + R)}\, e^{\Omega(P)t + \Omega_3(P)}$$

$$\psi_4 = -i\,\frac{\theta(\int^P \omega + Vt + D + R)\,\theta(D)}{\theta(\int^P \omega + D)\,\theta(Vt + D + R)}\, e^{\Omega(P)t + \Omega_3(P)} \;, \qquad (3.24)$$

where

$$R = (r,0,r)\;, \quad \epsilon = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \;.$$

The proof is a straightforward corollary of the analytical properties of $\theta(\int^P \omega)$, $\Omega(P)$, $\Omega_3(P)$ displayed above and of the Baker–Akhiezer function described in Sect. 3.4. Recall that now we do not take into account the second symmetry relation (3.9).

The expressions (3.24) for ψ_3 and ψ_4 enable us to calculate $M_3(t)$.

Lemma 3.

$$M_3 = -i\,\frac{\partial}{\partial t}\log\frac{\theta[\epsilon]\,(Vt + D + R)}{\theta(Vt + D + R)} \;.$$

Proof. From (3.23) we get

$$M_3 = \lim_{P \to \infty_+} \lambda(-i\psi_3(P) - \psi_4(P)) =$$

$$= -i\left[\frac{\frac{\partial}{\partial k}\ \theta[\epsilon]\left(\int^P \omega + Vt + D + R\right)}{\theta[\epsilon]\left(\int^P \omega + Vt + D + R\right)} - \frac{\frac{\partial}{\partial k}\ \theta\left(\int^P \omega + Vt + D + R\right)}{\theta\left(\int^P \omega + Vt + D + R\right)}\right]_{|P=\infty_+} ,$$

where $k = \lambda^{-1}$ is a local parameter at $P = \infty_+$. The derivative $\partial/\partial k$ may be replaced by $\partial/\partial t$ due to the usual reciprocity law for $d\Omega$ and ω.

The integrand in (3.20) turns out to be an exact derivative and thus we get

$$q_1(t) = \alpha\,\frac{\theta[\epsilon]\,(Vt+D+R)}{\theta(Vt+D+R)},\quad q_2(t) = \beta\,\frac{\theta(Vt+D+R)}{\theta[\epsilon]\,(Vt+D+R)}, \qquad (3.25)$$

where the constants of integration α, β are still to be determined.

3.6 *The Prym Condition*

It is now time to take into account the second symmetry condition (3.9), which is best done in the resultant formulae for the solutions. Substituting (3.24, 25) into (3.18), we have the coefficients of L_0:

$$(L_0)_{13} = -2S_{13} = -2\alpha\,\frac{\theta(Vt+D)\ \theta[\epsilon]\,(D+R)}{\theta(Vt+D+R)\,\theta(D)}$$

$$(L_0)_{42} = -2iS_{31}q_2^{-1} = \frac{2i}{\beta}\,a^2\,\frac{\theta(Vt+D+2R)\,\theta(D)}{\theta(Vt+D+R)\,\theta[\epsilon]\,(D+R)}$$

$$(L_0)_{24} = -2iS_{23} = -2i\beta\,\frac{\theta[\epsilon](Vt+D)\ \theta[\epsilon]\,(D+R)}{\theta[\epsilon](Vt+D+R)\,\theta(D)}$$

$$(L_0)_{31} = 2iS_{41}q_1^{-1} = -\frac{2a^2}{\alpha}\,\frac{\theta[\epsilon](Vt+D+2R)\,\theta(D)}{\theta[\epsilon](Vt+D+R)\,\theta[\epsilon]\,(D+R)}\ .$$

The relation (3.9) implies $(L_0)_{13} = -(L_0)_{42}$, $(L_0)_{24} = -(L_0)_{31}$, which gives

$$\alpha\beta = \mathrm{i}a^2 \frac{\theta(D)^2\,\theta(Vt+D+2R)}{\theta(D+R)^2\,\theta(Vt+D)} = \mathrm{i}a^2 \frac{\theta(D)^2\,\theta[\epsilon](Vt+D+2R)}{\theta(D+R)^2\,\theta[\epsilon](Vt+D)} \ .$$

For this equality to hold identically, the theta functions depending on t must cancel out:

$$\theta(Vt+D+2R) = c\ \theta(Vt+D)$$

with some constant c . Since $\overset{*}{\pi} V = -V$, $\overset{*}{\pi} R = R$ and, moreover,

$$\theta(-u) = \theta(u) = \theta(\overset{*}{\pi} u) \ ,$$

this implies

$$D = P-R\ , \quad \overset{*}{\pi} P = -P \ .$$

These relations give

$$\alpha = \Delta a \frac{\theta(P-R)}{\theta[\epsilon](P)}\, , \quad \beta = \frac{\mathrm{i}a}{\Delta} \frac{\theta(P-R)}{\theta[\epsilon](P)} \ ,$$

$$M_+ = \frac{2\mathrm{i}a}{\Delta} \frac{\theta[\epsilon](Vt+P-R)}{\theta[\epsilon](Vt+P)}\, , \quad M_- = 2\mathrm{i}\Delta a \frac{\theta(Vt+P-R)}{\theta(Vt+P)} \ ,$$

where the constant Δ is still to be determined.

3.7 *The Poisson Vector*

Still to be calculated is the Poisson vector $p(t)$, using (3.22); recall that $p_1^2 + p_2^2 + p_3^2 = 1$. In view of (3.21), we can write $\Psi(0)$ as

$$\Psi(0) = \mathscr{L}\begin{bmatrix} \mathscr{A} & \sigma_3\mathscr{A}\sigma_1 \\ \mathscr{B} & \sigma_3\mathscr{B}\sigma_1 \end{bmatrix}\mathscr{R} ,$$

where

$$\mathscr{L} = \operatorname{diag}\left[\frac{\Delta}{\theta(Vt+P)}, \frac{i}{\Delta\theta[\epsilon](Vt+P)}, -\frac{1}{\theta[\epsilon](Vt+P)}, \frac{i}{\theta(Vt+P)}\right] ,$$

$$\mathscr{R} = a\,\theta(P-R) \times$$

$$\operatorname{diag}\left[\frac{e^{\Omega(0_-)t}}{\theta(\int^{0_-}\omega+P-R)}, \frac{e^{\Omega(0_+)t}}{\theta(\int^{0_+}\omega+P-R}, \frac{e^{\Omega(0_+)t}}{\theta(\int^{0_+}\omega+P-R)}, \frac{e^{\Omega(0_-)t}}{\theta(\int^{0_-}\omega+P-R)}\right] ,$$

$$\mathscr{A} = \begin{bmatrix} \theta(\int^{0_-}\omega+Vt+P-R) & \theta(\int^{0_+}\omega+Vt+P-R) \\ \theta[\epsilon](\int^{0_-}\omega+Vt+P-R) & -\theta[\epsilon](\int^{0_+}\omega+Vt+P-R) \end{bmatrix} ,$$

$$\mathscr{B} = \begin{bmatrix} \theta[\epsilon](\int^{0_-}\omega+Vt+P) & -\theta[\epsilon](\int^{0_+}\omega+Vt+P) \\ \theta(\int^{0_-}\omega+Vt+P) & \theta(\int^{0_+}\omega+Vt+P) \end{bmatrix} . \tag{3.26}$$

To verify these formulae we recall that $\psi_j(0_\pm^{(1)}) = (-1)^j\psi_j(0_\pm^{(2)})$ and $e^{\Omega_3(0_\pm)} = \pm a$. Also, it can easily be shown that

$$\int^{0_\pm}\omega = \tfrac{1}{2}R \pm C , \quad \pi^* C = -C . \tag{3.27}$$

This and the relation $\theta(\mathrm{u}) = \theta(-\overset{*}{\pi}\mathrm{u})$ imply

$$\mathscr{B} = \sigma_1 \mathscr{A} \ . \tag{3.28}$$

Therefore (3.22) can be written as

$$\mathrm{L}_{-1} = -\mathscr{L}\,\mathrm{W}\begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}\mathrm{W}^{-1}\,\mathscr{L}^{-1}$$

with

$$\mathrm{W} = \begin{bmatrix}\mathscr{A} & \sigma_3\mathscr{A}\,\sigma_1\\ \sigma_1\mathscr{A} & \sigma_3\sigma_1\mathscr{A}\,\sigma_1\end{bmatrix}, \quad \mathrm{W}^{-1} = \frac{1}{2}\begin{bmatrix}\mathscr{A}^{-1} & \mathscr{A}^{-1}\sigma_1\\ \sigma_1\mathscr{A}^{-1}\sigma_3 & -\sigma_1\mathscr{A}^{-1}\sigma_1\sigma_3\end{bmatrix}$$

(we have assumed that $\epsilon = 1$ in (3.22), [see (3.14)]). After simple calculations we find

$$\mathrm{L}_{-1} = -\mathscr{L}\begin{bmatrix}\mathrm{S}_1\sigma_1+\mathrm{S}_2\sigma_2 & \mathrm{S}_3\sigma_3\sigma_1\\ \mathrm{S}_3\sigma_1\sigma_3 & \mathrm{S}_1\sigma_1-\mathrm{S}_2\sigma_2\end{bmatrix}\mathscr{L}^{-1} ,$$

where the S_j are defined by

$$\sum \mathrm{S}_j\sigma_j = \mathscr{A}\sigma_3\mathscr{A}^{-1} \ .$$

By equating the matrix coefficients $(\mathrm{L}_{-1})_{32} = (\mathrm{L}_{-1})_{14}$, we finally get $\Delta^2 = 1$ and

$$\mathrm{M}_+ = \frac{\theta(\mathrm{Vt}+\mathrm{P})}{\theta[\epsilon](\mathrm{Vt}+\mathrm{P})}(\mathrm{S}_1+i\mathrm{S}_2), \quad \mathrm{M}_- = \frac{\theta[\epsilon](\mathrm{Vt}+\mathrm{P})}{\theta(\mathrm{Vt}+\mathrm{P})}(\mathrm{S}_1-i\mathrm{S}_2) \ ,$$

$$M_3 = -\Delta\, S_3 \ .$$

We can now sum up our calculations.

Theorem 4. *The general solution of the equations of motion for the Kowalewski top is given by*

$$\begin{aligned}
M_+ &= 2ia\, \frac{\theta[\epsilon](Vt+P-R)}{\theta[\epsilon](Vt+P)} \\
M_- &= 2ia\, \frac{\theta(Vt+P-R)}{\theta(Vt+P)} \\
M_3 &= -i\, \frac{\partial}{\partial t} \log \frac{\theta[\epsilon]\,(Vt+P)}{\theta(Vt+P)} \\
p_+ &= 2\, \frac{\theta(Vt+P)}{\theta[\epsilon]\,(Vt+P)}\ \frac{AB}{AD+BC} \\
p_- &= 2\, \frac{\theta[\epsilon]\,(Vt+P)}{\theta(Vt+P)}\ \frac{CD}{AD+BC} \\
p_3 &= \frac{BC-AD}{AD+BC}\ ,
\end{aligned} \tag{3.29}$$

where

$$\epsilon = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad a = e^{\Omega_3(0_+)} \ ,$$

$$A = \theta\Big(\int_{\infty_+}^{0_-} \omega + Vt + P\Big)\,, \quad B = \theta\Big(\int_{\infty_+}^{0_+} \omega + Vt + P\Big) \ ,$$

$$C = \theta[\epsilon]\left(\int_{\infty_+}^{0_-}\omega+Vt+P\right), \quad D = \theta[\epsilon]\left(\int_{\infty_+}^{0_+}\omega+Vt+P\right) .$$

3.8 *The Geometry of the Liouville Tori*

The remaining indeterminacy in (3.29) for the dynamical variables (the change of sign $\Delta \longrightarrow -\Delta$ or the permutation $0_+ \longleftrightarrow 0_-$) reflects the freedom in reconstructing the Lax matrix (3.7) from the algebraic data. It is easily verified that this freedom amounts to conjugation $L \longrightarrow ULU^{-1}$ by a matrix U of the form

$$U = \mathrm{diag}(1,1,-1,-1) .$$

This is equivalent to a renormalization of the Baker–Akhiezer function $\psi \longrightarrow U\psi$ and induces a symmetry of the Kowalewski top:

$$M \longrightarrow BM, \quad p \longrightarrow -Bp, \quad B = \mathrm{diag}(-1,-1,1) . \tag{3.30}$$

Clearly, (3.30) leaves the Hamiltonian invariant but changes the sign of $f_2 = (pM)$. Recall that only the square $(pM)^2$ is a spectral invariant.

We may summarize the situation as follows.

Theorem 5. *If* $(pM) \neq 0$, *the common level surface of the spectral invariants* H, K, $(pM)^2$ *consists of two components (Liouville tori) each of which is an affine part of an Abelian variety isomorphic to* $\mathrm{Prym}_\pi X$. *These components are permuted by the transformation* (3.30).

If $(pM) = 0$, the curve E degenerates into a rational curve which is a two–sheeted cover of the z–plane. The curve X is given by the equation

$$(\mu^2 - d_1(z))^2 = 4(z^2 - 2Hz + 1 + H^2 - \tfrac{K}{4})$$

and has genus 2. In the variables $u = \sqrt{2}\,\mu zx$, $x = \frac{1}{2}(\mu^2 - \frac{1}{z})$ it takes on the usual hyperelliptic form

$$u^2 = x(x^2 + 2Hx + \tfrac{K}{4})(x^2 + 2Hx + \tfrac{K}{4} - 1)\ .$$

Notice that it is different from the Kowalewski curve (3.5) with $(pM) = 0$. The fact that there are various hyperelliptic curves associated with the Kowalewski top which are different from the classical Kowalewski curve was pointed out in [22]. The motion of the top linearizes on the Jacobians of these curves which are isogeneous to one another.

For $(pM) = 0$ the mapping of the Liouville torus to Jac X becomes an unramified two–sheeted covering. The corresponding theta functinal formulae are presented in [2].

3.9 *Reduction of Two–Dimensional Theta Functions*

Since the Kowalewski flow on Jac X is parallel to the Prym variety of the covering $X \longrightarrow E$, it s desirable to express the dynamics entirely in terms of the theta functions related to this Prym variety. The Prymian has polarization (2,1) and its period matrix is [51]

$$\Pi = \begin{bmatrix} 2\int_{b_1}(\omega_1+\omega_3) & \int_{b_2}(\omega_1+\omega_3) \\ 2\int_{b_1}\omega_2 & \int_{b_2}\omega_2 \end{bmatrix}\ .$$

Let $\frac{1}{2}B_0$ be the periof of E

$$B_0 = \int_{b_1} (\omega_1 - \omega_3) \ .$$

We write the Prym vectors V and P entering in (3.29), and C defined by (3.27) as

$$V = (\tfrac{v_1}{2}, v_2, \tfrac{v_1}{2}), \ \ P = (\tfrac{p_1}{2}, p_2, \tfrac{p_1}{2}), \ \ C = (\tfrac{c_1}{2}, c_2, \tfrac{c_1}{2})$$

and denote

$$w = (v_1t+p_1, v_2t+p_2), \quad c = (c_1, c_2) \ .$$

Then we have the following expressions for the theta functions occurring in (3.29)

$$\theta(Vt+P) = \theta\begin{bmatrix}0&0\\0&0\end{bmatrix}(w\,|\,\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(0\,|\,B_0) + \theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(w\,|\,\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(0\,|\,B_0) \ ,$$

$$\theta(Vt+P-R) = \theta\begin{bmatrix}0&0\\0&0\end{bmatrix}(w\,|\,\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(2r\,|\,B_0) + \theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(w\,|\,\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(2r\,|\,B_0) \ ,$$

$$\theta(Vt+P+\int_{\infty_+}^{0_\pm}\omega) = \theta\begin{bmatrix}0&0\\0&0\end{bmatrix}(w\pm c\,|\,\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(r\,|\,B_0) + \theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(w\pm c\,|\,\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(r\,|\,B_0) \ ,$$

$$\theta[\epsilon](Vt+P) = \theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(w\,|\,\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(0\,|\,B_0) + \theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(w\,|\,\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(0\,|\,B_0) \ ,$$

$$\theta[\epsilon](Vt+P-R) = \theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(w\,|\,\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(2r\,|\,B_0) + \theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(w\,|\,\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(2r\,|\,B_0) \ ,$$

$$\theta[\epsilon](Vt+P+\int_{\infty_-}^{0_\pm}\omega) = \theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(w\pm c|\Pi)\theta\begin{bmatrix}0\\0\end{bmatrix}(r|B_0)+\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(w\pm c|\Pi)\theta\begin{bmatrix}1\\0\end{bmatrix}(r|B_0) \ .$$

Remark. Adding to W a period of the form

$$\left[\begin{matrix}2&0\\0&1\end{matrix}\middle|\Pi\right]\begin{bmatrix}M\\N\end{bmatrix}, \quad M,N \in \mathbb{Z}^2$$

does not change the solutions (3.29). This shows that the mapping of the Liouville torus to $\mathrm{Prym}_\pi X$ is one–to–one, as was already mentioned in Theorem 5.

4. The Goryachev–Chaplygin Top

Now the system under consideration is a special case of the motion of a heavy rigid body with a fixed point, discovered by Goryachev and Chaplygin in 1900 [26]. It represents a symmetric top with the principal moments of inertia satisfying $I_1^{-1} : I_2^{-1} : I_3^{-1} = 1 : 1 : 1/4$ and the centre of mass located in the equatorial plane. The Hamiltonian of the Goryachev–Chaplygin top (GCT) is given by

$$H = \tfrac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - 2p_1 \ , \tag{4.1}$$

where M is the angular momentum and p is the field strength vector in the moving frame (Sect. 2).

The system (4.1) admits an extra integral of motion provided that the Casimir function f_2 (2.1) for the Poisson brackets (2.3) vanishes

$$M_1p_1 + M_2p_2 + M_3p_3 = 0 \ . \tag{4.2}$$

A more general system described by

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2 + 4\gamma M_3) - 2p_1$$

is called the Goryachev–Chaplygin gyrostat (GCG). It is also integrable if $(Mp) = 0$ [27]. We mention also two papers where GCT is studied in a different way. In [30] the R–matrix technique is used to solve both the classical and the quantum problems. In [31] the geometry of the complexified Liouville tori for GCT is thoroughly studied using the general technique developed in [21]. In particular,a close connection is established in [31] between GCT and the periodic Toda lattice with three particles.

Here we follow the paper [3]. Let us note that (compared with [3]) similar but slightly more complicated formulae for these solutions were obtained in [45].

4.1 *The Lax Pair for the Goryachev–Chaplygin Top*

There is an interesting connection between GCT and the Kowalewski top (KT) on the Lax representation level.

An important observation of [3] is that, by removing the first column and the first row of the Lax matrix (3.7) we get a Lax matrix for GCG. Clearly, we get

$$L = i\begin{bmatrix} \frac{2}{3}\gamma & \frac{P_3}{\lambda} & -M_+ \\ -\frac{P_3}{\lambda} & -2M_3-\frac{4}{3}\gamma & -2\lambda-\frac{P_+}{\lambda} \\ -M_- & \frac{P_-}{\lambda}+2\lambda & 2M_3+\frac{2}{3}\gamma \end{bmatrix}. \qquad (4.3)$$

Put

$$A = i \begin{bmatrix} -3M_3 - \frac{2}{3}\gamma & 0 & -M_+ \\ 0 & -2M_3 - \frac{2}{3}\gamma & -2\lambda \\ -M_- & 2\lambda & 2M_3 + \frac{4}{3}\gamma \end{bmatrix} . \tag{4.4}$$

The the Lax equation is equivalent to the Hamiltonian equation with the Hamiltonian (4.1), provided that the constraint (4.2) is satisfied.

For future use we introduce the notation

$$L = L_{-1}\lambda^{-1} + L_0 + L_1\lambda$$

for the coefficients of the Lax matrix (4.3).

The Lax representation with a spectral parameter for GCT and GCG permits us to apply the powerful machinery of algebraic geometry to solve the equations of motion. In the following we shall consider only the first case, i.e., put $\gamma = 0$. Formulae for the general case may be easily obtained in quite the same way.

4.2 *The Spectral Curve*

Let $\hat{X}$ denote the spectral curve given by the equation $\det(L(\lambda) - \mu I) = 0$. The symmetry relation

$$L(-\lambda) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} L(\lambda) \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \tag{4.5}$$

gives rise to an involution on $\hat{X}$

$$\tau : (\lambda,\mu) \longrightarrow (-\lambda,\mu) \ .$$

It is natural to conisder the quotient curve $X = \hat{X}/\tau$ given by

$$\mu^3 + \mu(2H - 4z - \tfrac{1}{z}) - 2iG = 0 \qquad z = \lambda^2 \ ,$$

where $H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - 2p_1$ is the Hamiltonian and $G = M_3(M_1^2 + M_2^2) + 2M_1p_3$ is the Goryachev–Chaplygin integral. It is equivalent to the Chaplygin curve [26]

$$y^2 = (\mu^3 + 2H\mu - 2iG)^2 - 16\mu^2 \ , \quad y = 8z\mu - \mu^3 - 2H\mu + 2iG \ .$$

Note that we always assume $(Mp) = 0$, $p^2 = 1$.

The spectral curve $\hat{X}$ is a three–sheeted covering of the λ–plane Λ and also is a double cover of $X = \hat{X}/\tau$

$$\begin{array}{cccccc} (\lambda,\mu) & \hat{X} & \xrightarrow{\quad 2:1 \quad} & X = \hat{X}/\tau & (\mu,z) \\ & {\scriptstyle 3:1}\big\downarrow & & \big\downarrow{\scriptstyle 3:1} & \\ (\lambda) & \Lambda & \xrightarrow{\quad 2:1 \quad} & \mathbb{C} & (z) \end{array}$$

We denote the point of $\hat{X}$ with $\lambda = 0$ and $\lambda = \infty$ in the following way:

$\lambda = 0$	0^{I}	$\mu = 0$	$\lambda = \infty$	∞^{I}	$\mu = 0$
	0^{II}	$\mu \sim -\lambda^{-1}$		∞^{II}	$\mu \sim -2\lambda$
	0^{III}	$\mu \sim \lambda^{-1}$		∞^{III}	$\mu \sim 2\lambda$.

X is a three–sheeted covering of the z–plane. We denote the points with $z = 0$ and $z = \infty$ by $0_1, 0_2$, and ∞_1, ∞_2, in such a way that $0_2, \infty_2$ are the branch points of the covering $X \longrightarrow \mathbb{C} \ni z$. This covering is unramified at $0_1, \infty_1$ and $\mu(0_1) = \mu(\infty_1) = 0$.

The function $\lambda = \sqrt{z}$ is double–valued on X and changes sign when analytically continued along a closed path which intersects a certain contour $\mathscr{L}$. Here $\mathscr{L}$ is a contour connecting the points 0_1 and ∞_1 and determined by the covering $\hat{X} \longrightarrow X$. Glueing two copies of X along $\mathscr{L}$, we obtain $\hat{X}$. The condition

$$\mu \sim -2\lambda \qquad \infty_2 , \qquad\qquad \mu \sim -\lambda^{-1} \qquad 0_2$$

uniquely fixes $\mathscr{L}$ and the branch of λ.

4.3 *Analyticity Properties of the Baker–Akhiezer Function*

Our main goal is to construct explicitly the Baker–Akhiezer function $\psi(P) = (\psi_1, \psi_2, \psi_3)^T$ which is analytic on $\hat{X}$ and satisfies

$$L\psi = \mu\psi , \quad \psi_t = A\psi . \tag{4.6}$$

We may assume that ψ satisfies the symmetry relation (4.5)

$$\psi(\tau P) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \psi(P) , \qquad P \in \hat{X} . \tag{4.7}$$

Hence, the component ψ_2 may be regarded as a single–valued function on X, while ψ_1 and ψ_3 are double–valued on X and change sign when analytically coninued along a closed path intersecting $\mathscr{L}$. We may assume that ψ_i are defined on $X \setminus \mathscr{L}$ and satisfy the symmetry relation

$$\psi_i^+(P) = (-1)^i \psi_i^-(P) \qquad (4.8)$$

for P belonging to the cut $\mathscr{L}$. In other words, ψ_1, ψ_3 acquire a factor $(-1)^{<\gamma, \mathscr{L}>}$ upon a circuit of γ. Here $<\gamma, \mathscr{L}>$ is the intersection number.

Let us define a matrix–valued function

$$\Psi(\lambda) = (\psi(P^I), \psi(P^{II}), \psi(P^{III})) \ ,$$

where P^I, P^{II}, P^{III} are the inverse images of λ with respect to the mapping $\hat{X} \longrightarrow \Lambda$. We mark them so that

$$P^{I,II,III} \longrightarrow \infty^{I,II,III} \qquad \lambda \longrightarrow \infty \ ,$$

$$P^{I,II,III} \longrightarrow 0^{I,II,III} \qquad \lambda \longrightarrow 0$$

hold. The function $\Psi(\lambda)$ is defined on the domain $U = U_0 \cup U_\infty$ which is a union of two simply connected domains with the points $\lambda = 0$ and $\lambda = \infty$ respectively. These domains also do not contain the branch points of the covering $\hat{X} \longrightarrow \Lambda$ and are invariant with respect to the involution $\lambda \longrightarrow -\lambda$.

The reduction (4.7) can be rewritten in terms of Ψ:

$$\Psi(-\lambda) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \Psi(\lambda) \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} . \qquad (4.9)$$

To understand this, it is necessary to note that $\tau 0^{\mathrm{II}} = 0^{\mathrm{III}}$, $\tau\infty^{\mathrm{II}} = \infty^{\mathrm{III}}$, $\tau 0^{\mathrm{I}} = 0^{\mathrm{I}}$, $\tau\infty^{\mathrm{I}} = \infty^{\mathrm{I}}$.

According to (4.6,8) it is natural to determine the asymptotics of $\Psi(\lambda)$ at $\lambda \longrightarrow \infty$ and $\lambda \longrightarrow 0$ as

$$\underset{\lambda \longrightarrow \infty}{\Psi} = (\Phi + S\lambda^{-1} + \ldots)\begin{bmatrix} 1 & & \\ & e^{-2\lambda t} & \\ & & e^{2\lambda t} \end{bmatrix}\begin{bmatrix} \lambda & & \\ & 1 & \\ & & 1 \end{bmatrix},$$

$$\underset{\lambda \longrightarrow 0}{\Psi} = T\begin{bmatrix} \lambda^{-1} & & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

Then the reduction (4.8) gives

$$\begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} S \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} = S \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} T \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = T \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}.$$

The coefficients of $L(\lambda) = L_{-1}\lambda^{-1} + L_0 + L_1\lambda$ are related to the matrices Φ, S, T in these expansions by

$$L_{-1} = -T\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix} T^{-1}, \quad L_0 = -2\left[S\Phi^{-1}, \begin{bmatrix} 0 & & \\ & 0 & i \\ & -i & 0 \end{bmatrix}\right], \tag{4.10}$$

$$L_1 = 2\Phi \begin{bmatrix} 0 & & \\ & -1 & \\ & & 1 \end{bmatrix} \Phi^{-1} ,$$

which gives

$$\Phi = \begin{bmatrix} q & & \\ & 1 & 1 \\ & -i & i \end{bmatrix} , \quad \frac{q_t}{q} = -3iM_3 .$$

In the usual way all these analytical properties can be reformulated for the vector function ψ on X .

With a suitable normlaisation, the Baker–Akhiezer function has the following properties which characterize it completely.

(1.) *ψ is analytic on $X \backslash \mathscr{L}$, satisfies the symmetry relations* (4.8) *on $\mathscr{L}$ and is meromorphic on $X \backslash \infty_2$.*

(2.) *In the neighbourhood of the points $0_1, \infty_1, \infty_2$, ψ has the following asymptotic behaviour:*

$$\psi \sim \begin{bmatrix} 0(\lambda^{-1}) \\ 0(1) \\ 0(\lambda^{-1}) \end{bmatrix} \quad \textit{for} \quad P \longrightarrow 0_1$$

$$\psi \sim \begin{bmatrix} q\lambda + 0(\lambda^{-1}) \\ 0(1) \\ 0(\lambda^{-1}) \end{bmatrix} \quad \textit{for} \quad P \longrightarrow \infty_1 ,$$

$$\psi \sim \left[\begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} + 0(\lambda^{-1})\right] \exp(-2\lambda t) \quad \textit{for} \quad P \longrightarrow \infty_2 \ .$$

(3.) *The divisor of poles of* ψ $\mathscr{D} = P_1 + P_2$ *has degree* 2 *and does not depend on* t .

(4.) *The normalization constant* q *above satisfies the differential equation* $q_t/q = -3iM_3$; *hence,*

$$q = \alpha \exp(-3i \int^t M_3 \, dt) \ .$$

The Baker–Akhiezer function ψ with these poperties satisfies (4.6), where L and A are almost the Lax matrices for GCT, with only the condition

$$(L_{-1})_{12} = -(L_{-1})_{21} \tag{4.11}$$

not being automatically fulfilled. (This last condition will be imposed in the last stage of the computation. As we shall see, it amounts to a suitable choice of the integration constant α .)

It is useful to present the expressions (4.10) in more detail

$$M_3 = -S_{32} - iS_{22}, \ M_+ = 2S_{12}, \ M_- = 2S_{21}/q \ ,$$

$$(L_{-1})_{12} = -\frac{T_{12}}{T_{22}}, \ (L_{-1})_{21} = \frac{T_{22}T_{31}}{T_{11}T_{32} - T_{31}T_{12}}, \tag{4.12}$$

$$p_+ = i\,\frac{T_{22}T_{11}}{T_{31}T_{12} - T_{11}T_{32}}, \ p_- = i\,\frac{T_{32}}{T_{22}} \ .$$

4.4 *Construction of the Baker–Akhiezer Function*

To write down explicit formulae for the function ψ_1, ψ_2 and ψ_3, we must define a number of standard objects on X . Let $\Omega_3(P)$, $\Delta(P)$ and $\Omega(P)$ be the normalized Abelian integrals of the third and the second kind, respectively, which are uniquely specified by their behaviour in the neighbourhoods of the points ∞_1, ∞_2, 0_2

$$\begin{array}{llll} P & e^{\Omega_3(P)} & e^{\Delta(P)} & \Omega(P) \\ \infty_1 & a & \lambda^2+0(1) & f \\ \infty_2 & \lambda+b+0(\lambda^{-1}) & d\lambda^{-1}+0(\lambda^{-2}) & -2\lambda+0(\lambda^{-1}) \\ 0_2 & c\lambda+0(\lambda^{-2}) & e & 0(1)\ . \end{array} \tag{4.13}$$

Let us denote

$$R = \int_b d\Omega_3\ ,\ \ \Delta = \int_b d\Delta\ ,\ \ V = \int_b d\Omega\ . \tag{4.14}$$

There are some useful relations between the different constants in (4.13) and (4.14). Comparing the singularities, we get

$$\lambda^2/\mu = e^{3\Omega_3(P)} e^{2\Delta(P)}\ , \tag{4.15}$$

which implies $3R + 2\Delta \equiv 0$ modulo the periods. Let us choose the paths $[\infty_2, 0_2]$, $[\infty_1, \infty_2]$ such that an exact equality holds; i.e.,

$$3R + 2\Delta = 0\ . \tag{4.16}$$

Also, using (4.15) we get

$$e^{\Delta(P)} = \frac{d}{\lambda}\left(1 - \frac{3b}{\lambda} + \ldots\right), \quad P \longrightarrow \infty_2 \ ,$$

where b is the constant term in the expansion of e^{Ω_3} at ∞_2 (4.13). Using the general properties of Abelian integrals we obtain

$$f = 3b \quad , \quad d = ae \ . \tag{4.17}$$

Choose $D \in \mathrm{Jac}\, X$ such that the divisor of zeros of $\theta(\int^P \omega + D)$ on X is precisely $\mathscr{D}$, the divisor introduced in the definition of ψ above. We are now in a position to write down the explicit formulae for ψ.

Theorem 6. *The function* ψ *is given by the following formulae:*

$$\psi_1 = \frac{q}{a\lambda} \frac{\theta(\int^P \omega + Vt + D - \frac{1}{2}R)\ \theta(D + \frac{3}{2}R)}{\theta(\int^P \omega + D)\ \ \theta(Vt + D + R)} \exp(\Omega(P)t + \Omega_3(P) + \Delta(P) - ft)$$

$$\psi_2 = \frac{\theta(\int^P \omega + Vt + D)\ \ \theta(D)}{\theta(\int^P \omega + D)\ \ \theta(Vt + D)} \exp(\Omega(P)t) \tag{4.18}$$

$$\psi_3 = -\frac{i}{\lambda} \frac{\theta(\int^P \omega + Vt + D + R)\ \ \theta(D)}{\theta(\int^P \omega + D)\ \ \theta(Vt + D + R)} \exp(\Omega(P)t + \Omega_3(P)) \ .$$

The expression for the function ψ_1 can also be written in a different form,

$$\psi_1(P) = q(\varphi(P) - i\varphi(\infty_2)\psi_3(P)) \ ,$$

$$\varphi(P) = b\lambda \frac{\theta(\int^P \omega+Vt+D-R) \quad \theta(D+\frac{3}{2}R)}{\theta(\int^P \omega+D) \quad \theta(Vt+D+\frac{1}{2}R)} \exp(\Omega(P)t-\Omega_3(P)-ft) \ . \tag{4.19}$$

4.5 *Formulae for Dynamical Variables*

Substituting asymptotics of ψ_2, ψ_3 at ∞_2 into (4.22), we obtain

$$iM_3 = -\frac{1}{2}\frac{\partial}{\partial t}\log\frac{\theta(Vt+D+R)}{\theta(Vt+D)} - b$$

$$q = \alpha \exp(3bt) \left[\frac{\theta(Vt+D+R)}{\theta(Vt+D)}\right]^{3/2} ,$$

where we have used the form $\int^P \omega = \frac{1}{2\lambda} V + \ldots$ of the Abel transform near ∞_2.

Now we must satisfy the last condition of (4.11). To obtain the expressions for $(L_{-1})_{12}$ and $(L_{-1})_{21}$, we use (4.18,19), respectively. We get

$$(L_{-1})_{12} = -\frac{\psi_1(0_2)}{\psi_2(0_2)} = -\frac{ce}{a} qe^{-ft} \frac{\theta(Vt+D+\frac{1}{2}R)\ \theta(Vt+D)\theta(D+\frac{3}{2}R)}{\theta^2(Vt+D+R)\ \theta(D)}$$

$$(L_{-1})_{21} = -\frac{\psi_2(0_2)}{\varphi(0_2)q} = -\frac{c}{a}\frac{1}{q} e^{ft} \frac{\theta(Vt+D+R)\ \theta(Vt+D+\frac{1}{2}R)\theta(D)}{\theta^2(Vt+D)\theta(D+\frac{3}{2}R)}$$

This implies that

$$a^2 = -\frac{1}{e}\frac{\theta^2(D)}{\theta^2(D+\frac{3}{2}R)} .$$

To compute $p_\pm$ we also use both (4.18,19) for ψ,

$$p_- = i\,\frac{\psi_3(0_2)}{\psi_2(0_2)}, \quad p_+ = \varphi(\infty_2)\,\frac{\psi_2(0_2)}{\varphi(0_2)} .$$

Finally, taking into account (4.16, 17), we obtain the following theorem:

Theorem 7. *The general solution of the GCT is given by the following formulae:*

$$M_+ = 2i\sqrt{e}\,\frac{\theta(Vt+D-\frac{1}{2}R)}{\theta(Vt+D)}\left[\frac{\theta(Vt+D+R)}{\theta(Vt+D)}\right]^{1/2}$$

$$M_- = -2i\sqrt{e}\,\frac{\theta(Vt+D+\frac{3}{2}R)}{\theta(Vt+D+R)}\left[\frac{\theta(Vt+D)}{\theta(Vt+D+R)}\right]^{1/2}$$

$$M_3 = \frac{i}{2}\frac{\partial}{\partial t}\log\frac{\theta(Vt+D+R)}{\theta(Vt+D)} + bi$$

$$p_+ = c\,\frac{\theta(Vt+D-R)\quad\theta(Vt+D+R)}{\theta^2(Vt+D)} \tag{4.20}$$

$$p_- = c\,\frac{\theta(Vt+D+2R)\quad\theta(Vt+D)}{\theta^2(Vt+D+R)}$$

$$p_3 = -\sqrt{e}\,\frac{c}{a}\,\frac{\theta(Vt+D+\frac{1}{2}R)}{[\theta(Vt+D)\,\theta(Vt+D+R)]^{1/2}} .$$

The square roots in (4.20) are quite unusual. Their presence is predicted by Painlevé analysis of the equations of motion, which shows that the leading powers of singularities in t are half integers [31]. The sign change of the square root in (4.20) leads to the transformation $M_1 \longrightarrow -M_1$, $M_2 \longrightarrow -M_2$, $p_3 \longrightarrow -p_3$ preserving the equations of motion.

The paths $[\infty_2,\infty_1]$, $[\infty_2,0_2]$ are already fixed (4.16). Constants a, c and e are defined by the integrals upon these very paths.

5. Integration of the Lax Representations with the Spectral Parameter on Elliptic Curve. XYZ Landau–Lifshitz Equation

All other tops considered below possess the Lax representations with the spectral parameter varying on elliptic curve. In this section we describe the integration process in this case, using the papers [4, 32]. The Lax representations of all examples considered below are of the matrix dimension 2×2. The general theory for an arbitrary matrix case is constructed in [33].

We use the uniformization of the spectral parameter suggested in [34]

$$w_1(u) = \frac{1}{\mathrm{sn}(u,k)},\ w_2(u) = \frac{\mathrm{dn}(u,k)}{\mathrm{sn}(u,k)},\ w_3(u) = \frac{\mathrm{cn}(u,k)}{\mathrm{sn}(u,k)}\ ,$$

$$w_\alpha^2 - w_\beta^2 = J_\beta - J_\alpha\ ,\ (J_1,J_2,J_3) = (0,k^2,1)\ .$$

Here sn, cn, dn are the Jacobi elliptic functions of the module k. The variable u varies on the torus $\hat{E}$ which is a parallelogram with the lattice $4K$, $4iK'$ (here K is the complete elliptic integral of the module k). Let us denote E the "quarter" of $\hat{E}$ – the torus with the lattice $2K$, $2iK'$.

The general form of the Lax representation with the spectral parameter on elliptic curve in the case of 2×2 matrix dimension is as follows:

$$L(u) = \sum_{\alpha=1}^{3} \sum_{s=1}^{N} \sum_{k=1}^{N_s} L_\alpha^{s,k}(t) f_\alpha^k(u-u_s)\sigma_\alpha \tag{5.1}$$

$$A(u) = \sum_{\alpha=1}^{3} \sum_{s=1}^{N} \sum_{k=1}^{N_s} A_\alpha^{s,k}(t) f_\alpha^k(u-u_s)\sigma_\alpha$$

$$f_\alpha^K(u) = \begin{cases} \dfrac{w_1 w_2 w_3}{w_\alpha} \left[\dfrac{w_1^2+w_2^2+w_3^2}{3} \right]^n (u) & k = 2n+2 \\[2ex] w_\alpha \left[\dfrac{w_1^2+w_2^2+w_3^2}{3} \right]^n (u) & k = 2n+1 \end{cases} .$$

The functions $f_\alpha^k(u)$ have a pole of the k–th order at the point $u=0$ and satisfy the important reduction

$$\begin{aligned} f_\alpha^k(u+2K)\sigma_\alpha &= \sigma_3 f_\alpha^k(u)\sigma_\alpha\sigma_3 , \\ f_\alpha^k(u+2iK')\sigma_\alpha &= \sigma_1 f_\alpha^k(u)\sigma_\alpha\sigma_1 . \end{aligned} \tag{5.2}$$

The functions $f_\alpha^k(u-u_s)$ generalize the function $1/(\lambda-\lambda_s)$ to the elliptic case.

The matrices $L(u)$, $A(u)$ (5.1) obey the symmetry relations (5.2). This implies that the spectral curve X

$$\mu^2 = \det L(u) \tag{5.3}$$

is a two–sheeted covering of the torus E . Let us choose the canonical basis of cycles in the natural way. It is shown in Fig. 2. The projections of the cycles a_1, b_1 on E form the canonical basis of cycles of E corresponding to the shifts on $2iK'$ and $2K$ respectively.

Let us define also the necessary Prym differentials (for details see [51], [32]). Denote

$$\nu_1 = \tfrac{1}{2}(\omega_1+\omega_{n+1}), \quad \nu_1 = \omega_i \qquad i = 2,\dots,n$$

the differentials odd with respect to the involution $\pi : (\mu,u) \longrightarrow (-\mu,u)$. These differentaisl differ from the canonical Prym differentials by the normalization. Their period matrix

$$\Pi_{ij} = \int_{b_j} d\nu_i \qquad\qquad i,j = 1,\dots,n$$

is simply relate with the canonical period matrix Π of the Prym variety $\mathrm{Prym}_\pi X$

$$\Pi = \begin{bmatrix} 1/2 & \\ & I \end{bmatrix} \Pi \begin{bmatrix} 1/2 & \\ & I \end{bmatrix} 2 \; . \tag{5.4}$$

It is this matrix which defines the corresponding theta function we use below.

Let us denote u_i^+, $u_i^- = \pi u_i^+$ the poins of X with projections $X \longrightarrow E$ equal to u_i .

The normalized Prym integrals of the second kind $\Omega(P)$

$$\pi^* d\Omega = -d\Omega , \quad \int_{a_j} d\Omega = 0 \qquad\qquad j = 1,\dots,n$$

are determined by the asymptotics at the poles

$$\Omega(P) = \pm \sum_{k}^{N_s} a_{s,k}(u-u_s)^{-k} + 0(1)\ , \quad P \longrightarrow u_s^{\pm}\ .$$

Here $a_{s,k}$ are constants. We denote

$$V_j = \int_{b_j} d\Omega \qquad\qquad j = 1,\dots,n$$

the b–period vector.

Theorem 8. *The Baker–Akhiezer function corresponding to the Lax pair* (5.1) *is given by*

$$\psi_1 = \frac{\theta(\int_{P_0}^{P} \nu + Vt + D \mid \Pi)}{\theta(\int_{P_0}^{P} \nu + D \mid \Pi)} \exp(\int_{P_0}^{P} d\Omega t)$$

$$\psi_2 = \frac{\theta(\int_{P_0}^{P} \nu + Vt + D + \Delta \mid \Pi)}{\theta(\int_{P_0}^{P} \nu + D \mid \Pi)} \exp(\int_{P_0}^{P} d\Omega t) \tag{5.5}$$

$$\nu = (\nu_1,\dots,\nu_n)\ ,\ \ \Delta = \pi i(1,0,\dots,0) = \int_{a_1} \nu\ ,\ \ D \in \mathbb{C}^n\ .$$

The complete proof of this theorem is given in the paper [4]. Here we only remark that the Baker–Akhiezer function (5.5) satisfies the reduction corresponding to (5.2). Indeed, for the analytical continuation along the cycles a_1 and b_1 we have

$$M_{b_1}\psi(P) = \sigma_3\psi(P)\ ,\ \ M_{a_1}\psi(P) = \sigma_1\psi(P)m(P)\ ,$$

where $m(P)$ is the following function:

$$m(P) = \frac{\theta\left(\int_{P_0}^{P}\nu + D \mid \Pi\right)}{\theta\left(\int_{P_0}^{P}\nu + D + \Delta \mid \Pi\right)}\ .$$

In the neighbourhood of the point u_s the matrix function of $u \in E$

$$\Psi(u) = (\psi(u^+), \psi(u^-))$$

is well–defined. Here $u^{\pm}$ are the two preimages of the point u with respect to the projection $X \longrightarrow E$. They are uniquely determined by the conditions $u^{\pm} \longrightarrow u_s^{\pm}$ when $u \longrightarrow u_s$. Let us also consider the diagonal matrix

$$\hat{\mu} = \begin{bmatrix} \mu(u^+) & \\ & \mu(u^-) \end{bmatrix}\ .$$

As usual, the following equalities are valid :

$$L\Psi = \Psi\hat{\mu}\ ,\ \ \Psi_t = A\Psi\ . \tag{5.6}$$

Substitution of the asymptotics of Ψ and $\hat{\mu}$ near u_s

$$\Psi(u) = (\Phi_s + 0(u-u_s))\exp\Big(\sum_{k}^{N_s} a_{s,k}(u-u_s)^{-k} t\sigma_3\Big)$$

$$\hat{\mu} = \ell\sigma_3(u-u_s)^{-N_s} + \ldots \qquad\qquad u \longrightarrow u_s$$

in (5.6) determines $L_\alpha^{s,k}$ and $A_\alpha^{s,k}$. In particular, for the first coefficients we have

$$L_1^{s,N_s} = \ell\,\frac{C_sD_s - A_sB_s}{A_sD_s - B_sC_s}, \quad L_2^{s,N_s} = -i\ell\,\frac{C_sD_s + A_sB_s}{A_sD_s - B_sC_s},$$

$$L_3^{s,N_s} = \ell\,\frac{A_sD_s + B_sC_s}{A_sD_s - B_sC_s}, \quad \Phi_s = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}, \tag{5.7}$$

$$\ell^2 = \sum_\alpha (L_\alpha^{s,N_s})^2 .$$

$$\Phi_s\sigma_3\Phi_s^{-1} = \frac{1}{a_{s,N_s}}\sum_\alpha A_\alpha^{s,N_s}\sigma_\alpha = \frac{1}{\ell}\sum_\alpha L_\alpha^{s,N_s}\sigma_\alpha .$$

Calculating Φ_s and reducing A_s, B_s, C_s, D_s by common multipliers we see that L_α^{s,N_s} are given by the expressions (5.7), where

$$A_s = \theta(Vt+D+\epsilon_s \mid \Pi) \qquad\qquad B_s = \theta(Vt+D+\epsilon_s+r_s \mid \Pi)$$

$$C_s = \theta(Vt+D+\epsilon_s+\Delta \mid \Pi) \qquad\qquad D_s = \theta(Vt+D+\epsilon_s+r_s+\Delta \mid \Pi) ,$$

$$\epsilon_s = \int_{P_0}^{u_s^+} \nu\,, \qquad r_s = \int_{u_s^+}^{u_s^-} \nu\,.$$

Here the projection of the integration path in r_s on E should be homologically equivalent to zero. The elliptical integral $u = \int du$ calculated along the path of ϵ_s should be equal to

$$\int_{P_0}^{u_s^+} du = u_s - P_0 \pmod{4K, 4iK'}$$

modulo period lattice of the "big" torus E.

The solutions presented above may be considered as the finite–gap solutions of integrable non–linear equations with the Lax representations with elliptic spectral parameter. To construct such an equation, we should introduce a new variable x with respect to which the Baker–Akhiezer function satisfies the similar equation $\Psi_x = B\Psi$. Here $B(u)$ is the matrix elliptic function of the same structure as $A(u)$, i.e., the reductions (5.2) are valid for $B(u)$ also. The compatibility condition

$$B_t - A_x + [B,A] = 0$$

gives the non–linear integrable equation.

The additional condition that ψ is the eigenfunction of some matrix $L(u)$ means that ψ is an analytic function on X which is two–sheeted covering of E. All finite–gap solutions are obtained by the choice of all possible $L(u)$ or, equivalently, of all possible two–sheeted coverings of E.

The most important example of an equation of this kind is the completely anisotropic (XYZ) Landau–Lifshitz equation

$$S_t = [S, S_{xx}] + [S, IS], \quad S_1^2 + S_2^2 + S_3^2 = 1 \ .$$

Here the square brackets denote the vector product and IS – the vector with the coordinates (I_1S_1, I_2S_2, I_3S_3). This equation describes the non–linear waves in ferromagnetics. The zero curvature representation for it was found in 1979 by Sklyanin and Borovik [34]:

$$B(u) = -i\rho \sum_{\alpha=1} S_\alpha w_\alpha(u)\sigma_\alpha$$

$$A(u) = 2i\rho^2 \sum_\alpha \frac{w_1w_2w_3}{w_\alpha}(u) S_\alpha\sigma_\alpha - \sum_{\alpha\beta\gamma} w_\alpha(u)\,[S,S_x]_\alpha\,\sigma_\alpha \ , \qquad (5.8)$$

$$\rho = \tfrac{1}{2}\sqrt{I_3-I_1}\ , \quad k = \sqrt{\frac{I_2-I_1}{I_3-I_1}}\ , \quad I_1 < I_2 < I_3 \ .$$

The corresponding Ψ–function has the following singularity at $u = 0$:

$$\Psi(u) = (\Phi+0(u))\exp(-i\rho x\sigma_3 \tfrac{1}{u} + 2i\rho^2 t\sigma_3 \tfrac{1}{u^2}) \ .$$

Finally, we obtain the following:

Theorem 9 [4]. *The finite–gap solutions of the Landau–Lifshitz equation are given by*

$$S_1 = \frac{CD-AB}{AD-BC}, \quad S_2 = -i\,\frac{CD+AB}{AD-BC}, \quad S_3 = \frac{AD+BC}{AD-BC},$$

$$A = \theta(Ux+Vt+D \mid \Pi) \qquad B = \theta(Ux+Vt+D+r \mid \Pi)$$
$$C = \theta(Ux+Vt+D+\Delta \mid \Pi) \qquad D = \theta(Ux+Vt+D+r+\Delta \mid \Pi) \,.$$

Here all the parameters are determined by an arbitrary Riemann surface X *which is a two-sheeted cover of* E . *The vectors* U *and* V *are the* b*-period vectors of the normlaized Prym differentials of the second kind*

$$U_i = \int_{b_i} d\Omega_1 \,, \qquad \Omega_1 \longrightarrow \mp(i\rho u^{-1}+0(1))$$
$$\qquad\qquad\qquad\qquad\qquad\qquad u \longrightarrow 0^{\pm}$$
$$V_i = \int_{b_i} d\Omega_2 \,, \qquad \Omega_2 \longrightarrow \pm(2i\rho^2 u^{-2}+0(1)) \,.$$

The integral r *is equal to*

$$r = \int_{0^+}^{0^-} \nu$$

and the path of integration should be fixed in such a way that for an elliptic integral the equality $\int_{0^+}^{0^-} du \equiv 0 \ (\mathrm{mod}\ 4K,\ 4iK')$ *holds.*

6. Curves of Lower Genera. The Euler Case and the Neumann System

For the curves of small genera $(n = 1,2)$ the formulae of the previous section can be simplified. For this purpose we use the well–known addition formula for theta functions

$$\theta(z_1\,|\,\Pi)\,\theta(z_2\,|\,\Pi) = \sum_{\delta\in\frac{1}{2}\mathbb{Z}^n/2\mathbb{Z}^n} \theta\begin{bmatrix}\delta\\0\end{bmatrix}(z_1+z_2\,|\,2\Pi)\,\theta\begin{bmatrix}\delta\\0\end{bmatrix}(z_1-z_2\,|\,2\Pi)\ ,$$

where the sum is taken over all n–dimensional vectors δ with the coordinates $0, \frac{1}{2}$.

For $n = 1$ we have

$$L_1^{s,N_s} = \ell\,\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(z)\,\theta\begin{bmatrix}1\\0\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1\\1\end{bmatrix}(z)\,\theta\begin{bmatrix}1\\1\end{bmatrix}(r_s)}\ ,$$

$$L_2^{s,N_s} = -i\ell\,\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(z)\,\theta\begin{bmatrix}0\\0\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1\\1\end{bmatrix}(z)\,\theta\begin{bmatrix}1\\1\end{bmatrix}(r_s)}\ , \tag{6.1}$$

$$L_3^{s,N_s} = \ell\,\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(z)\,\theta\begin{bmatrix}0\\1\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1\\1\end{bmatrix}(z)\,\theta\begin{bmatrix}1\\1\end{bmatrix}(r_s)}\ ,$$

$$z = 2Vt + 2D + 2\epsilon_s + r_s\ .$$

Here and below in this section we use the notation $\theta\begin{bmatrix}\epsilon_1\\\epsilon_2\end{bmatrix}(x) = \theta\begin{bmatrix}\epsilon_1\\\epsilon_2\end{bmatrix}(x\,|\,2\Pi)$. For $n = 2$ the formulae are more complicated

$$L_1^{s,N_s} = -\ell\,\frac{\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(z)\,\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(r_s)+\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}(z)\,\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1&0\\1&0\end{bmatrix}(z)\,\theta\begin{bmatrix}1&0\\1&0\end{bmatrix}(r_s)+\theta\begin{bmatrix}1&1\\1&0\end{bmatrix}(z)\,\theta\begin{bmatrix}1&1\\1&0\end{bmatrix}(r_s)}$$

$$L_2^{s,N_s} = -i\ell\,\frac{\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(z)\,\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(r_s)+\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(z)\,\theta\begin{bmatrix}0\,1\\0\,0\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\,\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(r_s)+\theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z)\,\theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(r_s)} \tag{6.2}$$

$$L_3^{s,N_s} = \ell\,\frac{\theta\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(z)\,\theta\begin{bmatrix}0\,0\\1\,0\end{bmatrix}(r_s)+\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(z)\,\theta\begin{bmatrix}0\,1\\1\,0\end{bmatrix}(r_s)}{\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(z)\,\theta\begin{bmatrix}1\,0\\1\,0\end{bmatrix}(r_s)+\theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(z)\,\theta\begin{bmatrix}1\,1\\1\,0\end{bmatrix}(r_s)}\;.$$

Let us now consider the Euler top and the Neumann system, which have the simplest elliptic L–A pairs. The L–A pair for the equations of motion of the Euler top

$$M_t = [M, IM]\ ,\quad I = \mathrm{diag}(I_1, I_2, I_3)\ ,\quad \rho = \tfrac{1}{2}\sqrt{I_3 - I_1}\ ,\quad |M| = 1$$

is as follows:

$$L = -i\rho \sum_\alpha M_\alpha w_\alpha(u)\sigma_\alpha$$

$$A = 2i\rho^2 \sum_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(u) M_\alpha \sigma_\alpha\ , \qquad k = \sqrt{\frac{I_2 - I_1}{I_3 - I_1}}\ .$$

The spectral curve is given by the equation (5.3) and corresponds to the case $n = 1$. The solutions are given by the expressions (6.1), where $\ell = 1$ and $\Omega \longrightarrow \pm 2i\rho^2 u^{-2}$, $u \longrightarrow 0^\pm$.

The solutions of the Landau–Lifshitz equation independent of t are the solutions of the Neumann system [37]

$$S_{tt} + IS = \lambda(t)S\,, \qquad S^2 = 1\ . \qquad (6.3)$$

It was solved by C. Neumann [38] with the help of the separation of variables method. The equation (6.3) was considered in connection with the finite–gap potentials [39, 40, 41]. In particular, the generalization of the system (6.3) to a higher dimension case was solved. The L–A pair for the system (6.3) is equal to

$$L = 2i\rho^2 \sum_{\alpha} \frac{w_1 w_2 w_3}{w_\alpha}(u)\, S_\alpha \sigma_\alpha - i\rho \sum_{\alpha\beta\gamma} w_\alpha(u) S_\beta S_{t\,\gamma} \epsilon_{\alpha\beta\gamma} \sigma_\alpha\ ,$$

$$A = -i\rho \sum_{\alpha} S_\alpha w_\alpha(u) \sigma_\alpha\ .$$

The function *det* $L(u)$ is even and has a pole of the fourth order at the point $u = 0$. Therefore, the spectral curve corresponds to the case $n = 2$, and possesses an involution $\tau u = -u$. We denote by $u = p_1, p_2, q_1, q_2$ the branch points of the covering $X \longrightarrow E$. The Prym differentials are odd with respect to the involution

$$\overset{*}{\tau}\,\nu = -\nu\ .$$

For the vector r we have

$$r = \int_{\ell} \nu = -\int_{\tau\ell} \nu = -r + \int_{a_2} \nu \Rightarrow r = \begin{bmatrix} 0 \\ \pi i \end{bmatrix}\ ,$$

since $\tau\ell = \ell - a_2$ (Fig. 2). The one–half of theta constants $\theta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(r\,|\,2\Pi)$ in (6.2) becomes equal to zero. Finally we obtain the following formulae:

$$S_1 = -\frac{\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(z\,|\,2\Pi)\,\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}(z\,|\,2\Pi)\,\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}}$$

$$S_2 = -i\,\frac{\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(z\,|\,2\Pi)\,\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}(z\,|\,2\Pi)\,\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}}$$

$$S_3 = \frac{\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}(z\,|\,2\Pi)\,\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}(z\,|\,2\Pi)\,\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}}\,,$$

where $\theta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix} = \theta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(0\,|\,2\Pi)$,

$$z = 2Vt+D,\ \ V_n = \int_{b_n} d\Omega,\ \ \Omega \longrightarrow \mp i\rho u^{-1},\ \ u \longrightarrow 0^{\pm}\ .$$

The Prym differentials $2\nu_1, 2\nu_2$ in this case are the holomorphic differentials of the Riemann surface $X/\pi\tau$ of the genus 2. The involution π is a hyperelliptic involution of $X/\pi\tau$ with $0, K, iK', K+iK', p=p_1\equiv-p_2, q=q_1\equiv -q_2$ being the fixed points of π. It is easy to see that $2\nu_1, 2\nu_2$ are normalized, so the matrix 2Π is exactly the period matrix of $X/\pi\tau$.

7. Manakov and Clebsch Cases

It was remarked in the paper [37] that the one–phase solutions (depending on the combination $x+vt$) of the Landau–Lifshitz equation and of the asymmetric ciral

0(3)–field equation are the solutions of the Clebsch and Manakov cases of integrability respectively. In this way the Lax representation for these tops come from known zero curvature representation for the Landau–Lifshitz equation (5.8) and asymmetric ciral 0(3)–field equation [35].

The Manakov Case. The Lax representation is as follows:

$$L_M(u) = \sum_\alpha \left\{ S_\alpha w_\alpha(u-\kappa) + T_\alpha w_\alpha(u+\kappa) \right\} \sigma_\alpha/2i \ ,$$

$$A_M(u) = c_1 A_M^{(1)} + c_2 A_M^{(2)} \ , \tag{7.1}$$

$$A_M^{(1)} = \sum S_\alpha w_\alpha(u-\kappa)\sigma_\alpha/2i \ ,$$

$$A_M^{(2)} = -\sum \left\{ S_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(u-\kappa) + T_\alpha w_\alpha(2\kappa) w_\alpha(u-\kappa) \right\} \sigma_\alpha/2i \ .$$

The Lax equation (1.1) with the matrices (7.1) describes the Hamiltonian system with the Poisson bracket (2.7) and the Hamiltonian

$$H = c_1 H_1 + c_2 H_2 \ , \ H_1 = \sum w_\alpha S_\alpha T_\alpha \ , \ w_\alpha \equiv w_\alpha(2\kappa) \ ,$$

$$H_2 = \tfrac{1}{2} \sum \left(-w_\alpha^2 (S_\alpha^2 + T_\alpha^2) + 2 \frac{w_1 w_2 w_3}{w_\alpha} S_\alpha T_\alpha\right) \ . \tag{7.2}$$

We see that the spectral curve corresponds to $n = 2$. The general solution of the Manakov case is given by (6.2):

$$S_\alpha = \epsilon_\alpha S \frac{\theta[i_\alpha](z_0+2Vt)\theta[i_\alpha](r+\delta)+\theta[j_\alpha](z_0+2Vt)\theta[j_\alpha](r+\delta)}{\theta[m](z_0+2Vt)\theta[m](r+\delta)+\theta[n](z_0+2Vt)\theta[n](r+\delta)} ,$$

$$T_\alpha = \epsilon_\alpha T \frac{\theta[i_\alpha](z_0+2Vt)\theta[i_\alpha](r-\delta)+\theta[j_\alpha](z_0+2Vt)\theta[j_\alpha](r-\delta)}{\theta[m](z_0+2Vt)\theta[m](r-\delta)+O[n](z_0+2Vt)\theta[n](r-\delta)}$$

$$\epsilon_1 = -1, \ \epsilon_2 = -i, \ \epsilon_3 = 1 ,$$

$$[i_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ [i_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ [i_3] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$[j_1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ [j_2] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ [j_3] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[m] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ [n] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} ,$$

$$r = \int_{-\kappa^+}^{\kappa^-} \nu , \ \delta = \int_{\kappa^+}^{-\kappa^+} \nu , \ z_0 \in \mathbb{C}^2 , \ V_i = \int_{b_i} d\Omega .$$

Here S and T are constants (2.8) and the normalized abelian integral Ω is determined by the asymptotics

$$\int d\Omega \longrightarrow \pm \frac{1}{2i} S \left[-\frac{c_2}{(u-x)^2} + \frac{c_1}{u-x} + \dots \right] , \ u \longrightarrow \kappa^\pm .$$

The Clebsch Case. The Lax representation is as follows:

$$L_c(u) = \sum \left\{ p_\alpha \frac{w_1 w_2 w_3}{w_\alpha} + M_\alpha w_\alpha \right\} \frac{\sigma_\alpha}{2i} ,$$

$$A_c(u) = d_1 A_c^{(1)} + d_2 A_c^{(2)} \ ,$$

$$A_c^{(1)} = \sum p_\alpha w_\alpha \sigma_\alpha / 2i \ ,$$

$$A_c^{(2)} = \sum \left\{ p_\alpha w_\alpha (w^2 + J_\alpha - 2J) + M_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \right\} \frac{\sigma_\alpha}{2i} \ , \tag{7.3}$$

$$w^2 = (w_1^2 + w_2^2 + w_3^2)/3 \ , \ J = (J_1 + J_2 + J_3)/3 \ , \ w_\alpha = w_\alpha(u) \ .$$

It is a Hamiltonian system with the Poisson bracket (2.3) and the Hamiltonian

$$H = d_1 H_1 + d_2 H_2 \ , \ H_1 = \tfrac{1}{2} \sum (J_\alpha p_\alpha^2 + M_\alpha^2) \ ,$$

$$H_2 = \tfrac{1}{2} \sum \left[\frac{J_1 J_2 J_3}{J_\alpha} p_2^2 - J_\alpha M_\alpha^2 \right] \ . \tag{7.4}$$

The formulae for solutions are obtained by the isomorphism [5, 36] of this case and the Manakov case:

$$p_\alpha = w_\alpha(x)(S_\alpha - T_\alpha) \ , \ M_\alpha = \frac{w_1 w_2 w_3}{w_\alpha}(x)\,(S_\alpha + T_\alpha) \ ,$$

$$c_1 = \frac{w_1^2 w_2^2 + w_1^2 w_3^2 + w_2^2 w_3^2}{2 w_1 w_2 w_3}(d_1 - w_1^2 d_2) + 2 w_1 w_2 w_3 d_2 \ ,$$

$$c_2 = d_1 - w_1^2 d_2 \ , \ w_\alpha = w_\alpha(x) \ .$$

Direct integration of the L–A pair (7.3) by the technique of Sect. 5 yields the formulae

(6.2) for p_α, where $\ell = p$ (2.4), $r = \int_{0^+}^{0^-} \nu$ and vector V is determined by the normalized integral with the singularity

$$\int d\Omega \longrightarrow \pm \frac{1}{2i}\left[\frac{d_2}{u^3}p + \frac{d_2}{u^2}\frac{p^M}{p^2} - \frac{d_1}{u}p + \dots\right], \quad u \longrightarrow 0^\pm .$$

These latest formulae were obtained by Kötter [42] (see also [9]) for $d_2 = 0$. The expressions for M_α obtained in this way are more complicated and we do not present them here.

Remark. Adding to Vt a period vector of the form

$$\left[\begin{matrix}1 & 0\\ 0 & 1\end{matrix} \,\middle|\, \Pi \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right] \begin{bmatrix}N\\ M\end{bmatrix}, \quad N,M \in \mathbb{Z}^2 \tag{7.5}$$

does not change the solutions. Let us normalize the period matrix (7.5) changing a basis in $\mathbb{C}^2$. We obtain that the period lattice is given by the normalized matrix

$$\left[\begin{matrix}2 & 0\\ 0 & 1\end{matrix} \,\middle|\, 2\Pi\right], \tag{7.6}$$

where Π and Π are connected by (5.4). The matrix (7.6) is exactly the period matrix of the Prym variety $Prym_\pi(X)$. It shows that the mapping of the Liouville torus to $Prym_\pi(X)$ is one–to–one. This fact was established in [34, 32].

Remark. All integrable cases considered in the present and next sections depend on 6 arbitrary parameters. We obtain additional parameters adding the Casimir functions f_1, f_2, g_1, g_2 to Hamiltonians (7.2,4). Furthermore, the bracket (2.3) is invariant with respect to the transformation $p_\alpha \longrightarrow ap_\alpha$ (for all α together), which changes a Hamiltonian. We also remark that for any integral K of the Clebsch top (as well as of the first Steklov case of integrability [see below])

$$\sum_\alpha \partial K / \partial J_\alpha$$

is an integral of motion. Therefore, the transformation $J_\alpha \longrightarrow J_\alpha + \Delta$ (for all α together) preserves integrability. Combined with the transformation $p_\alpha \longrightarrow ap_\alpha$ mentioned above it guarantees integrability of the Clebsch and 1-st Steklov cases with arbitrary J_α.

8. Steklov Cases

The integrable Steklov case of motion of rigid body in liquid was solved by Kötter [44]. In his paper he used implicitly the Lax representation with an elliptic spectral parameter [28]. Various modifications of the Lax pairs for the Steklov cases were suggested in [45], [46], [5].

The Second Steklov Case. It possesses the following Lax representation:

$$L_{II}(u) = \sum \left\{ S_\alpha w_\alpha(u) + \frac{1}{2} T_\alpha (w_\alpha(u-\kappa) + w_\alpha(u+\kappa)) \right\} \frac{\sigma_\alpha}{2i} ,$$

$$A_{II}(u) = c_1 A_{II}^{(1)} + c_2 A_{II}^{(2)} , \tag{8.1}$$

$$A_{II}^{(1)} = 2\sum S_\alpha \frac{w_1w_2w_3}{w_\alpha}(u)\frac{\sigma_\alpha}{2i} \ ,$$

$$A_{II}^{(2)} = \sum T_\alpha \left\{ w_\alpha(u-\kappa) - w_\alpha(u+\kappa) \right\} \frac{\sigma_\alpha}{2i} \ .$$

The Lax equation (1.1) with the matrices (8.1) describes the Hamiltonian system with the Poisson bracket (2.7) and the Hamiltonian

$$H = c_1H_1 + c_2H_2 \ ,$$

$$H_1 = \sum (w_2^2(\kappa)S_\alpha^2 - 2\frac{w_1w_2w_3}{w_\alpha}(\kappa)S_\alpha T_\alpha) \ ,$$

$$H_2 = \sum(-\frac{w_1w_2w_3}{w_\alpha^2}(\kappa)\, T_\alpha^2 + 2w_\alpha(\kappa)S_\alpha T_\alpha) \ .$$

The L–A pair satisfies the reduction

$$L(-u) = -L(u) \ , \ \ A(-u) = A(u) \ . \tag{8.2}$$

Note that the factor E/i $(iu = -u)$ is a rational curve. The matrix L, multiplied by $w_1w_2w_3(u)$

$$L \longrightarrow w_1w_2w_3L$$

becomes a function on E/i. So the Steklov cases (see also the first Steklov case below) possesses the Lax representations with a rational spectral parameter.

The spectral curve X (5.3) corresponds to $n = 3$ of Sect. 5. It also has an involution

$$\tau : (\mu,u) \longrightarrow (-\mu,-u) \ ,$$

which is a corollary of (8.2). This involution has two fixed points 0^+ and 0^- with $u=0$ (Fig. 3). The factor X/τ is a curve of genus 2. The involution π changing the sheets of the covering $X \longrightarrow E$ is the hyperelliptic involution of X/τ. Its fixed points are $p_1, q_1, p_2, K, iK', K+iK'$.

We shall specify the parameters determining the Baker–Akhiezer function to satisfy the reduction

$$\psi(\tau P) = \psi(P) \ . \tag{8.3}$$

For the L–A pair of the ψ–function, satisfying (8.3) the reduction (8.2) is automatically fulfilled.

One can always choose a canonical basis of cycles such that (Fig. 3)

$$\tau a_1 = a_4, \ \ \tau b_1 = b_4, \ \ \tau a_2 = a_3, \ \ \tau b_2 = b_3 \ .$$

The Prym differentails $\nu = (\nu_1,\nu_2,\nu_3)^T$ (Sect. 5) satisfy the equality

$$\overset{*}{\tau}\nu = T\nu \ , \ \ T = \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} \ .$$

The asymptotics at the singularity points of μ and of the normalized abelian integral of the second kind Ω, determining the velocity vector V, are as follows:

$$\mu \longrightarrow \pm \frac{S}{2iu} \qquad \Omega \longrightarrow \pm c_1 \frac{S}{iu^2} \qquad P \longrightarrow 0^{\pm}$$

$$\mu \longrightarrow \pm \frac{T}{4i(u-\kappa)} \qquad \Omega \longrightarrow \pm c_2 \frac{T}{2i(u-\kappa)} \qquad P \longrightarrow \kappa^{\pm}$$

$$\mu \longrightarrow \mp \frac{T}{4i(u+\kappa)} \qquad \Omega \longrightarrow \pm c_2 \frac{T}{2i(u+\kappa)} \qquad P \longrightarrow -\kappa^{\pm} \ .$$

Hence, the equality

$$\tau^* d\Omega = d\Omega$$

holds. For the b–periods we have

$$\begin{matrix} T \ \Pi \, T = \Pi \\ V = TV \end{matrix} \ \Rightarrow \ \Pi = \begin{bmatrix} \alpha & \beta & \beta \\ \beta & \gamma & \delta \\ \beta & \delta & \gamma \end{bmatrix} , \quad V = \begin{bmatrix} v_1 \\ v_2 \\ v_2 \end{bmatrix} . \tag{8.4}$$

Let us fix the fixed point of τ as the starting point in all integrals $P_0 = 0^+$ or 0^-. The symmetry of the period matrix gives

$$\theta(Tx \mid \Pi) = \theta(x \mid \Pi) \ .$$

It, in turn, yields

$$\theta\Big(\int_{P_0}^{\tau P} \nu + Vt + D \,\Big|\, \Pi\Big) = \theta\Big(T\Big(\int_{P_0}^{\tau P} \nu + Vt + D\Big) \,\Big|\, \Pi\Big) =$$

$$\theta\Big(\int_{P_0}^{P} \nu + Vt + D \,\Big|\, \Pi\Big)$$

if the vector D is also symmetric

$$D = TD \ . \tag{8.5}$$

Thus we obtain that (8.3) is equivalent to (8.5).

The even part of $Prym_\pi(X)$ with respect to τ is a two–dimensional Abelian torus. We see that the flow Vt is restricted to this torus. It is, therefore, natural to present solutions in terms of the two–dimensional theta functions. For this purpose we use the reduction technique of theta functions [32]. Let us make a substitution of the theta function's summation variable

$$m = N(n+\delta)\ ,\quad N = \begin{bmatrix} 1 & & \\ & 1 & -1 \\ & 1 & 1 \end{bmatrix} \ .$$

Here $n \in \mathbb{Z}^3$, $\delta = (\delta_1,\delta_2,\delta_3) = \{(0,0,0), (0,1/2,1/2)\}$. The matrix $N^T \Pi N$ consists of two blocks

$$N^T \Pi N = \begin{bmatrix} \Pi_+ & 0 \\ 0 & \Pi_- \end{bmatrix},\ \Pi_+ = \begin{bmatrix} \alpha & 2\beta \\ 2\beta & 2(\gamma+\delta) \end{bmatrix},\ \Pi_- = 2(\gamma-\delta) \ .$$

Hence, the following equality is valid:

$$<\Pi m,m> + 2<x,m> = <\Pi_+(n_1+\delta_1,n_2+\delta_2),(n_1+\delta_1,n_2+\delta_2)> +$$

$$+ 2\langle(x_1,x_2+x_3),\,(n_1+\delta_1,\,n_2+\delta_2)\rangle +$$

$$+ \langle \Pi_-(n_3+\delta_3),n_3+\delta_3)\rangle + 2\,\langle x_3-x_2,n_3+\delta_3\rangle \ .$$

It, in turn, gives the representation of the 3–dimensional theta function in terms of 2–dimensional and 1–dimensional theta functions:

$$\begin{aligned}\theta(x\,|\,\Pi) = {}& \theta\begin{bmatrix}0\ 0\\0\ 0\end{bmatrix}((x_1,x_2+x_3)\,|\,\Pi_+)\,\theta\begin{bmatrix}0\\0\end{bmatrix}(x_3-x_2\,|\,\Pi_-) + \\ & + \theta\begin{bmatrix}0\ 1\\0\ 0\end{bmatrix}((x_1,x_2+x_3)\,|\,\Pi_+)\,\theta\begin{bmatrix}1\\0\end{bmatrix}(x_3-x_2\,|\,\Pi_-) \ .\end{aligned} \tag{8.6}$$

The structures (8.4,5) of the vectors V and D prove that the 1–dimensional theta functions in (8.6) are constants.

The First Steklov Case. The corresponding L–A pair and the Hamiltonian are as follows:

$$L_I(u) = \sum \left\{ p_\alpha w_\alpha \left(w^2 + \frac{J_\alpha - J}{2}\right) + \tfrac{1}{2} M_\alpha w_\alpha \right\} \frac{\sigma_\alpha}{2i} \ ,$$

$$A_I(u) = d_1 A_I^{(1)} + d_2 A_I^{(2)} \ , \ \ A_I^{(1)} = -2 \sum p_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \frac{\sigma_\alpha}{2i} \ ,$$

$$A_I^{(2)} = \sum \left\{ 2p_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \left(w^2 + J - \frac{J_\alpha}{2}\right) + M_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \right\} \frac{\sigma_\alpha}{2i} \ , \ \ w_\alpha = w_\alpha(u) \ ,$$

$$H = d_1 H_1 + d_2 H_2 \ , \ \ H_1 = \tfrac{1}{2} \sum \left(\left(J_\alpha^2 + 2\,\frac{J_1 J_2 J_3}{J_\alpha}\right) p_\alpha + 2 J_\alpha p_\alpha M_\alpha - M_\alpha^2 \right) \ ,$$

$$H_2 = \frac{1}{2}\sum\left(J_\alpha(J_1^2+J_2^2+J_3^2-J_\alpha^2)p_\alpha^2+2\,\frac{J_1J_2J_3}{J_\alpha}\,p_\alpha M_\alpha + J_\alpha M_\alpha^2\right) . \tag{8.7}$$

The formulae for solutions can be easily obtained using the isomorphism [5] of this case with the second Steklov case:

$$p_\alpha = S_\alpha, \; M_\alpha = S_\alpha(w_\alpha^2(\kappa)-3w^2(\kappa))-2T_\alpha\,\frac{w_1w_2w_3}{w_\alpha}(\kappa) \; ,$$

$$c_1 = d_1-w^2(\kappa)d_2 \, , \; c_2 = -w_1w_2w_3(\kappa)d_2 \; .$$

Direct integration of the L–A pair (8.7) yields the same formula for p_α.

Remark. Adding to $vt = (v_1,v_2)^T t$ a period vector of the form

$$\left[\begin{array}{cc|cc} 1 & 0 & 2\alpha & 2\beta \\ 0 & 1 & 2\beta & \gamma+\delta \end{array}\right]\begin{bmatrix} N \\ M \end{bmatrix} , \; N,M \in \mathbb{Z}^2 \tag{8.8}$$

does not change the solutions. Since $2\nu_1$, $\nu_2+\nu_3$ are the normlaized holomorphic differentials of the Riemann surface X/τ, the matrix (8.8) is the period matrix of X/τ. It shows that the mapping of the Liouville torus to $Jac(X/\tau)$ is one–to–one. This fact was established in another way in [21].

9. Complete Description of the Motion in the Rest Frame

Up to now we have described the motion of tops in the moving frame attached to the body. But for a complete description of the rotation it is necessary to describe it in the rest frame.

It is convenient to use the isomorphism of an algebra of vectors in $\mathbb{R}^3$ with a vector multiplicaton and an algebra of traceless 2×2 matrices with a commutation operation

$$\begin{aligned} X = (X_1,X_2,X_3) &\longmapsto X = \sum_\alpha X_\alpha \frac{\sigma_\alpha}{2i} \\ [X \times Y] &\longmapsto [X,Y] \ . \end{aligned} \tag{9.1}$$

Everywhere below X means the matrix (9.1). The coordinates X and X' of vector in the moving and the rest frames, respectively are connected by the transformation

$$X = GX'G^{-1} \tag{9.2}$$

with some 2×2 matrix G . Our aim is to determine this connection matrix.

The equations of motion of a heavy rigid body about a fixed point in the moving frame attached to the body are as follows:

$$M_t = [M,(-G_tG^{-1})] + [p,L] \ , \ p_t = [p,(-G_tG^{-1})] \ . \tag{9.3}$$

Here $L = \Sigma L_\alpha \sigma_\alpha / 2i$ and L_α are the constant coordinates of the center of mass in the moving frame with an origin at the fixed point. Comparing (2.1) with (9.3) we get

$$G_tG^{-1} = -\sum_\alpha \frac{\partial H}{\partial M_\alpha} \frac{\sigma_\alpha}{2i} \ . \tag{9.4}$$

We also fix the third axis of the rest frame, assuming it to be the gravity vector. Combined with (9.2) it gives

$$G\sigma_3 G^{-1} = \Sigma\, p_\alpha \sigma_\alpha \; . \tag{9.5}$$

An arbitrary solution of (9.4) satisfying (9.5) may differ by a constant diagonal gauge factor

$$G \longrightarrow GC \; , \quad [C, \sigma_3] = 0 \; . \tag{9.6}$$

The remaining freedom (9.6) corresponds to the so far unspecified two axes of the rest frame.

The motion of the rigid body in liquid is described in a similar way. In this case the vectors

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) \; , \qquad \Omega_\alpha = \partial H / \partial M_\alpha \; ,$$

$$v = (v_1, v_2, v_3) \; , \qquad v_\alpha = \partial H / \partial p_\alpha$$

are the angular and translation velocities of the rigid body in the moving frame attached to the body [9]. As above, for the heavy tops the rotation of the fixed frame to the moving frame is determined by the matrix G satisfying (9.4). Let us choose the third axis of the rest frame coinciding with the impulse p (it is constant). Then for G we have

$$G\sigma_3 G^{-1} = \frac{1}{p} \Sigma\, p_\alpha \sigma_\alpha \; . \tag{9.7}$$

The remaining freedom in G is the same as for the heavy tops (9.6).

The velocity of translation movement in the rest frame $v' = (v'_1, v'_2, v'_3)$ is equal to

$$\sum_\alpha v'_\alpha \sigma_\alpha = G^{-1} \sum_\alpha v_\alpha \sigma_\alpha G = G^{-1} \sum_\alpha \frac{\partial H}{\partial p_\alpha} \sigma_\alpha G \ .$$

To find $G(t)$ we still have to solve the linear differemtial equation (9.4). It turns out, however, that the Baker–Akhiezer functions contain more information than the Euler–Poisson equations themselves and allows us to find $G(t)$ without solving (9.4).

The Kowalewski Top.

Let us consider the equation $\Psi_t = A\Psi$ at $\lambda = 0$. Observe that $A(\lambda = 0)$ decomposes into two 2×2 blocks, which essentially coincide with the angular velocity. In particular, the matrix

$$\varphi = \begin{bmatrix} \psi_1(0_-) & \psi_1(0_+) \\ \psi_3(0_-) & \psi_3(0_+) \end{bmatrix}$$

satisfies

$$\varphi_t = -\frac{1}{2i} \sum \frac{\partial H}{\partial M_\alpha} \sigma_\alpha \varphi = -\frac{1}{2i} (M_1\sigma_1 + M_2\sigma_2 + 2M_3\sigma_3)\varphi \ .$$

From (3.26) we find

$$\varphi = a\,\theta(P-R)\begin{bmatrix} \dfrac{1}{\theta(Vt+P)} & 0 \\ 0 & -\dfrac{1}{\theta[\epsilon](Vt+P)} \end{bmatrix} \times$$

$$\mathscr{A}\begin{bmatrix} \dfrac{e^{bt}}{\theta(\int_{\infty_+}^{0_-}\omega+P-R)} & 0 \\ 0 & \dfrac{e^{-bt}}{\theta(\int_{\infty_+}^{0_+}\omega+P-R)} \end{bmatrix},$$

where $\pm b = \int_{\infty_+}^{0_\mp} d\Omega$ and $\mathscr{A}$ is given by (3.26) [see (3.28)]

$$\mathscr{A} = \begin{bmatrix} \theta(\int_{\infty_+}^{0_-}\omega+Vt+P) & \theta(\int_{\infty_+}^{0_+}\omega+Vt+P) \\ \theta[\epsilon]\,(\int_{\infty_+}^{0_-}\omega+Vt+P) & -\theta[\epsilon]\,(\int_{\infty_+}^{0_+}\omega+Vt+P) \end{bmatrix}.$$

It is easily checked that the time evolution of the Poisson vector p is given by

$$p(t) = \frac{1}{2i}\,\varphi\sigma_3\,\varphi^{-1}\ .$$

So reducing φ by the constant right factor, we have

$$G = \begin{bmatrix} \dfrac{1}{\theta(Vt+P)} & \\ & -\dfrac{1}{\theta[\epsilon](Vt+P)} \end{bmatrix}\mathscr{A}\begin{bmatrix} e^{bt+b_0} & \\ & e^{-bt-b_0} \end{bmatrix},$$

where $b_0 = const$.

By inverting, we also obtain the evolution of the top in the rest frame. For example, the motion of the symmetry axis of the top in the rest frame is given by

$$\Sigma L'_\alpha \sigma_\alpha = \begin{bmatrix} e^{-bt-b_0} & \\ & e^{bt+b_0} \end{bmatrix} \mathscr{A}^{-1} \sigma_3 \mathscr{A} \begin{bmatrix} e^{bt+b_0} & \\ & e^{-bt-b_0} \end{bmatrix} ,$$

where L'_α are the coordinates of the unit vector directed along the axis of the top.

For the Clebsch and Steklov cases we restrict ourselves to the case $d_{II} = 0$, i.e., $H = H_I$ for both systems.

The Clebsch Case.

Substituting the asymptotics

$$\Psi(u) = (\Phi + Qu + 0(u^2))\exp\left[\frac{\sigma_3}{2i}\frac{p}{u}t\right]$$

into the equations

$$L\Psi = \Psi\hat{\mu}, \quad \Psi_t = A_I\Psi$$

we obtain

$$p\Phi\sigma_3\Phi^{-1} = \sum p_\alpha\sigma_\alpha$$

$$\sum M_\alpha\sigma_\alpha = [Q\Phi^{-1}, \sum p_\alpha\sigma_\alpha] + \frac{pM}{p^2}\sum p_\alpha\sigma_\alpha \tag{9.8}$$

$$2i\,\Phi_t\Phi^{-1} = [\sum p_\alpha\sigma_\alpha, Q\Phi^{-1}] .$$

From these formulae we get

$$\Phi_t\Phi^{-1} = -\frac{1}{2i}\sum M_\alpha\sigma_\alpha + \frac{pM}{p^2}\sum p_\alpha s_\alpha \ . \tag{9.9}$$

The equalities (9.8,9) show that $G(t)$ satisfying (9.4,7) can be easily obtained using Φ with the help of multiplication by the right factor

$$G = \Phi \exp\left(-\frac{\sigma_3}{2i}\frac{pM}{p}t\right)$$

(we consider the case $H = H_I$). Finally, we have

$$G = \begin{bmatrix} \theta(Vt+D\mid\Pi) & \theta(Vt+D+r\mid\Pi) \\ \theta(Vt+D+\Delta\mid\Pi) & \theta(Vt+D+r+\Delta\mid\Pi) \end{bmatrix} \begin{bmatrix} e^{bt+b_0} & \\ & e^{-bt-b_0} \end{bmatrix}, \tag{9.10}$$

$$b = -\frac{1}{2i}\frac{pM}{p} + \frac{1}{2}\int_{0^-}^{0^+} d\Omega \ .$$

Remark. We ignored this fact, but in reality the constructed Ψ–function satisfies the equation $\psi_t = A\psi + a(\lambda,t)\psi$ (Sect. 1), since it was determined up to a scalar factor depending on t. Nevertheless the connection (9.2) of the bases with (9.8) is valid since (9.2) is invariant with respect to this multiplication.

The Steklov Case.

The analogous expressions for the Steklov case are as follows:

$$\Psi(u) = (\Phi+Su+Qu^2+0(u^3))\exp\left[\frac{\sigma_3}{2i}\left[-\frac{2p}{u^2}t\right]\right] \ ,$$

$$\hat{\mu} = \left[\frac{p}{u^3} + \frac{pM}{2p}\frac{1}{u} + 0(u) \right] \frac{\sigma_3}{2i} ,$$

$$p\Phi\sigma_3\Phi^{-1} = \Sigma\, p_\alpha\sigma_\alpha ,$$

$$[\Sigma\, p_\alpha\sigma_\alpha, Q\Phi^{-1}] = \frac{pM}{2p^2}\Sigma(p_\alpha\sigma_\alpha - \frac{1}{2}M_\alpha\sigma_\alpha)$$

$$2i\Phi_t\Phi^{-1} = -2[\Sigma\, p_\alpha\sigma_\alpha, Q\Phi^{-1}] + \Sigma\, p_\alpha\sigma_\alpha(J - J_\alpha) =$$
$$= \Sigma(M_\alpha - J_\alpha p_\alpha)\sigma_\alpha + \Sigma\, p_\alpha\sigma_\alpha(J - \frac{pM}{p^2})$$

$$G = \Phi \exp(-\frac{\sigma_3}{2i}(Jp - \frac{pM}{p})t) .$$

The final result in this case is given by the same formula (9.10) as for the Clebsch case. The difference is that in the Steklov case the theta functions in (9.10) are 3–dimensional (Sect. 8) and the constant b is determined by the slightly different expression

$$b = -\frac{1}{2i}(Jp - \frac{pM}{p}) + \frac{1}{2}\int_{0_-}^{0_+} d\Omega .$$

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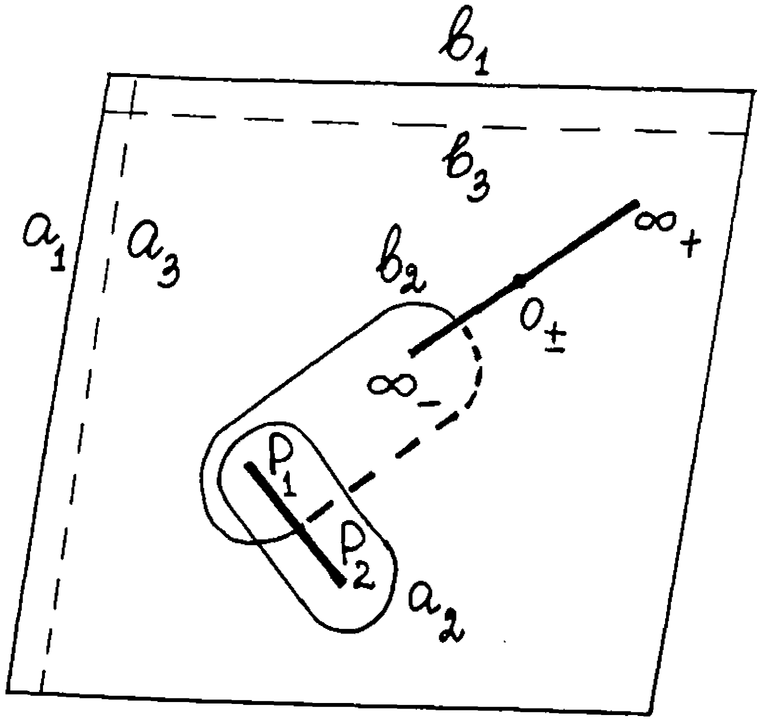

Fig 1. Shows a plane model of the elliptic curve E . The cycles a_i, b_i are depicted relative to the representation of X as a two–sheeted cover of E : continuous lines show parts of the cycles on the upper sheet while dotted lines show their parts on the lower sheet.

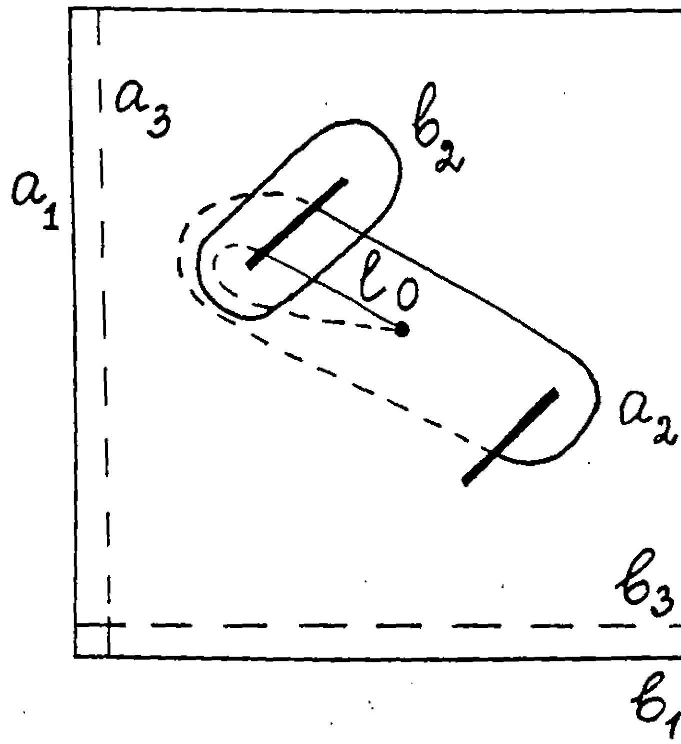

Fig. 2

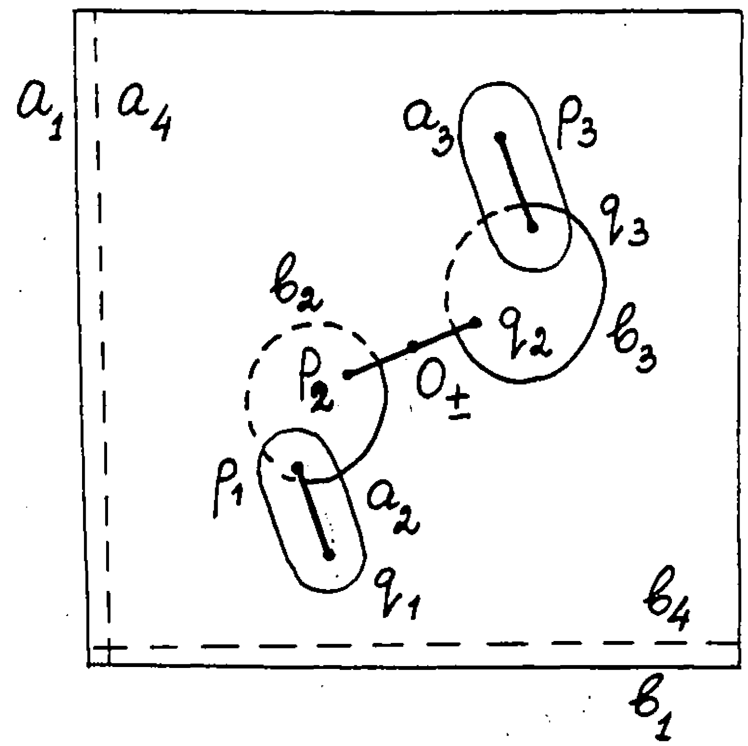

Fig. 3

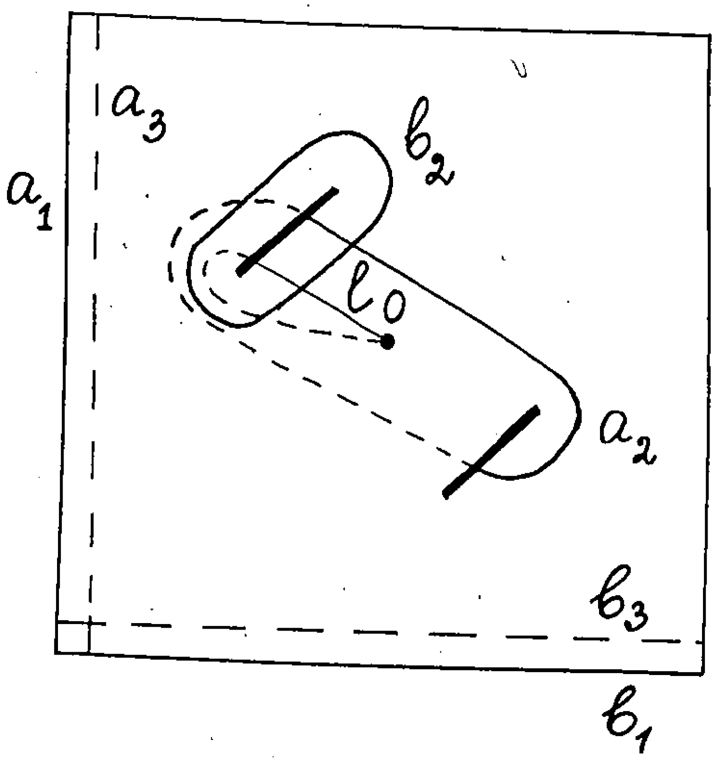

Fig. 2

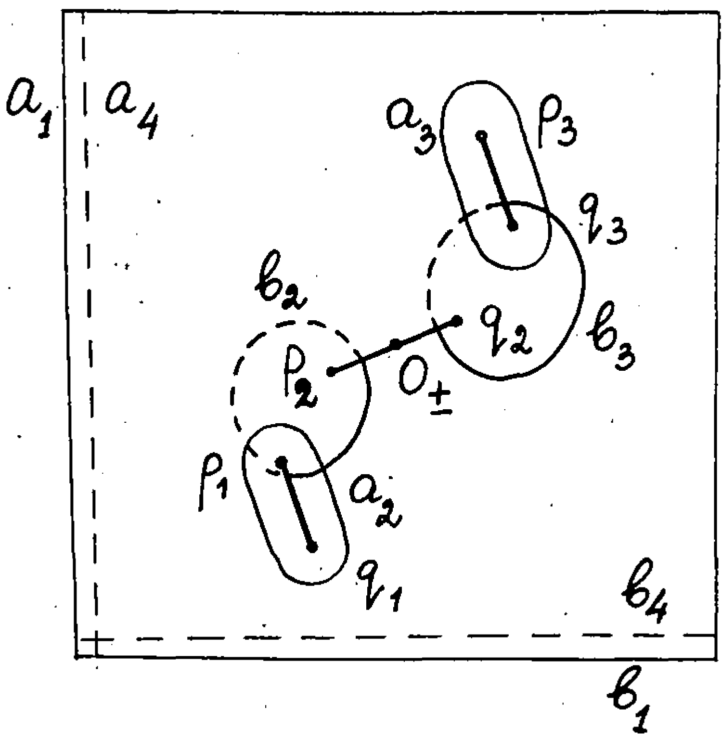

Fig. 3