A note on R.J. Sacker Theorem.

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Abstract

Sacker's Theorem asserts that if $k \in \mathbb{N}$,

$$p_n \in (0,2),\ p_{n+k} = p_n \text{ for all the } n \in \mathbb{N}_0$$

then the Ricker equation $x_{n+1} = x_n \exp(p_n - x_n)$, $x_0 = x \in (0, +\infty)$ has a globally asymptotically stable k-periodic solution $\{x_n^*\}_{n=0}^{+\infty}$ and

$$\frac{1}{k}\sum_{i=0}^{k-1} x_i^* = p^* := \frac{1}{k}\sum_{i=0}^{k-1} p_i$$

I prove here that above assertions of Sacker's theorem remain true if we replace the condition $p_n \in (0,2)$ by a weaker condition $p_n \in [0,2]$. I obtain here also an effective estimates of the speed of convergence of arbitrary solution of the Ricker equation to the limit periodic solution.

0. Foreword. The Ricker equation has been studied in connection with some biological problems. In [1], R.J. Sacker studies the k-periodic Ricker equation

$$x_{n+1} = R_{p_n}(x_n) = x_n \exp(p_n - x_n),\ x_0 = x \in (0, +\infty),\ p_{n+k} = p_n \tag{1}$$

for $n \in \mathbb{N}_0$, and proves the following theorem.

Theorem A. *Assume $p_n \in (0,2)$. Then the Ricker equation (1) has a globally asymptotically stable k-periodic solution x_n^*. Moreover*

$$\frac{1}{k}\sum_{i=0}^{k-1} x_i^* = p^* := \frac{1}{k}\sum_{i=0}^{k-1} p_i. \tag{2}$$

Some constants, which determine the speed of convergence of arbitrary solution to the limit periodic solution appear in [1] as ineffective constants thanks to the use of Borel's Lemma. In this note I replace the Sacker's "Uniformity Lemma" in his proof by some direct considerations, wich allow obtain an effective estimate of the mentioned above speed. I prove also that after replacement the condition $p_n \in (0,2)$ by more week condition $p_n \in [0,2]$

all the above assertion of Sacker's Theorem remain true. More precisely I prove the following

Theorem 1 . *Let* $k \in \mathbb{N}$,

$$p_n \in [0,2],\ p_{n+k} = p_n \textit{ for all the } n \in \mathbb{N}_0.$$

Let

(3) $$A = \min_{n \in [0,k) \cap \mathbb{N}_0} p_n,\ B = \max_{n \in [0,k) \cap \mathbb{N}_0} p_n,$$

$$p^* := \frac{1}{k} \sum_{i=0}^{k-1} p_i.$$

Then the Ricker equation (1) has a globally asymptotically stable k-periodic solution $\{x_n^*\}_{n=0}^{+\infty}$, *and (2) holds.*

Moreover, let $\{x_n\}_{n=0}^{+\infty}$ *be a solution of the equation (1), and* $x_0 > 0$.

Let $b = \max(x_0, \exp(B-1)), a = min(x_0,\ A,\ b\exp(A-b))$. *The following cases exhaust all the possibilities:*

(a) in the case $0 < A \le B < 2$, $n \in \mathbb{N}_0$ *we have*

(4) $$|x_n - x_n^*| \le (\lambda_0)^{\max(0,(n-3)/4)} \frac{b}{a}(b-a)\max(|, |b-1|, 1),$$

with

(5) $$0 < \lambda_0 = \max(e^2/4 - 1,\ |a-1|, \exp(B-2)) < 1;$$

(b) in the case $0 < A < B = 2$, *we take*

$$C \in (\max(A,1), 2),\ h_2(C) = (C-1)(C(\exp(2-C)-1),$$

(6) $$0 < \lambda_1(C) = \max(e^2/4 - 1,\ |a-1|, C-1, h_2(C)).$$

Then $\lambda_1(C) < 1$, *and*

(7) $$|x_n - x_n^*| \le (\lambda_1(C))^{\max(0,(n-k-3)/(k+2))} \frac{b}{a}(b-a)(b-1),$$

where $n \in \mathbb{N}_0$;

(c) in the case $0 = A \le B < 2$ *we have*

(8) $$|x_n - x_n^*| \le (\lambda_2)^{\max(0,(n-k-2)/4)} \frac{b}{a}(b-a)\max(|, |b-1|, 1),$$

where $n \in [k-1, +\infty) \cap \mathbb{N}_0$,

(9) $$0 < \lambda_2 = \max(e^2/4 - 1,\ |a\exp((-k)a) - 1|, \exp(B-2)) < 1;$$

(d) in the case $A = 0$, $B = 2$, *we take*

$$C \in (1,2),\ h_2(C) = (C-1)(C(\exp(2-C)-1),$$

(10) $$0 < \lambda_3(C) =$$

$$\max(e^2/4 - 1,\ |a\exp(-ka) - 1|, C - 1, h_2(C).)$$

Then $\lambda_3(C) < 1$, *and*

(11) $$|x_n - x_n^*| \le (\lambda_3(C))^{\max(0,(n-2k-2)/(k+2))} \frac{b}{a}(b-a)(b-1),$$

where where $n \in [k-1, +\infty) \cap \mathbb{N}_0$;

(e) in the case $A = B = 0$, *we have the inequality*

(12) $$x_n < \frac{x_0}{1 + nx_0}$$

(f) in the case $A = B = 2$, *if* $\varepsilon \in (0, 1]$

$$n \in [C_1(x_0, \varepsilon), +\infty) \cap \mathbb{N},$$

where

$$C_1(x_0, \varepsilon) = max(0, \ln(1/x_0), \ln(1/x_1)) +$$

$$3 + \frac{\ln(1/\varepsilon)}{\ln(1/(1 - \varepsilon^2/12))},$$

then

$$|x_n - 2| < \varepsilon.$$

The Theorem 1 is proved for the cases (a) and (b) respectively in the sections 1 and 2, for the cases (c) and (d) – in the section 3, for the cases (e) and (f) respectively in the sections 4 and 5.

Remark. The estimate in (4) is independent from k, but estimate in (7) depends from k. This is an essential difference between these two estimates.

1. Proof of the Theorem 1 in the case (a) The iteration of (1) give the equality

(13) $$x_{n+m} = R_{p_n}(x_n) = x_n \exp\left(\sum_{i=0}^{m-1}(p_{n+i} - x_{n+i})\right),$$

where $m \in \mathbb{N}$.

Lemma 1. *Let* $B \in [0, +\infty)$, $A \in [0, B]$. *Assume* $A \le p \le B$. *Let*

$$b \in [\exp(B-1), +\infty),\ a \in [0, min[A, b\exp(A-b))],\ x \in [a, b].$$

Then $R_p(x) := x\exp(p - x) \in [a, b]$.

Proof. Since

$$R_p'(x) := (1-x)\exp(p-x),\ R_p''(x) := (x-2)\exp(p-x),$$

it follows that

$$R_p(x) \le \exp(B-1) \le b,$$

$$R_p(x) \ge \min(R_p(a), R_p(b)) = \min(a\exp(A-a), b\exp(A-b)) \ge a.$$

■

Let

$$h_p(x) = (R_p(x) - 1)(x - 1). \tag{14}$$

Lemma 2. *If $x \in [0,1]$, $C \in [1,2]$, and $p \in [0,2]$, then*

$$|h_p(x)| \le \max\left(\frac{e^2}{4} - 1,\, |x-1|\right). \tag{15}$$

If $C \in [1,2]$, $x \in [1,C]$ and $p \in [0,2]$, then $h_2(x)$ increases together with increasing $x \in [1,2]$,

$$0 \le h_2(x) \le h_2(C) \le h_2(2) = 1,\ 2\exp(-2) - 1 \le \tag{16}$$

$$|h_p(x)| \le \max\left(1 - \frac{2}{e^2}, \exp(p-2)h_2(C)\right) \le \max\left(1 - \frac{2}{e^2}, \exp(p-2)\right),$$

and, if $C \in [1,2)$, then

$$0 \le h_2(C) < 1. \tag{17}$$

Proof. We have

$$h_p(x) = (x^2 - x)\exp(p - x) - x + 1,\ h_p'(x) = \tag{18}$$

$$-(x^2 - 3x + 1)\exp(p - x) - 1,\ h_p''(t) = (x^2 - 5x + 4)\exp(p - x).$$

If $x \in [0,1]$, then, clearly,

$$u(x) := -\frac{1}{4}\exp(2 - x) + 1 - x \le \tag{19}$$

$$-x(1-x)\exp(2-x) + 1 - x = h_p(x) \le 1 - x.$$

Further we have

$$u''(t) < 0,\ -1 < -\frac{\exp(2)}{4} + 1 = u(0) = \tag{20}$$

$$\min(u(0), u(1)) \le u(x) \le h_p(x).$$

The inequality (15) follows from (19) and (20). If $x \in [1,2]$, then, in view of the equalities (18), $h_p''(x) < 0$ for $x \in [1,2]$; hence

$$h_p(x) \ge \min(h_p(1),\, h_p(2)) \ge 2\exp(p-2) - 1 \ge \tag{21}$$

$$2\exp(-2) - 1, \text{ if } x \in [1,2].$$

On the other hand, in view of (18), $h_p'(x)$ decreases together with increasing of $x \in [1,2]$, $h_2'(1) = e - 1$, $h_2'(2) = 0$; hence $h_2'(x) > 0$, and $h_2(x)$ increases together with increasing $x \in (1,C)$, $0 \le h_2(x) \le h_2(C) \le h_2(2) = 1$. Therfore, if $x \in [1,C]$, then

$$h_p(x) = \exp(p-2)h_2(x) \le \exp(p-2)h_2(C) \le \exp(p-2), \tag{22}$$

and, if $C \in [1,2)$, then $0 \le h_2(C) < 1$. Clearly, the inequalities (16) follow from (21) and (22). ■

Corollary 1. *If $x \in [0,2]$ and $p \in [0,2]$, then*

$$|h_p(x)| \le 1. \tag{23}$$

Proof. The inequality (23) directly follows from (15) and (16) ■

Corollary 2. *Let $B \in (0,2)$, $A \in (0,B]$, $A \le p_n \le B$ for all the $n \in \mathbb{N}_0$, $b = \exp(B-1)$, $a = min(A, b\exp(A-b))$. If $x \in [a,2]$, then*

$$|h_{p_n}(x)| \le \lambda_0,$$

where λ_0 have been specified in (5).

Proof. Since $\exp(2)/4 - 1 > 0,84 > 0,73 > 1 - 2\exp(-2)$, it follows that the assertion of the Corollary directly follows from (15) and (16). ■

Corollary 3. *Let*

$$B \in (1,2],\ A \in (0,B),\ C \in (\max(1,A),B),\ A \le p \le B$$

for all the $n \in \mathbb{N}_0$, $b = \exp(B-1)$, $a = min(A, b\exp(A-b))$. If $x \in [a,C]$, then

$$\lambda_1 < 1,\ \textit{and}\ |h_p(x)| \le \lambda_1,$$

where λ_1 have been specified in (6).

Proof. Since $\exp(2)/4 - 1 > 0,84 > 0,73 > 1 - 2\exp(-2)$, it follows that the assertion of the Corollary directly follows from (15) and (16). ■

Let

$$g_{i,0}(x) = x,\ \text{and}\ g_{i,n}(x) = R_{p_{n-1+i}}(g_{i,n-1}(x)), \tag{24}$$

where $i \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, and let

$$a_{i,n} = \min_{x\in[a,b]}(g_{i,n}(x)), b_{i,n} = \min_{x\in[a,b]}(g_{i,n}(x)), \tag{25}$$

where $i \in \mathbb{N}_0$, $n \in \mathbb{N}_0$.

Clearly,

$$a_{i+k,n} = a_{i,n},\ b_{i+k,n} = b_{i,n},\ g_{i+k,n}(x) = g_{i,n}(x), \tag{26}$$

where $i \in \mathbb{N}_0$, $n \in \mathbb{N}_0$.

Lemma 3. *Let*

$$i \in \mathbb{N}_0,\ n \in \mathbb{N}_0,\ m \in \mathbb{N}_0.$$

Then

$$g_{i,n+m}(x) = g_{i+n,m}(g_{i,n}(x)). \tag{27}$$

Proof. Clearly, if $m = 0$ then (27) holds. Suppose that $m \in \mathbb{N}$, and let the equality (27) holds for $m-1$ instead of m. Then,

$$g_{i+n,m}(g_{i,n}(x)) = R_{p_{m-1+i+n}}(g_{i+n,m-1}(g_{i,n}(x))) =$$

$$R_{p_{m-1+i+n}}(g_{i,n+m-1}(x)) = g_{i,n+m}(x).$$

■

Clearly,

$$g'_{i,n}(x) = (1 - g_{i,n-1}(x))\frac{g_{i,n(x)}}{g_{i,n-1}(x)}g'_{i,n-1}(x)) \tag{28}$$

for any $n \in \mathbb{N}$. The iteration of (28) give the equality

$$g'_{i,n}(x) = \frac{g_{i,n}(x)}{x}\prod_{\kappa=0}^{n-1}(1 - g_{i,\kappa}(x)). \tag{29}$$

for any $n \in \mathbb{N}_0$.

Lemma 4. *Let*

$$B \in (0, 2),\ A \in (0, B],\ A \le p_n \le B \textit{ for all the } n \in \mathbb{N}_0.$$

Let $b \ge \exp(B-1)$, $a = \min(A, b\exp(A-b))$, $a \le x \le b$ *Let* $\{x_n\}_{n=0}^{+\infty}$ *be a solution of the equation (1). Let further* $\{m_1,\ m_2\} \subset \mathbb{N}_0$ *and* $m_1 < m_2$. *Let, finally,* $i \in \mathbb{N}_0$, *and*

$$g_{i,n}(x) \in [a, 2] \textit{ for } n \in [m_1,\ m_2) \cap \mathbb{Z}.$$

Then

$$\prod_{n=m_1}^{m_2} |g_{i,n}(x) - 1| \le (\lambda_0)^{[(m_2-m_1+1)/2]} \tag{30}$$

Proof. Let us consider the pairs

$$g_{i,m_2-2\kappa+2}(x),\ g_{m_2-2\kappa+1}(x),\ \text{where;}\, \kappa \in \mathbb{N},\ m_2 - 2\kappa + 1 \ge m_1,$$

i.e.

$$\kappa \le m_3 := [(m_2 - m_1 + 1)/2] \le (m_2 - m_1 + 1)/2.$$

Then

$$\prod_{\kappa=m_1}^{m_2} |g_{i,n}(x) - 1| =$$

$$\left(\prod_{i=1}^{m_3} |g_{i,m_2-2\kappa+2}(x) - 1||g_{m_2-2\kappa+1}(x) - 1|\right) \times$$

$$\prod_{n=m_1}^{m_2-2m_3} |g_{i,n}(x) - 1|,$$

According to the Corollary 2 of the Lemma 2,

$$|g_{i,m_2-2\kappa+2} - 1||g_{i,m_2-2i+1} - 1| =$$

$$|h_{p_{i+m_2-2\kappa+1}}(g_{i,m_2-2\kappa+1})| \le \lambda_0$$

for $i = 1$, ..., m_3 and, clearly,

$$\prod_{n=m_1}^{m_2-2m_3} |g_{i,n}(x) - 1| \le 1.$$

Hence (30) holds. ■

Lemma 5. *Let are fulfilled all the conditions of the Lemma 4.*
Let further $i \in \mathbb{N}_0$, $n \in \mathbb{N}_0$. *Then*

$$\prod_{\kappa=0}^{n} |g_{i,\kappa}(x) - 1| \leq \tag{31}$$

$$\leq (\lambda_0)^{\max(0,(n-1)/4))} \max(|b-1|, 1)$$

Proof. If $n = 0$ then, clearly, (31) holds. Let $n \in \mathbb{N}$. Let $n_0 = 0$, if $x_0 > 2$ and $n_0 = -1$ if $x_0 \leq 2$. Let $n \in \mathbb{N}$ and let

$$\mathfrak{N}(n) = \{\nu \in [0, n-1] \cap \mathbb{N} \colon g_{i,\nu}(x) > 2\}.$$

Let the set $\mathfrak{N}(n)$ consists of s elements. If $s > 0$, then let n_1, ..., n_s are all the elements in the set $\mathfrak{N}(\mathfrak{n})$, and let $n_i < n_j$, if $0 \leq i < j \leq s$. Let $n = n_{s+1}$. Since $g_{i,\nu+1}(x_0) < 2$ for any $\nu \in \mathfrak{N}(n)$, it follows that $n_i - n_{i-1} \geq 2$ if $s \geq 1$ and $i = 1, ..., s$. Clearly, $n_{s+1} - n_s = n - n_s \geq n - (n-1) = 1$,

$$n = n_{s+1} = n_0 + \sum_{i=1}^{s+1} (n_i - n_{i-1} \geq -1 + 2s + 1, s \leq n/2,$$

Further we have

$$\prod_{\kappa=0}^{n} |g_{i,\kappa}(x) - 1| = \tag{32}$$

$$\left(\prod_{j=0}^{s} \left(\prod_{\kappa=n_j+1}^{n_{j+1}} |g_{i,\kappa}(x) - 1| \right) \right) \times$$

$$\prod_{\kappa=0}^{n_0} |g_{i,\kappa}(x) - 1|.$$

If we take in the Lemma 3 $m_1 = n_j + 1$, $m_2 = n_{j+1}$ then we obtain the inequality

$$\prod_{\kappa=n_j+1}^{n_{j+1}} |g_{i,\kappa}(x) - 1| \leq (\lambda_0)^{[(n_{j+1}-n_j)/2]},$$

for $j = 0$, ..., s. Since $[(n_{j+1} - n_j)/2] \geq (n_{j+1} - n_j)/2 - 1/2$, it follows that

$$\sum_{j=0}^{s} [(n_{j+1} - n_j)/2] \geq ((n - n_0) - (s+1))/2 \geq (n - 1 - n_0 - s)/2 \geq (n-2)/4.$$

Therefore

$$\left(\prod_{j=0}^{s} \left(\prod_{i=n_j+1}^{n_{j+1}} |x_i - 1| \right) \right) \leq (\lambda_0)^{(n-2)/4)} \tag{33}$$

Clearly,

$$\prod_{\kappa=0}^{n_0} |g_{i,\kappa}(x)-1| \le \max(|x-1|,1) \le \max(|b-1|,1) \tag{34}$$

The inequality (30) follows from the inequalities (33) and (34). ■

Lemma 6. *Let* $B \in (0,2)$, $A \in (0,B]$

$$A \le p_n \le B \text{ for all the } n \in \mathbb{N}_0,$$

$b \ge \exp(B-1)$, $0 < a \le \min(A, b\exp(A-b))$, $a \le x \le b$. *Let* $n \in \mathbb{N}$. *Then*

$$|g'_{i,n}(x)| \le (\lambda_0)^{(\max(0,(n-3)/4)} \frac{b}{a} \max(|b-1|,1), \tag{35}$$

where λ_0 *have been specified in (5)*

Proof. Assertion of the Lemma follows directly from the Lemma 1, Lemma 5 and (29).

Let

$$f_0(x) = g_{0,k}(x) - x.$$

Since, according to the Lemma 1, $f_0(a) \ge 0$, $f_0(b) \le 0$, it follows that there exists a $x_0^{**} \in [a,b]$ such that $f_0(x_0^{**}) = 0$; let is fixed such x_0^{**}. Let $\{x_n^*\}_{n=0}^{+\infty}$ be a solution of the equation (1) with initial values $x_0^* = x_0^{**}$. This solution is $k-$*periodic*, and, according to the Lemma 1, $x_n^* \in [a,b]$ for any $n \in \mathbb{N}_0$. Clearly, $x_n = g_{0,n}(x_0)$, $x_n^* = g_{0,n}(x_0^*)$ for any $n \in \mathbb{N}_0$.for $i = 0, ..., k-1$. Therefore

$$|x_n - x^*_{k\{n/k\}}| = |x_n - x_n^*| =$$
$$|g'_{0,n}((1-\theta)x_n + \theta x_n^*)||x - x_0^*| \le$$
$$(b-a)|g'_{0,n}((1-\theta)x_n + \theta x_n^*)|,$$

where $\theta \in (0,1)$. and (4) follows from (35) now. ■.

The Theorem 1 in the case (a) is proved.

2. Proof of the Theorem 1 in the case (b) Let

$$C \in (\max(A,1),2). \tag{36}$$

Since $h_2(x)$ increases with increasing $x \in [1,2]$, and (16) holds, it follows that

$$0 \le h_2(x) \le h_2(C) < 1, \text{ if } x \in [1,C]. \tag{37}$$

Lemma 7. *Let*

$$k \in \mathbb{N},\ 0 < A < B = 2,\ A \le p_n \le B,\ p_{n+k} = p_n$$

for all the $n \in \mathbb{N}_0$. *Let further*

$$b \ge \exp(B-1),\ 0 < a \le \min(A, b\exp(A-b)),\ x \in [a,b].$$

Let (3) and (36) hold. Then $k \ge 2$ *and*

$$\{m \in [0,k] \cap \mathbb{Z} \colon g_{i,n+m}(x) < C\} \ne \emptyset \tag{38}$$

for any i and n in $\mathbb{N}_0$.

Proof. The inequality $k \geq 2$ directly follows from the relations $0 < A < B = 2$ and from (3). If (38) isn't true, then

$$g_{i,m}(x) > C \tag{39}$$

for any $m = 0, ..., k-1$ In view of the equalities (3), there exists $m \in [1, k] \cap \mathbb{N}$ such that $p_{i+m} = A$. Then

$$g_{i,m}(x) = R_{p_{i+m-1}}(g_{i,m-1}(x)) = g_{i,m-1}(x)\exp(A - g_{i,m-1}(x)).$$

In view of (39) and (36), $1 < C \leq g_{i,m-1}(x)$. Since $R_{p_{i+m-1}}(x)$ decreases together with increasing of $x \in [1, +\infty)$ it follows that

$$g_{i,m}(x) = g_{i,m-1}(x)\exp(A - g_{i,m-1}(x)) \leq C\exp(A - C) < C.$$

So we obtain a contradiction with (39). This proves the Lemma. ∎

Lemma 8. *Let $p_n \in [0, 2]$ for all the $n \in \mathbb{N}_0$ Let $\{x_n\}_{n=0}^{+\infty}$ be a solution of equation (1). Let further $\{m_1, m_2\} \subset \mathbb{N}_{\not{\ }}$ and $m_1 < m_2$. Let further*

$$x_n \in [a, 2] \text{ for } n \in [m_1, m_2) \cap \mathbb{Z}. \tag{40}$$

Then

$$\prod_{n=m_1}^{m_2} |x_n - 1| \leq 1. \tag{41}$$

Proof. Let $m = m_2 + 1 - m_1$. If $m = 1$, then

$$\prod_{n=m_1}^{m_2} |x_n - 1| = |x_{m_1} - 1| \leq 1. \tag{42}$$

If $m = 2$, then assertion of the Lemma coincides with assertion of the Corollary 1 of The Lemma 2. Let $m \geq 3$ and let assertion of the Lemma is true for all $\mu \in [1, m-1] \cap \mathbb{Z}$. Then it is true for $\mu = m - 2 \geq 1$. Therefore

$$\prod_{n=m_1}^{m_2-2} |x_n - 1| \leq 1, \tag{43}$$

and, accpording to assertion of the Corollary 1 of The Lemma 2,

$$\prod_{n=m_2-1}^{m_2} |x_n - 1| \leq 1, \tag{44}$$

In view of (43) and (44) assertion of the Lemma is true for $\mu = m$. ∎.

Lemma 9. *Let*

$$k \in \mathbb{N},\ 0 < A < B = 2,\ A \leq p_n \leq B,\ p_{n+k} = p_n$$

for all the $n \in \mathbb{N}_0$,

$$b \geq \max(\exp(B - 1))],\ 0 < a \leq \min(A, b\exp(A - b)),\ a \leq x \leq b,$$

(45) $$\max(A,1) < C < 2,$$

the equalities (3) take place, $x \le 2.$ and $m \in [k+1,+\infty] \cap \mathbb{N}$.

Then (for given $i \in \mathbb{N}_0$, $n \in \mathbb{N}_0$ and $x \in (0,2]$)

there exists $\nu \in [0,k+1] \cap \mathbb{N}_0$ such that

(46) $$\prod_{\kappa=0}^{\nu} |1 - g_{i,n+\kappa}(x)| \le \lambda_1(C)) < 1,$$

and

(47) $$g_{i,n+\nu+1}(x) \le 2,$$

where $\lambda_1(C)$ specified in (6).

Proof. In view of (38), let

(48) $$m_0 = \{m \in [0,k] \cap \mathbb{Z} \colon g_{i,n+m}(x) < C\}$$

The product

$$\left(\prod_{\kappa=0}^{m_0-1} |1 - g_{i,n+\kappa}(x)|\right)$$

is an empty product, if $m_0 = 0$, and, according to the Lemma 8, this product is not bigger, than 1, if $m_0 \ge 1$. Hence,

(49) $$\prod_{\kappa=0}^{m_0} |1 - g_{i,n+\kappa}(x)| =$$

$$|1 - g_{i,n+m_0}(x)| \times$$

$$\prod_{\kappa=0}^{m_0-1} |1 - g_{i,n+\kappa}(x)| \le$$

$$|1 - g_{i,n+m_0}(x)| \le C - 1 \le \lambda_1(C).$$

If $g_{i,n+m_0+1}(x) \le 2$, then, we can take $\nu = m_0$ and for this ν assertion of the Lemma is true. If $g_{i,n+m_0+1}(x) > 2$, then we have

(50) $$\prod_{\kappa=0}^{m_0+1} |1 - g_{i,n+\kappa}(x)| =$$

$$|1 - g_{i,n+m_0}(x)||1 - g_{i,n+m_0+1}(x)| \times$$

$$\prod_{\kappa=0}^{m_0-1} |1 - g_{i,n+\kappa}(x)| \le$$

$$|1 - g_{i,n+m_0}(x)||1 - g_{i,n+m_0+1}(x)| \le \lambda_1(C),$$

according to the Lemma 2; further we have

$$g_{i,n+m_0+2}(x) = R_{p_{i+n+m_0+1}}(g_{i,n+m_0+1}(x)) =$$

$$\exp(p_{i,n+m_0+1})(g_{i,n+m_0+1}(x)) \exp(-g_{i,n+m_0+1}(x)) <$$

$$\exp(p_{i,n+m_0+1})2\exp(-2) \leq 2.$$

Therefore the assertion of the Lemma is true for $\nu = m_0 + 1$ in this case. ■

If the conditions of the Lemma 9 are fulfilled, then we denote by $\nu^{\sim}(i,n,x)$ the smallest number amomg of all the ν, which satisfies the reqirements of the aserrtion of this Lemma. Let

$$\nu_0^{\wedge\wedge}(x) = -1, \text{ if } x \in [0,2],$$

and

$$\nu_0^{\wedge\wedge}(x) = 0, \text{ if } x > 2.$$

Then $g_{0,\nu_0^{\wedge\wedge}(x)+1}(x) \in [0,2]$. According to the Lemma 9, there exists an increasing sequence $\{\nu^{\wedge}(m,x)\}_{m=0}^{+\infty}$ with the following properties:

(I)

$$\nu^{\wedge}(m,x) \in [-1,+\infty) \cap \mathbb{Z} \text{ for all the } m \in \mathbb{N}_{\not{r}},$$

(II)

$$\nu^{\wedge}(0,x) = \nu_0^{\wedge\wedge}(x),$$

(III)

$$g_{0,\nu^{\wedge}(m,x)+1}(x) \in [0,2] \text{ for all the } m \in \mathbb{N}_{\not{r}},$$

(IV)

$$\nu^{\wedge}(m,x) + 1 \leq \nu^{\wedge}(m+1,x) \leq \nu^{\wedge}(m,x) + k + 2, \tag{51}$$

(V)

$$\prod_{\kappa=\nu^{\wedge}(m-1,x)+1}^{\nu^{\wedge}(m,x)} |1 - g_{0,\kappa}(x)| \leq \lambda_1(C),$$

where $m \in \mathbb{N}$ and $\lambda_1(C)$ is specivied in (7). For example, the sequence, produced by recurrence equation

$$\nu^{\wedge}(0,x) = \nu_0^{\wedge\wedge}(x),\ \nu^{\wedge}(m+1,x) =$$

$$\nu^{\sim}(i,n,y),$$

with $i = n = \nu^{\wedge}(m,x) + 1, y = g_{0,\nu^{\wedge}(m,x)}(x)$ has the properties (I) – (V).

Let $n \in \mathbb{N}_0$. Then, clearly, $n \geq \nu_0^{\wedge}(x)$. Let

$$\mu = \mu(n,x) = \max\{m \in \mathbb{N}_0 \colon n \geq \nu^{\wedge}(m,x)\}$$

Then

$$\nu^{\wedge}(\mu(n,x),x) \leq n < \nu^{\wedge}(\mu(n,x)+1,x) \leq \nu^{\wedge}(\mu(n,x),x) + k + 2,$$

and

$$0 \leq n - \nu^{\wedge}(0,x)(x) \leq$$

$$\nu^{\wedge}(\mu(n,x)+1,x) - \nu^{\wedge}(0,x) =$$

$$\sum \kappa = 0^{\mu(n,x)}(\nu^{\wedge}(\kappa+1,x) - \nu^{\wedge}(\kappa,x)) \leq (\mu(n,x)+1)(k+2).$$

Hence $\mu(n,x) \ge \max(0, n/(k+2)-1) = \max(0,(n-k-2)/(k+2))$. Therefore

$$\prod_{\kappa=0}^{n} |1-g_{0,\kappa}(x)| = \left(\prod_{\kappa=0}^{\nu_0^\wedge(x)} |1-g_{0,\kappa}(x)|\right) \times$$

$$\left(\prod_{\kappa=\nu_0^\wedge(x)+1}^{\nu_\mu^\wedge(x)} |1-g_{0,\kappa}(x)|\right) \times$$

$$\left(\prod_{\kappa=\nu_\mu^\wedge(x)+1}^{n} |1-g_{0,\kappa}(x)|\right).$$

Clearly,

$$\left(\prod_{\kappa=0}^{\nu_0^\wedge(x)}\right) \le (b-1),$$

the product

$$\prod_{\kappa=\nu^\wedge(\mu,x)+1}^{n} |1-g_{0,\kappa}(x)|$$

is an empty product if $\mu(n,x)=n$, and, according to the Lemma 8, it is not bigger then 1, if $\mu(n,x)<n$. Finally,

$$\prod_{\kappa=\nu_0^\wedge(x)+1}^{\nu^\wedge(\mu,x)} |1-g_{0,\kappa}(x)| =$$

$$\prod_{m=1}^{\mu}\left(\prod_{\kappa=\nu^\wedge(m-1,x)+1}^{\nu^\wedge(m,x)} |1-g_{0,\kappa}(x)|\right) \le \lambda_1^{\mu(n,x)} \le \lambda_1^{\max(0,(n-k-2)/(k+2))}.$$

Hence,

$$\prod_{\kappa=0}^{n} |1-g_{0,\kappa}(x)| \le (b-1)\lambda_1^{\max(0,(n-k-2)/(k+2)}, $$

(52) $$|g'_{0,n}(x)| \le \frac{b}{a}(b-1)(\lambda_1(C))^{\max(0,(n-k-3)/(k+2)},$$

where $n \in \mathbb{N}_0$ and we have (7). The Theorem 1 in the case (b) is proved.

3. Proof of the Theorem 1 in the cases (c) and (d)

Lemma 10. *Let*

$$k \in \mathbb{N},\ 0 = A < B < 2,\ A \le p_n \le B,\ p_{n+k} = p_n$$

for all the $n \in \mathbb{N}_0$. *Let*

$$b \ge \exp(B-1),\ 0 < a \le \min(b\exp(A-b),\ p^*),$$

and (3) takes place. Then $k \ge 2$ *and*

(53) $$\{m \in [1,k] \cap \mathbb{Z}\colon\ g_{i,m}[a,b]) \subset [a,b]\} \ne \emptyset$$

for any $i \in \mathbb{N}_0$.

Proof. The inequality $k \geq 2$ directly follows from the equality $A = 0$, inequality $B > 0$ and (3) . Clearly, $g_{i,m}([0, b]) \subset [0, b]$. Therefore we must prove that there exsists $m_i \in [1, k] \cap \mathbb{Z}$ such that $g_{i,m_i}([a, b]) \subset [a, +\infty)$. The contrary means that

$$g_{i,m}([a, b]) \not\subset [a, +\infty). \tag{54}$$

for any $m = 1, ..., k$. Then $g_{i,m}([a, b]) = [a_{i,m}, b_{i,m}]$, where

$$a_{i,m} < a, b_{i,m} \leq b \tag{55}$$

for any $m = 1, ..., k$. We want to prove that

$$a_{i,m} = g_{i,m}(a) \tag{56}$$

for any $m = 0, ..., k$. Since $g_{i,0}(x) = x$, it follows that (56) holds for $m = 0$. Suppose that $m \in [1, k] \cap \mathbb{N}$, and (56) holds for $m - 1$. If $b_{i,m-1} \leq 1$, then the function $R_{p_{i+m-1}}(x)$ increases on $[a_{i,m-1}, b_{i,m-1}]$; hence

$$g_{i,m}(x) = R_{p_{i+m-1}}(g_{i,m-1}(x)) \geq R_{p_{i+m-1}}(a_{i,m-1}) =$$

$$R_{p_{i+m-1}}(g_{i,m-1}(a)) = g_{i,m}(a).$$

If $1 < b_{i,m-1} \leq b$, then the function $R_{p_{i+m-1}}(x)$ increases on $[a_{i,m-1}, 1]$ and decreases on $[1, b_{i,m-1}]$. Therefore, in view of (55),

$$a > a_{i,m} = \min_{x \in [a,b]} g_{i,m}(x) = \min_{x \in [a,b]} R_{p_{i+m-1}}(g_{i,m-1}(x)) =$$

$$\min(R_{p_{i+m-1}}(a_{i,m-1}), R_{p_{i+m-1}}(b_{i,m-1}).$$

Since

$$R_{p_{i+m-1}}(b_{i,m-1}) \geq R_{p_{i+m-1}}(b)) \geq b \exp(-b)) \geq a,$$

it follows that

$$a > a_{i,m} = \min_{x \in [a,b]} g_{i,m}(x) = R_{p_{i+m-1}}(a_{i,m-1}) =$$

$$R_{p_{i+m-1}}(g_{i,m-1}(a)) = g_{i,m}(a).$$

So, the contrary to the assertion of the Lemma means that $g_{i,m}(a) < a$ for all the $m = 1, ..., k$. Since $k \geq 2$, it follows that

$$\sum_{m=0}^{k-1}(p_{i+m} - g_{i,m}(a)) = kp^* - \sum_{m=0}^{k-1} g_{i,m}(a)) > kp^* - ka \geq 0$$

In view of (13),

$$g_{i,k}(a) = a \exp\left(\sum_{m=0}^{k-1}(p_{i+m} - g_{i,m}(a))\right) \geq \tag{57}$$

$$a \exp(kp^* - ka).$$

So we obtain a contradiction with (55). This proves the Lemma. ∎

In view of (53) let

(58) $$\nu^{\vee}(i) = \min(\{m \in [1,k] \cap \mathbb{Z}\colon g_{i,m}[a,b]) \subset [a,b]\}$$

for any $i \in \mathbb{N}_0$. In view of (58),

(59) $$g_{i,\nu^{\vee}(i)}[a,b]) \subset [a,b]\}$$

Lemma 11. Let $\{i,n\} \subset \mathbb{N}_0$, and let

$$g_{i,n}([a,b]) \subset [a,b]\}$$

Then

(60) $$g_{i,n+\nu^{\vee}(i+n)}([a,b]) \subset [a,b].$$

for any $i \in \mathbb{N}_0$.

Proof. In view of (27),

(61) $$g_{i,n+\nu^{\vee}(i+n)}(x) = g_{i+n,\nu^{\vee}(i+n)}(g_{i,n}(x)).$$

Therefore

$$g_{i+n,\nu^{*}(i+n)}(g_{i,n}([a,b])) \subset g_{i+n,\nu^{\vee}(i+n)}([a,b]) \subset [a,b].$$

■

In view of (61), let

(62) $$\nu^{*}(i,0) = 0, \nu^{*}(i,n) = \nu^{*}(i,n-1) + \nu^{\vee}(i + \nu^{*}(i,n-1)),$$

for any $n \in \mathbb{N}$.

In view of (58), (62),

(63) $$\nu^{*}(i,n) \geq n,$$

for any $n \in \mathbb{N}_0$.

According to the Lemma 11,

(64) $$g_{i,\nu^{*}(i,n)}([a,b]) \subset [a,b].$$

In view of (62),

(65) $$1 \leq \nu^{*}(i,n) - \nu^{*}(i,n-1) \leq k,$$

Lemma 12. *If* $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$, *then*

(66) $$\nu^{*}(i,n+m) = \nu^{*}(i,n) + \nu^{*}(i + \nu^{*}(i,n), m).$$

Proof. We apply induction on m. Clearly, (66) holds for m=0. In view of (62),

$$\nu^{*}(i,1) = \nu^{\vee}(i),$$

(67) $$\nu^{*}(i,n+1) = \nu^{*}(i,n) + \nu^{\vee}(i + \nu^{*}(i,n)) =$$

$$\nu^*(i,n) + \nu^*(i + \nu^*(i,n), 1)$$

Let $m \in \mathbb{N}$, $m \geq 2$ and let assertion of the Lemma holds for $m-1$. Let further $i_1 = i + \nu^*(i,n)$ Then, in view of (62) and inductive hypothesis

$$\nu^*(i, n+m) = \tag{68}$$

$$\nu^*(i, n+m-1) + \nu^\vee(i + \nu^*(i, n+m-1)) =$$
$$\nu^*(i, n+m-1) + \nu^\vee(i + \nu^*(i, n+m-1)) =$$
$$\nu^*(i, n+m-1) + \nu^*(i + \nu^*(i, n+m-1), 1),$$

According to inductive hypothesis,

$$\nu^*(i, n+m-1) = \nu^*(i,n) + \nu^*(i + \nu^*(i,n), m-1) = \tag{69}$$

$$\nu^*(i,n) + \nu^*(i_1, m-1).$$

Therefore

$$\nu^*(i + \nu^*(i, n+m-1), 1) = \tag{70}$$

$$\nu^*(i + \nu^*(i,n) + \nu^*(i_1, m-1), 1) =$$
$$\nu^*(i_1 + \nu^*(i_1, m-1), 1).$$

In view of (68) – (70) and inductive hypothesis,

$$\nu^*(i, n+m) = \nu^*(i,n) + \tag{71}$$

$$\nu^*(i_1, m-1) + \nu^*(i_1 + \nu^*(i_1, m-1), 1) = \nu^*(i,n) +$$
$$\nu^*(i_1, m) = \nu^*(i,n) + \nu^*(i + \nu^*(i,n), m)$$

■

Lemma 13. *If* $n \in \mathbb{N}_0$, *then*

$$g_{i,\nu^*(i,n)}([a,b]) \subset [a,b] \tag{72}$$

Proof. Since $\nu^*(i,0) = 0$, it follows that the assertion of the Lemma is true for $n = 0$. Suppose that $n \in \mathbb{N}$, and the assertion of the lemma is true for $n-1$. Hence $g_{i,\nu^*(i,n-1)}([a,b]) \subset [a,b]$. Let $i^* = i + \nu^*(i, n-1)$, and, in view of (62), let

$$m^* = \nu^*(i,n) - \nu^*(i,n-1) = \nu^\vee(i + \nu^*(i,n-1)) = \nu^\vee(i^*);$$

Then, in view of (58),

$$g_{i^*,\nu^\vee(i^*)}([a,b]) \subset [a,b]\}.$$

Hence,

$$g_{i,\nu^*(i,n)}([a,b]) = g_{i,\nu^*(i,n-1)+m^*}([a,b]) =$$
$$g_{i+\nu^*(i,n-1),m^*}(g_{i,\nu^*(i,n-1)}([a,b])) =$$
$$g_{i^*,\nu^\vee(i^*)}(g_{i,\nu^*(i,n-1)}([a,b])) \subset [a,b].$$

■ **Lemma 14.** *Let*

$$k \in \mathbb{N},\, 0 = A < B < 2,\, A \le p_n \le B,\, p_{n+k} = p_n$$

for all the $n \in \mathbb{N}_0$. *Let* $x > 0$,

$$b \ge \max(\exp(B-1))],\, 0 < a \le \min(b\exp(A-b),\, p^*),$$

and (3) takes place.

If $n_1 \in \mathbb{N}_0$, $n_2 \in \mathbb{N}$,

$$a < g_{i,n}(a),\, g_{i,n+m}(a) < a \text{ for } m = 1,\, ...,\, n_2, \tag{73}$$

then

$$g_{i,n+m}(a) \ge a\exp(-ma) \tag{74}$$

Proof. We apply induction on m. If $g_{i,n}(a) \ge 1$, then

$$g_{i,n+1}(a) = R_{p_{i+n}}(g_{i,n}(a)) \ge b\exp(p-b) \ge a,$$

what is impossible, according to (73). Hence,

$$g_{i,n+1}(a) = R_{p_{i+n}}(g_{i,n}(a)) \ge R_{p_{i+n}}(a) \ge a\exp(-a).$$

If $n_2 = 1$, then the Lemma is proved. Let $n_2 \ge 2, m \in [2, n_2] \cap \mathbb{N}$, and let the inequality (74) holds for $m-1$ instead of m. Then

$$g_{i,n+m}(a) = R_{p_{i+n+m-1}}(g_{i,n+m-1}(a)),$$

and $g_{i,n+m-1}(a) < a < 1$. Therefore

$$g_{i,n+m}(a) \ge R_{p_{i+n+m-1}}(a\exp(-(m-1)a)) =$$

$$a\exp(-a(m-1) + p_{i+n+m-1} - a\exp(-(m-1)a) \ge a\exp(-am).$$

■

Corollary 1. *If* $\nu^*(i,n) < n_1 < \nu^*(i,n+1)$, *then* $g_{i,n_1}(a) \ge a\exp(-ak)$

Proof. The assertion of the Corollary follows from the inequalities

$$g_{i,\nu_i^*(n)}(a) \ge a,\; g_{i,\nu_i^*(n+1)}(a) \ge a,$$

$$g_{i,\nu_i^*(\kappa)}(a) < a \text{ for } \kappa \in (\nu_i^*(n), \nu_i^*(n+1)) \cap \mathbb{N}_0,$$
$$\nu_i^*(n+1)) - \nu_i^*(n) \le k+1,$$

and from the Lemma 10.

Corollary 2. The inequality

$$g_{i,\kappa}(a) \ge a\exp(-ak))$$

holds for any $\kappa \in [i, +\infty) \cap \mathbb{N}_0$.

Proof. Since $\nu^*(i,0) = 0$, $\nu^*(i,n+1) - \nu^*(i,n) \ge 1$, it follows that

$$\cup_{n=0}^{+\infty}[\nu_i^*(n), \nu_i^*(n+1)) = [0, +\infty).$$

Hence, if $\kappa \in \mathbb{N}_0$, then $\kappa \in [\nu^*(i,n), \nu^*(i,n+1)) \cap \mathbb{N}_0$ for some $n \in \mathbb{N}_0$. According to the Corollary 1,

$$g_{i,\kappa}(a) \geq a \exp(-ak).$$

■

Lemma 15. *Let* $k \in \mathbb{N}$,

$$B \in (0,2),\ A = 0\ (\textit{hence}\ p^* > 0),\ A \leq p_n \leq B, p_{n+k} = p_n$$

for all the $n \in \mathbb{N}_0$, *and (3) holds. Let further*

$$b \geq \exp(B-1),\ 0 < a \leq \min(p^*,\ b\exp(A-b)),\ a \leq x \leq b.$$

Let $n \in \mathbb{N}_0$. *Then*

$$|g'_{i,n}(x)| \leq (\lambda_2)^{\max(0,(n-3)/4))} \frac{b}{a} \max(|b-1|, 1), \tag{75}$$

where λ_2 *have been specified in (9).*

Proof. It is sufficient to repeat the Proof of the Lemma 6 with λ_2 instead of λ_0 ■

Lemma 16. *Let* $k \in \mathbb{N}$,

$$B \in (0,2],\ A = 0\ (\textit{hence}\ p^* > 0,\ A \leq p_n \leq B, p_{n+k} = p_n$$

for all the $n \in \mathbb{N}_0$, *and (3) holds. Let further*

$$b \geq \exp(B-1),\ 0 < a \leq \min(p^*,\ b\exp(A-b)),\ a \leq x \leq b, C \in (1,2).$$

Let $n \in \mathbb{N}_0$. *Then*

$$|g'_{0,n}(x)| \leq \frac{b}{a}(b-1)(\lambda_3(C))^{\max(0,(n-k-3)/(k+2))}, \tag{76}$$

where $\lambda_3(C)$ *have been specified in (10).*

Proof. It is sufficient to repeat the Proof of the inequality (52) with the value $\lambda_3(C)$ instead of $\lambda_1(C)$ ■

Clearly, there is two numbers n_1 and n_2 in the set $\{0, ..., k\}$ such that $n_1 < n_2$ and

$$k\{\nu^*(0,n_1)/k\} = k\{\nu^*(0,n_2)/k\}, \tag{77}$$

where, as usually, $\{x\}$ denotes the fractional part of x.

Let $m_0 = \nu^*(0,n_2) - \nu^*(0,n_1)$, $i_0 := \nu^*(0,n_1)$ Since $n_1 \leq k-1$ and $\nu^*(0,\mu+1) - \nu^*(0,mu) \leq k+1$ for any $\mu \in \mathbb{N}_0$, it follows that

$$i_0 \leq k^2,\ m_0 \leq (k+1)k \tag{78}$$

If we take in (66) $i = 0$, $n = i_0$, $m = n_2 - n_1$, then we obtain the equality

$$m_0 = \nu^*(i_0, n_2 - n_1)$$

In view of (77), $q := m_0/k \in \mathbb{N}$. Since $m_0 \leq (k+1)k$, it follows that $q \leq k+1$. According to the Lemma 13,

$$g_{i_0,m_0)}([a,b]) \subset [a,b] \tag{79}$$

Hence, as in section 1, the map $x \mapsto g_{i_0,m_0)}(x)$, $x \in [a,b]$ has a fixpoint $x^{**}_{i_0}$. Let

$$\{x^*_n\}_{n=i_0}^{+\infty} \tag{80}$$

be a solution of the equation

$$x_{n+1} = R_{p_n}(x_n) = \tag{81}$$

$$x_n \exp(p_n - x_n),\ x_n \in (0,+\infty), n \in [i_0,+\infty) \cap \mathbb{N}_0$$

with initial values $x^*_{i_0} = x^{**}_{i_0} \in [a,b]$. This solution i qk-periodic. Let $\{x_n\}_{n=0}^{+\infty}$ be a solution of the equation (1) with $x_0 \in [a,b]$. Snce $i_0 = \nu^*(0,n_1)$, it follows that $x_{i_0} \in [a,b]$. Therefore, in the same art, as (4) and (7) are deduced, we obtain

$$|x_n - x^*_n| \leq (\lambda_2)^{\max(0,n-i_0-3)/4)} \frac{b}{a}(b-a)\max(|b-1|,1), \tag{82}$$

if $B \in (0,2)$, $A \in [0,B)$, $n \in [i_0,+\infty) \cap \mathbb{N}_0$, and λ_2 is specified in (9),

(83)

$$|x_n - x^*_n| \leq (\lambda_3(C))^{\max(0,(n-i_0-k-3)/(k+2))} \frac{b}{a}(b-a)\max(|b-1|,1),$$

if

$$B \in (0,2],\ A \in [0,B),\ C \in (\max(A,1),2),\ n \in [i_0,+\infty) \cap \mathbb{N}_0,$$

and $\lambda_3(C) \in (0,1)$ is specified in (10).

We will prove now that the sequence (80) is not omly qk-periodic but also qk-periodic.

Lemma 17. *There exists $c \in (0,a]$ such that $g_{0,k}(c) \geq c$.*

Proof. Let us consider the sequence

$$\{g_{0,kn}(a)\}_{n=0}^{+\infty}. \tag{84}$$

The contrary to the assertion of the Lemma means that this sequense is decreasing. In fact, if $\{g_{0,k(n+1)}(a) \geq g_{0,kn}(a)$, the we can take $c = g_{0,kn}(a)$, because then

$$g_{0,k}(c) = g_{0,k}(g_{0,kn}(a)) = \{g_{0,k(n+1)}(a) \geq g_{0,kn}(a) = c$$

So, the sequence (refeq:3bc) is decreasing. According to th Corollary 2 of the Lemma 14, $g_{0,kn}(a) \geq a\exp(-ka)$. Hence

$$c_0 = \lim_{n\to\infty}(g_{0,kn}(a)) \geq a\exp(-ka) > 0.$$

Then

$$g_{0,k}(c_0) = \lim_{n\to\infty}(g_{0,k}(g_{0,kn}(a))) =$$

$$\lim_{n\to+\infty}(g_{0,k(n+1)}(a)) = c_0.$$

■

Lemma 18. *There exists $y \in (0, b]$ such that $g_{0,k}(y) = y$.*

Proof. According to the Lemma 17, there exists $c \in (0, a]$ such that

$$g_{0,k}(c) \geq c.$$

If $g_{0,k}([c, b]) \subset [c, b]$ the map $x \to g_{0,k}(x)$ has a fixed point in $[c, b]$.

The opposite case means that $g_{0,k}(d) \notin [c, b]$ for some $d \in [c, b]$.

Since $g_{0,k}([c, b]) \subset (0, b]$, it follows that $g_{0,k}(d) < c \leq d$.

Let now $\psi(x) = g_{0,k}(x) - x$. Then $\psi(c) \geq 0$, $\psi(d) < 0$. Hence there exists $y \in [c, d)$ such that $\psi(y) = 0$, and $g_{0,k}(y) = y$. ■

According to the Lemma 18, there exists $y \in (0, b)$ such that $g_{0,k}(y) = y$; we fix such y; let $a_1 = \min(y, a)$ and let $\{y_n\}_{n=0}^{+\infty}$ be the solution of the equation (1) with $y_0 = y$. We replace now in the previous considerations the segment $[a, b]$ by segment $[a_1, b]$ and see that

$$|y_n - x_n^*| \leq C\lambda^n, \tag{85}$$

where C is come positive constant, $\lambda \in (0, 1)$, and $n \in [i_0, +\infty) \cap \mathbb{N}_0$. Each $n \in \mathbb{N}_0$ $n = r_1 + kr_2 + qk\nu$, where

$$r_1 \in [0, k-1] \cap \mathbb{N}_0, r_2 \in [0, q-1] \cap \mathbb{N}_0,$$

and $\nu \in \mathbb{N}_0$. Therefore the inequality (85) can be rewritten in the form

$$|y_{r_1} - x^*_{r_1+kr_2+qk\nu}| \leq C\lambda^{r_1+kr_2+qk\nu}. \tag{86}$$

Therefore

$$x^*_{r_1+k(r_2+q\nu)} = y_{r_1},$$

where $\{s = r_2 + ql,\ \nu\} \subset \mathbb{N}_0$,

$$r_1 \in [0, k-1] \cap \mathbb{N}_0, r_2 \in [0, q-1] \cap \mathbb{N}_0,$$

$$n = r_1 + k(r_2 + q\nu) \in [i_0, +\infty) \cap \mathbb{N}_0.$$

Consequently, the sequence (81) is not only qk-periodic, but also k-priodic, and

$$x_n^* = y_n \text{ for } n \in [i_0, +\infty) \cap \mathbb{N}_0.$$

Since x_n^* is undefined, if $n \in [0, i_0) \cap \mathbb{N}_0$, we let

$$x_n^* := y_n \text{ if } n \in [0, i_0) \cap \mathbb{N}_0.$$

Therefore

$$x_n^* = y_n \text{ for } n \in [0, +\infty) \cap \mathbb{N}_0 \tag{87}$$

In view of (26), we can replace $i_0 \leq k^2$, and by $k\{i_0/k\} \leq k-1$; therefore in view of (87), (82), (83), that

$$|x_n - x_n^*| \leq (\lambda_2)^{\max(0,(n-k-2)/4)} \frac{b}{a}(b-a)\max(|b-1|, 1), \tag{88}$$

if $B \in (0,2)$, $A \in [0,B)$, $n \in [k-1,+\infty) \cap \mathbb{N}_0$ and λ_2 have been specified in (9),

(89)
$$|x_n - x^*_{k\{n/k\}}| \le (\lambda_3(C))^{\max(0,(n-2k-2)/(k+2))} \frac{b}{a}(b-a)\max(|b-1|,1),$$

if

$$B \in (0,2],\ A \in [0,B),\ C \in (\max(A,1),2),\ n \in [k-1,+\infty) \cap \mathbb{N}_0;$$

and $\lambda_3(C) \in (0,1)$ have been specified in (10). ■ The Theorem 1 in the cases (c) and (d) is proved

3. Proof of the Theorem 1 in the case (e).

Lemma 19. *Let $\{x_n\}_{n=0}^{+\infty}$ be an arbitrary sequence, which satisfies to conditios*

$$0 \le x_{n+1} \le \frac{x_n}{1+x_n} \textit{ where } n \in \mathbb{N}_0.$$

Then

$$x_n \le \frac{x_0}{1+nx_0} \textit{ for any } n \in \mathbb{N}_0. \tag{90}$$

Proof. Assertion of the Lemma is true for $n=0$. Supppose that $n \in \mathbb{N}$ and that assertion of the Lemma is true for $n-1$ instead of n. Then

$$0 \le x_{n-1} \le \frac{x_0}{1+(n-1)x_0} \text{ and } 0 \le x_n \le \frac{x_{n-1}}{1+x_{n-1}}. \tag{91}$$

Since the function $x/(1+x)$ is increasing on $(0,+\infty)$, it follows from (91) that

$$0 \le x_n \le \frac{x_{n-1}}{1+x_{n-1}} \le\le \frac{x_0/(1+(n-1)x_0)}{1+x_0/(1+(n-1)x_0)} = x_0/(1+nx_0).$$

■

Since in the case (e) we have $x_{n+1} = x_n/\exp(x_n) \le x_0/(1+nx_0)$, it follows from the Lemma 19 that the assertion of the Theorem 1 is true in this case . ■ The Theorem 1 in the case (e) is proved.

4. Proof of the Theorem 1 in the case (f).

Let $\{x_n\}_{n=0}^{+\infty}$ be an arbitrary solution of the equation (1) Then $x_n \in (0,e]$ for all the $n \in \mathbb{N}$. Clearly $x_n \in [1,3]$ for some $n \in \mathbb{N}_0$, if and only if the iequality $|\eta_n| \le 1$ holds for $\eta_n = 2 - x_n$.

Let $|\eta| \le 1$. Then

$$(2-\eta)\exp(\eta) - 2 = \sum_{n=1}^{+\infty} \left(\frac{2}{n!} - \frac{1}{(n-1)!}\right)\eta^n =$$

$$\eta - \sum_{n=3}^{+\infty} \frac{n-2}{n!}\eta^n =$$

$$\eta\left(1 - \sum_{n=3}^{+\infty} \frac{n-2}{n!}\eta^{n-1}\right) =$$

$$\eta\left(1-\eta^2\left(\frac{1}{6}+\eta\sum_{n=4}^{+\infty}\frac{n-2}{n!}\eta^{n-4}\right)\right),$$

$$\frac{1}{6}+\eta\sum_{n=4}^{+\infty}\frac{n-2}{n!}\eta^{n-4}\geq$$

$$\frac{1}{6}-\sum_{n=4}^{+\infty}\frac{n-2}{n!}(-1)^n>$$

$$\frac{1}{6}-\frac{1}{12}=\frac{1}{12},$$

and

$$|(2-\eta)\exp(\eta)-2|\leq|\eta|(1-\eta^2/12)\leq\eta. \tag{92}$$

Therfore, if $|\eta_n|\leq 1$, then, in view of (92),

$$|\eta_{n+1}|\leq|\eta_n|(1-(\eta_n)^2/12)\leq\eta_n\leq 1, \tag{93}$$

and $|\eta_{n+1}|<\eta_n$, if $0<|\eta_n|\leq 1$. Let is given $\varepsilon\in(0,1]$.

If $|\eta_{n_0}|<\varepsilon$ for some $n_0\in\mathbb{N}_0$, then, in view of (93), $|\eta_{n_0+m}|\leq|\eta_{n_0}|<\varepsilon$ for all the $m\in\mathbb{N}_0$. If $1\geq|\eta_n|\geq\varepsilon>0$ for $n\in[n_0,n_0+m]\cap\mathbb{N}_0$, for some n_0 and m in $\mathbb{N}_0$, then, in view of (93),

$$|\eta_{n+1}|\leq|\eta_n|(1-(\varepsilon)^2/12), \tag{94}$$

$$\varepsilon\leq|\eta_{n_0+m}|\leq|\eta_{n_0}|(1-(\varepsilon)^2/12)^m\leq(1-(\varepsilon)^2/12)^m, \tag{95}$$

and

$$m\leq\frac{\ln(1/\varepsilon)}{\ln(1/(1-\varepsilon^2/12))}.$$

Hence, if $|\eta_{n_0}|\leq 1$ for some $n_0\ in\mathbb{N}_0$, then

$$|\eta_n|<\varepsilon,\text{ for all the }n\in[n_0+C_0(\varepsilon),+\infty),$$

where

$$C_0(\varepsilon)=1+\frac{\ln(1/\varepsilon)}{\ln(1/(1-\varepsilon^2/12))}.$$

The function $R_2(x)=x\exp(2-x)$ decreases with increasing $x\in[1,+\infty)$ and maps bijectively $[1,2]$ onto $[2,e]$; it maps bijectively the half-interval $[1,+\infty)$ onto $(0,e]$ also. Let $w(x)$ be the inverse map to the map

$$x\mapsto R_2(x),x\in[1,+\infty].$$

Let $\gamma_1=w(1)$. Since $R_2(3)=3/e\in(1,2)$ it follows that $\gamma_1>3$, and the function $R_2(x)=x\exp(2-x)$ maps bijectively $(3,\gamma_1]$ onto $[1,3/e)\subset[1,3]$; therefore if $x_0\in(3,\gamma_1]$, then $x_1\in[1,3]$.

If $x_0>\gamma_1$, then $x_1\in(0,1)$.

If $x_n\in(0,1)$ for some $n\in\mathbb{N}_0$, then each $m\in\mathbb{N}_0$ such that $x_\nu\in(0,1)$ for all the $\nu\in[n,n+m]\cap\mathbb{N}_0$ satisfies to the inequalities $1>x_{n+m}\geq x_n\exp(m)$ and $m<\ln(1/x_n)$; therefore

$$m_1=\max\{m\in\mathbb{N}_0\colon x_\nu\in(0,1)\text{ for }\nu\in[n,n+m]\cap\mathbb{N}_0\}\leq\ln(1/x_n),$$

and, if $m_2 = m_1 + 1$, then $x_{m_2} \in [1, e] \subset [1, 3]$.

Consequently, for each $x_0 \in (0, +\infty)$ there exists

$$n_0 = n_0(x_0) \in [0, max(0, \ln(1/x_0), \ln(1/x_1)) + 2$$

such that $x_{n_0} \in [1, 3]$.

Therefore

$$|x_n - 2| < \varepsilon \text{ if } n \in [C_1(\varepsilon), +\infty) \cap \mathbb{N},$$

where

$$C_1(\varepsilon) = max(0, \ln(1/x_0), \ln(1/x_1)) + \\ 3 + \frac{\ln(1/\varepsilon)}{\ln(1/(1 - \varepsilon^2/12))}.$$

■ The Theorem 1 is proved.

References.

[1] R. J.Sacker.,2007, A note on periodic Ricker maps. Journal of Difference Equations and Applications, Vol 13, No 1, January 2007, 89-92.