On behavior of the Bergman tau-function near the boundary of Hurwitz space

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1 Introduction

In this paper we study the asymptotical behavior of the Bergman tau-function introduced in ([5], [6], see also the survey [7]) near different components of the boundary of the highest stratum of Hurwitz space.

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The total Hurwitz space $\mathcal{H}_{g,N}$ consists of meromorphic functions of degree N on Riemann surfaces of genus g. This space can be naturally stratified according to multiplicities of poles and multiplicities of the critical points of the meromorphic functions. By $\mathcal{H}_{g,N}(1,\dots,1)$ we shall denote the moduli space of meromorphic functions with N simple poles and $2N+2g-2$ simple critical values on compact Riemann surfaces of the genus g.

Denote by $\mathcal{H}$ an arbitrary stratum of the Hurwitz space which consists of meromorphic functions

$$\lambda : \mathcal{L} \longrightarrow \mathbb{C}P^1$$

on compact Riemann surfaces $\mathcal{L}$ of genus g, such that the multiplicities of poles and multiplicities of critical points of the function λ are fixed (the union of all such strata coincides with the total Hurwitz space $\mathcal{H}_{g,N}$). The number of poles (counting multiplicities) of the function $\mathcal{L}$ equals N; the number of critical points (counting multiplicities) equals $2N+2g-2$. We denote the poles of function $\mathcal{L}$ by $\infty^{(1)},\dots,\infty^{(K)}$, and critical points by $P_1,\dots,P_M$. Corresponding critical values $\lambda_m := \lambda(P_m)$ can be used as local coordinates on $\mathcal{H}$.

Let us also introduce the covering $\tilde{\mathcal{H}}$ of the stratum $\mathcal{H}$ which consists of triples $(\mathcal{L},\lambda,\{a_\alpha,b_\alpha\})$, where $(\mathcal{L},\lambda)\in\mathcal{H}$, and $\{a_\alpha,b_\alpha\}$ is a canonical basis in homologies of $\mathcal{L}$. On the space $\tilde{\mathcal{H}}$ the critical values $\lambda_1,\dots,\lambda_M$ can also be used as local coordinates.

Choosing some canonical basis in homologies of $\mathcal{L}$ introduce the canonical meromorphic bidifferential $W(\cdot,\cdot)$ on $\mathcal{L}$ corresponding to this basis and the Bergman projective connection S_B via the asymptotical relation

$$W(x(P),x(Q)) = \left(\frac{1}{(x(P)-x(Q))^2} + \frac{1}{6}S_B(x(P)) + o(1)\right) dx(P)\,dx(Q)$$

as $Q \to P$, where x is a holomorphic local parameter on $\mathcal{L}$.

The Bergman tau-function τ is defined as a holomorphic solution to the system

$$\frac{\partial \log \tau}{\partial \lambda_m} = -\frac{1}{6}\mathrm{res}|_{P_m}\frac{S_B - S_{d\lambda}}{d\lambda} \tag{1.1}$$

where $d\lambda$ is the differential of function λ; $S_{d\lambda}$ is the projective connection defined by Schwarzian derivative $S_{d\lambda} = \{\lambda(P), x(P)\}$, where $x(P)$ is an arbitrary local parameter. The difference of two projective connections, S_B and $S_{d\lambda}$, is a meromorphic quadratic differential.

Solutions of system (1.1) are defined up to multiplication by an arbitrary constant, after analytical continuation of a solution of (1.1) along a closed loop in $\tilde{H}$ it gains a constant (i. e. $\{\lambda_1,\dots,\lambda_M\}$-independent) multiplicative twist. This means that the Bergman tau-function is defined as a section of some line bundle over $\tilde{H}$. In what follows we consider the Bergman tau-function only locally, i. e. over some contractible neighborhood of a chosen point in $\tilde{H}$.

The Bergman tau-function is an object of primary interest in the theory of Hurwitz spaces, Frobenius manifolds, isomonodromic deformations, Hermitian two-matrix models and spectral theory of Riemann surfaces. It appears as the isomonodromic tau-function of semisimple Frobenius manifolds related Hurwitz spaces, in a factorization formula for the determinant of the Laplacian in the Poincaré metric on a compact Riemann surface, it also enters the expression for the genus one contribution to free energy in Hermitian two-matrix model. Conjecturally, its modulus square essentially coincides with the properly regularized determinant of the Laplacian corresponding to the metric $|d\lambda|^2$ of the infinite volume on $\mathcal{L}$.

The Bergman tau-function if non-singular and non-vanishing everywhere inside corresponding stratum of the Hurwitz space (although in fact this is not a function, but a section of a holomorphic line bundle over the stratum, the notion of zero and pole of τ is well-difined).

Denote the multiplicity of a critical points P_m by k_m. In a neighbourhood of P_m we introduce *the distinguished local parameter*, x_m, $m = 1, \dots, M$ via

$$x_m(P) = (\lambda(P) - \lambda(P_m))^{1/(k_m+1)} .$$

Let us introduce the prime-form $E(P, Q)$ and the multiplicative holomorphic $g(1-g)/2$-differential

$$\mathcal{C}(P) = \frac{1}{\mathcal{W}[v_1, \dots, v_g](P)} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \Theta(K^P)}{\partial z_{\alpha_1} \dots \partial z_{\alpha_g}} v_{\alpha_1} \dots v_{\alpha_g}(P) , \tag{1.2}$$

where

$$\mathcal{W}(P) := \det_{1\le\alpha,\beta\le g} ||w_\beta^{(\alpha-1)}(P)|| \tag{1.3}$$

is the Wronskian determinant of holomorphic differentials at the point P; Θ is the theta-function built from the matrix $\mathbf{B}$ of b-periods of the Riemann surface $\mathcal{L}$.

The (locally unique, up to a multiplication with an arbitrary constant) solution of the system (1.1) was found in [5]:

Theorem 1 *The Bergman tau-function on the stratum $\mathcal{H}$ of the Hurwitz space is given by the following formula:*

$$\tau(\mathcal{L}, w) = \mathcal{F}^{2/3} e^{-\frac{\pi i}{6}\langle \mathbf{r}, \mathbf{B}\mathbf{r}\rangle} \prod_{m,n,\, m<n} \{E(D_m, D_n)\}^{d_m d_n/6} \tag{1.4}$$

where the function

$$\mathcal{F} := [d\lambda(P)]^{\frac{g-1}{2}} e^{-\pi i \langle \mathbf{r}, K^P\rangle} \left\{ \prod_{m=1}^{M} [E(P, D_m)]^{\frac{(1-g)d_m}{2}} \right\} \mathcal{C}(P) \tag{1.5}$$

is independent of P; the integer vector $\mathbf{r}$ is defined by the equality

$$\mathcal{A}\big((d\lambda)\big) + 2K^P + \mathbf{B}\mathbf{r} + \mathbf{q} = 0 ; \tag{1.6}$$

$\mathbf{q}$ *is another integer vector, $(d\lambda) \equiv \sum_m d_m D_m$ is the divisor of the differential $d\lambda$ (the multiplicities d_k are constant within a given stratum $\mathcal{H}$); the initial point of the Abel map $\mathcal{A}$ coincides with P and all the paths are chosen inside the same fundamental polygon $\widehat{\mathcal{L}}$. If one (or both) arguments of the prime-form coincides with some point of divisor $(d\lambda)$, the prime-form is computed with respect to the distinguished local parameter at this point, given by $x_m(P) := (\lambda(P) - \lambda(D_m))^{1/(d_m+1)}$.*

Globally, τ is a section of a line bundle over $\mathcal{H}$. However, here we consider τ locally on $\mathcal{H}$. Our goal is to understand the behavior of τ on various boundary components of the largest stratum $\mathcal{H}_{g,N}(1, \dots, 1)$, which consists of functions with simple poles and simple critical points.

When all critical points of function λ are simple, the distinguished local parameter is given by $x_m(P) = \sqrt{\lambda(P) - \lambda_m}$, the basic cycles on $\mathcal{L}$ can be always chosen such that $\mathbf{r} = 0$ and the definition (1.1) can be written as follows: can be rewritten as follows:

$$\frac{\partial \log \tau}{\partial \lambda_m} = -\frac{1}{12} S_B(x_m)|_{x_m=0}, \quad m = 1, \dots, M . \tag{1.7}$$

where $S_B(x_m)|_{x_m=0}$ is the Bergman projective connection computed at the point P_m with respect to the distinguished local parameter x_m.

In this paper we consider the Bergman tau-function on the space $\mathcal{H}_{g,N}(1,\dots,1)$ (i.e the stratum of highest dimension of the total Hurwitz space), and study its behavior on various components of the boundary of this space. The divisor $(d\lambda)$ for the stratum $\mathcal{H}_{g,N}(1,\dots,1)$ is given by:

$$(d\lambda) = \sum_{m=1}^{M} P_m - 2\sum_{n=1}^{N} \infty^{(n)}$$

In that case the expression (1.4) for τ looks as follows:

$$\tau = \mathcal{F}^{2/3} \left\{ \frac{\left\{\prod_{m,n=1}^{M} E(P_m,P_n)\right\} \left\{\prod_{k,l=1}^{N} E^4(\infty^{(l)},\infty^{(k)})\right\}}{\prod_{m=1}^{M}\prod_{k=1}^{N} E^2(P_m,\infty^{(k)})} \right\}^{1/6} \tag{1.8}$$

where

$$\mathcal{F} = [d\lambda(P)]^{(g-1)/2} \mathcal{C}(P) \left\{ \frac{\prod_{m=1}^{M} E(P,P_m)}{\prod_{k=1}^{N} E^2(P,\infty^{(k)})} \right\}^{(1-g)/2} \tag{1.9}$$

The boundary of $\mathcal{H}_{g,N}(1,\dots,1)$ consists of the following components:

1. The component arising when two simple branch points (say, P_1 and P_2) glue forming a double point, while corresponding Riemann surface is split into a union of two non-singular Riemann surfaces $\mathcal{L}_+$ and $\mathcal{L}_-$ (of genera g_+ and g_- respectively such that $g_+ + g_- = g$) "glued" along a node connecting points P_1^+ and P_1^- belonging to $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively ("separating case"). The functions λ_+ and λ_- on $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively, arising in the limit, do not have critical points on $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively.

2. The component arising when two simple branch points (say, P_2 and P_1) glue forming a double point, while corresponding Riemann surface does not split into a union of two non-singular Riemann surfaces ("non-separating case"). In this way we get a Riemann surface $\mathcal{L}_0$ of genus $g-1$ with a node connecting points P_1^+ and P_1^-. The function λ_0 on $\mathcal{L}_0$ arising as a limit of function λ on $\mathcal{L}$ does not have critical points at P_1^+ and P_1^-.

3. The "caustic", where two simple critical points of function λ coincide to form a critical point of multiplicity 2. The caustic $\mathcal{H}_{g,N}(1,\dots,1)$ is itself a stratum of the total Hurwitz space. Riemann surfaces corresponding to this component are non-singular.

4. The component arising when one of critical points of function λ and two simple poles of λ tend to each other. This component coincides with the stratum of total Hurwitz space where all critical points of function λ are simple, one pole has multiplicity two, and all other poles are simple. Riemann surfaces corresponding to this boundary component are non-singular, too.

Consider the first type of boundary, formed by singular Riemann surfaces consisting of two components ("separating case"). Let critical points, say P_1 and P_2, of the meromorphic function λ collide in such a way that the Riemann surface $\mathcal{L}$ degenerates to a nodal Riemann surface with components $\mathcal{L}^+$ of genus g^+ and $\mathcal{L}^-$ of genus g^- ($g^+ + g^- = g$; one can think about pinching a trivial cycle on the surface $\mathcal{L}$ passing through the colliding critical points; see the precise description of the degeneration below). Assume that a canonical basis $\{a_\alpha, b_\alpha\}$ on $\mathcal{L}$ can be represented as the union of two canonical bases $\{a_\alpha^+, b_\alpha^+\}$ and $\{a_\alpha^-, b_\alpha^-\}$ on $\mathcal{L}^+$ and $\mathcal{L}^-$, respectively.

The meromorphic function λ on $\mathcal{L}$ gives rise to two meromorphic functions λ_+ and λ_- on $\mathcal{L}^+$ and $\mathcal{L}^-$ respectively. Therefore, one gets two elements $(\mathcal{L}^\pm, \lambda_\pm, \{a_\alpha^\pm, b_\alpha^\pm\})$ of the Hurwitz spaces $\tilde{H}_{g^\pm, N^\pm}(1, \dots, 1)$.

Then one has the following asymptotics

$$\tau(\mathcal{L}, \lambda, \{a_\alpha, b_\alpha\}) \sim C(\lambda_1 - \lambda_2)^{1/4}\, \tau_+(\mathcal{L}^+, \lambda_+, \{a_\alpha^+, b_\alpha^+\})\, \tau_-(\mathcal{L}^-, \lambda_-, \{a_\alpha^-, b_\alpha^-\})\,,$$

as $s \to 0$, where C is a constant; $\lambda_1 := \lambda(P_1)$; $\lambda_2 := \lambda(P_2)$ are critical values and $\tau_\pm$ are the Bergman tau-functions on the Hurwitz spaces $\tilde{H}_{g^\pm, N^\pm}(1, \dots, 1)$.

In particular, one can observe that the asymptotical behavior of the Bergman tau-function resembles that of the Faltings δ-invariant:

$$e^{\delta(M_t)} \sim |t|^{-\frac{4g^+g^-}{g}} e^{\delta(M^+)} e^{\delta(M^-)}\,,$$

where M_t is a family of genus g Riemann surfaces degenerating as $t \to 0$ to a nodal Riemann surface with components $M^\pm$ of genus $g^\pm$ (see [11], [4]).

Consider the second case, when in the limit $P_2 \to P_1$ the Riemann surface $\mathcal{L}$ turns into a Riemann surface $\mathcal{L}_0$ of genus $g-1$ with a node at P_1. Let us assume that the basic cycle a_g on $\mathcal{L}$ is chosen to encircle the branch cut $[P_1, P_2]$, i.e. in the limit $P_2 \to P_1$ the cycle a_g encircles the nodal point P_1.

The asymptotics of Bergman tau-function in this limit has the following form:

$$\tau(\mathcal{L}, \lambda; \{a_\alpha, b_\alpha\}) \sim C(\lambda_1 - \lambda_2)^{1/4} \tau_0(\mathcal{L}_0, \lambda_0; \{a_1, b_1, \dots, a_{g-1}, b_{g-1}\})\,. \tag{1.10}$$

In the third case (caustic), as $P_2 \to P_1$, we get the following asymptotics of the tau-function:

$$\tau(\mathcal{L}, \lambda, \{a_\alpha, b_\alpha\}) \sim C(\lambda_2 - \lambda_1)^{1/12}\, \tau_0(\mathcal{L}_0, \lambda_0, \{a_\alpha, b_\alpha\}) \tag{1.11}$$

where $\tau_0(\mathcal{L}_0, \lambda_0, \{a_\alpha, b_\alpha\})$ is the Bergman tau-function on the caustic $\mathcal{H}^C$ i.e. on the stratum of Hurwitz space where all branch points are simple, except the point P_1^0 on $\mathcal{L}_0$ obtained from gluing P_2 and P_1, which has multiplicity two, C is a constant

In the fourth case, when a critical point P_1 is glued with two poles $\infty^{(1)}$ and $\infty^{(2)}$ of λ, we get a non-singular Riemann surface $\mathcal{L}_0$ of the same genus g; the function λ turns into meromorphic function λ_0 on $\mathcal{L}_0$ with double pole at the point $\infty^{(1)}$ on $\mathcal{L}_0$ obtained by gueing P_1 with $\infty^{(1)}$ and $\infty^{(2)}$. Denoting by τ_0 the Bergman tau-function on the stratum of Hurwitz space corresponding to meromorphic functions with all simple critical points, one double pole and other simple poles, we get the asymptotics as $\lambda_1 \to \infty$:

$$\tau(\mathcal{L}, \lambda, \{a_\alpha, b_\alpha\}) \sim C\lambda_1^{1/4}\, \tau_0(\mathcal{L}_0, \lambda_0, \{a_\alpha, b_\alpha\}) \tag{1.12}$$

where C is a constant.

2 Monodromies of Bergman tau-function

The tau-function on an arbitrary stratum of Hurwitz space defined either by differential equations (1.1) or by explicit formulas (1.4), (1.5) depends on the choice of canonical basis of cycles on $\mathcal{L}$. The following theorem describes the transformation law of τ under symplectic change of the canonical basis.

Theorem 2 *Let two canonical bases of cycles on $\mathcal{L}$ are related by a symplectic matrix:*

$$\begin{pmatrix} \tilde{\mathbf{b}} \\ \tilde{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \tag{2.1}$$

where $\mathbf{a} = (a_1, \dots, a_g)^t$, $\mathbf{b} = (b_1, \dots, a_g)^t$ Then corresponding Bergman tau-functions are related as follows:

$$\frac{\tau(\mathcal{L}, \lambda, \{\tilde{a}_\alpha, \tilde{b}_\alpha\})}{\tau(\mathcal{L}, \lambda, \{a_\alpha, b_\alpha\})} = \epsilon \det(C\mathbf{B} + D) \tag{2.2}$$

where $\mathbf{B}$ is the matrix of b-periods of $\mathcal{L}$; ϵ is a root of unity: $\epsilon^{24} = 1$.

Proof. The canonical bidifferential $\tilde{W}$ corresponding to the basis (2.1) looks as follows ([3], p.10):

$$\tilde{W}(P, Q) = W(P, Q) - 2\pi i \langle (C\mathbf{B} + D)^{-1} C\mathbf{w}(P), \mathbf{w}(Q) \rangle \tag{2.3}$$

where $\mathbf{w}(P) = (w_1(P), \dots, w_g(P))^t$ is the column vector of basic holomorphic differentials normalized with respect to basis (a_α, b_α). Therefore, the difference between Bergman projective connections $\tilde{S}_B$ as S_B is the holomorphic quadratic differential given by:

$$(\tilde{S}_B - S_B)(dx(P))^2 = -12\pi i \sum_{\alpha,\beta=1}^{g} [(C\mathbf{B} + D)^{-1} C]_{\alpha\beta} w_\alpha(P) w_\beta(P) \tag{2.4}$$

According to the definition of the Bergman tau-function (1.1), we have

$$\frac{\partial}{\partial \lambda_m} \log \frac{\tau(\mathcal{L}, \lambda, \{\tilde{a}_\alpha,\, \tilde{b}_\alpha\})}{\tau(\mathcal{L}, \lambda, \{a_\alpha,\, b_\alpha\})} = 2\pi i \mathrm{res}\Big|_{P=P_m} \left\{ \sum_{\alpha,\beta=1}^{g} [(C\mathbf{B} + D)^{-1} C]_{\alpha\beta} \frac{w_\alpha(P) w_\beta(P)}{d\lambda(P)} \right\} \tag{2.5}$$

Using the Rauch variational formulas for matrix of b-periods (see [6]):

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial \lambda_m} = 2\pi i \mathrm{res}\Big|_{P=P_m} \frac{w_\alpha(P) w_\beta(P)}{d\lambda(P)} \tag{2.6}$$

we can rewrite (2.5) as follows:

$$\frac{\partial}{\partial \lambda_m} \log \frac{\tau(\mathcal{L}, \lambda, \{\tilde{a}_\alpha,\, \tilde{b}_\alpha\})}{\tau(\mathcal{L}, \lambda, \{a_\alpha,\, b_\alpha\})} = \frac{\partial}{\partial \lambda_m} \mathrm{logdet}(C\mathbf{B} + D) \tag{2.7}$$

which implies (2.2) for some constant ϵ.

To prove that ϵ is a root of unity one should make use of the formula (1.4), (1.5) for the tau-function. Under change of canonical basis of cycles the prime-form transforms as follows ([3], formula (1.20)):

$$\tilde{E}(P, Q) = E(P, Q) \exp \left\{ \pi i \langle (C\mathbf{B} + D)^{-1} C \int_P^Q \mathbf{w}, \int_P^Q \mathbf{w} \rangle \right\} . \tag{2.8}$$

The differential $\mathcal{C}(P)$ transforms as follows ([3], formula (1.23)):

$$\tilde{\mathcal{C}}(P) = \epsilon' \{\det(C\mathcal{B} + D)\}^{3/2} \mathcal{C}(P)$$

$$\times \exp\left\{ -\pi i \langle \mathbf{B} \mathbf{r}_0,\, \mathbf{r}_0 \rangle + \pi i \langle (C\mathbf{B} + D)^{-1} C K^P,\, K^P \rangle - 2\pi i \langle (C\mathbf{B} + D)^{-1} K^P, \mathbf{r}_0 \rangle \right\} \tag{2.9}$$

where ϵ' is a root of unity of eighth degree; the vectors $\mathbf{r}_0$, $\mathbf{q}_0 \in (\mathbf{Z}/2)^g$ are defined by (see Lemma 1.5 of [3]):

$$\tilde{K}^P = [(C\mathbf{B} + D)^{-1}]^t K^P + \tilde{\mathbf{B}}\mathbf{r}_0 + \mathbf{q}_0 \ . \tag{2.10}$$

Substituting these transformation laws into (1.4), (1.5), we get (2.2); the factor ϵ appears due to presence of ϵ' is (2.9), and sign uncertainty in the choice of distinguished local parameters at branch points and infinities.

3 Families of degenerating surfaces and asymptotical formulas

We construct several one-parametric families of Riemann surfaces degenerating as the parameter tends to zero.

Let $\mathcal{L}^+$ and $\mathcal{L}^-$ be two compact Riemann surfaces of genus g^+ and g^-, $g^\pm \geq 0$. Choose points $P_\pm \in \mathcal{L}^\pm$ and their open neighborhoods $D^\pm \subset \mathcal{L}^\pm$ such that for a certain choice of holomorphic local parameters $\lambda^\pm$ on $\mathcal{L}^\pm$ one has $D^\pm = \{P \in \mathcal{L}^\pm : |\lambda^\pm(P)| < 1\}$ and $\lambda^\pm(P^\pm) = 0$. Define the map $\lambda : D^+ \cup D^- \to \mathbb{C}$ setting $\lambda(P) = \lambda^\pm(P)$ if $P \in D^\pm$.

Using these data we construct three families of degenerating Riemann surfaces of genus $g^- + g^+$.

Case I. Let s be a complex number, $|s| < 1$ and let $P^\pm(s)$ be the points in $D^\pm$ such that $\lambda(P^\pm(s)) = s$.

Cut the discs $D^\pm$ along the (oriented) straight segments $[P^\pm, P^\pm(s)]$ and glue the surfaces $\mathcal{L}^+$ and $\mathcal{L}^-$ along these cuts identifying a point P on the left shore of the "+"-cut with the point Q ($\lambda^+(P) = \lambda^-(Q)$) on right shore of the "−"-cut and vice versa; the resulting topological real 2-d surface can be turned into a compact Riemann surface $\mathcal{L}_s$ of genus $g = g^- + g^+$ in a usual way (one chooses the local parameter near the left endpoint P of the cut as $\zeta(Q) = \sqrt{\lambda(Q)}$, near the right endpoint $P(s)$ the local parameter is $\zeta(Q) = \sqrt{\lambda(Q) - \lambda(P(s))}$, the choice of the local parameter at other points of $\mathcal{L}_s$ is obvious).

Case Ia. This family is constructed similarly to Cases I, the only difference is the position of cuts inside the disks $D^\pm$: choose a complex number t, $|t| < 1$ and introduce the cuts inside the discs $D^\pm$ connecting the points $\lambda = \sqrt{t}$ and $\lambda = -\sqrt{t}$; after the same gluing of the shores of these cuts as in case I we get the family $\mathcal{L}_t$ of degenerating compact Riemann surfaces.

Case Ib. This family is obtained similarly to Cases I and Ia, but instead of gluing the disks along the cuts we use the standard "plumbing construction" (see [2]). Choose t, $|t| < 1$ delete from the discs $D^\pm$ the smaller discs $|\lambda^\pm| \leq |t|$ and glue the obtained annuli, $A^\pm$, identifying points $P \in A^+$ and $Q \in A^-$ such that $\lambda^+(P)\lambda^-(Q) = t$. After this gluing the surfaces $\mathcal{L}^\pm$ turn into a single Riemann surface $\mathcal{L}'_t$ of genus $g^- + g^+$.

In what follows we derive asymptotical formulas (as $s \to 0$) for basic holomorphic objects (the normalized holomorphic differentials, the canonical meromorphic differential, the prime-form, etc) on the Riemann surfaces constructed in case I.

The asymptotical formulas (as $t \to 0$) for case Ib were first derived in [2]. In [12] it was claimed that all the formulas from [2] are incorrect and new ones were proved. Our analysis (in particular, see an Example below) shows that formulas from [2] (as well as Fay's proofs of these formulas) are applicable in cases Ia. As it was explained to us by Richard Wentworth (private communication) Fay in fact makes a mistake when considering case Ib: his "pinching parameter" depends in its turn on deformation parameter and this results in additional terms in asymptotical expansions which were lost in [2]. In case Ia the pinching parameter is independent of deformation parameter and Fay's scheme works perfectly.

The case of our concern, I, is very similar to case Ia (the pinching parameter, λ in equation (3.18) below, is independent of the deformation parameter s) and we give here the proofs of all the asymptotical formulas for it. Mainly we use the methods similar to those of Fay (where they are applicable); although we have chosen to follow the pretty elementary analytical methods of Yamada (avoiding Grauert's theorem and sheaf cohomologies from [2], [9]) when introducing a holomorphic family of Abelian differentials on $\mathcal{L}_s$ and studying the analytical properties of the coefficients in the Laurant expansions in the pinching zone.

3.1 Example

We start with the following simple statement. Let $\mathcal{L}$ be the two-fold branched covering of the Riemann sphere $\mathbb{P}^1$ with branch points λ_1, λ_2. Let $P \in \mathcal{L}$ and λ be the projection of P on $\mathbb{P}^1$. Then the map

$$P \mapsto \delta = \sqrt{\frac{\lambda - \lambda_1}{\lambda - \lambda_2}}$$

is the biholomorphic isomorphism of $\mathcal{L}$ and $\mathbb{P}^1$. Applying to δ the fractional linear transformation $\delta \mapsto \gamma = \frac{\lambda_2 - \lambda_1}{\delta - 1} + \lambda_2$, we get the isomorphism

$$P \mapsto \gamma = \lambda + \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)} \tag{3.1}$$

of $\mathcal{L}$ and $\mathbb{P}^1$ which is more convenient for our future purposes.

Now let $t > 0$ and $\lambda_1 = -\sqrt{t}$, $\lambda_2 = \sqrt{t}$. When $t \to 0$ the Riemann sphere $\mathcal{L}_t = \mathcal{L}$ degenerates to the singular Riemann surface with two components, Riemann spheres S_+ and S_-, attached to each other at the point 0. So, our situation is exactly that described in Case Ia.

Let $\omega_t(\cdot, \cdot)$, $\omega_-(\cdot, \cdot)$ and $\omega_+(\cdot, \cdot)$ be the canonical meromorphic bidifferentials on $\mathcal{L}_t$, S_- and S_+ respectively.

Recall now that according to Fay ([2], formula (49), p. 41) for two spheres glued via plumbing construction (this Case Ib and not the case we deal with at the moment!) one has the asymptotics:

$$\omega_t(\lambda(P), \lambda(Q)) = \begin{cases} \omega_\pm(\lambda(P), \lambda(Q)) + \frac{t}{4}\omega_\pm(\lambda(P), 0^\pm)\omega_\pm(\lambda(Q), 0^\pm) + O(t^2), & \text{if } P, Q \in S_\pm \\ -\frac{t}{4}\omega_\pm(\lambda(P), 0^\pm)\omega_\mp(\lambda(Q), 0^\mp) + O(t^2) & \text{if } P \in S_\pm, Q \in S_\mp \end{cases} \tag{3.2}$$

(In [2] the minus sign in the last line is lost.)

Let P, Q be two points of the covering $\mathcal{L}_t$ lying on the same sheet (say S_+) with projections λ and μ; assume for simplicity that λ and μ are real and positive.

Using the uniformization map (3.1), one can write the following asymptotics for the canonical meromorphic differential on $\mathcal{L}$:

$$\omega_t(\lambda, \mu) = \frac{d\gamma(\lambda) d\gamma(\mu)}{(\gamma(\lambda) - \gamma(\mu))^2} = \frac{(1 + \frac{\lambda}{\sqrt{\lambda^2 - t}})(1 + \frac{\mu}{\sqrt{\mu^2 - t}})}{[\lambda - \mu + \sqrt{\lambda^2 - t} - \sqrt{\mu^2 - t}]^2} d\lambda\, d\mu =$$

$$= \frac{d\lambda\, d\mu}{(\lambda - \mu)^2} + \frac{t}{4\lambda^2\mu^2} d\lambda\, d\mu + O(t^2) d\lambda\, d\mu \tag{3.3}$$

as $t \to 0+$ which agrees with Fay's formula (49).

(We remind the reader that the canonical bidifferential ω_+ on S_+ (as well as on S_-) is $\frac{d\lambda\, d\mu}{(\lambda - \mu)^2}$ and, therefore, $\omega_\pm(\lambda(P), 0^\pm)\omega_\pm(\lambda(Q), 0^\pm) = \frac{1}{\lambda^2\mu^2} d\lambda d\mu$.)

If $P \in S_+$, $Q \in S_-$ then all the "μ"- square roots in (3.3) change their sign and we arrive at the second case of Fay's expansion.

3.2 Asymptotical formulas.

Here we deal with Case I, assuming that the genera of the surfaces $\mathcal{L}^{\pm}$ are greater than zero, the important (nonstable) case of genus zero components will be considered separately (see Proposition 1 below).

Denote the part of the Riemann surface $\mathcal{L}_s$ which came from the discs $D^{\pm}$ after the gluing procedure by $\mathcal{U}$. The domain $\mathcal{U}$ is an open (topological) annulus and the map λ can be considered as defined on $\mathcal{U}$. The map

$$\lambda : \mathcal{U} \to \{\lambda \in \mathbb{C} : |\lambda| < 1\} \tag{3.4}$$

defines a two-sheeted covering of the disc $\{|\lambda| < 1\}$ ramified over $\lambda = 0$ and $\lambda = s$, whereas the map

$$\mathcal{U} \ni P \mapsto X = \lambda - \frac{s}{2} + \sqrt{\lambda(\lambda - s)} \tag{3.5}$$

is a well-defined biholomorphic bijection (of course, the value of the square root depends on to which disk, D^+ or D^-, the point P belongs; one also has to fix a branch of the square root, say, for the disk D^+ with the cut between 0 and s, there are two choices and we make one once and forever).

(It should be noted that map (3.5) (being appropriately extended) uniformizes the two-sheeted covering of the Riemann sphere branched over the points 0 and s. The image of the point at infinity of the first sheet is ∞, the image of the point at infinity of the second one is 0.)

For sufficiently small s the annulus

$$\mathbb{A}_s = \{P : \frac{|s|^2}{4} < |X| < 1\}$$

belongs to $\mathcal{U}$. Moreover, the boundary curve $|X| = 1$ lies in a small vicinity of the circle $|\lambda| = 1/2$ of the "+"-sheet of the covering (3.4), whereas the boundary curve $|X| = |s|^2/4$ lies in a small vicinity of the circle $|\lambda| = 1/2$ of the "$-$"-sheet.

The following two lemma are analogs of Yamada's Theorem 1 and Lemma 1 ([12], p. 116) for the family $\mathcal{L}_s$. We follow the proofs of Yamada making necessary (in fact, rather minor) modifications.

Lemma 1 *Let $v_{\pm}$ be holomorphic differentials on $\mathcal{L}^{\pm}$. There exists a holomorphic differential w_s on $\mathcal{L}_s$ such that for any ρ, $\sqrt{|s|} < \rho < 1$ holds the inequality*

$$||w_s - v_+||_{\Omega_\rho^+} + ||w_s - v_-||_{\Omega_\rho^-} \leq C(\rho)|s|, \tag{3.6}$$

where

$$\Omega_\rho^+ = \mathcal{L}^+ \setminus \{P \in D^+ : |X(P)| \leq \rho\}$$
$$\Omega_\rho^- = \mathcal{L}^- \setminus \{P \in D^- : |X(P)| \geq |s|^2/(4\rho)\}\,.$$

Here as usual, the L_2-norm of a one-form in a subdomain Ω of a Riemann surface is defined via

$$||u||_\Omega = \int\int_\Omega u \wedge \overline{*u}\,.$$

Remark. The curves $|X| = \sqrt{|s|}$ and $|X| = |s|^{3/2}/4$ belong to small vicinities of the circles $|\lambda| = \sqrt{|s|}/2$ lying on the "+" and "-" sheets of the covering (3.4) respectively.

Proof.

Let $\int_0^\lambda u_+ = \sum_{n=1}^\infty \alpha_n \lambda^n$ near P^+; after passing to coordinate X,

$$\lambda = \frac{X}{2} + \frac{s}{2} + \frac{s^2}{8X},$$

we get

$$f_+(\lambda) = \int_0^\lambda u^+ = \sum_{n=1}^{\infty} a_n^+(s) X^n + a_0(s) + \sum_{n=-\infty}^{-1} a_n^-(s) X^n,$$

where

$$a_n^+(s) = \alpha_n(1/2^n + O(s)); \quad a_0(s) = O(s); \quad a_n^-(s) = O(s^{2|n|}),$$

as $s \to 0$.

Analogously, from the expansion the expansion $f_-(\lambda) = \int_0^\lambda u_- = \sum_{n=1}^\infty \beta_n \lambda^n$ near P^- one gets

$$\int_0^\lambda u_- = \sum_{n=1}^{\infty} b_n^+(s) X^n + b_0(s) + \sum_{n=-\infty}^{-1} b_n^-(s) X^n,$$

where

$$b_n^+(s) = \beta_n(1/2^n + O(s)); \quad b_0(s) = O(s); \quad b_n^-(s) = O(s^{2|n|}),$$

as $s \to 0$.

Now, [12], we are to construct a sequence, $\Phi_s^{(k)}$, of C^1-forms on $\mathcal{L}_s$ coinciding with $v_\pm$ in $\Omega_\rho^\pm$ and such that

$$||\Phi_s^{(k)} - i * \Phi_s^{(k)}||^2 \le O(s^2) + 1/k\,. \tag{3.7}$$

For harmonic function h_s in the annulus $\{|s|^2/4\rho \le |X| \le \rho\}$ with boundary values f_- and f_+ one has the relation

$$\frac{1}{2\pi}\int\int_{|s|^2/4\rho \le |X| \le \rho} (|\partial_X h_s|^2 + |\partial_{\bar X} h_s|^2) \frac{|dX \wedge \overline{dX}|}{2} =$$

$$= \sum_{n=1}^{\infty} \frac{n|b_{-n}^- - a_{-n}^-|^2}{\rho^{2n} - (\frac{|s|^2}{4\rho})^{2n}} + \sum_{n=1}^{\infty} \frac{n|b_n^+ - a_n^+|^2}{\rho^{-2n} - (\frac{|s|^2}{4\rho})^{-2n}} + \frac{|b_0 - a_0|^2}{2\log(\frac{\rho^2}{|s|^2/4})} = O(s^2)\,. \tag{3.8}$$

It can be shown (say, via polynomial interpolation along radii directions) that one can change the function h_s in small vicinities of boundary circles $|X| = \rho$ and $|X| = |s^2|/4\rho$ obtaining the function $h_s^{(k)}$ such that

$$\int\int_{|s|^2/4\rho \le |X| \le \rho} (|\partial_X (h_s - h_s^{(k)})|^2 + |\partial_{\bar X} (h_s - h_s^{(k)})|^2) \frac{|dX \wedge \overline{dX}|}{2} \le \frac{1}{k} \tag{3.9}$$

and the 1-form

$$\Phi_s^{(k)} = \begin{cases} v_\pm \ \text{ in } \ \Omega_\rho^\pm, \\ d(h_s^{(k)}) \ \text{ in } \ \mathcal{L}_s \setminus (\Omega_\rho^+ \cup \Omega_\rho^-) \end{cases} \tag{3.10}$$

is C^1-smooth. Since the operator $\mathrm{Id} - i*$ kills the $(1,0)$-forms, the inequality (3.7) follows from (3.8) and (3.9).

Decomposing $(\mathrm{Id} - i*)\Phi_s^{(k)}$ into (L_2-orthogonal!) sum of a harmonic one-form ω_h, an exact form ω_e and a co-exact form ω_e^* (see [1], Chapter V; here "exact form" means a form belonging to the L_2-closure of the space of smooth exact forms), we observe that the left part of the equation

$$\Phi_s^{(k)} - \omega_e = i * \Phi_s^{(k)} + \omega_h + \omega_e^*$$

is a closed form, whereas its left part is co-closed, therefore, both are harmonic by virtue of Weyl's Lemma (see [1], Chapter V).

Now, applying to the harmonic form $\Phi_s^{(k)} - \omega_e$ the operator $\frac{1}{2}(\mathrm{Id} + i*)$ one gets a holomorphic one-form

$$\Psi_s^{(k)} = \frac{1}{2}(\mathrm{Id} + i*)[\Phi_s^{(k)} - \omega_e]\,.,$$

which coincides with $v_\pm + \frac{1}{2}(\mathrm{Id} + i*)\omega_e$ in $\Omega_\rho^\pm$. Therefore,

$$||\Psi_s^{(k)} - v_+||^2_{\Omega_\rho^+} + ||\Psi_s^{(k)} - v_-||^2_{\Omega_\rho^-} \le \frac{1}{4}||\omega_e + i*\omega_e|| \le \frac{1}{2}||\omega_e|| \le \frac{1}{2}||\Phi_s^{(k)} - i*\Phi_s^{(k)}|| \tag{3.11}$$

and

$$||\Psi_s^{(k)} - v_+||^2_{\Omega_\rho^+} + ||\Psi_s^{(k)} - v_-||^2_{\Omega_\rho^-} \le O(s^2) + \frac{1}{k}$$

by virtue of (3.7).

Choosing from the sequence $\{\Psi_s^{(k)}\}_{k\ge1}$ a converging subsequence (uniform L_2-boundedness of holomorphic forms on a compact Riemann surface implies uniform boundedness of their coefficients) and passing to the limit $k \to \infty$ we get a holomorphic 1-form w_s with all the needed properties. □

Remark 1 Actually a stronger variant of Lemma 1 is true: the differentials $v_\pm$ can be meromorphic with poles lying outside of $D^\pm$. In this case the differential w_s is also meromorphic and have the same singularities as $v_\pm$.

Now choose on $\mathcal{L}^\pm$ a canonical basis of cycles $\{a_\alpha^\pm, b_\alpha^\pm\}_{\alpha=1,\dots,g^\pm}$ such that none of the cycles intersects the disk $D^\pm$. Let also $\{u_\alpha^\pm\}_{\alpha=1,\dots,g^\pm}$ be the corresponding basis of normalized differentials.

The set of cycles $\{a_\alpha, \beta_\alpha\}_{\alpha=1,\dots,g^++g^-} = \{a_1^+,\dots,a_{g^+}^+, a_1^-,\dots,a_{g^-}^-; b_1^+,\dots,b_{g^+}^+, b_1^-,\dots,b_{g^-}^-\}$ forms a canonical basis on the Riemann surface $\mathcal{L}_s$. Let $\{v_\alpha^{(s)}\}_{\alpha=1,\dots,g^-+g^+}$ be the corresponding basis of normalized holomorphic differentials on $\mathcal{L}_s$.

Let also $w_\alpha^{(s)}$ be a holomorphic one form on $\mathcal{L}_s$ which is constructed in Lemma 1 when one takes $(v_+, v_-) = (v_\alpha^+, 0)$ for $\alpha = 1,\dots,g^+$ and $(v_+, v_-) = (0, v_{\alpha-g^+}^-)$ for $\alpha = g^+ + 1,\dots,g^+ + g^-$.

The corresponding a-period matrix $\mathbb{P} = ||\oint_{a_\alpha} w_\beta^{(s)}||_{\alpha,\beta=1,\dots,g^++g^-}$ satisfies

$$\mathbb{P} = I_{g^++g^-} + O(s)$$

as $s \to 0$ due to Lemma 1. This immediately implies the following lemma.

Lemma 2 *The basis $\{v_\alpha^{(s)}\}_{\alpha=1,\dots,g^++g^-}$ of normalized holomorphic differentials on $\mathcal{L}_s$ satisfies*

$$(v_1^{(s)},\dots,v_{g^++g^-}^{(s)}) = (I_{g^-+g^+} + O(s))(w_1^{(s)},\dots,w_{g^++g^-}^{(s)}), \tag{3.12}$$

in particular, all the differentials $v_\alpha^{(s)}$ are uniformly (with respect to s) bounded in, say, $\mathcal{L}_s \setminus \{P \in \mathcal{L}_s, |\lambda(P)| < 1/4\}$.

Laurent expansion for basic holomorphic differentials. Writing the differential $v_\alpha^{(s)}$ as $v_\alpha^{(s)}(X)dX$ in the local parameter $X = \lambda - \frac{s}{2} + \sqrt{\lambda(\lambda - s)}$ and expanding the coefficient $v_\alpha^{(s)}(\cdot)$ in the Laurent series in the annulus $|s|^2/4 < |X| < 1$, one gets

$$v_\alpha^{(s)}(X)dX = (\sum_{n>o}\gamma_{-n}(s)X^{-n} + \sum_{n\ge0}\gamma_n(s)X^n)dX\,. \tag{3.13}$$

Observe that $dX = \frac{X d\lambda}{\sqrt{\lambda(\lambda-s)}}$ and for $n \geq 0$ one has

$$X^n dX = \frac{\left(\lambda - s/2 + \sqrt{\lambda(\lambda-s)}\right)^{n+1}}{\sqrt{\lambda(\lambda-s)}} d\lambda = \left\{ \sum_{k=0}^{n+1} p_k(s)\lambda^k + \frac{1}{\sqrt{(\lambda(\lambda-s)}} \sum_{k=0}^{n+1} q_k(s)\lambda^k \right\} d\lambda \qquad (3.14)$$

with some polynomials $p_k(s), q_k(s)$. On the other hand, since

$$(\lambda - s/2 + \sqrt{\lambda(\lambda-s)})(\lambda - s/2 - \sqrt{\lambda(\lambda-s)}) = s^2/4\,,$$

for $n > 0$ one has

$$X^{-n} dX = \frac{4^n}{s^{2n}} \frac{\left(\lambda - s/2 - \sqrt{\lambda(\lambda-s)}\right)^n \left(\lambda - s/2 + \sqrt{\lambda(\lambda-s)}\right)}{\sqrt{\lambda(\lambda-s)}} d\lambda =$$

$$= \frac{1}{s^{2n-2}} \left\{ \sum_{k=0}^{n-1} \tilde{p}_k(s)\lambda^k + \frac{1}{\sqrt{\lambda(\lambda-s)}} \sum_{k=0}^{n-1} \tilde{q}_k(s)\lambda^k \right\} d\lambda \qquad (3.15)$$

with some polynomials $\tilde{p}_k(s), \tilde{q}_k(s)$.

For $n > 0$ one has

$$\gamma_{-n}(s) = \frac{1}{2\pi i} \int_{|X|=|s|^2/4} v_\alpha^{(s)}(X) X^{n-1} dX = \frac{1}{2\pi i} \int_{\Gamma_-} v_\alpha^{(s)}(\lambda) \left(\lambda - s/2 + \sqrt{\lambda(\lambda-s)}\right)^{n-1} d\lambda =$$

$$= \int_{\Gamma_-} O(1) \times O(s^{2n-2}) d\lambda = O(s^{2n-2}) \qquad (3.16)$$

as $s \to 0$ (the contour Γ_- over which goes the last integration lies in a small vicinity of the circle $|\lambda| = 1/2$ of the "-"-sheet; the factor $v_\alpha^{(s)}(\lambda)$ is uniformly bounded on this contour with respect to s by virtue of Lemma 2).

In the same manner for $n \geq 0$ one has

$$\gamma_n(s) = \frac{1}{2\pi i} \int_{|X|=1} \frac{v_\alpha^{(s)}(X)}{X^{n+1}} dX = \frac{1}{2\pi i} \int_{\Gamma_+} \frac{v_\alpha^{(s)}(\lambda) d\lambda}{\left(\lambda - s/2 + \sqrt{\lambda(\lambda-s)}\right)^{n+1}} = O(1) \qquad (3.17)$$

(The contour Γ_+ lies in a small vicinity of the circle $|\lambda| = 1/2$ of the +-sheet, the factor $v_\alpha^{(s)}(\lambda)$ is uniformly bounded by virtue of Lemma 2, the denominator of the integrand is close to 1.)

Now from (3.13), (3.14) and (3.15) together with the estimates (3.17) and (3.16) one gets the expansion

$$v_\alpha^{(s)}(\lambda) d\lambda = \sum_{k=0}^{\infty} a_k(s)\lambda^k \, d\lambda + \frac{1}{\sqrt{\lambda(\lambda-s)}} \sum_{k=o}^{\infty} b_k(s)\lambda^k \, d\lambda, \qquad (3.18)$$

where the coefficients a_k, b_k are *analytic near* $s = 0$. This expansion is valid in the zone $\{|s|^2/4 < |X| < 1\}$ (the latter for small s is close to the set $\{P \in \mathcal{L}_s : |\lambda(P)| \leq 1/2\}$).

Remark 2 Expansion (3.18) is a complete analog of Fay's expansion stated on page 40 of [2] for deformation family Ib. However, it is important here that in (3.18) the parameter λ is s-independent whereas in expansion from [2] the pinching parameter χ depends on deformation parameter. The latter fact was missed by Fay when he wrote his asymptotical expansions (in particular, his last formula on page 40 of [2] should contain more terms at the right hand side) ([10]).

Main asymptotical formulas for basic holomorphic differentials and the canonical meromorphic bidifferential. Let $W, W_\pm$ be the canonical meromorphic bidifferentials on $\mathcal{L}_s$ and $\mathcal{L}^\pm$ respectively.

Theorem 3 *For $\alpha = 1, \dots, g^+$ one has the asymptotics as $s \to 0$*

$$v_\alpha^{(s)}(P) = \begin{cases} u_\alpha^+(P) + \frac{s^2}{16} u_\alpha^+(P_+) W_+(P, P_+) + o(s^2) & \text{if } P \in L^+ \setminus D^+ \subset \mathcal{L}_s \\ -\frac{s^2}{16} u_\alpha^+(P_+) W_-(P, P_-) + o(s^2) & \text{if } P \in \mathcal{L}^- \setminus D^- \subset \mathcal{L}_s\,. \end{cases} \tag{3.19}$$

For $\alpha = g^+ + k$, $k = 1, \dots, g^-$ one has

$$v_\alpha^{(s)}(P) = \begin{cases} u_k^-(P) + \frac{s^2}{16} u_k^-(P_-) W_-(P, P_-) + o(s^2) & \text{if } P \in L^- \setminus D^- \subset \mathcal{L}_s \\ -\frac{s^2}{16} u_k^-(P_-) W_+(P, P_+) + o(s^2) & \text{if } P \in \mathcal{L}^+ \setminus D^+ \subset \mathcal{L}_s\,. \end{cases} \tag{3.20}$$

Here the values of differentials at the points $P_\pm$ are calculated in the local parameter λ, the values of differentials at $P \in \mathcal{L}^\pm \setminus D^\pm \subset \mathcal{L}_s$ are calculated in an arbitrary local parameter inherited from $\mathcal{L}^\pm$ (of course, the same for the l. h. s. and the r. h. s.)

Theorem 4 *For the canonical meromorphic differential on $\mathcal{L}_s$ one has the following asymptotics as $s \to 0$:*

$$W(R, S) = \begin{cases} W_+(R, S) + \frac{s^2}{16} W_+(R, P_+) W_+(S, P_+) & \text{if } R, S \in \mathcal{L}^+ \setminus D^+ \subset \mathcal{L}_s, \\ -\frac{s^2}{16} W_+(R, P_+) W_-(S, P_-) & \text{if } R \in \mathcal{L}^+ \setminus D^+ \subset \mathcal{L}_s;\ S \in \mathcal{L}^- \setminus D^- \subset \mathcal{L}_s, \\ W_-(R, S) + \frac{s^2}{16} W_-(R, P_-) W_-(S, P_-) & \text{if } R, S \in \mathcal{L}^- \setminus D^- \subset \mathcal{L}_s\,. \end{cases} \tag{3.21}$$

Proof. Observe that $\lim_{s\to 0} \sqrt{\lambda(P)(\lambda(P) - s)} = \pm\lambda(P)$ if $P \in D^\pm \setminus [0, s] \subset \mathcal{L}_s$. Let $\alpha = 1, \dots, g^+$. Taking two points in $\mathcal{U}$ with $\lambda(P) = \lambda$ and sending $s \to 0$ in (3.18), one gets

$$u_\alpha^+(\lambda) d\lambda = \left(\sum_{k=0}^{\infty} a_k(0) \lambda^k + \sum_{k=0}^{\infty} b_k(0) \lambda^{k-1} \right) d\lambda$$

for the point on the "+"-sheet and

$$0 = \sum_{k=0}^{\infty} a_k(0) \lambda^k - \sum_{k=o}^{\infty} b_k(0) \lambda^{k-1}$$

for the point on the "−"-sheet. This implies the relations

$$b_0(0) = 0 \tag{3.22}$$

and

$$\frac{u_\alpha^+(P_+)}{2} = a_0(0) = b_1(0). \tag{3.23}$$

For $P \in D^+$ one has

$$\frac{1}{s}(v_\alpha^{(s)} - v_\alpha^{(0)}) = \sum_{k\geq 0}^{\infty} \frac{a_k(s) - a_k(0)}{s} \lambda^k \, d\lambda +$$

$$= \sum_{k\geq 0} \left\{ \frac{b_k(s)-b_k(0)}{s} \frac{\lambda^k}{\sqrt{\lambda(\lambda-s)}} + b_k(0)\lambda^{k-1} \frac{\frac{\lambda}{\sqrt{\lambda(\lambda-s)}}-1}{s} \right\} d\lambda = \tag{3.24}$$

$$= \left\{ \sum_{k=0}^{\infty} a'_k(0)\lambda^k + \sum_{k=0}^{\infty} b'_k(0)\lambda^{k-1} + \frac{1}{2}\sum_{k=0}^{\infty} b_k(0)\lambda^{k-2} + O(s) \right\} d\lambda\,.$$

Since $b_0(0) = 0$, the limit of the left hand side of (3.24) as $s \to 0$ is a meromorphic differential on $\mathcal{L}^+$ with a single pole at P_+, therefore, it is a holomorphic differential, i. e.

$$b'_0(0) + \frac{1}{2} b_1(0) = 0\,. \tag{3.25}$$

Moreover, since all the a-periods of this differential vanish it equals to zero.

Then, again for a point on the "+"-sheet, we have

$$\frac{1}{s^2}(v_\alpha^{(s)} - v_\alpha^{(0)}) = \frac{1}{s^2} \left[\sum_{k\geq 0} (a_k(0) + s a'_k(0) + \frac{s^2}{2} a''_k(0) + O(s^3))\lambda^k + \right.$$

$$\left. \sum_{k\geq 0} (b_k(0) + s b'_k(0) + \frac{s^2}{2} b''_k(0) + O(s^3))\lambda^{k-1}(1 + \frac{s}{2\lambda} + \frac{3}{8}\frac{s^2}{\lambda^2} + O(s^3)) - \sum_{k\geq 0} a_k(0)\lambda^k - \sum_{k\geq 0} b_k(0)\lambda^{k-1} \right] d\lambda$$

Since s-linear term in the braces vanishes, the limit of this expression as $s \to 0$ equals to

$$\left[\sum_{k=0}^{\infty} \frac{a''_k(0)}{2}\lambda^k + \frac{b''_k(0)}{2}\lambda^{k-1} + \frac{3}{8} b_k(0)\lambda^{k-3} + \frac{b'_k(0)}{2}\lambda^{k-2} \right] d\lambda.$$

Thus the limit is a meromorphic differential on $\mathcal{L}^+$ with a single pole of *the second order* ($b_0(0) = 0!$); the corresponding Laurent coefficient is

$$\frac{3}{8} b_1(0) + \frac{b'_0(0)}{2} = \frac{b_1(0)}{8} = \frac{1}{16} u_\alpha^+(P_+)$$

due to (3.23) and (3.25). All the a-periods of this differential vanish, therefore, it coincides with

$$\frac{1}{16} u_\alpha^+(P_+) W_+(\,\cdot\,, P_+)$$

and the first asymptotics in (3.19) is proved.

The other asymptotics of Theorem 3 can be proved in a similar way. Theorem 4 follows from Theorem 3 (see [2] p. 41 for a short explanation of this implication). □

It is also possible to prove Theorem 4 independently: one starts from the generalization of Lemma 1 given in Remark 1, using this generalization with, say, $v_- = 0$ and $v_+ = W_+(\,\cdot\,, Q)$ with $Q \in \mathcal{L}^+ \backslash D^+$, one establishes expansion (3.18) for one-form $W(\,\cdot\,, Q)$ exactly in the same manner as it was done for a basic holomorphic differential. Repeating the proof of Theorem 3 with $W(\,\cdot\,, Q)$ instead of $v_\alpha^{(s)}$ we arrive to the asymtotics stated in Theorem 4. This implies the following proposition.

Proposition 1 *Theorem 4 remains true when the components $\mathcal{L}^+$ and $\mathcal{L}^-$ are permitted to be of genus* 0.

The following proposition gives the asymptotics of other type than given in Theorem 4: now one of the arguments of the canonical meromorphic bidifferential lies inside the pinching zone (being one of the two endpoints of the cut).

Proposition 2 *Let a point P lies on the surface $\mathcal{L}^\pm$ far from the pinching zone and let $P_r = \lambda^{-1}(s)$ and $P_l = \lambda^{-1}(0)$ be the critical points of the map $\lambda : \mathcal{U} \to \{\lambda : |\lambda| < 1\}$. Then*

$$W(P, P_r) = \frac{\sqrt{s}}{2} W_\pm(P_\pm, P) + O(s^{3/2}), \tag{3.26}$$

$$W(P, P_l) = -i\frac{\sqrt{s}}{2} W_\pm(P_\pm, P) + O(s^{3/2}), \tag{3.27}$$

as $s \to 0$. Here the differentials are calculated in the local parameters related to corresponding branched coverings: i. e. $\sqrt{\lambda(\cdot) - s}$ at P_r, $\sqrt{\lambda(\cdot)}$ at P_l; $\lambda^\pm(\cdot)$ at $P_\pm$ and an arbitrary local parameter inherited from $\mathcal{L}^\pm$ at P.

Proof. For the 1-form $W(\,\cdot\,, P)$ one has the expansion (3.18) with $b_0(0) = 0$, $b_0'(0) + \frac{1}{2}b_1(0) = 0$ and $b_1(0) = a_0(0) = \frac{1}{2}W_\pm(P_\pm, P)$.

Now substituting in this expansion $\lambda = s + t^2$, $d\lambda = 2t\,dt$ setting $t = 0$ and then sending $s \to 0$ we get (3.26). Substituting $\lambda = t^2$, $d\lambda = 2t\,dt$, setting $t = 0$ and sending $s \to 0$, we get (3.27). $\square$

Example: Degenerating families of branched coverings. In what follows we shall be mainly interest in the following specific construction, where the families $\mathcal{L}_s$ arise. Let

$$\lambda : \mathcal{L} \to \mathbb{C}P^1$$

be a ramified covering of the Riemann sphere constructed as follows: one takes N copies of the Riemann sphere $\mathbb{C}P^1$ cut along a broken line without selfintersections and with vertices at $M = 2N + 2g - 2$ chosen points $\lambda_1, \dots, \lambda_M$. Choose also a sequence of transpositions $\sigma_0 = Id, \sigma_1, \dots, \sigma_{M-1}, \sigma_M = Id$ from S_N acting transitively on $\{1, \dots, N\}$ and such that for any $i = 1, \dots, M$ the transposition $\mu_i = \sigma_{i-1}\sigma_i^{-1}$ is a transposition of exactly two elements. Gluing the different shores of the cuts on the spheres to each other as prescribed by transpositions σ_i one gets a compact Riemann surface $\mathcal{L}$ of genus g and a natural projection $\lambda : \mathcal{L} \to \mathbb{C}P^1$ with (simple) ramification points $P_1, \dots, P_M \in \mathcal{L}$ lying over $\lambda_1, \dots, \lambda_M \in \mathbb{C}P^1$.

Let $1 \le N_+ < N$ and $N_- = N - N_+$. Assume now that the set of transpositions $\{\sigma_i\}$ has the following property: for some *odd* k the set $\sigma_0, \sigma_1, \dots, \sigma_{k-2}$ acts transitively on some subset $A_+ \subset \{1, \dots, N\}$ having N_+ elements and does not move the elements of its complement $A_- = \{1, \dots, N\} \setminus A_+$, whereas the set $\sigma_{k+2}, \dots, \sigma_M$ acts transitively on A_- and does not move the elements of A_+.

Consider now a one parametric family $\mathcal{L}_s$ of compact Riemann surfaces of genus g assuming that in the above constructions the projections $\lambda_1, \dots, \lambda_k = 0, \lambda_{k+2}, \dots \lambda_M$ are fixed and $\lambda_{k+1} = \lambda_k + s = s$ (one has to assume that the broken line $[\lambda_1, \dots, \lambda_k, \lambda_k + s, \lambda_{k+2}, \dots, \lambda_M]$ has no selfintersections for all s sufficiently close to 0).

Obviously, we get a family of the type I. Namely, the surface $\mathcal{L}^+$ of genus $g^+ = (k - 2N_+ + 2)/2$ is glued from the N_+ copies of the Riemann sphere numbered by the indices from the set A_+ and cut along the broken line $[\lambda_1, \dots, \lambda_{k-1}]$ as prescribed by transpositions $\sigma_0, \dots, \sigma_{k-2}, Id$, whereas the surface $\mathcal{L}^-$ of genus $g^- = g - g^+$ is glued from N_2 copies of the Riemann sphere numbered by the indices from A_- and cut along the broken line $[\lambda_{k+2}, \lambda_{k+3}, \dots, \lambda_M]$ as prescribed by the transpositions $Id, \sigma_{k+2}, \dots, \sigma_M$.

Thus constructed Riemann surfaces $\mathcal{L}_s$ and $\mathcal{L}^{\pm}$ are realized as branched coverings $\lambda : \mathcal{L}_s \to \mathbb{C}P^1$ and $\lambda_{\pm} : \mathcal{L}^{\pm} \to \mathbb{C}P^1$. Suppose the transposition $\sigma_{k-1}\sigma_k^{-1}$ interchanges the elements $p \in A_+$ and $q \in A_-$.

Then the point P_+ is the point from the preimage $\lambda_+^{-1}(0)$ which lies on the p-th copy of the Riemann sphere (the "p-th sheet" of the covering λ_+), whereas the point P_- is the point from the preimage $\lambda_-^{-1}(0)$ lying on the q-th sheet of the covering λ_-.

Denote by P_j $j = 1, \dots, M$ the critical points of the map (meromorphic function) λ corresponding to critical values $\lambda_1, \dots, \lambda_M$. The colliding critical values are λ_k and $\lambda_{k+1} = \lambda_k + s$. The corresponding (colliding as $s \to 0$) critical points are P_k and P_{k+1}.

4 Asymptotical behavior of Bergman tau-function: separating case

4.1 Genus zero

First we recall the explicit expression for the Bergman tau-function on the Hurwitz space $H_{0,N}(1, \dots, 1)$ obtained in [6] (see also [8]) for a short proof).

Let the (equivalence class of) covering $\lambda : \mathcal{L} \to \mathbb{C}P^1$ belongs to the space $H_{0,N}(1, \dots, 1)$. Enumerate the sheets of this covering from 1 to N. There exists a unique biholomorphic map uniformization map $U : \mathcal{L} \to \mathbb{C}P^1$ such that $U(P) = \lambda(P) + o(1)$ as $P \to \infty^{(1)}$, where $\infty^{(1)}$ is the point from the preimage $l^{-1}(\infty)$ lying on the first sheet of the covering $\mathcal{L}$.

Let $x_m = \sqrt{\lambda - \lambda_m}$ be a local parameter in a vicinity of the ramification point P_m and $\zeta_k = \frac{1}{\lambda}$ be the local parameter near the point $\infty^{(k)}$ belonging to the preimage $\lambda^{-1}(\infty)$ and lying on the k-th sheet. The meromorphic differential dU can be written as $f_m(x_m)dx_m$ near P_m and as $h_k(\zeta_k)d\zeta_k$ near ∞^k. The functions f_m, h_k, $m = 1, \dots, M$; $k = 2, \dots, N$ are holomorphic at 0. The function h_1 has the second order pole at 0.

Proposition 3 *(see [6]). The Bergman tau-function τ on $H_{0,N}(1, \dots, 1)$ is given by the following expression:*

$$\tau^{12} = \frac{\prod_{k=2}^N [h_k(0)]^2}{\prod_{m=1}^M f_m(0)} \,. \tag{4.1}$$

It turns out that expression (4.1) can be rewritten in a nicer form which seems to be much more natural. To a covering $\lambda : \mathcal{L} \to \mathbb{C}P^1$ there corresponds a meromorphic quadratic differential $(d\lambda)^2$ having double zeroes at $P_1, \dots, P_M$ and the fourth order poles at $\infty^{(1)}, \dots, \infty^{(N)}$. The quadratic differential

$$Q = \frac{\prod_{k=1}^N W^2(\,\cdot\,, \infty^{(k)})}{\prod_{m=1}^M W(\,\cdot\,, P_m)} \tag{4.2}$$

has the same zeroes and poles as $(d\lambda)^2$. (We remind the reader that for genus zero coverings one has $2N - M = 2$.) Therefore the ratio

$$R = \frac{Q}{(d\lambda)^2} \tag{4.3}$$

is a holomorphic function on $\mathcal{L}$ and, therefore, a constant.

Proposition 4 *For the Bergman tau-function on the space $H_{0,N}(1, \dots, 1)$ one has the relation*

$$\tau^{12} = R\,. \tag{4.4}$$

Proof. The expression $Q(P)/(d\lambda(P))^2$ is independent of $P \in \mathcal{L}$. Sending $P \to \infty^{(1)}$ one gets the right hand side of (4.1). $\square$

Having this proposition and the asymptotical expansions for the canonical meromorphic bidifferential established in the previous section, one can easily write an asymptotical formula for the Bergman tau-function on $H_{0,N}(1,\dots,1)$ when two critical points of the covering map λ collide and the Riemann sphere $\mathcal{L}$ degenerates to two Riemann spheres attached to each other.

Theorem 5 *Let the (equivalence class) of the covering $\lambda : \mathcal{L} \to \mathbb{C}P^1$ belong to $H_{0,N}(1,\dots,1)$ and let the covering $\lambda : \mathcal{L} \to \mathbb{C}P^1$ degenerate as explained in the example at the end of the previous section. Then the arising coverings $\lambda_+ : \mathcal{L}^+ \to \mathbb{C}P^1$ and $\lambda_- : \mathcal{L}^- \to \mathbb{C}P^1$ belong to Hurwitz spaces $H_{0,N_+}(1,\dots,1)$ and $H_{0,N_-}(1,\dots,1)$ respectively. Denote by $\tau_\pm$ the Bergman tau-function on $H_{0,N_\pm}(1,\dots,1)$. Then one has the asymptotics*

$$\tau(\mathcal{L},\lambda) \sim s^{1/4}\tau_+(\mathcal{L}^+,\lambda_+)\tau_-(\mathcal{L}^-,\lambda_-) \tag{4.5}$$

as $s \to 0$.

Proof. Take some $P \in \mathcal{L}$ far from the pinching zone (assume for definiteness that $P \in \mathcal{L}^+$) and rewrite (4.3) as

$$R = \frac{\prod_{k=1}^{N^+} W^2(P,\infty_+^{(k)})\prod_{k=1}^{N_-} W^2(P,\infty_-^{(k)})}{W(P,P_r)W(P,P_l)\prod_{m=1}^{M_+} W(P,P_m^+)\prod_{m=1}^{M_-} W(P,P_m^-)} \tag{4.6}$$

where we have separated the critical points $P_1,\dots,P_M$ of the map λ into P_r, P_l (these are colliding points: $\lambda(P_l)=0$, $\lambda(P_r)=s$), the points P_m^+ belonging to the part $\mathcal{L}^+$ and the points P_m^- belonging to $\mathcal{L}^-$; the points over infinities $\infty^{(k)}$ are separated into the points $\infty_+(k)$ and $\infty_-^{(k)}$ belonging to $\mathcal{L}^+$ and $\mathcal{L}^-$ respectively. Using Theorem 4 and Proposition 2, we obtain that this expression is equivalent as $s \to 0$ to

$$\frac{\prod_{k=1}^{N_+} W_+^2(P,\infty_+(k))\prod_{k=1}^{N_-}\frac{s^4}{16^2}W_+^2(P,P_+)W_-^2(\infty_-^{(k)},P_-)}{\frac{\sqrt{s}}{2}W_+(P,P_+)\frac{1}{2i}\sqrt{s}W_+(P,P_+)\prod_{m=1}^{M_+}W_+(P,P_m^+)\prod_{m=1}^{M_-}(-\frac{s^2}{16})W_+(P,P_+)W_-(P_-,P_m^-)}$$

which by virtue of the relations $M_\pm = 2N_\pm - 2$ can be rewritten as

$$Cs^3\frac{\prod_{k=1}^{N_+} W_+^2(P,\infty_+^{(k)})}{\prod_{m=1}^{M_+} W_+(P,P_m^+)}\frac{\prod_{k=1}^{N_-} W_-^2(P_-,\infty_-^{(k)})}{\prod_{m=1}^{M_-} W_-(P_-,P_m^-)},$$

where C is a moduli (i. e. $\{\lambda_1,\dots,\lambda_M\}$) independent constant. The latter expression is nothing but the 12-th power of the r. h. s. of (4.5). $\square$

4.2 Genus one

Let now the covering $\lambda : \mathcal{L} \mapsto \mathbb{C}P^1$ belong to $H_{1,N}(1,\dots,1)$. Let two critical points of λ collide in such a way that the surface L degenerates to a singular surface with two components, an elliptic curve $\mathcal{L}^+$ and a Riemann sphere $\mathcal{L}^-$, attached to each other; we again assume that this degeneration goes as explained in the example at the end of the previous section and we keep the same notation as in that example.

First, we recall the explicit expression for the Bergman tau-function in genus one.

Proposition 5 *(see [6] and [8]) Choose a canonical basis $\{a_1, b_1\}$ of cycles on the elliptic curve $\mathcal{L}$. Let σ be the b-period of the normalized holomorphic differential v on $\mathcal{L}$ (i. e. $\oint_{a_1} v = 1$ and $\oint_{b_1} v = \sigma$). Let $v = f_m(x_m)dx_m$ near P_m and $v = h_k(\zeta_k)d\zeta_k$ near $\infty^{(k)}$. Then*

$$\tau^{12}(\mathcal{L}, \lambda, \{a_1, b_1\}) = [\theta_1'(0|\sigma)]^8 \frac{\prod_{k=1}^N [h_k(0)]^2}{\prod_{m=1}^M f_k(0)}\,, \tag{4.7}$$

where θ_1 is the first Jacobi's theta-function.

Theorem 6 *Let the (equivalence class) of the covering $\lambda : \mathcal{L} \to \mathbb{C}P^1$ belong to $H_{1,N}(1, \dots, 1)$ and let the covering $\lambda : \mathcal{L} \to \mathbb{C}P^1$ degenerate as just explained. Then the arising coverings $\lambda_+ : \mathcal{L}^+ \to \mathbb{C}P^1$ and $\lambda_- : \mathcal{L}^- \to \mathbb{C}P^1$ belong to Hurwitz spaces $H_{1,N_+}(1, \dots, 1)$ and $H_{0,N_-}(1, \dots, 1)$ respectively. Denote by τ_+, τ_- the corresponding Bergman tau-functions. Assume also that the cycles $\{a_1, b_1\}$ on the curve $\mathcal{L}$ are chosen to lie on $\mathcal{L}^+$. Then one has the asymptotics*

$$\tau(\mathcal{L}, \lambda, \{a_1, b_1\}) \sim s^{1/4} \tau_+(\mathcal{L}^+, \lambda_+, \{a_1, b_1\}) \tau_-(\mathcal{L}^-, \lambda_-) \tag{4.8}$$

as $s \to 0$.

Proof. First, recall the following relation between the canonical meromorphic bidifferential and the normalized differential:

$$2\pi i v(P) = \int_{b_1} W(Q, P) dP. \tag{4.9}$$

Due to Theorem 3,

$$\theta_1'(0|\sigma) \to \theta_1'(0|\sigma_+)$$

as $s \to 0$, where σ_+ is the b-period of the normalized differential on the curve $\mathcal{L}^+$ (with respect to the basis $\{a_1, b_1\}$). Using (4.9) and the asymptotics from Theorem 4 and Proposition 2, we obtain that

$$\tau^{12} = c_1 [\theta_1'(0|\sigma)]^8 \frac{\prod_{k=1}^{N_+} \left[\oint_{b_1} W(\lambda, \infty_+^{(k)}) d\lambda\right]^2}{\prod_{m=1}^{M_+} \oint_{b_1} W(\lambda, P_m^+) d\lambda} \frac{1}{\oint_{b_1} W(\lambda, P_l) d\lambda} \frac{1}{\oint_{b_1} W(\lambda, P_r) d\lambda} \frac{\prod_{k=1}^{N_-} \left[\oint_{b_1} W(\lambda, \infty_-^{(k)}) d\lambda\right]^2}{\prod_{m=1}^{M_-} \oint_{b_1} W(\lambda, P_m^-) d\lambda} \sim$$

$$\sim c_2 \left\{ [\theta_1'(0|\sigma_+)]^8 \frac{\prod_{k=1}^{N_+} \left[\oint_{b_1} W_+(\lambda, \infty_+^{(k)}) d\lambda\right]^2}{\prod_{m=1}^{M_+} \oint_{b_1} W_+(\lambda, P_m^+) d\lambda} \right\} \frac{1}{s \left[\oint_{b_1} W_+(\lambda, P_+) d\lambda\right]^2} \times$$

$$s^{4N_- - 2M_-} \left(\oint_{b_1} W_+(\lambda, P_+) d\lambda \right)^{2N_- - M_-} \frac{\prod_{k=1}^{N_-} \left[W_-(P_-, \infty_-^{(k)})\right]^2}{\prod_{m=1}^{M_-} W_+(P_-, P_m^-)}\,,$$

where c_1, c_2 are some moduli independent constants. Since $2N_- - M_- = 2$ (the $\mathcal{L}^-$ has genus 0!), this implies (4.8). $\square$

4.3 Case of higher genus

4.3.1 Some additional asymptotics

Here we find asymptotics for quantities entering the expression for the Bergman tau-function in higher genus. We start from the asymptotics for the prime form. First, recall the following expression, relating the prime form, $E(x,y)$, to the canonical meromorphic differential on an arbitrary compact Riemann surface of genus g (see [2], p. 26):

$$\frac{\theta(\int_x^y \vec{v} - e)\theta(\int_x^y \vec{v} + e)}{\theta^2(e)E^2(x,y)} = W(x,y) + \sum_{i,j=1}^{g} \frac{\partial^2 \log\theta(e)}{\partial z_i \partial z_j} v_i(x)v_j(y), \tag{4.10}$$

where $\vec{v} = (v_1, \dots, v_g)^t$ is a column of basic holomorphic differentials, e is an arbitrary vector from $\mathbb{C}^n$.

¿From this expression taken together with the asymptotics for the basic holomorphic differentials and the canonical meromorphic bidifferential one easily derives the following asymptotics for the prime form on the family $\mathcal{L}_s$.

- $$E^2(P,Q) = E^2_{\pm}(P,Q) + o(1) \tag{4.11}$$

  as $s \to 0$, here the points P, Q belong to $\mathcal{L}^{\pm}$ and are far from the pinching zone, $E_{\pm}(P,Q)$ is the prime form on $\mathcal{L}^{\pm}$, all the prime forms are calculated in local parameters near P and Q inherited from $\mathcal{L}^{\pm}$;

- $$E^2(P,Q) = -\frac{16}{s^2} E^2_{\pm}(P, P_{\pm}) E^2_{\mp}(Q, P_{\mp}) + O(\frac{1}{s}) \tag{4.12}$$

  if $P \in \mathcal{L}^{\pm}$ and $Q \in \mathcal{L}^{\mp}$;

- $$E^2(P, P_r) = \frac{2}{\sqrt{s}} E^2_{\pm}(P, P_{\pm}) + O(\sqrt{s}), \qquad E^2(P, P_l) = \frac{2i}{\sqrt{s}} E^2_{\pm}(P, P_{\pm}) + O(\sqrt{s}), \tag{4.13}$$

  if $P \in \mathcal{L}^{\pm}$, the local parameter at P_l is $\sqrt{\lambda}$, the local parameter near P_r is $\sqrt{\lambda - s}$.

¿From now on we use the following notation $\Delta(s) = \frac{4}{s}$ and denote by a single letter ϵ different unitary constants ("phase factors", ($|\epsilon| = 1$) which may appear as additional factors in some of our formulas; the concrete values of these factors are of no interest for us.

The next two quantities whose asymptotics we need are defined as follows (see [3], (1.13) and (1.17)):

$$\sigma(P,Q) = \exp\left\{ -\sum_{k=1}^{g} \int_{a_k} v_k(x) \log\frac{E(x,P)}{E(x,Q)} \right\}, \tag{4.14}$$

and

$$C(P) = \frac{\theta(\int_P^{Q_1} \vec{v} \cdots + \int_P^{Q_g} \vec{v} + K_P) \prod_{i<j}^{g} E(Q_i, Q_j) \prod_{i=1}^{g} \sigma(Q_i, P)}{\det(v_i(Q_j)) \prod_{i=1}^{g} E(P, Q_i)} \tag{4.15}$$

where $Q_1, \dots Q_g$ are arbitrary points of $\mathcal{L}$ (expression (4.15) is independent of the choice of these points) and K_P is the vector of Riemann constants.

Using asymptotics for the prime-form (4.11–4.13) and the basic holomorphic differentials one easily obtains from (4.14) the following asymptotics as $s \to 0$:

$$\sigma(P,Q) \sim \sigma_{\pm}(P,Q) \left[\frac{E_{\pm}(Q,P_{\pm})}{E_{\pm}(P,P_{\pm})}\right]^{g^{\mp}} , \tag{4.16}$$

for $P, Q \in \mathcal{L}^{\pm}$;

$$\sigma(P,Q) \sim \epsilon \sigma_{\pm}(P,P_{\pm})\sigma_{\mp}(P_{\mp},Q) \frac{[E_{\mp}(P_{\mp},Q)]^{g^{\pm}}}{[E_{\pm}(P,P_{\pm})]^{g^{\mp}}} [\Delta(s)]^{g^{\pm}-g^{\mp}} , \tag{4.17}$$

if $P \in \mathcal{L}^{\pm}$, $Q \in \mathcal{L}^{\mp}$;

$$\sigma(P_r,Q) \sim \epsilon\sigma(P_l,Q) \sim \epsilon\sigma_{\pm}(P_{\pm},Q)\, [E_{\pm}(P_{\pm},Q)]^{g^{\mp}}\, [\Delta(s)]^{(3g^{\mp}-g^{\pm})/4} \tag{4.18}$$

if $Q \in \mathcal{L}^{\pm}$.

The asymptotics of (4.15) is a bit more tricky to obtain and we give more details. First choose the points $\{Q_i\}$ in such a way that g^+ of them, $R_1, \ldots, R_{g^+}$ belong to $\mathcal{L}^+$ and the other g^- points, $S_1, \ldots, S_{g^-}$, belong to $\mathcal{L}^-$. Then, assuming for definiteness $P \in \mathcal{L}^+$, one has as $s \to 0$

$$\theta(\int_P^{Q_1} \vec{v} \cdots + \int_P^{Q_g} \vec{v} + K_P|\mathbb{B}) \sim$$

$$\theta\left(\int_P^{R_1} \begin{pmatrix}\vec{v}_+\\ \vec{0}\end{pmatrix} + \cdots + \int_P^{R_{g^+}} \begin{pmatrix}\vec{v}_+\\ \vec{0}\end{pmatrix} + g_- \int_P^{P_+} \begin{pmatrix}\vec{v}_+\\ 0\end{pmatrix} + \int_{P_-}^{S_1} \begin{pmatrix}\vec{0}\\ \vec{v}_-\end{pmatrix} + \cdots + \int_{P_-}^{S_{g^-}} \begin{pmatrix}\vec{0}\\ \vec{v}_-\end{pmatrix} + \right.$$

$$\left. + \begin{pmatrix} K_P^+ - g^- \int_P^{P_+} \vec{v}_+ \\ K_{P_-}^- \end{pmatrix} \Big| \mathrm{diag}(\mathbb{B}^+, \mathbb{B}^-)\right) =$$

$$= \theta_+(\int_P^{R_1} \vec{v}_+ + \cdots + \int_P^{R_{g^+}} \vec{v}_+ + K_P^+)\ \theta_-(\int_{P_-}^{S_1} \vec{v}_- + \cdots + \int_{P_-}^{S_{g^-}} \vec{v}_- + K_{P_-}^-) . \tag{4.19}$$

Now using the asymptotics for the prime form and σ, we see that the numerator of (4.15) (with the just made choice of $Q_1, \ldots, Q_g$) is equivalent to

$$\epsilon\, \theta_+(\int_P^{R_1} \vec{v}_+ + \cdots + \int_P^{R_{g^+}} \vec{v}_+ + K_P^+)\ \theta_-(\int_{P_-}^{S_1} \vec{v}_- + \cdots + \int_{P_-}^{S_{g^-}} \vec{v}_- + K_{P_-}^-) \prod_{i<j} E_+(R_i,R_j) \prod_{i<j} E_-(S_i,S_j)$$

$$\left\{\prod_{i=1}^{g^+}\prod_{j=1}^{g^-} E_+(R_i,P_+)E_-(S_j,P_-)\right\} [\Delta(s)]^{g^+g^-} \prod_{i=1}^{g^+} \sigma_+(R_i,P) \frac{\{E_+(P,P_+)\}^{g^+g^-}}{\left\{\prod_{j=1}^{g^+} E_+(R_j,P_+)\right\}^{g^-}}$$

$$[\sigma_+(P_+,P)]^{g^-} [E_+(P_+,P)]^{(g^-)^2} [\Delta(s)]^{g^-(g^- - g^+)} \prod_{j=1}^{g^-} \frac{\sigma_-(S_j,P_-)}{\{E_-(S_j,P_-)\}^{g^+}} ,$$

whereas the denominator of (4.15) is equivalent to

$$\epsilon \left\{\prod_{i=1}^{g^+} E_+(P,R_i)\right\} [E_+(P,P_+)]^{g^-} \left\{\prod_{j=1}^{g^-} E_-(P_-,S_j)\right\} [\Delta(s)]^{g^-} \mathrm{det}(v_i^+(R_j))\mathrm{det}(v_i^-(S_j)) .$$

So, after rearranging the terms and numerous cancellations, one gets the asymptotics

$$C(P) \sim \epsilon C_{\pm}(P) C_{\mp}(P_{\mp}) \{E_{\pm}(P, P_{\pm})\}^{g^{\mp}(g^{\pm}+g^{\mp}-1)} \{\sigma_{\pm}(P_{\pm}, P)\}^{g_{\mp}} \Delta(s)^{[g^{\mp}]^2 - g^{\mp}} \tag{4.20}$$

if $P \in \mathcal{L}^{\pm}$.

Now we are able to calculate the asymptotics of the Bergman tau-function in higher genus. First recall the explicit expression for the tau-function found in [5]. Let a pair $(\mathcal{L}, \lambda)$ belong to the Hurwitz space $H_{g,N}(1, \dots, 1)$ and $g \geq 2$. Let

$$(d\lambda) = \sum_{k=1}^{M+N} d_k D_k$$

be the divisor of the meromorphic differential $d\lambda$. (One has $d_k = 1$ if D_k is a critical point of λ and $d_k = -2$ if D_k is a (simple) pole of λ and, therefore, a pole of $d\lambda$ of order 2. One has the relation $d_1 + \dots + d_{M+N} = M - 2N = 2g - 2$, so $M + N = 3N + 2g - 2$. Denote by $\mathcal{A}_P(\cdot)$ the Abel map with the base point P and let K^P be the vector of Riemann constants. Then one has the relation

$$\mathcal{A}((d\lambda)) + 2K^P + \mathbb{B}\mathbf{r} + \mathbf{q} = 0 \tag{4.21}$$

with some integer vectors $\mathbf{r}$ and $\mathbf{q}$.

Remark 3 Let us emphasize that in order to define the vector K^P and the Abel map $\mathcal{A}_P$ (as well as the prime-form and the left hand side of expression (4.10)) one has to introduce the system of cuts on the surface $\mathcal{L}$ in such a way that the integration $\int_x^y \vec{v}$ is well-defined for any x, y belonging to the surface $\mathcal{L} = \mathcal{L}_s$ dissected along the cuts. We choose the usual symplectic basis of homologies $\{a_\alpha^{\pm}, b_\alpha^{\pm}\}_{\alpha=1,\dots,g^{\pm}}$ on $\mathcal{L}^{\pm}$, take curves representing this basis and dissect the $\mathcal{L}^{\pm}$ along these curves. The resulting dissected surface $\mathcal{L}^{\pm}$ is homeomorphic to a sphere with $g^{\pm}$ holes, whereas the surface $\mathcal{L}_s$ dissected along the same curves is homeomorphic to a sphere with g holes. Notice that the boundary of any hole is the trivial cycle ($a_\alpha^{\pm} + b_\alpha^{\pm} - a_\alpha^{\pm} - b_\alpha^{\pm} = 0$) and, therefore, the $\int_x^y \vec{v_{\pm}}$ and $\int_x^y \vec{v}$ are well-defined on the corresponding dissected surfaces.

Proposition 6 *(see [5], Theorem 2.9) Let $g > 1$. Then one has the following explicit expression for the Bergman tau-function on the $H_{g,N}(1, \dots, 1)$:*

$$\tau^{-6}(\mathcal{L}, \lambda) = e^{2\pi i <\mathbf{r}, K^P>} C^{-4}(P) \prod_{k=1}^{M+N} [\sigma(D_k, P)]^{d_k} \{E(D_k, P)\}^{(g-1)d_k} , \tag{4.22}$$

where P is an arbitrary point of $\mathcal{L}$ and the integer vector $\mathbf{r}$ is defined by (4.21).

Now we are able to prove our main Theorem.

Theorem 7 *For the Bergman tau-function $\tau(\mathcal{L}, \lambda)$ on the Hurwitz space $H_{g,N}(1, \dots, 1)$, $g > 1$ one has the asymptotics*

$$\tau(\mathcal{L}_s, \lambda) \sim s^{1/4} \tau_+(\mathcal{L}^+, \lambda_+) \tau_-(\mathcal{L}^-, \lambda_-) \tag{4.23}$$

as $s \to 0$.

Proof. The following lemma immediately follows from the definition of the vector of the Riemann constants,

$$K_\beta^P = \frac{1}{2} + \frac{\mathbb{B}_{\beta\beta}}{2} - \sum_{\alpha=1,\alpha\neq\beta}^{g} \int_{a_\alpha} \left(v_\alpha \int_P^x v_\beta \right),$$

and Theorem 3.

Lemma 3 *One has the asymptotics*

$$K^P \sim \begin{pmatrix} K_+^P - g^- \int_P^{P_+} \vec{v}_+ \\ K_-^{P_-} \end{pmatrix} \quad , \tag{4.24}$$

as $s \to 0$, where K_+^P and $K_-^{P_-}$ are the vectors of Riemann constants for the surfaces $\mathcal{L}^+$ and $\mathcal{L}^-$ with the base points P and P_- respectively.

Assume that the point P lies on the component $\mathcal{L}^+$. Using Lemma 1, one can pass to the limit $s \to 0$ in the equation (4.21). This results in the relations

$$\mathcal{A}_P^+((d\lambda_+)) + 2K_+^P + \mathbb{B}^+\mathbf{r}^+ + \mathbf{q}^+ \tag{4.25}$$

and

$$\mathcal{A}_{P_-}^-((d\lambda_-)) + 2K_-^{P_-} + \mathbb{B}^-\mathbf{r}^- + \mathbf{q}^+ , \tag{4.26}$$

where $\mathbf{r} = (\mathbf{r}^+, \mathbf{r}^-)$, $\mathbf{q} = (\mathbf{q}^+, \mathbf{q}^-)$ and $\mathcal{A}^\pm$ is the Abel map on $\mathcal{L}^\pm$.

Now one has

$$\tau^{-6}(\mathcal{L}, \lambda) \sim \epsilon e^{2\pi i < \mathbf{r}^+,\ K_+^P >} e^{2\pi i < \mathbf{r}^-,\ K_-^{P_-} >} e^{-2\pi i g^- < \mathbf{r}^+,\ \int_P^{P_+} \vec{v}_+ >}$$

$$\{C_+(P)\}^{-4}\{C_-(P_-)\}^{-4}\{E_+(P,P_+)\}^{4g^-(1-g)}\{\sigma_+(P_+,P)\}^{-4g^-}[\Delta(s)]^{4(g^- - (g^-)^2)}$$

$$\prod_{k=1}^{M_+ + N_+} [\sigma_+(D_k^+, P)]^{d_k^+} \left[\frac{E_+(P,P_+)}{E_+(D_k^+, P_+)} \right]^{g^- d_k^+}$$

$$\left[\sigma_+(P_+,P)\{E_+(P_+,P)\}^{g^-}[\Delta(s)]^{(3g^- - g^+)/4} \right]^2$$

$$\prod_{k=1}^{M_- + N_-} \left\{ \sigma_-(D_k^-, P_-)\sigma_+(P_+,P) \frac{[E_+(P_+,P)]^{g^-}}{[E_-(D_k^-,P_-)]^{g^+}} [\Delta(s)]^{g^- - g^+} \right\}^{d_k^-}$$

$$\prod_{k=1}^{M_+ + N_+} \{E_+(D_k^+, P)\}^{(g-1)d_k^+} \left[[\Delta(s)]^{1/4} E(P,P_+) \right]^{2(g-1)} \prod_{k=1}^{M_- + N_-} \{\Delta(s) E_+(P,P_+) E_-(D_k^-, P_-)\}^{(g-1)d_k^-}$$

with $\sum d_k^+ = 2g^+ - 2$, $\sum d_k^- = 2g^- - 2$, $g = g^+ + g^-$. Observe that $\Delta(s)$ enters the above expression with power

$$4(g^- - (g^-)^2) + \frac{3g^- - g^+}{2} + (g^- - g^+)(2g^- - 2) + \frac{g-1}{2} + (g-1)(2g^- - 2) = \frac{3}{2},$$

all the factors $E_+(P,P_+)$ cancel out $(4g^-(1-g)+g^-(2g^+-2)+2g^-+g^-(2g^--2)+2(g-1)+(g-1)(2g^--2)=0)$ and the remaining terms can be rearranged into the product of

$$e^{2\pi i<\mathbf{r}^+,\ K_+^P>}C_+^{-4}(P)\prod_{k=1}^{M_++N_+}[\sigma_+(D_k^+,P)]^{d_k^+}\left\{E_+(D_k^+,P)\right\}^{(g^+-1)d_k^+}\ , \tag{4.27}$$

$$e^{2\pi i<\mathbf{r}^-,\ K_-^{P_-}>}C_-^{-4}(P_-)\prod_{k=1}^{M_-+N_-}[\sigma_-(D_k^-,P_-)]^{d_k^-}\left\{E_-(D_k^-,P_-)\right\}^{(g^--1)d_k^-}\ . \tag{4.28}$$

and

$$e^{-2\pi i g^-<\mathbf{r}^+,\ \int_P^{P_+}\vec{v}_+>}\left\{[\sigma_+(P_+,P)]^{-2}\frac{\prod_{k=1}^{M_++N_+}[E_+(D_k^+,P)]^{d_k^+}}{\prod_{k=1}^{M_++N_+}[E_+(D_k^+,P_+)]^{d_k^+}}\right\}^{g^-}\ . \tag{4.29}$$

According to Lemma 2.11 from [5], the expression in the braces in (4.29) is nothing but $e^{2\pi i<\mathbf{r}^+,\ \mathcal{A}_P^+(P_+)>}$ and, therefore, the expression (4.29) equals one; expressions (4.27) and (4.28) coincide with $\tau_+^{-6}(\mathcal{L}^+,\lambda_+)$ and $\tau_-^{-6}(\mathcal{L}^-,\lambda_-)$ respectively. $\square$

5 Asymptotics of Bergman tau-function under degeneration: non-separating case

This case differs from the separating case only in some technical details (see [2], [12]). The final result (asymptotics (1.10)) can be proved in the same manner as asymptotics (4.23) in the separating case. In order to shorten the text we shall present here only an explicit calculation for the simplest example: a two-sheeted covering with $2g+2$ branch points, two of them collide.

Consider a hyperelliptic curve $\mathcal{L}_g$ of genus g:

$$w^2=\prod_{m=1}^{2g+2}(\lambda-\lambda_m)\equiv P_{2g+2}(\lambda) \tag{5.1}$$

and its degeneration $\mathcal{L}_g^0$ via pinching of a_g which encircles λ_{2g+1} and λ_{2g+2}:

$$w^2=(\lambda-\lambda_0)^2\prod_{m=1}^{2g}(\lambda-\lambda_m)\equiv(\lambda-\lambda_0)^2P_{2g}(\lambda) \tag{5.2}$$

Corresponding non-degenerate curve we denote by $\mathcal{L}_{g-1}$:

$$w_0^2=\prod_{m=1}^{2g}(\lambda-\lambda_m) \tag{5.3}$$

One has the following formula for the Bergman tau-function in hyperelliptic case in genus g (see [6]):

$$\tau_g=\det\mathcal{A}_g\prod_{m<n}(\lambda_n-\lambda_m)^{1/4} \tag{5.4}$$

where

$$\mathcal{A}_{ij}=\oint_{a_j}\frac{\lambda^{i-1}}{w}\ ,\qquad i,j=1,\dots,g \tag{5.5}$$

Theorem 8 *In the limit λ_{2g+1}, $\lambda_{2g+2} \to \lambda_0$ the Bergman tau-function behaves as follows:*

$$\tau_g(\lambda_1, \dots \lambda_{2g+2}) \to C(\lambda_{2g+1} - \lambda_{2g+2})^{1/4} \tau_{g-1}(\lambda_1, \dots \lambda_{2g}) \tag{5.6}$$

Proof. Under degeneration the Wronskian behaves as follows:

$$\prod_{m<n\ m,n=1}^{2g+2} (\lambda_n - \lambda_m)^{1/4} \to (\lambda_{2g+1} - \lambda_{2g+2})^{1/4} \left\{ \prod_{m=1}^{2g} (\lambda_0 - \lambda_m)^{1/2} \right\} \left\{ \prod_{m<n\ m,n=1}^{2g} (\lambda_n - \lambda_m)^{1/4} \right\} \tag{5.7}$$

Consider $\det \mathcal{A}_g$. For the first row one has

$$\oint_{a_g} \frac{\lambda^k d\lambda}{P_{2g+2}(\lambda)} \to 2\pi i \frac{\lambda_0^k}{P_{2g}(\lambda_0)}\,, \tag{5.8}$$

since $\sqrt{P_{2g+2}(\lambda)} \to (\lambda - \lambda_0)\sqrt{P_{2g}(\lambda)}$ and integral over a_g is computed via the residue at λ_0. Now, multiply after degeneration the first column of $\mathcal{A}_g$ with λ_0^{g-1}, the second column with λ_0^{g-2} etc, we get all the entries of the last row equal to λ_0^{g-1}; $\det \mathcal{A}_g$ gains factor $2\pi i \lambda_0^{-(1+\cdots+g-1)}\sqrt{P_{2g}(\lambda_0)}$. The row number k in the new matrix is given by

$$\left(\oint_{a_k} \frac{\lambda_0^{g-1} d\lambda}{(\lambda - \lambda_0) P_{2g}(\lambda)}\,,\ \oint_{a_k} \frac{\lambda \lambda_0^{g-2} d\lambda}{(\lambda - \lambda_0) P_{2g}(\lambda)}\,, \dots,\ \oint_{a_k} \frac{\lambda^{g-1} d\lambda}{(\lambda - \lambda_0) P_{2g}(\lambda)} \right)\,.$$

Now subtract the 2nd column from the first; the third from the second etc. This kills the factors $(\lambda - \lambda_0)$ in all denominators except the last column. But the last row has now all zero entries except the last one equal to λ_0^{g-1}. As a result, $\det \mathcal{A}$ factorizes to the product of $\det \mathcal{A}_{g-1}$, factor $2\pi i/\sqrt{P_{2g}(\lambda_0)}$ and some power of -1:

$$\det \mathcal{A}_g \to \pm \frac{2\pi i (-1)^g}{\sqrt{P_{2g}(\lambda_0)}} \det \mathcal{A}_{g-1} \tag{5.9}$$

Multiplying that with Wronskian, we see that $\sqrt{P_{2g}(\lambda_0)}$ cancels out and we come to the asymptotics stated in the theorem.

6 Asymptotics of Bergman tau-function on caustic

The caustic $\mathcal{H}_C$ is the component of the boundary of Hurwitz space $\mathcal{H}(1, \dots, 1)$ where two simple critical points of the function λ collide to form a critical point of multiplicity 2. The complex structure of Riemann surface $\mathcal{L}$ remains non-degenerated in such limit. The caustic is itself a stratum of Hurwitz space which corresponds to meromorphic functions such that all poles are simple, and all critical points except one are also simple. The remaining critical point has multiplicity 2. Thus using formula 1.4 we can introduce on $\mathcal{H}_C$ its Bergman tau-function τ_C.

We are going to prove the following theorem:

Theorem 9 *On the caustic, when the branch point P_2 tends to P_1 to form a double branch point P_0, we get the following asymptotics of the Bergman tau-function (1.8) in terms of corresponding critical values λ_1 and λ_2:*

$$\tau(\mathcal{L}, \lambda, \{a_\alpha, b_\alpha\}) \sim C(\lambda_2 - \lambda_1)^{1/12}\, \tau_C(\mathcal{L}_0, \lambda_0, \{a_\alpha, b_\alpha\}) \tag{6.1}$$

where C is a constant; $\tau_C(\mathcal{L}_0, \lambda_0, \{a_\alpha, b_\alpha\})$ is the Bergman tau-function on caustic i.e. on the stratum $\mathcal{H}_C$ of Hurwitz space where all branch points are simple, except the multiplicity two critical point P_0 on the limiting Riemann surface $\mathcal{L}_0$, obtained from gluing P_2 and P_1. The function τ_C is defined by (1.4) with divisor

$$(d\lambda)_C = 2P_0 + \sum_{m=3}^{M} P_m - 2\sum_{k=1}^{N} \infty^{(k)} . \tag{6.2}$$

To study asymptotical behaviour of all ingredients of the tau-function (1.8) in the limit $\lambda_2 \to \lambda_1$ we need to introduce a local parameter on $\mathcal{L}$ which behaves "well" in the limit. Obviously, the local parameters $\sqrt{\lambda(P) - \lambda_1}$ or $\sqrt{\lambda(P) - \lambda_2}$ are not suitable for that purpose, since they do not tend to a local parameter on the limiting Riemann surface $\mathcal{L}_0$ with double branch point at P_0. The distinguished local parameter near P_0 which is used to define the tau-function τ_C on $\mathcal{H}_C$ according to (1.4) is $x_0(P) = (\lambda(P) - \lambda_0)^{1/3}$.

Consider a neighbourhood of $\mathcal{L}$ containing both points P_1 and P_2. Such a neighbourhood can be identified with a neighbourhood of the same branch points on the three-sheeted genus zero Riemann surface $\mathcal{L}_1$, which has simple branch points at λ_1 (where sheets number 1 and 2 are glued) and λ_2 (where sheets number 2 and 3 are glued). The Riemann surface $\mathcal{L}_1$ can be biholomorphically mapped to the Riemann sphere. The meromorphic function $\lambda(\gamma)$ on $\mathbf{CP}^1$ corresponding to the branch covering $\mathcal{L}_1$ looks as follows:

$$\lambda(\gamma) = \frac{\gamma^3}{3} - \frac{\gamma^2}{2}\beta + \lambda_1 \tag{6.3}$$

where $\beta := [6(\lambda_1 - \lambda_2)]^{1/3}$. We have $d\lambda/d\gamma = \gamma(\gamma - \beta)$, i.e. the critical points are $\gamma = 0$ and $\gamma = \beta$. Corresponding critical values are given by λ_1 and λ_2, respectively.

Identifying a neighbourhood containing the branch points λ_1 and λ_2 on $\mathcal{L}_1$ with a neighbourhood D on $\mathcal{L}$ containg the branch points P_1 and P_2, we get a biholomorphic map (6.3) of a domain D_0 in γ-plane to the neighbourhood $D \subset \mathcal{L}$. In the limit $\lambda_2 \to \lambda_1$ i.e. $\beta \to 0$ we have $\lambda(\gamma) = \gamma^3/3 + \lambda_1$, i.e. $\gamma = [3(\lambda - \lambda_1)]^{1/3}$. Therefore γ can be used as a local parameter on $\mathcal{L}$ in a neighbourhood containing both points P_1 and P_2 both for $P_2 \neq P_1$ and after the limit $P_2 \to P_1$.

Computation of the behaviour of Bergman tau-function in the limit essentially gives rise to computation of Jacobian of change of variable from distinguished local parameters to γ (before and after the limit).

To find the asymptotics of the Bergman tau-function

Lemma 4 *In the limit $P_2 \to P_1$ the following asymptotics hold:*

$$E(P, P_1) = 3^{1/4}2^{-1/6}(\lambda_2 - \lambda_1)^{1/12}\{E_C(P, P_0) + o(1)\} \tag{6.4}$$

$$E(P, P_2) = 3^{1/4}2^{-1/6}(\lambda_1 - \lambda_2)^{1/12}\{E_C(P, P_0) + o(1)\} \tag{6.5}$$

where $P \in \mathcal{L}$ is a point not coinciding with P_1 and P_2 such that in the limit $P_2 \to P_1$ the value $\lambda(P)$ is kept constant; The prime-form $E_C(P, P_0)$ on $\mathcal{L}_0$ is computed at P_0 with respect to distinguished local parameter $(\lambda - \lambda_0)^{1/3}$.

$$E(P_1, P_2) = \left(\frac{3}{2}\right)^{1/2}(\lambda_2 - \lambda_1)^{1/2}(1 + o(1)) \tag{6.6}$$

Proof. Consider (6.4). By definition of $E(P, P_1)$ we have:

$$E(P,P_1) := E(P,Q)\left\{d\sqrt{\lambda(Q)-\lambda_1}\right\}^{1/2}\Big|_{Q=P_1} = E(P,Q)\{d\gamma(Q)\}^{1/2}\Big|_{Q=P_1}\left\{\frac{d\sqrt{\lambda(Q)-\lambda_1}}{d\gamma(Q)}\right\}^{1/2}\Big|_{Q=P_1} \tag{6.7}$$

A simple computation shows that

$$\frac{d\sqrt{\lambda(Q)-\lambda_1}}{d\gamma(Q)}\Big|_{Q=P_1} = \frac{\beta^{1/2}}{\sqrt{2}} \tag{6.8}$$

On the other hand, since in our limit $\gamma(Q) \to [3(\lambda-\lambda_1)]^{1/3}$, we have

$$E(P,Q)\{d\gamma(Q)\}^{1/2}\Big|_{Q=P_1} \to 3^{1/6}E_C(P,P_1) \tag{6.9}$$

By substitution of (6.8) and (6.9) into (6.7), we get (6.4). In the same way we prove (6.5).

Consider (6.6). We have

$$E(P_1,P_2) = \left[E(P,Q)\left\{d\sqrt{\lambda(P)-\lambda_1}\right\}^{1/2}\left\{d\sqrt{\lambda(Q)-\lambda_2}\right\}^{1/2}\right]\Big|_{P=P_1\,,\,Q=P_2} \tag{6.10}$$

This relation can be rewritten as follows:

$$E(P_1,P_2) = \left[E(P,Q)\,\{d\gamma(P)d\gamma(Q)\}^{1/2}\right]\Big|_{\gamma(P)=0\,,\,\gamma(Q)=\beta}$$

$$\times\left\{\frac{d\sqrt{\lambda(P)-\lambda_1}}{d\gamma(P)}\right\}^{1/2}\Big|_{P=P_1}\left\{\frac{d\sqrt{\lambda(Q)-\lambda_2}}{d\gamma(Q)}\right\}^{1/2}\Big|_{Q=P_2} \tag{6.11}$$

The standard asymptotics of the prime-form on diagonal implies the asymptotics

$$\left[E(P,Q)\,\{d\gamma(P)d\gamma(Q)\}^{1/2}\right]\Big|_{\gamma(P)=0\,,\,\gamma(Q)=\beta} = \beta(1+o(1)) \tag{6.12}$$

as $\beta \to 0$. Substituting (6.12) and (6.8) into (6.11), we get (6.6).

□

To get the asymptotics (6.1) one has to substitute (6.4) and (6.5) into (1.8), (1.9) and take into account that the caustic tau-function τ_C is given by (1.4), (1.5) (where we assume that the basic cycles on $\mathcal{L}$ are chosen such that $\mathbf{r}=0$) with divisor (6.2).

The constant C in (6.1) is equal to $(-1)^{(1-g)/36}(3/2)^{1/12}$.

7 Asymptotics of Bergman tau-function when a critical point and two simple poles form a second order pole

Consider the limit when two simple poles and a simple critical point of function λ coincide. If we use function λ to realize the Riemann surface $\mathcal{L}$ as a branched covering, and assume that sheets number 1 and 2 are glued at P_1, this limit corresponds to $\lambda_1 \to \infty$; then points $\infty^{(1)}$ and $\infty^{(2)}$ glue to form a pole of second degree of function λ, which we denote by $\infty^{(0)}$. In this way we get a Riemann surface $\mathcal{L}_0$

with moromorphic function $\lambda_0(P)$ which has one double pole at $\infty^{(0)}$; all other poles remain simple, as well as all remaining critical points of function λ.

We notice here that the limit when a critical point tends to one simple branch point only is impossible. Geometrically this is obvious by considering the Riemann surface as a branch covering: if a branch point λ_1 tends to ∞, corresponding points at infinity must glue. This also follows from the fact that degree of divisor $(d\lambda)$ should remain equal $2g-2$.

Theorem 10 *In the limit $\lambda_1 \to \infty$ we the following asymptotics holds:*

$$\tau(\mathcal{L}, \lambda, \{a_\alpha, b_\alpha\}) \sim C\, \lambda_1^{1/4}\, \tau_0(\mathcal{L}_0, \lambda_0, \{a_\alpha, b_\alpha\}) \tag{7.1}$$

where τ_0 is the Bergman tau-function on the stratum of Hurwitz space which corresponds to all simple critical points, one double pole and other simple poles of function λ; the divisor of differential $d\lambda_0$ looks as follows:

$$(d\lambda_0) = \sum_{m=2}^{M} P_m - 3\infty^{(0)} - 2\sum_{k=3}^{N} \infty^{(k)} \ . \tag{7.2}$$

The canonical basis of cycles $\{a_\alpha,\, b_\alpha\}$ on $\mathcal{L}_0$ is naturally inherited from the canonical basis of cycles on $\mathcal{L}$; C is a constant.

In analogy to the treatment of caustic, to consider the limit $\lambda_1 \to \infty$ one has to introduce on $\mathcal{L}$ an appropriate local parameter γ which covers a neighbourhood of $\mathcal{L}$ containg points P_1, $\infty^{(1)}$ and $\infty^{(2)}$. The map from a domain in γ-plane to such a neighbourhood looks as follows: follows:

$$\lambda(\gamma) = \left[\frac{\gamma^2}{2} - b\gamma\right]^{-1} \tag{7.3}$$

where parameter b is related to λ_1 by $\lambda_1 = -2/b^2$ i.e.

$$b = \sqrt{-\frac{2}{\lambda_1}}$$

The point $\gamma = 0$ is mapped to $P = \infty^{(1)}$; the point $\gamma = b$ is mapped to λ_1 and point $\gamma = 2b$ is mapped to $\infty^{(2)}$. In the limit $\lambda_1 \to \infty$ we have $b = 0$ and

$$\gamma = 1/\sqrt{2\lambda} \tag{7.4}$$

i.e., up to the factor $1/\sqrt{2}$ in this limit γ coincides with distinguished local parameter at $\infty^{(0)}$.

For finite λ_1 we have the following expressions for derivatives of distinguished local parameters at P_1, $\infty^{(1)}$ and $\infty^{(2)}$ with respect to γ:

$$\frac{d\sqrt{\lambda - \lambda_1}}{d\gamma}\Big|_{P=P_1} = -i\frac{\lambda_1}{\sqrt{2}} \tag{7.5}$$

$$\frac{d(1/\lambda)}{d\gamma}\Big|_{P=\infty^{(1)}} = -i\sqrt{\frac{2}{\lambda_1}} \tag{7.6}$$

$$\frac{d(1/\lambda)}{d\gamma}\Big|_{P=\infty^{(2)}} = i\sqrt{\frac{2}{\lambda_1}} \tag{7.7}$$

Using these formulas one can find asymptotics of all ingredients of the tau-function (1.8) as $\lambda_1 \to \infty$:

Lemma 5 *The following asymptotics hold as* $\lambda_1 \to \infty$*:*

$$E(P,P_1) = \frac{\sqrt{-i}}{\sqrt{2}}\lambda_1^{1/2}\{E(P,\infty^{(0)}) + o(1)\}\,, \tag{7.8}$$

$$E(P,\infty^{(1)}) = \sqrt{-i}\lambda_1^{-1/4}\{E(P,\infty^{(0)}) + o(1)\}\,, \tag{7.9}$$

$$E(P,\infty^{(2)}) = \sqrt{i}\lambda_1^{-1/4}\{E(P,\infty^{(0)}) + o(1)\}\,, \tag{7.10}$$

where P *is an arbitrary point of* $\mathcal{L}$ *such that* $\lambda(P)$ *is independent of* λ_1 *and* $P \neq \infty^{(1,2)}$*.*

$$E(\infty^{(1)},\infty^{(2)}) = \pm\frac{4i}{\lambda_1}(1 + o(1)) \tag{7.11}$$

$$E(P_1,\infty^{(1)}) = \pm\frac{\sqrt{2}}{\lambda_1^{1/4}}(1 + o(1)) \tag{7.12}$$

$$E(P_1,\infty^{(2)}) = \pm\frac{\sqrt{2}}{\lambda_1^{1/4}}(1 + o(1)) \tag{7.13}$$

Proof. Consider (7.8). We have

$$E(P,P_1) = \left[E(P,Q)\{d\gamma(Q)\}^{1/2}\right]\Big|_{Q=P_1}\left\{\frac{d\sqrt{\lambda(Q)-\lambda_1}}{d\gamma(Q)}\right\}^{1/2}\Big|_{Q=P_1}$$

In the limit $\lambda_1 \to \infty$ we get from (7.4):

$$\left[E(P,Q)\{d\gamma(Q)\}^{1/2}\right]\Big|_{Q=P_1} \to 2^{1/4}E(P,\infty^{(0)})\,. \tag{7.14}$$

Combining this limit with (7.5), we get (7.8).

In the same way we deduce (7.9) and (7.10) from (7.6) and (7.7), respectively.

Consider (7.11). By definition we have

$$E(\infty^{(1)},\infty^{(2)}) = \left[E(P,Q)\{d(1/\lambda(P))d(1/\lambda(Q))\}^{1/2}\right]\Big|_{P=\infty^{(1)}\,,\,Q=\infty^{(2)}}$$

$$\times\left[E(P,Q)\{d\gamma(P))d(\gamma(Q))\}^{1/2}\right]\Big|_{P=\infty^{(1)}\,,\,Q=\infty^{(2)}}\left\{\frac{d(1/\lambda)}{d\gamma}\Big|_{\gamma=0}\frac{d(1/\lambda)}{d\gamma}\Big|_{\gamma=2b}\right\}^{1/2}$$

which leads to (7.11) if we use the asymptotics of the prime-form on diagonal:

$$\left[E(P,Q)\{d\gamma(P))d(\gamma(Q))\}^{1/2}\right]\Big|_{\gamma(P)=0\,,\,\gamma(Q)=2b} \sim 2b$$

as $b \to 0$ (recall that $b^2 = -2/\lambda_1$ and $\gamma(\infty^{(1)} = 0$ and $\gamma(\infty^{(2)} = 2b$, and asymptotics (7.4), (7.6) and (7.7).

Similarly, recalling that $\gamma(P_1) = b$ and using the asymptotics

$$\left[E(P,Q)\{d\gamma(P))d(\gamma(Q))\}^{1/2}\right]\Big|_{\gamma(P)=0\,,\,\gamma(Q)=b} \sim b$$

as $b \to 0$, together with (7.5), (7.6) and (7.7), we get asymptotics (7.12) and (7.13).

□

To get the Theorem 10 one has to substitute the asymptotics obtained in Lemma 5 into (1.8), (1.9) and make use of the formulas (1.4), (1.5) with divisor (7.2) and $\beta r = 0$ for the tau-function τ_0 from (7.1).

□

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