GRADIENT HOMOTOPIES OF GRADIENT VECTOR FIELDS

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Abstract. Consider two gradient vector fields on on the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ and nowhere vanishing in $\mathbf{S}^{n-1}$. It is shown that if they are homotopic, then they are gradient homotopic. The same result holds for Hamiltonian vector fields.

1. Introduction and statement of results.

Assume that we have a continuous vector field v definied on the closed unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ and having no zeroes in $\mathbf{S}^{n-1}$. We are interested in finding a topological criterion which assures v has zeroes in $\mathbf{B}^n$. By standard facts from the homotopy theory (see for example [N]) we know that if v is not homotopic (in the space of all continuous vector fields on $\mathbf{B}^n$ nowhere vanishing in $\mathbf{S}^{n-1}$) to a constant vector field (i.e. $\deg(v\ |_{\mathbf{S}^{n-1}}) \neq 0$) then v must have zeroes in $\mathbf{B}^n$. The aim of this paper is to study whether we can obtain a better result if we restrict ourselves to some classes of vector fields. For exemple, if we know that v is gradient or Hamiltionian, could we deduce from some 'homotopic' properties of v that it has zeroes in $\mathbf{B}^n$ even if $\deg(v\ |_{\mathbf{S}^{n-1}}) = 0$. The most natural approach to this problem is to consider gradient (or Hamiltonian homotopies).

Definition 1. *Two gradient vector fields $grad\, f, grad\, g$ on $\mathbf{B}^n$ nowhere vanishing in $\mathbf{S}^{n-1}$ are said to be gradient homotopic if there exists a C^1-function $F(x,t)$ on $\mathbf{B}^n \times \mathbf{I}$ such that:*

(1) $F(*,0) \equiv f,\ F(*,1) \equiv g$

(2) $grad_x F$ *has no zeroes in* $\mathbf{S}^{n-1} \times \mathbf{I}$.

Proposition 1. *Let $v = grad\, f$, where $f \in C^1(\mathbf{B}^n)$, and let v be nowhere zero in $\mathbf{B}^n$. Then v is gradient homotopic to a constant vector field.*

Proof. We can assume that f is C^{infty}. By the Hadamard Lemma

$$f(x) = f(0) + \sum_{i=1}^{n} x_i \cdot g_i(x),$$

where g_i are C^{infty} and $v(0) = (g_1(0), \dots, g_n(0))$. Then

$$F(x,t) = f(0) + \sum_{i=1}^{n} x_i \cdot g_i(tx)$$

is the required homotopy between v and $v(0)$.

□

The problem which we consider is following: If two gradient vector fields are homotopic, are they necessarily gradient homotopic? If the answer were negative, it would mean that considering gradient homotopies we obtain a better criterion for vanishing of $v = grad\, f$ at some point than using the ordinary ones. But we will show that the answer is positive.

THEOREM 1. *Assume that we have on $\mathbf{B}^n$ two gradient vector fields nowhere vanishing in $\mathbf{S}^{n-1}$ and homotopic (in the space of all continuous vector fields on $\mathbf{B}^n$ nowhere vanishing in $\mathbf{S}^{n-1}$). Then, they are gradient homotopic.*

Let $n = 2k$. We call a vector field v Hamiltonian if $v_i = -\partial H/\partial x_{k+i}, v_{k+i} = \partial H/\partial x_i$, for $i = 1, 2, \dots, k$, and some $H \in C^1(\mathbf{B}^n)$. Hamiltonian vector fields occurs naturally in Mechanics (see e.g.[A]). For Hamiltonian vector fields one can define Hamiltonian homotopies in an analogous way as in the gradient case. Note that a vector field v is Hamiltonian iff $v_i = -w_{k+i}$ and $v_{k+i} = w_i$, $i = 1, 2, \dots, k$, for some gradient vector field w, so Theorem 1 implies:

THEOREM 2. *Assume that we have on $\mathbf{B}^n$ two Hamiltonian vector fields nowhere vanishing in $\mathbf{S}^{n-1}$ and homotopic (in the space of all continuous vector fields on $\mathbf{B}^n$ nowhere vanishing in $\mathbf{S}^{n-1}$). Then, they are Hamiltonian homotopic.*

NOTATION AND REMARKS.

1. *In this paper the character* $\mathbf{I}$ *is reserved for the interval* $[0,1]$.
2. *By a* C^{infty} *map (function, vector field) on a subset of an affine space we mean the restriction to this set of a* C^{infty} *map (function, vector field) defined in an open neighbourhood of the set.*
3. *We often replace continuous vector fields or maps defined on submanifolds of affine spaces by their* C^{infty} *approximations. We can do it thank to the approximation theorems (see e.g. [***H***] Theorem 3.5 or [***M***]). The obtain vector field (map) is homotopic to the original one. This principle will be also used to approximate homotopies, as maps defined on* $manifold \times \mathbf{I}$, *or elements of homotopy groups, as maps defined on spheres. For example, if we have an element of the k-th homotopy group of the space of continuous vector field defined on* $\mathbf{B}^n$, *then it can be represented by a* C^{infty} *map from* $\mathbf{S}^k \times \mathbf{B}^n$ *to* $\mathbf{R}^n$.

2.Proof of Theorem 1

Let us consider the following spaces:

$\mathcal{V} = \mathcal{V}(n) = \{v = (v_1, ..., v_n) \in (C^0(\mathbf{B}^n))^n;\ v(x) \neq 0 \text{ for } x \in \mathbf{S}^{n-1}\}$,

$\tilde{\mathcal{V}} = \tilde{\mathcal{V}}(n) = \{(v,h) \in C^0(\mathbf{S}^{n-1}; T\mathbf{S}^{n-1}) \times C^0(\mathbf{S}^{n-1}); (v(x), h(x)) \neq 0 \text{ for } x \in \mathbf{S}^{n-1}\}$

with C^0-topology (where $C^0(\mathbf{S}^{n-1}; T\mathbf{S}^{n-1})$ denotes the space of continuous sections of the bundle $T\mathbf{S}^{n-1}$),

$\mathcal{A} = \mathcal{A}(n) = \{f \in C^1(\mathbf{B}^n);\ grad\, f \in \mathcal{V}\}$

with C^1-topology, and

$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}(n) = \{(g,h) \in C^1(\mathbf{S}^{n-1}) \times C^0(\mathbf{S}^{n-1}); (grad\, g, h) \in \tilde{\mathcal{V}}\}$

with mixed $C^1 \times C^0$-topology.

Theorem 1 says that the gradient map $\mathbf{grad} : \mathcal{A} \to \mathcal{V}$ induces an injection on connected components. The spaces $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{V}}$ of (gradient) vector fields restricted to $\mathbf{S}^{n-1}$ are used in the inductive step. Let $\mathbf{n}$ denote the outward pointed unit normal vector field on $\mathbf{S}^{n-1}$.

Lemma 1. *The map $\varphi : \mathcal{A} \to \tilde{\mathcal{A}}$ defined by*

$$\varphi(f) = (f, \partial f/\partial \mathbf{n}) \mid_{\mathbf{S}^{n-1}}$$

is a homotopy equivalence. The same holds for $\tilde{\varphi} : V \to \tilde{V}$, given by

$$\tilde{\varphi}(v) = (v', h),$$

where $v \mid_{\mathbf{S}^{n-1}} = v' + h \cdot \mathbf{n}$.

Proof. Let $\rho : \mathbf{I} \to \mathbf{I}$ be a C^{infty} function equal to 0 in a neighbourhood of 0 and to 1 in a neighbourhood of 1. Define $\psi : \tilde{\mathcal{A}} \to \mathcal{A}$ by the formula

$$\psi(g, h)(x) = \rho(\|x\|) \cdot \Big(g(x/\|x\|) + (\|x\| - 1) \cdot h(x/\|x\|)\Big)$$

Then it is easy to check that:

(1) $\varphi \circ \psi = id_{\tilde{\mathcal{A}}}$

(2) $\psi \circ \varphi \sim id_{\mathcal{A}}$ by the homotopy $t \cdot (\psi \circ \varphi)(f) + (1 - t) \cdot f$,

which gives the result for φ. The proof for $\tilde{\varphi}$ is similar.

□

Proposition 2. *The map $\iota : \tilde{\mathcal{A}} \to \tilde{\mathcal{V}}$ which sends (g, h) to $(grad\, g, h)$ induces one-to-one correspondence between the connected components of these spaces.*

Proof. First we show the injectivity of $\pi_0(\iota) : \pi_0(\tilde{\mathcal{A}}) \to \pi_0(\tilde{\mathcal{V}})$ (the induced map on the sets of connected components) by induction on n.

Case n=2. It is the special case (for many reasons, one of them is that $T\mathbf{S}^1$ is a trivial bundle). Let s a parameterization by arc length on $\mathbf{S}^1$. Then, the vector field $\partial/\partial s$ is a nowhere zero section of $T\mathbf{S}^1$.

The space $\mathcal{A}' = \{(v, h) \in \tilde{\mathcal{V}}; \int_{\mathbf{S}^1} v\, ds = 0\}$ (with C^0-topology) is just the image of ι and the induced map $\tilde{\mathcal{A}} \to \mathcal{A}'$ is a homotopy equivalence. In fact, $\tilde{\mathcal{A}}$ is homeomorphic to $\mathcal{A}' \times \mathbf{R}$ by $(g, h) \to ((grad\, g, h), g(0))$ (the inverse is $((v, h), c) \to (c + \int_0^s v, h))$.

Consider $\mathcal{A}' \subset \tilde{\mathcal{V}}$. The connected components $\tilde{\mathcal{V}}_i$ of $\tilde{\mathcal{V}}$ are classified by the topological degree $deg(v \cdot \partial/\partial s + h \cdot \mathbf{n}) = i \in \mathbf{Z}$.

Lemma 2. *For $i \neq 1$ the inclusion $\mathcal{A}' \cap \tilde{\mathcal{V}}_i \subset \tilde{\mathcal{V}}_i$ is a homotopy equivalence.*

Proof. Let $(v, h) \in \tilde{\mathcal{V}}_i$. Consider the continuous functions $v_+(s) = max_{s \in \mathbf{S}^1}(v(s), 0)$, $v_-(s) = min_{s \in \mathbf{S}^1}(v(s), 0)$ and their integrals $c_+(v)$, $c_-(v)$ over $\mathbf{S}^1$. Note that c_+ and c_- are continuous functions of v and they nowhere vanish (since $i \neq 1$). Therefore, $\mathcal{A}'_i$ is a strong deformation retract of $\tilde{\mathcal{V}}_i$, by the deformation

$$((v, h), t) \longrightarrow t \cdot (v, h) + (1 - t) \cdot ((L(v) \cdot v_+ + (L(v))^{-1} \cdot v_-), h),$$

where $L(v) = (-c_-(v)/c_+(v))^{1/2}$.

□

For $i = 1$ the above proof does not work. Take $(v_0, h_0) \in \mathcal{A}'_1 = \mathcal{A}' \cap \tilde{\mathcal{V}}_1$. Since $\int_{\mathbf{S}^1} v = 0$, v_0 has zeroes, and turning the circle we can assume that $v_0(0) = 0$. Hence $h_0(0) \neq 0$. After perturbing (v_0, h_0) in $\mathcal{A}'_1$, we can also assume that it equals $(sin, 1)$ just near 0. We shall show that it is in fact homotopic (in $\mathcal{A}'_1$) to $(sin, 1)$. By the arguments from the proof of Lemma 2, it suffices to find such a homotopy in $\{(v, h) \in \tilde{\mathcal{V}}_1; min(v) < 0 \text{ and } max(v) > 0\}$.

The map $\alpha = (v_0, h_0) : \mathbf{S}^1 \to \mathbf{R}^2 \setminus 0 \simeq \mathbf{C} \setminus 0$ has degree 0 (since $v_0 + h_0 {\cdot} \mathbf{n}$ has degree 1), so it can be lifted to the map $\tilde{\alpha} : \mathbf{S}^1 \to \mathbf{C}$ i.e. $exp \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(0) = \pi i/2$. Let $\gamma : \mathbf{S}^1 \to \mathbf{C}$ satisfies $exp \circ \gamma = (sin, 1)$ and $\gamma(0) = \pi i/2$. Then,

$$exp(t \cdot \alpha + (1 - t) \cdot \gamma)$$

is a homotopy with required properties.

□

Inductive step. Let $(g, h) \in \tilde{\mathcal{A}} = \tilde{\mathcal{A}}(n)$. We will deform (g, h) in $\tilde{\mathcal{A}}$ in order to obtain an element of $\tilde{\mathcal{A}}$ of the most simple form. The first step is to deform g to a Morse function (see e.g. **[H]**). So, assume that g is a Morse function and divide the set of its critical points into the set P on which h is positive and Q on which h is negative (h nowhere vanishes on the set of critical points of g).

Lemma 3. *Let $(g,h) \in \tilde{\mathcal{A}}$ be as above. Then*

$$deg(grad\, g + h\cdot \mathbf{n}) = (-1)^n + \sum_{p\in P}(-1)^{i(p)},$$

where $i(p)$ denotes the index of g at p.
(If $P = \emptyset$, then we mean the right-hand side of the formula equal to $(-1)^n$)

Proof. Assume that $c = max_{x\in \mathbf{S}^{n-1}} h(x)$ is greater than zero. Consider on $U = \{x \in \mathbf{R}^n; 1 \le \|x\| \le 2\}$ the vector field

$$w(x) = v(x/\|x\|) - 2c\cdot(\|x\| - 1)\cdot \mathbf{n}(x/\|x\|),$$

where $v = grad\, g + h\cdot\mathbf{n}$. Note that $w\mid_{\mathbf{S}^{n-1}} \equiv v$ and $deg(w\mid_{\mathbf{S}_2^{n-1}}) = (-1)^n$, where $\mathbf{S}_2^{n-1}$ denotes the sphere of radius 2. The point $x \in U$ satisfies $w(x) = 0$ if and only if $x/\|x\| \in P$ and $\|x\| = \{h(x/\|x\|)/(2c)\} + 1$. The local index of w at such x equals $-(-1)^{i(x/\|x\|)}$. Therefore, by the properties of the index [H] we have

$$deg(w \mid \mathbf{S}_2^{n-1}) - deg(w \mid \mathbf{S}^{n-1}) = -\sum_{p\in P}(-1)^{i(p)}$$

and the lemma follows for $c > 0$. If $c \le 0$, then $deg(v) = (-1)^n$ and $P = \emptyset$. This ends the proof.

□

We continue the process of perturbing (g,h) using isotopies of $\mathbf{S}^{n-1}$. Fix $p \in P$. Since $n \ge 3$, we can move P into a small neighbourhood of p without changing Q. Afterwards we move Q into $\mathbf{S}^{n-1} \cap \{x \in \mathbf{R}^n; x_n \le -1/2\}$ and P into $\mathbf{S}^{n-1} \cap \{x \in \mathbf{R}^n; x_n \ge 1/2\}$. Since h is positive on P and negative on Q we can change it linearly by $(t\cdot h + (1-t)\cdot x_n)$ to a new $h \equiv x_n$. We call (g,h) of such form to be normal.

Assume that we have two elements (g_1,h_1) and (g_2,h_2) of $\tilde{\mathcal{A}}$ homotopic in $\tilde{\mathcal{V}}$. Then,

$$deg(grad\, g_1 + h_1\cdot\mathbf{n}) = deg(grad\, g_2 + h_2\cdot\mathbf{n}).$$

By the above, we can assume that $(g_i,h_i), i = 1,2;$ are normal (in particular $h_1 \equiv h_2 \equiv x_n$). Let $\varphi : \mathbf{B}^{n-1} \to \mathbf{S}_-^{n-1}$ be a diffeomorphism onto the southern hemishere and let

$P(i)\,(i=1,2)$ denotes the set of critical points of g_i on which $h_i \equiv x_n$ is positive. By Lemma 3

$$\sum_{p\in P(1)} (-1)^{i(p)} = \sum_{p\in P(2)} (-1)^{i(p)}.$$

Consequently, $grad\,(g_1 \circ \varphi)$ is homotopic to $grad\,(g_2 \circ \varphi)$ in $\mathcal{V}(n-1)$ and, by the inductive assumption and Lemma 1, they are also gradient homotopic. Let $G : \mathbf{B}^{n-1} \times \mathbf{I} \to \mathbf{R}$ be such a homotopy. Consider on $(\mathbf{S}^{n-1}_{-} \times \mathbf{I}) \cup (\mathbf{S}^{n-1}_{-} \times \{0,1\})$ the function equal to $G \circ \varphi^{-1}$ on $\mathbf{S}^{n-1}_{-} \times \mathbf{I}$, to g_1 on $\mathbf{S}^{n-1} \times \{0\}$ and to g_2 on $\mathbf{S}^{n-1} \times \{1\}$. By **[H]**, it can be extended to a C^{infty} function G' on $\mathbf{S}^{n-1} \times \mathbf{I}$ and now (G', x_n) is the required homotopy. This ends the proof of injectivity of $\pi_0(\iota)$ and hence,by Lemma 1, the proof of the Theorem 1.

The surjectivity of $\pi_0(\iota)$ follows from Lemma 1 and the following lemma.

Lemma 4. *For each $d \in \mathbf{Z}$ there is $f \in \mathcal{A}$ such that $deg(grad\, f\, |_{\mathbf{S}^{n-1}}) = d$.*

Proof Let $\rho : \mathbf{I} \to \mathbf{I}$ be a C^{infty} function equal to 0 near 0 and 1 near 1. For $n = 2$ and $d \neq 1$ we define $f_d(z) := \rho(\|z\|)Re(z^{1-d})$, where $z \in \mathbf{B}^2 \subset \mathbf{R}^2 \simeq \mathbf{C}$. By the Cauchy-Riemann formula $d(z^{1-d})/dz = (\partial f/\partial x, -\partial f/\partial y)$ near $\mathbf{S}^1$. Therefore, $deg(grad\, f\, |_{\mathbf{S}^{n-1}}) = d$. For $d = 1$ we put $f_1(x,y) = x^2 + y^2$.

Assume $n > 2$. Define $f(x_1, x_2, \ldots, x_n) = f_d(x_1, x_2) + x_3^2 + \ldots + x_n^2$. Then $grad\, f = (\partial f_d/\partial x_1, \partial f_d/\partial x_2, 2x_3, \ldots, 2x_n)$ and by the properties of suspension (see e.g. **[H]**,**[S]**)

$$deg(grad\, f\, |_{\mathbf{S}^{n-1}}) = deg(grad\, f_d\, |_{\mathbf{S}^1}) = d$$

□

3. Questions

Assume that we have a continuous family $v : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{R}^n$ of continuous vector fields on $\mathbf{B}^n$ such that the restriction $\tilde{v}$ of v to $\partial(\mathbf{B}^n \times \mathbf{B}^m) = \mathbf{S}^{n-1} \times \mathbf{B}^m \cup \mathbf{B}^n \times \mathbf{S}^{m-1}$ nowhere vanishes. If $\tilde{v}$ gives a nontrivial element in $\pi_{n+m-1}(\mathbf{R}^n \setminus 0) = \pi_{n+m-1}(\mathbf{S}^{n-1})$ (i.e. v is not homotopic (in the space of all families of vector fields as above) to a constant (family of) vector field(s) $v(x,y) \equiv w \in \mathbf{R}^n \setminus 0$), then v must have zeroes (see [N]). As in the case of one vector field one can ask whether in the case of v gradient (i.e. $v(x,y) = grad_x f(x,y)$ for some $f \in C^1(\mathbf{B}^n \times \mathbf{B}^m)$) there is a better invariant which shows that v has zeroes. First we note that simply by repeating the proof of Proposition 1 we obtain:

Proposition 3. *Let $v(x,y) = grad_x f$ be as above. If v has no zeroes in $\mathbf{B}^n \times \mathbf{B}^m$, then v is gradient homotopic (i.e. in the space of all families of gradient vector fields nowhere vanishing in $\mathbf{B}^n \times \mathbf{B}^m$)) to a constant vector field.*

Question 1. *Are two families of gradient vector fields gradient homotopic, if they are homotopic ?*

This question seems to be much more difficult than the problem we have considered earlier. If the answer is negative, it means that there exist some new obstruction for families of gradient vector fields which can be used in looking for their zeroes. Question 1 is closely related to the homotopic properties of the map $\mathbf{grad} : \mathcal{A} \to \mathcal{V}$. We have shown (Proposition 2 and Lemma 1) that $\pi_0(\mathbf{grad})$ is a bijection.

Fix $* = x_1 \in \mathcal{A}$ and $* = \partial/\partial x_1 \in \mathcal{V}$.

Question 2. *Is $\pi_m(\mathbf{grad}) : \pi_m(\mathcal{A}, *) \to \pi_m(\mathcal{V}, *)$ injective ?*

Proposition 4. *If $\pi_m(\mathbf{grad})$ is injectieve, then every family of gradient vectors fields (on $\mathbf{B}^n$) which is homotopic to a constant vector field is gradient homotopic to a constant vector field. The converse holds for $m < n - 1$.*

Proof Take any family $v(x,y) = grad_x f(x,y)$ of gradient vector fields. We can assume that f is smooth. By the Hamadard Lemma

$$f(x,y) = f(0,y) + \sum_{i=1}^{n} x_i \cdot g_i(x,y),$$

where g_i's are smooth and $grad_x f(0,y) = (g_1(0,y), g_2(0,y), ..., g_n(0,y))$. If N is sufficiently large, then $v(x,y) \neq 0$ for $(\|x\| + \|y\|^{2N}) \geq 1$, so the homotopy

$$F(x,y,t) = (1-t)f(0,y) + \sum_{i=1}^{n} x_i \cdot g_i \left((1-t)x + t(1-\|y\|^{2N})x, y\right)$$

joins f with

$$h(x,y) = \sum_{i=1}^{n} x_i \cdot g_i \left((1-\|y\|^{2N})x, y\right)$$

But the gradient (with respect to x) of h for $y \in \mathbf{S}^{m-1}$ does not depend on x and equals $w(y) = (g_1(0,y), g_2(0,y), ..., g_n(0,y))$. So, we have just proved that any family of gradient vector fields is gradient homotopic to a family which does not depend on x for $y \in \mathbf{S}^{m-1}$. The same argument works for the homotopies of gradient vector fields and so, it is easy to see that the homotopy class of $w : \mathbf{S}^{m-1} \to \mathbf{R}^n \setminus 0$ depends only on the gradient homotopy class of $v(x,y) = grad_x f(x,y)$.

Assume that $\pi_m(\mathbf{grad})$ is injective and take a $v(x,y) = grad_x f(x,y)$ homotopic to a constant vector field. We can assume that f is smooth and that v restricted to $\mathbf{B}^n \times \mathbf{S}^{m-1}$ does not depend on y. We denote this map as above by w. We know that v is homotopic to a constant vector field say $\partial/\partial x_1$. Let $V : (\mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I}, \partial(\mathbf{B}^n \times \mathbf{B}^m) \times \mathbf{I}) \to (\mathbf{R}^n, \mathbf{R}^n \setminus 0)$ be a such homotopy. By the same arguments as at the begining of the proof, we can change V in such a way that $V \mid_{\mathbf{B}^n \times \mathbf{S}^{m-1} \times \mathbf{I}}$ does not depend on y (in particular it is gradient with respect to x).

Let $\tilde{f} : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{R}$ be defined as follows:

(1) $\tilde{f}(x,y) = f(x,2y)$ if $\|y\| \leq 1/2$

(2) $grad_x \tilde{f}(x,y) = V(x, y/\|y\|, 2\|y\| - 1)$ if $\|y\| \geq 1/2$

(3) $\tilde{f} \mid \mathbf{B}^n \times \mathbf{S}^{m-1} \equiv x_1$.

Then $\tilde{f}$ is gradient homotopic to f and V can be considered as a(n) (ordinary) homotopy between $grad_x \tilde{f}$ and $\partial/\partial x_1$.

The function $\tilde{f}$ induces, via the identification $(\mathbf{S}^m, *) \simeq (\mathbf{B}^m/\mathbf{S}^{m-1}, \mathbf{S}^{m-1})$, the function $h : \mathbf{B}^n \times \mathbf{S}^m \to \mathbf{R}$ satisfying $h(x, *) \equiv x_1$. So, h definies $\sigma_h \in \pi_m(\mathcal{A}, *)$ which gives (by V, see above) a trivial element of $\pi_m(\mathcal{V}, *)$. By the assumption σ_h is trivial in $\pi_m(\mathcal{A}, *)$. Let $H : \mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I} \to \mathbf{R}$ be a homotopy between h and the constant map. Let $\alpha : \mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I} \to \mathbf{B}^n \times (\mathbf{B}^m/\mathbf{S}^{m-1}) \times \mathbf{I} = \mathbf{B}^n \times \mathbf{S}^m \times \mathbf{I}$ be the canonical projection. Then $H \circ \alpha$ is a gradient homotopy between $\tilde{f}$ and a constant vector field.

Now we assume that $m < n$ and every family of gradient vector fields which is homotopic to a constant vector field is gradient homotopic to a constant vector field. Let $h : (\mathbf{S}^m, *) \to (\mathcal{A}, *)$ be such that $grad \circ h$ is homotopic to a constant map. We can treat h as a function on $\mathbf{B}^n \times \mathbf{S}^m$. By the identification $(\mathbf{S}^m, *) \simeq (\mathbf{B}^m/\mathbf{S}^{m-1}, \mathbf{S}^{m-1})$ h gives a function f on $\mathbf{B}^n \times \mathbf{B}^m$ such that $v(x, y) = grad_x f(x, y)$ is homotopic, by a homotopy $V : \mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I} \to \mathbf{R}^n$ to a constant vector field. By the assumption, we can find a gradient homotopy $F : \mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I} \to \mathbf{R}$ such that

(1) $F(x, y, 0) \equiv f(x, y)$ (in particular $F(x, y, 0) \equiv x_1$ for $\|y\| = 1$),

(2) $F(x, y, 1) \equiv x_1$

(3) $grad_x F(x, y, t) \neq 0$ if $(x, y) \in \partial(\mathbf{B}^n \times \mathbf{B}^m)$.

In order to prove the proposition it sufficies to change F in such a way that it equals identically x_1 for $\|y\| = 1$.

Using the Hadamard Lemma, as at the begining of the proof, we change F in such a way that $grad_x F(x, y, t)$ does not depend on x for $\|y\| = 1$ and still equals x_1 on $\mathbf{B}^n \times \mathbf{S}^{m-1} \times \{0, 1\}$. Put $w(y, t) = grad_x F(x, y, t)$ for $\|y\| = 1$. Adding to F a function not depending on x we change it such that now $F(0, y, t) \equiv 0$. This implies $F(x, y, t) = \langle x, w(y, t)\rangle$ if $\|y\| = 1$. By the assumption on m and n, there exists a homotopy $W : \mathbf{S}^{m-1} \times I \times I \to \mathbf{R}^n \setminus 0$ between w and $\partial/\partial x_1$ which equals identically $\partial/\partial x_1$ on $\mathbf{S}^{m-1} \times \{0, 1\}$. Now, in order to obtain a homotopy with required properties, it sufficies to glue F with $\langle x, w(y, t)\rangle$. One can do it, for example, as follows. Choose $\alpha : \mathbf{I} \to \mathbf{I} \times \mathbf{I}$ a parametrization of the union of three sides of the square, for example

$$\alpha(t) = \begin{cases} (0, 3t) & \text{if } t \leq 1/3 \\ (3t - 1, 1) & \text{if } 1/3 \leq t \leq 2/3 \\ (1, 3 - 3t) & \text{if } 2/3 \leq t\,. \end{cases}$$

Now the formula for the homotopy can be written as follows

$$\tilde{F}(x,y,t)=\begin{cases} F(x,(1+\alpha_2(t))\cdot y,\alpha_1(t)) & \text{if } \|y\|\le(1+\alpha_2(t))^{-1} \\ \langle x,\ W(y/\|y\|,(1+\alpha_2(t))\cdot\|y\|-1,\alpha_1(t))\rangle & \text{otherwise .} \end{cases}$$

□

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REFERENCES

[A] V.I.Arnold, "Mathematical Methods of Classical Mechanics", Springer-Verlag, New York, 1978

[H] M.W.Hirsh, "Differential Topology", Springer-Verlag, New York Heidelberg Berlin, 1976

[M] J.W.Milnor, "Topology from the differentiable viewpoint", Univ.Press Virginia, Charlottesville, 1965

[N] L.Nirenberg, "Topics in Nonlinear Functional Analysis", N.Y.U. Lecture Notes, 1974

[S] R.M.Switzer, "Algebraic Topology - Homotopy and Homology", Springer-Verlag, New York Heidelberg Berlin, 1975

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