FAMILIES OF AFFINE PLANES: THE EXISTENCE OF A CYLINDER

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Abstract. Given a family of complex affine planes, we show that it is trivial over a Zariski open subset of the base. The proof relies upon a relative version of the contraction theorem.

Introduction

The varieties X, S, U, V considered below are assumed being smooth, quasi-projective and defined over $\mathbb{C}$. By a *cylinder over* U we mean a Cartesian product $U \times \mathbb{C}^k$ where $k > 0$. We say that a family $f : X \to S$ of quasi-projective varieties *contains a cylinder* if, for some Zariski open subset S_0 of S, there is a commutative diagram

$$\begin{tikzcd} f^{-1}(S_0) \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & S_0 \times \mathbb{C}^k \arrow[dl, "pr_1"] \\ & S_0 & \end{tikzcd}$$

where φ is an isomorphism.

The main result of this paper is the following

Theorem 0.1. *A smooth family $f : X \to S$ with general fibers isomorphic to $\mathbb{C}^2$ contains a cylinder $S_0 \times \mathbb{C}^2$.*

A similar result is well-known for families of relative dimension 1 [BhaDut, EakHei, KamMiy, KamWri, Miy1, Miy2, MiySug, Sug]. In relative dimension 2 the existence of a cylinder was an assumptions in the previous work of [AsaBha, Kam, Miy2, Sat]. For higher relative dimensions, it makes a part of Conjecture 3.8.3 in [DolVei].

Theorem 0.1 proves Lemma III of [Kal] which is one of the principal ingredients in the proof of the following result.

Theorem [Kal]. *A polynomial p on $\mathbb{C}^3$ with general fibers isomorphic to $\mathbb{C}^2$ is a variable of the polynomial algebra $\mathbb{C}^{[3]}$. In particular, all its fibers are isomorphic to $\mathbb{C}^2$.*

Section 1 is devoted to a relative version of the contraction theorem over an affine base, which we need in sections 2 and 3 where we prove Theorem 0.1.

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1. A contraction theorem

The main result of this section (Theorem 1.3) is a relative version of the classical Castelnuovo-Enriques-Kodaira contraction theorem. In the analytic setting it follows from the Moishezon-Nakano-Fujiki theorem (see [AncTom, Kod, Moi, FujNak, Nak] and especially [Fuj, Rem. 3]), whereas in the projective setting it follows from the theorem on contraction of extremal rays as given in [KMM, Thm. 3-2-1] (cf. also [Art, Maz, Cor, Ish]). Actually, the particular version that we need is much simpler, so we provide a proof along the lines of the Castelnuovo-Enriques-Kodaira contraction theorem [Kod, Appendix], [GriHar, Sect. 4.1, p. 154].

As usual, for an algebraic variety X its structure sheaf is denoted by $\mathcal{O}_X$. If L is a line bundle on X and Y is a subvariety of X then $\mathcal{O}_Y(L)$ denotes the sheaf of germs of section of L over Y. We begin with the following lemma (cf. [Gra, §7.6], [Pet, Thm. 4.7]).

Lemma 1.1. *Let $\rho : E \to S$ be a smooth proper morphism of smooth quasi-projective varieties with fibers $E_s := \rho^{-1}(s)\,(s \in S)$. For a line bundle L on E we denote $\mathcal{L} := \mathcal{O}_E(L)$ and $\mathcal{L}_s := \mathcal{O}_{E_s}(L)$. Suppose that*

(o) $$H^q(E_s, \mathcal{L}_s) = 0 \qquad \forall s \in S, \quad \forall q \geq 1.$$

Then for any Zariski open affine subset $S_0 \subset S$ and $E_{S_0} := \rho^{-1}(S_0)$ we have

(a) $$H^q(E_{S_0}, \mathcal{L}) = 0 \qquad \forall q \geq 1\,,$$

and

(b) *for every point $s \in S_0$ the restriction homomorphism*

$$H^0(E_{S_0}, \mathcal{L}) \to H^0(E_s, \mathcal{L}_s)$$

is surjective. Furthermore,

(c) *the sheaf $\rho_*\mathcal{L}$ is locally free and generated by a vector bundle, say, $\xi(L)$ over S with fibers $\xi(L)_s = H^0(E_s, \mathcal{L}_s)$, $s \in S$.*

Proof. (a) Note that by [Ha, Prop. III.9.2.c] $\mathcal{L}$ is a flat $\mathcal{O}_S$-module. In virtue of the assumption (o), for every $s \in S$ and every $q \geq 1$ the natural homomorphism

$$R^q\rho_*\mathcal{L} \otimes_{\mathcal{O}_S} k(s) \to H^q(E_s, \mathcal{L}_s) = 0$$

is an isomorphism, and the coherent sheaf $R^q\rho_*\mathcal{L}$ is locally free [Gra, Thm. 4], [Ha, Thm. III.12.11.a, Ex. II.5.8.c], [Dan, Prop. II.3.7] (here $k(s) \simeq \mathbb{C}$ denotes the residue field of a closed point $s \in S$). Thus we have

$$R^q\rho_*\mathcal{L} = 0 \qquad \forall q \geq 1\,.$$

The Leray spectral sequence gives now isomorphisms

$$H^q(E, \mathcal{L}) \simeq H^q(S, \rho_*\mathcal{L}) \qquad \forall q \geq 1 \tag{1}$$

[Pet, (5.16)]. For a Zariski open affine subset $S_0 \subset S$, by Serre's vanishing theorem [Ha, Thm. III.3.7] we have

$$H^q(S_0, \rho_*\mathcal{L}) = 0 \qquad \forall q \geq 1,$$

which together with (1) implies (a).
(b) Since $R^1\rho_*\mathcal{L} = 0$, for every point $s \in S_0$ the homomorphism

$$\rho_*\mathcal{L} \otimes_{\mathcal{O}_{S_0}} k(s) \to H^0(E_s, \mathcal{L}_s) \tag{2}$$

is surjective, whence it is an isomorphism [Ha, Thm. III.12.11]. On the other hand, since S_0 is affine we have an isomorphism

$$(\rho_*\mathcal{L})|_{S_0} \simeq H^0(E_{S_0}, \mathcal{L})^{\sim} \tag{3}$$

where $M^{\sim}$ denotes the $\mathcal{O}_{S_0}$-module generated by an $H^0(S_0, \mathcal{O}_{S_0})$-module M [Ha, Prop. III.8.5]. Now (2) and (3) yield (b).
(c) By (o) we have $h(s) := \dim H^0(E_s, \mathcal{L}_s) = \chi(E_s, \mathcal{L}_s)$ where the Euler characteristic is locally constant on S [Gra, Thm. 5], [Dan, Prop. II.3.8]. Now the isomorphism in (2) and [Ha, Ex. II.5.8.c] imply that $\rho_*\mathcal{L}$ is a locally free sheaf with $\rho_*\mathcal{L} \simeq \mathcal{O}_S(\xi(L))$ where $\xi(L)_s = H^0(E_s, \mathcal{L}_s)$, $s \in S$. The proof is completed. □

Corollary 1.2. *The statements of Lemma 1.1 remain true if one replaces the assumption* (o) *by any one of the following two:*
(o′) *For every $s \in S$ the line bundle $L|_{E_s} - K_{E_s}$ on the fiber E_s is ample, where K_{E_s} is the canonical bundle on E_s.*
(o″) *For every $s \in S$ and for each irreducible component C of the fiber E_s, $C \simeq \mathbb{P}^n$ $(n \geq 1)$ and $L|_C \simeq \mathcal{O}(l)$ with $l \geq 0$.*

Indeed, by the Kodaira-Nakano vanishing theorem any one of the conditions (o′) and (o″) implies (o) [GriHar, Sect.1.2, p. 154].

Theorem 1.3. *Let $\pi : V \to S$ be a smooth proper morphism of smooth quasi-projective varieties, and let $E \subset V$ be an irreducible smooth divisor (proper over S) which meets every fiber $V_s = \pi^{-1}(s)$ $(s \in S)$ transversally. Let $E_s := V_s \cap E = \bigcup_{i=1}^m C_{s,i}$ be the decomposition into irreducible components (ordered arbitrarily for every $s \in S$).*

(a) *Assume that for every $s \in S$ and each $i = 1, \dots, m$, $C_{s,i} \simeq \mathbb{P}^n$ $(n \geq 1)$ with the conormal bundle $J_{C_{s,i}}/J^2_{C_{s,i}} \simeq \mathcal{O}_{\mathbb{P}^n}(1)$, where $J_{C_{s,i}}$ is the ideal sheaf of $C_{s,i}$ in V_s.*[1] *Then there is a commutative diagram*

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & W \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

where W is a smooth quasi-projective variety, π' is a smooth morphism and φ is an S-contraction of the divisor E on V onto a smooth subvariety $A \subset W$ étale over S.

(b) *Let E' be another smooth divisor on V (proper over S) which meets every fiber V_s $(s \in S)$ and the divisor E transversally, and let $E'_s := E' \cap V_s = \bigcup_{j=1}^{m'} C'_{s,j}$ be the decomposition into irreducible components. Assume that for every $s \in S$ and for every pair i, j such that $C_{s,i} \cap C'_{s,j} \neq \emptyset$, this intersection becomes a hyperplane under an isomorphism $C_{s,i} \simeq \mathbb{P}^n$ as in (a). Then the restriction $\varphi|_{E'} : E' \to W$ is*

[1] In other words (by Kodaira's contraction theorem [Kod]) we assume that each irreducible component $C_{s,i}$ can be contracted in the fiber V_s into a smooth point.

an S-immersion onto a divisor (proper over S) whose singularities are at worst transversal intersections (along A) of several smooth branches.

Proof. (a) We may assume in the sequel that the base S is connected. Fix a very ample line bundle H on V. For an arbitrary point $s_0 \in S$, letting $L = mH$ with m sufficiently big, we may assume that the line bundle $L|_{V_{s_0}} - K_{V_{s_0}}$ on the fiber V_{s_0} over s_0 is ample, and so by the Kodaira vanishing theorem [GriHar, Sect.1.2] we have $H^q(V_{s_0}, \mathcal{L}_{s_0}) = 0 \quad \forall q \geq 1$ (here $\mathcal{L}_s := \mathcal{O}_{V_s}(L)$). Since $\mathcal{L}$ is a flat $\mathcal{O}_S$-module, by the semi-continuity theorem [Ha, Thm. III.12.8] it follows that

$$H^q(V_s, \mathcal{L}_s) = 0 \quad \forall q \geq 1 \tag{4}$$

for every point s in a neighborhood S_0 of the point s_0. Thus, for m_0 large enough and $L = L_0 := m_0 H$, (4) holds for every point $s \in S$.

Since the divisor E is irreducible, the monodromy of the smooth family $\pi|_E : E \to S$ acts transitively on the set of irreducible components $\{C_{s,i}\}_{i=1}^m$ of the fiber E_s ($s \in S$). Hence all these components (regarded as cycles of V) are algebraically (and then also numerically) equivalent. Thus $k := \deg(L_0|_{C_{s,i}})$ does not depend on s, i. Consider the (Cartier) divisors $L_j := L_0 + jE = m_0 H + jE$ on V ($j \in \mathbb{Z}$). Under our assumptions $\big(C_{s,i} \simeq \mathbb{P}^n$ and $[C_{s,i}]|_{C_{s,i}} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)\big)$ for every $s \in S$ and each $i = 1, \dots, m$ we have $L_j|_{C_{s,i}} \simeq \mathcal{O}_{\mathbb{P}^n}(l)$ with $l := k - j$, and so

$$H^q(E_s, (\mathcal{L}_j)_s) = 0 \qquad \forall q \geq 1, \quad \forall j = 0, \dots, k\,. \tag{5}$$

Now the same argument as in the proof of the Castelnuovo-Enriques-Kodaira theorem [Kod, Appendix], [GriHar, p. 477] shows that:

(i) *for every $j = 0, \dots, k$ and for every $s \in S$ the restriction map*

$$H^0(V_s, (\mathcal{L}_j)_s) \to H^0(E_s, (\mathcal{L}_j)_s)$$

is surjective,

(ii) *the linear system $|L_k|_{V_s}|$ of divisors on V_s is base point free, and*

(iii) *the associated morphism $\varphi_s : V_s \to \mathbb{P}^{h-1} = \mathbb{P}\big(H^0(V_s, (\mathcal{L}_k)_s)^*\big)$ (with $h := h^0(V_s, (\mathcal{L}_k)_s)$) yields a contraction of the irreducible components $C_{s,i}$ ($i = 1, \dots, m$) of the divisor $E_s \subset V_s$ into m distinct smooth points.*

For the convenience of readers we sketch this argument. For each $j = 1, \dots, k$ consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_{V_s}(L_{j-1}) \to \mathcal{O}_{V_s}(L_j) \to \mathcal{O}_{E_s}(L_j) \to 0$$

and the corresponding long exact cohomology sequence

$$0 \to H^0(V_s,\, (\mathcal{L}_{j-1})_s) \to H^0(V_s,\, (\mathcal{L}_j)_s) \to H^0(E_s,\, (\mathcal{L}_j)_s) \to H^1(V_s,\, (\mathcal{L}_{j-1})_s) \to \dots\,. \tag{6}$$

It follows from (5) and (6) that for every $q \geq 1$ the natural homomorphisms

$$H^q(V_s,\, (\mathcal{L}_0)_s) \to H^q(V_s,\, (\mathcal{L}_1)_s) \to \dots \to H^q(V_s,\, (\mathcal{L}_k)_s)$$

are surjective, and so by (4) all these groups vanish. (In particular,

$$H^q(V_s,\, (\mathcal{L}_{k-1})_s) = H^q(V_s,\, (\mathcal{L}_k)_s) = 0 \qquad \forall q \geq 1, \quad \forall s \in S\,.) \tag{7}$$

Now (6) implies (i).

Since the line bundle $L_k|_{V_s\setminus E_s} \simeq L_0|_{V_s\setminus E_s}$ is very ample, the restriction $\varphi_s|_{V_s\setminus E_s}$ gives an embedding. Furthermore, the restriction $L_k|_{E_s}$ is a trivial bundle, and so (ii) and (iii) easily follow.

By Lemma 1.1(c), (7) implies that the dimension $h := h^0(V_s, (\mathcal{L}_k)_s)$ is constant on S and $\xi(L_k) = \left(\bigcup_{s\in S} H^0(V_s, (\mathcal{L}_k)_s) \to S\right)$ is a rank h vector bundle on S. Thus, clearly,

$$\varphi : V \to \mathbb{P}(\xi(L_k)^*), \qquad \varphi\,|\,V_s = \varphi_s\,,$$

is a proper morphism onto a closed subvariety $W := \varphi(V)$ of (the total space of) the projective bundle $\mathbb{P}(\xi(L_k)^*)$; actually it consists of contracting the divisor $E \subset V$ onto a smooth subvariety $A \subset W$ étale (and m-sheeted) over S under the projection $\pi' := \mathrm{pr}|_W$ where $\mathrm{pr} : \mathbb{P}(\xi(L_k)^*) \to S$ is the standard projection. By [Ha, Prop. II.7.10(b)] $\mathbb{P}(\xi(L_k)^*)$ is a quasi-projective variety, whence so is W.

The morphism $\varphi : V \setminus E \to W \setminus A \hookrightarrow \mathbb{P}^{h-1}$ is an embedding (indeed, so is the morphism given by the line bundle $L_k|_{V\setminus E} \simeq L_0|_{V\setminus E}$). Hence $W \setminus A$ is a smooth variety. To show that the variety W itself is smooth, we proceed locally using local trivializations of the vector bundle $\xi(L_k)$. Fix a point $s \in S$ together with an affine neighborhood S_0 of s in S, and an index $i_0 \in \{1, \dots, m\}$. Since by (i) the restriction map $H^0(V_s, (\mathcal{L}_{k-1})_s) \to H^0(E_s, (\mathcal{L}_{k-1})_s)$ is surjective, we can find $n+1$ sections $\xi_{s,0}, \dots, \xi_{s,n} \in H^0(V_s, (\mathcal{L}_{k-1})_s)$ which are linearly independent when restricted to sections of $L_{k-1}|_{C_{s,i_0}} \simeq \mathcal{O}_{\mathbf{P}^n}(1)$. Fix also a section η_0 of the line bundle $[E]$ over V that vanishes on E, another one $\eta_{s,1} \in H^0(V_s, (\mathcal{L}_k)_s)$ which does not vanish on C_{s,i_0}, and a basis $\sigma_{s,j}$ $(j = 1, \dots, h)$ of $H^0(V_s, (\mathcal{L}_k)_s)$. Shrinking the neighborhood S_0 (if necessary) we may assume that $\sigma_{s,j} = \sigma_j|_{V_s}$ where $\sigma_j \in H^0(V_{S_0}, \mathcal{L}_k)$ $(j = 1, \dots, h)$ (see Lemma 1.1(b)) and for any point $s' \in S_0$ the restrictions $\sigma_j|_{V_{s'}} \in H^0(V_{s'}, (\mathcal{L}_k)_{s'})$ $(j = 1, \dots, h)$ still form a basis. Decomposing the sections $\eta_{s,1},\ (\eta_0|_{V_s}) \cdot \xi_{s,l} \in H^0(V_s, (\mathcal{L}_k)_s)$ $(l = 0, \dots, n)$ by the basis $\sigma_{s,j}$ $(j = 1, \dots, h)$ we may extend them to sections, say, $\eta_1,\ \xi_0, \dots, \xi_n \in H^0(V_{S_0}, \mathcal{L}_k)$ decomposed by the system $\sigma_j \in H^0(V_{S_0}, \mathcal{L}_k)$ $(j = 1, \dots, h)$ with the same coefficients. Then the ratios

$$z_0' := \frac{\xi_0}{\eta_1}, \dots, z_n' := \frac{\xi_n}{\eta_1} \tag{8}$$

can be pushed down to regular functions (say) $z_0, \dots, z_n$ in a neighborhood of the point $c_{s,i_0} := \varphi(C_{s,i_0}) \in A$ in W which give a local coordinate system on the fiber W_s with center at the point c_{s,i_0} (cf. [Kod, Appendix], [GriHar, p. 477]). Clearly, they still give a local coordinate system on the fiber $W_{s'}$ around the point $c_{s',i_0} := \varphi(C_{s',i_0}) \in A$ close enough to c_{s,i_0}. Thus if $(x_1, \dots, x_r)$ (with $r := \dim_{\mathbb{C}} S$) is a local coordinate system at the point $s \in S$, then $(x_1, \dots, x_r,\ z_0, \dots, z_n)$ define a local coordinate system on W with center at the point c_{s,i_0}, and the projection π' in these local coordinates is given as

$$(x_1, \dots, x_r,\ z_0, \dots, z_n) \longmapsto (x_1, \dots, x_r)\,.$$

This proves (a).

(b) Let M be an algebraic vector bundle on an algebraic variety Z, and let L, L' be two transversal vector subbundles of M, so that $M/(L\cap L') \simeq L/(L\cap L') \oplus L'/(L\cap L')$. Then we have $M/L \simeq L'/(L\cap L')$. Letting $M := TV|_{E\cap E'}$, $L := TE|_{E\cap E'}$, and $L' = TE'|_{E\cap E'}$, and using the above observation we obtain an isomorphism of the normal bundles

$$N_{(E\cap E')/E'} \simeq N_{E/V}. \tag{9}$$

Since by our assumption, for every $s \in S$ and for every pair (i,j) such that $C_{s,i} \cap C'_{s,j} \neq \emptyset$ we have $(C_{s,i}, C_{s,i} \cap C'_{s,j}) \simeq (\mathbb{P}^n, \mathbb{P}^{n-1})$ and $N_{C_{s,i}/V_s} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$, it follows from (9) that $N_{(C_{s,i} \cap C'_{s,j})/E'_s} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. This allows us to apply the argument as in (a) [2] replacing the pair (V, E) by the pair $(E', E \cap E')$. Hence the restriction $\varphi|_{E'}$ is an S_0-contraction of the divisor $E \cap E'$ on E'. The image $\varphi(E')$ is a divisor in W proper over S_0 and having only smooth branches[3]. Since for every $s \in S$ the fiber $E'_s := E' \cap V_s$ is a smooth divisor in V_s and $\varphi_s = \varphi|_{V_s} : V_s \to W_s$ is the blowing up with the (finite) smooth center $A_s := A \cap W_s$, it follows that the intersection of local branches of the (reduced) divisor $\varphi(E'_s) \subset W_s$ at any point of A_s is transversal. Therefore, the branches of the divisor $\varphi(E') \subset W$ which contain the center A meet also transversally each other and every fiber W_s $(s \in S_0)$. Now the proof is completed. □

2. Combinatorial constructions

We use below the following

2.1. Terminology and notation. Let $\pi : V \to S$ be a family of quasi-projective varieties. *Shrinking the base* means passing to a new family $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ where U is a Zariski open subset of S; usually we keep the same notation before and after shrinking the base.

By a *smooth family of quasi-projective varieties* we mean a smooth surjective morphism $f : X \to S$ of smooth quasi-projective varieties; hereafter the base S is supposed to be irreducible. Note that any quasi-projective family with a smooth total space can be made smooth by shrinking the base.

We say that a family $\bar f : V \to S$ is a *relative completion* of $f : X \to S$ if $\bar f$ is a proper morphism, $X \subset V$ is a Zariski open dense subset and $f = \bar f|_X$. It is *of simple normal crossing* (or simply SNC) *type* if $D := V \setminus X$ is a simple normal crossing divisor on V. If the family $\bar f : V \to S$ is smooth and each fiber $V_s := \bar f^{-1}(s)$, $s \in S$, meets the divisor D transversally along an SNC-divisor $D_s := D \cap V_s \subset V_s$, then we say that (V, D) is a *relative SNC-completion* of X. Clearly, any smooth relative completion with an SNC-divisor D can be reduced to a relative SNC-completion by shrinking the base.

Let $f : X \to S$ be a smooth family with all fibers isomorphic to $\mathbb{C}^2$, and let $\bar f : V \to S$ be its relative SNC-completion. Then for every point $s \in S$ the 'boundary divisor' D_s is a *rational tree* (on the smooth rational projective surface V_s). The latter means that each irreducible component $C_{s,i}$ of D_s is a smooth rational curve, and the *weighted dual graph* (say) Γ_s of D_s is a tree (e. g., see [Zai, §2]).

Let $v \in \Gamma_s$ be an *at most linear (-1)-vertex* of Γ_s (that is, the valence of v is at most 2 and the weight of v is -1). The Castelnuovo contraction of the corresponding irreducible (-1)-component of D_s leads again to an SNC-completion (V'_s, D'_s) of $X_s \simeq \mathbb{C}^2$. The dual graph Γ'_s of D'_s is obtained from Γ_s by the *blowing down* v. The inverse operation on graphs is called a *blowing up*. This blowing up (blowing down) is called *inner* if v is a linear vertex of Γ_s and *outer* if v is terminal. The graph Γ_s is called *minimal* if no contraction is possible; in this case it is linear [Ram]. All minimal linear

[2] which are also valid for $n = 0$.

[3] In fact, for $n \geq 2$ the image itself is smooth since then (assuming that E' is smooth) for any $i \in \{1, \dots, m\}$, $C_{s,i} \cap C'_{s,j} \neq \emptyset$ for at most one value of j.

graphs corresponding to minimal SNC-completions of $\mathbb{C}^2$ are described in [Ram] and [Mor]; we call them the *Ramanujam-Morrow graphs.*

Assume that $\bar{f} : (V, D) \to S$ as above is a proper and smooth SNC-family, and fix a base point $s_0 \in S$. There exists a smooth horizontal connection on V which is tangent along the boundary SNC-divisor D (indeed, it can be patched from local smooth connections tangent along D using a smooth partition of unity on V). This provides us with a *geometric monodromy representation*

$$\mu : \pi_1(S,\, s_0) \to \mathrm{Diff}\,(V_{s_0},\, D_{s_0})\,.$$

We denote by the same letter μ the induced *combinatorial monodromy representation* $\pi_1(S,\, s_0) \to \mathrm{Aut}\,\Gamma_{s_0}$.

For a vertex $v \in \Gamma_{s_0}$, let $O(v)$ be its μ-orbit. Clearly, two vertices $v,\, v' \in \Gamma_{s_0}$ belong to the same orbit if and only if, for a certain irreducible component $E = E(v)$ of the boundary divisor $D \subset V$, the corresponding irreducible components $C(v)$ and $C(v')$ of the curve $D_{s_0} \subset V_{s_0}$ are contained in $E_{s_0} := E \cap V_{s_0}$ (note that this fact is stable under shrinking the base). The next important lemma easily follows from Theorem 1.3.

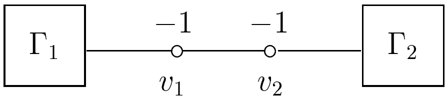
Lemma 2.1. *In the notation as above, let v be at most linear (-1)-vertex of the graph Γ_{s_0}. Assume that the orbit $O(v) \subset \Gamma_{s_0}$ of v does not contain a pair of neighbors in Γ_{s_0}. Then (possibly, after shrinking the base) there is a relative blowing down of the irreducible component $E(v)$ of the divisor D which gives again a relative SNC-completion of the family $f : X \to S$.*

2.2. **Equivariant contractions.** From now on we consider a smooth SNC-completion of $\mathbb{C}^2$ by an SNC-divisor (say) D_0 with a weighted dual graph Γ. Denote by $O(v)$ the orbit of a vertex v of Γ under the action on Γ of the full automorphism group $\mathrm{Aut}\,\Gamma$. If v is at most linear (-1)-vertex such that its orbit $O(v)$ does not contain a pair of neighbors in Γ, then all the vertices in $O(v)$ can be simultaneously contracted; we call this an *equivariant contraction* (or an *equivariant blowing down*). The main result of this subsection is the following proposition.

Proposition 2.2. *A graph Γ as above can be contracted to a Ramanujam-Morrow graph by means of equivariant contractions.*

The proof is done in Lemmas 2.3 and 2.4 below.

Lemma 2.3. *Let v_1 and v_2 be at most linear (-1)-vertices of Γ which are neighbors and belong to the same orbit (i.e., $v_2 \in O(v_1)$). Draw Γ as follows:*



Then the following statements hold.

(a) $\alpha(\Gamma_i) = \Gamma_j$ for any $\alpha \in \mathrm{Aut}\,\Gamma$ with $\alpha(v_i) = v_j,\;\; i, j \in \{1, 2\}$. Moreover, $O(v_1) = \{v_1, v_2\}$, and there is only one pair $\{v_1, v_2\}$ satisfying the assumptions of the lemma.

(b) If $w \in \Gamma_1$ is at most linear (-1)-vertex of Γ then the orbit $O(w)$ does not contain a pair of neighbors in Γ.

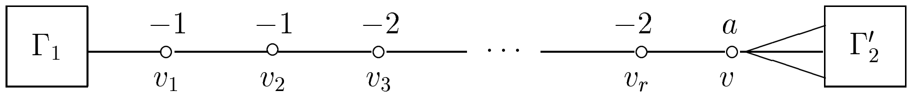
Proof. (a) Denote by $\mathrm{Br}_l(v_i)$ resp., $\mathrm{Br}_r(v_i)$ the left resp., the right-hand branch of Γ at v_i; thus $\mathrm{Br}_l(v_1) = \Gamma_1$ and $\mathrm{Br}_r(v_2) = \Gamma_2$. Since $\mathrm{card}\,\mathrm{Br}_l(v_2) = \mathrm{card}\,\mathrm{Br}_l(v_1) + 1$, for any $\alpha \in \mathrm{Aut}\,\Gamma$ with $\alpha(v_1) = v_2$ we have $\alpha(\mathrm{Br}_l(v_1)) = \mathrm{Br}_r(v_2)$, whence $\alpha(\Gamma_1) = \Gamma_2$, $\alpha(v_2) = v_1$ and $\alpha(\Gamma_2) = \Gamma_1$.

In particular, $\mathrm{card}\,\Gamma_1 = \mathrm{card}\,\Gamma_2$. Therefore, if $\beta \in \mathrm{Aut}\,\Gamma$ is such that $\beta(v_1) = v_1$ then $\beta(\Gamma_1) = \Gamma_1$, whence $\beta(v_2) = v_2$ and $\beta(\Gamma_2) = \Gamma_2$. This proves the first statement of (a).

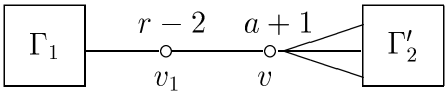
Let (v_1', v_2') be another pair of at most linear (-1)-neighbors of Γ which belong to the same orbit. As we have seen the both edges $[v_1, v_2]$ and $[v_1', v_2']$ of Γ divide Γ into two parts of equal cardinality, which is only possible if $[v_1, v_2] = [v_1', v_2']$. Thus the pair (v_1, v_2) is unique. It follows that $O(v_1) = \{v_1, v_2\}$ which proves (a). It also follows that $O(w) \cap O(v_1) = \emptyset$ which proves (b). ☐

Lemma 2.4. *Suppose that the graph Γ is not minimal. Then Γ has an at most linear (-1)-vertex w such that the orbit $O(w)$ does not contain a pair of neighbors.*

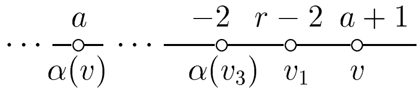
Proof. Let (v_1, v_2) be a pair of at most linear (-1)-neighbors of Γ which belong to the same orbit. Clearly, $\Gamma \neq \{v_1, v_2\}$, and so Γ contains a fragment

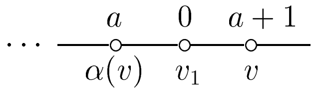


where either $a \neq -2$ or v is a branch vertex of Γ. Contracting the chain $(v_2, v_3, \ldots, v_r)$ we obtain the graph



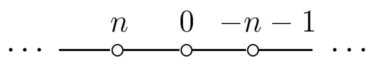
where the vertex v cannot be contracted. Assume that Γ does not contain at most linear (-1)-vertices other than v_1 and v_2. Then after this contraction the resulting graph is minimal (that is, a Ramanujam-Morrow graph). Show that this is impossible. Indeed, otherwise Γ would be a linear graph admitting an automorphism $\alpha \in \mathrm{Aut}\,\Gamma$ which interchanges v_1 and v_2 resp., Γ_1 and Γ_2. After the contraction as above, the image of Γ would contain one of the following fragments



$(r \geq 3)$,


$(r = 2)$.

But a Ramanujam-Morrow graph can have only one positively weighted vertex, and a neighbor of this vertex has zero weight. Hence $a = -1$ in the left-hand fragment above, which contradicts the minimality assumption. Furthermore, the only fragments of a Ramanujam-Morrow graph of length 3 including a zero vertex in the middle are of the form



where $n > 0$ [Ram, Mor, FleZai, §3.5]. Thus the both cases above are not possible. In virtue of Lemma 2.3 this concludes the proof. ☐

Proposition 2.2 and Lemma 2.1 yield

Corollary 2.5. *Let as above $f : X \to S$ be a smooth family of quasi-projective varieties with all fibers isomorphic to $\mathbb{C}^2$, and let $\bar{f} : V \to S$ be its relative SNC-completion. Then the boundary divisor $D = V \setminus X$ (possibly, after shrinking the base) can be contracted providing a new relative SNC-completion $\bar{f}^{\min} : V^{\min} \to S$, where for each $s \in S$ the dual graph $\Gamma_s^{\min}$ of the boundary divisor $D_s^{\min} := V_s^{\min} \setminus X_s$ is a Ramanujam-Morrow graph.*

We need below the following lemma from [FleZai].

Lemma 2.6. [FleZai, L. 3.7] *Let Γ be a Ramanujam-Morrow graph. Then Γ can be transformed, by a sequence of inner blowing ups and blowing downs, into one of the following graphs:*

$$\overset{1}{\circ} \qquad\qquad \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{n}{\circ}\ \ (n \neq -1) \qquad\qquad \overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{k-1}{\circ}\!\!-\!\!\!-\!\!\overset{-1}{\circ}\ \ (k \geq 1)\,. \qquad (*)$$

3. Proof of Theorem 0.1

The next proposition is the key point in the proof of Theorem 0.1.

Proposition 3.1. *Let the assumptions of Corollary 2.5 above be fulfilled. Then the family $f : X \to S$ (possibly, after shrinking the base) admits a relative SNC-completion $\tilde{f} : \tilde{V} \to S$ such that for every $s \in S$ the boundary divisor $\tilde{D}_s := \tilde{V}_s \setminus X_s$ is irreducible (and so isomorphic to $\mathbb{P}^1$).*

Proof. Let a relative SNC-completion $(V^{\min}, D^{\min})$ be as in Corollary 2.5. Note that for any Ramanujam-Morrow graph Γ except the following one:

$$\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}$$

we have $\operatorname{Aut}\Gamma = \{\mathrm{id}\}$. Let us deal with this exceptional case first. The edge of this graph (invariant under automorphisms) corresponds to a section (say) Σ of $D^{\min}$ over S (Σ is just the set of double points of the divisor $D^{\min}$). We can blow up $V^{\min}$ along Σ and then (possibly, after shrinking the base) blow down (according to Lemma 2.1) the proper transform(s) of (the irreducible components of) D to arrive at a new relative SNC-completion with only irreducible boundary divisors in fibers, as required.

In the other cases the absence of nontrivial automorphisms implies that

- each irreducible component E_i of the boundary divisor $D^{\min}$ meets every fiber $V_s^{\min}$ along an irreducible curve $C_{s,i}$, and
- the intersection of two such components $\Sigma_{ij} := E_i \cap E_j$ $(i \neq j)$ (if non-empty) is a (smooth) section.

These two properties are stable under blowing up with center at a section which is the intersection of two components of the boundary divisor, as well as under blowing down of a component of the boundary divisor which corresponds to an at most linear (-1)-vertex (it is defined correctly over S in virtue of Lemma 2.1).

The last observation and Lemma 2.6 imply that a relative SNC-completion $(V^{\min}, D^{\min})$ can be transformed (possibly, after shrinking the base) into another one with the dual graph Γ_{s_0} as in $(*)$. If we finally arrive at a relative SNC-completion with the dual graph $\Gamma = \Gamma_{s_0}$ as in the third case of $(*)$, then after blowing down the (-1)-curve in every fiber we obtain a relative SNC-completion with the dual graph as in the second case of $(*)$. In particular, we may assume that the singular locus Σ of the boundary divisor D is a section.

If $n = 0$ then we deal with the exceptional case which is already settled.

If $n > 0$ then we can proceed as before, performing first an inner relative blowing up with center at Σ and then an outer relative blowing down. After a sequence of n such 'elementary transformations' we get a relative SNC-completion with $n = 0$, and so we can finish the proof as above.

Finally, consider the case where $n \le -2$. In this case we have $\operatorname{Aut}\Gamma = \{\mathrm{id}\}$, and so the combinatorial monodromy of the family $\bar f : D \to S$ is trivial. Hence the divisor D consists of two smooth irreducible components, say, C_0 and C_1 with $C_{s,0}^2 = 0$ and $C_{s,1}^2 = n \le -2$ for every $s \in S$.

Suppose that there exists a section Σ' of $\bar f \,|\, C_0 : C_0 \to S$ disjoint with $\Sigma := C_0 \cap C_1$. The kind of elementary transformations appropriate in our case is blowing up with center at Σ' and then blowing down the proper transform of C_0 (by Theorem 1.3, this is possible after shrinking the base). Performing n such elementary transformations (this needs at each step the existence of a section as above), we arrive again at a relative SNC-completion of the second type with $n = 0$, and so we are done. Thus it remains to prove the following statement.

Claim. *After shrinking the base appropriately one can find a section Σ' of $\bar f \,|\, C_0 : C_0 \to S$ disjoint with $\Sigma := C_0 \cap C_1$.*

Proof of the claim. Letting in Lemma 1.1 $E = C_0$ and $L = [C_1]|_E$ (so that $L|_{E_s} = L|_{C_{s,0}} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ for every $s \in S$) and shrinking the base S to make it affine, by Corollary 1.2(b) we conclude that for every point $s \in S$ the restriction map

$$H^0(E, \mathcal{L}) \to H^0(E_s, \mathcal{L}_s) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

is surjective. Thus for any points $s_0 \in S$ and $z_0 \in C_{s_0,0} \setminus C_{s_0,1}$ there exists a section $\sigma \in H^0(E, \mathcal{L})$ with $\sigma^*(0) \cdot C_{s_0,0} = z_0$. The divisor $\Sigma' := \sigma^*(0)$ on $E = C_0$ (linearly equivalent to Σ) passes through the point z_0 and meets every fiber $E_s = C_{s,0}$ $(s \in S)$ transversally at one point. Clearly, $Z := \bar f(\Sigma \cap \Sigma') \not\ni s_0$ is a Zariski closed proper subset of the base S. The restrictions of the sections Σ and Σ' onto the Zariski open subset $S_0 := S \setminus Z$ of S are disjoint, as required. This proves the claim. ☐

Now the proof of Proposition 3.1 is completed. ☐

In virtue of Corollary 2.5 and Proposition 3.1 the proof of Theorem 0.1 is reduced to the following simple lemma. It is well known that any smooth family with fibers isomorphic to a projective space is locally trivial in the étale topology (and so it is a smooth Severi-Brauer variety) [Gro, Thm. I.8.2]. It is locally trivial in the Zariski topology if and only if this family (or, equivalently, its dual) admits local sections; Lemma 3.2 below provides a proof along the lines in [Gro, II, Sect. 0].

Lemma 3.2. *Let $\bar f : V \to S$ be a proper smooth family over a quasi-projective base with all fibers isomorphic to $\mathbb{P}^n$, and let D be an irreducible smooth divisor on V which meets every fiber V_s $(s \in S)$ transversally, with $D_s \simeq \mathbb{P}^{n-1}$ and $N_{D_s/V_s} \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. Then the family (V, D) is locally trivial in the Zariski topology.*

Proof. Fix an arbitrary point $s_0 \in S$. Shrinking the base S to an affine neighborhood of the point s_0, and letting in Lemma 1.1 $E = V$ and $L = [D]$, by Corollary 1.2 we will have that the restriction map

$$H^0(E, \mathcal{L}) \to H^0(E_{s_0}, \mathcal{L}_{s_0}) \simeq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$$

is surjective. Fix sections $\sigma_0, , \ldots, \sigma_n \in H^0(E, \mathcal{L})$ such that their restrictions to the fiber E_{s_0} are linearly independent and $\sigma_0^*(0) = D$. Shrinking the base further we may suppose that, for every fiber E_s ($s \in S$), the restrictions $\sigma_0|_{E_s}, \ldots, \sigma_n|_{E_s}$ are linearly independent as well. Then the morphism

$$\varphi : V \to S \times \mathbb{P}^n, \qquad z \longmapsto \big(\bar{f}(z),\, (\sigma_0(z) : \ldots : \sigma_n(z))\big)$$

yields a desired trivialization over S. $\square$

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