Max-Planck-Institut für Mathematik
Bonn

Membranes and higher groupoids

by

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Max-Planck-Institut für Mathematik
Preprint Series 2015 (14)

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March 8, 2015

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0 Introduction.

The goal of this paper is threefold.

(0.1) First, we generalize the classical result of C. Reutenauer ([41], §8.6.12) on the abelianization of the commutant of the free Lie algebra. Denote by $\mathrm{FL}(V)$ the free Lie algebra on a finite-dimensional vector space V over a field k of characteristic 0. Reutenauer's theorem (in its dual formulation) says that $H^1([\mathrm{FL}(V), \mathrm{FL}(V)], k)$ is identified with the space of formal germs of closed 2-forms on V, see formula (1.5.7) below. Our generalization (Theorems 1.5.2 and 1.5.10), uses a certain free dg-Lie algebra $\mathfrak{f}^\bullet(V)$, containing $\mathrm{FL}(V)$ as the degree 0 part, and (if formulated in the dual version similar to the above) produces the full de Rham complex of formal germs of forms, truncated in degrees ≥ 2. Instead of abelanization of the commutant, we use the so-called semiabelianization of $\mathfrak{f}^\bullet(V)$. We call a dg-Lie algebra $\mathfrak{g}^\bullet$, situated in degrees ≤ 0, *semiabelian*, if the commutators $\mathfrak{g}^i \otimes \mathfrak{g}^j \to \mathfrak{g}^{i+j}$ vanish for $i, j \leq -1$.

By Quillen's theory, dg-Lie $\mathbb{Q}$-algebras situated in degrees ≤ 0, correspond to all rational homotopy types. The class of semiabelian dg-Lie algebras corresponds to those homotopy types that are represented by strict ∞-groupoids of Brown-Higgins [7]. Indeed, for such homotopy types the Whitehead products $\pi_{i+1} \otimes \pi_{j+1} \to \pi_{i+j+1}$, vanish for $i, j \geq 1$. On the other hand, by Grothendieck's philosophy, all homotopy types should correspond to appropriately defined weak ∞-groupoids. The functor of taking the maximal semiabelian quotient of a dg-Lie algebra $\mathfrak{g}^\bullet \mapsto \mathfrak{g}^\bullet_{\mathrm{sab}}$ is therefore the analog of that of "strictification", from the category of "true" ∞-groupoids to that of Brown-Higgins ones. Our result can be thus seen as calculating the (Lie algebra version of the) nonabelian derived functor of this strictification procedure.

(0.2) Our second goal is to use the theory of 2-dimensional holonomy of connections with values in crossed modules of Lie groups and algebras, as developed by Baez-Schreiber [2], in order to construct a representation of

2-dimensional membranes in $\mathbb{R}^n$ by certain formal series-type data. This representation extends the classical construction of K.-T. Chen which associates to a path γ in $\mathbb{R}^n$ the noncommutative formal power series

$$E_\gamma(Z_1, ..., Z_n) \;=\; \sum_{d=0}^{\infty} \sum_{i_1,...,i_d=0}^{n} Z_{i_1}...Z_{i_d} \int_\gamma (dx_{i_1}, ..., dx_{i_d}),$$

the generating function for all the iterated integrals of the coordinate 1-forms dx_i. This series, being group-like, lies in $\widehat{G}_n^0$, the free prounipotent proalgebraic group on n generators. We extend $\widehat{G}_n^0$ to a crossed module of prounipotent groups $G_n^{\geq -1} = \{G_n^{-1} \xrightarrow{\partial} G_n^0\}$, which carries a natural connection over $\mathbb{R}^n$ with vanishing fake curvature in the sense of Breen-Messing [6]. The Lie algebra crossed module of $G_n^{\geq -1}$ is a natural completion of the truncation of $\mathfrak{f}^\bullet(V)$ in degrees ≥ -1. To each 2-brane σ we then associate (2.5.5) an element $\widehat{M}(\sigma) \in G_n^{-1}$, with $\partial\widehat{M}(\sigma) \in G_n^0$ being the Chen series corresponding to the boundary path of σ.

(0.3) Third, we generalize the construction of Baez-Schreiber [2] to that of p-dimensional holonomy for connections with values in *crossed complexes* of Lie groups. Crossed complexes correspond to Brown-Higgins ∞-groupoids with one object and consist of a non-abelian "head" in degrees 0 and -1 attached to an abelian "tail" in lower degrees. In this paper we consider only what should be understood as connections in trivial n-gerbes with a given structure n-groupoid. Our construction is based on the same techniques as in [2], namely a covariant generalization of Chen's iterated integrals in the presence of a background connection, see [26]. It gives a strict functor from an appropriately defined n-groupoid of unparametrized membranes in a manifold X to the n-groupoid corresponding to the crossed complex, see Theorem 3.3.6 .

Although Brown-Higgins n-groupoids are rather restrictive from the general homotopy-theoretic point of view, this seems to be the maximal generality in which we have a holonomy corresponding to a p-brane as a *geometric object*, without any extra data such as a choice of a parametrization or of a slicing into lower-dimensional membranes. A more general construction given by E. Getzler [21], depends on such choices but comes with a system of higher homotopies accounting for making a different choice. The usual homotopy-theoretic source of Brown-Higgins n-groupoids is provided not by single spaces but by filtered spaces [9]. Considering thin homotopies (reparametrizations, cancellations etc.) to pass from parametrized to unparametrized ("geometric") membranes can be seen as introducing a differential-geometric analog of the skeletal filtration.

The place of Brown-Higgins n-groupoids among more general homotopy types (strict 2-dimensional associativity, more solid geometric nature of higher holonomy with values in them) is somewhat similar to the place of commutative rings among all associative rings. It may be therefore interesting to study infinitesimal non-Brown-Higgins deformations of Brown-Higgins n-groupoids in geometric terms, similarly to noncommutative deformations of commutative algebras.

We then apply the theory of p-dimensional holonomy to extend the representation of 2-branes above to the case of p-branes in $\mathbb{R}^n$ with arbitrary p. For p-branes Σ whose boundary consists of a point, this representation associates to Σ the corresponding de Rham current (integration functional on polynomial p-forms). In general, this is a certain nonabelian twist of such de Rham representation. This construction can be seen as a nonlinear and nonabelian analog of Grassmann's geometric calculus: we represent p-branes in $\mathbb{R}^n$ by data constructed out of exterior powers $\Lambda^i(\mathbb{R}^n)$, $i \leq p$.

(0.4) A different generalization of Reutenauer's theorem was found by B. Feigin and B. Schoikhet [19]. Their result provides a relation of the free associative, not Lie, algebra with $\Omega^{2\bullet,\mathrm{cl}}$, the space of closed differential forms of arbitrary even degrees. Our Theorem 1.5.4 provides a generalization in a different direction, involving forms of all, not just even, degrees.

It seems that the dg-Lie algebra $\mathfrak{f}^\bullet(V)$, associated to a vector space V, admits a "curvilinear" analog which is a natural dg-Lie agebroid $\mathcal{P}_X^\bullet$ associated to any smooth (or complex or algebraic) manifold X and containing the free Lie algebroid $\mathcal{P}_X$ from [30] as the degree 0 part. We discuss this possibility in more detail in Remark 1.2.4. Note that nonlinear analogs of the Feigin-Shoikhet result have been subject of several recent works [17, 18, 28].

An approach to 2-dimensional holonomy somewhat different from that of J. Baez and U. Schreiber (and thus from the approach of the present paper) was developed by A. Yekutieli [48]. That approach is based on direct approximation by "Riemann products", not on iterated integrals. It can probably be applied to the p-dimensional holonomy (situation of Chapter 3) as well. An approach involving the holonomy of connections on the space of paths (and thus using iterated integrals, albeit implicitly) was developed by J. F. Martins and R. Picken [38].

(0.5) The three chapters of the paper correspond to the three main subjects outlined above. In the Appendix we provide a self-contained account of the main points of the theory of covariant iterated integrals.

I would like to thank P. Bressler, R. Mikhailov, B. Shoikhet and A. Yekutieli for useful discussions and/or correspondence. This work was supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan and by the Max-Planck-Institut für Mathematik, Bonn, Germany.

1 A resolution of an abelian Lie algebra.

1.1 The resolution.

Let k be a field of characteristic 0. We denote by $\mathrm{Vect}_k^{\mathbb{Z}}$ the symmetric monoidal category of $\mathbb{Z}$-graded k-vector spaces $V^\bullet = \bigoplus_{i\in\mathbb{Z}} V^i$, with the usual graded tensor product and the symmetry $V^\bullet \otimes W^\bullet \to W^\bullet \otimes V^\bullet$ given by the Koszul sign rule:

$$v \otimes w \longmapsto (-1)^{\deg(v)\deg(w)} w \otimes v.$$

By a *graded Lie k-algebra* we mean a Lie algebra object $\mathfrak{g}^\bullet$ in $\mathrm{Vect}_k^{\mathbb{Z}}$, so the bracket on $\mathfrak{g}^\bullet$ satisfies the antisymmetry and the Jacobi identity twisted, in the usual way, by the Koszul sign factors. In particular, for $V^\bullet \in \mathrm{Vect}_k^{\mathbb{Z}}$ we have the *free graded Lie algebra* $\mathrm{FL}(V^\bullet)$. By a *dg-Lie k-algebra* we mean a graded Lie k-algebra equipped with a differential d of degree $+1$, satisfying the graded Leibniz rule and such that $d^2 = 0$.

Let V be a finite-dimensional k-vector space (concentrated in degree 0). Let $Z_1, ..., Z_n$ be a basis of V. We will write $\mathrm{FL}(Z_1, ..., Z_n)$ for the free Lie k-algebra $\mathrm{FL}(V)$. For each nonempty subset $I = \{i_1 < ... < i_p\}$ of $\{1, ..., n\}$ we introduce a generator $Z_I = Z_{i_1,...,i_p}$ of degree $(-p+1)$. In the case $p = 1$ these generators will be identified with the Z_i above. Let $\mathfrak{f}^\bullet = \mathfrak{f}^\bullet(Z_1, ..., Z_n)$ be the free graded Lie k-algebra on all the generators Z_I. The degree 0 part of this algebra is $\mathrm{FL}(Z_1, ..., Z_n)$. One can express this more invariantly by saying that $\mathfrak{f}^\bullet = \mathfrak{f}^\bullet(V)$ is constructed from V as follows:

$$\mathfrak{f}^\bullet(V) = \mathrm{FL}\big(\Lambda^{\geq 1}(V)\big), \quad \deg(\Lambda^p(V)) = -p+1. \tag{1.1.1}$$

Indeed, the Z_I for $|I| = p$, form a basis in $\Lambda^p(V)$.

We introduce a derivation d of $\mathfrak{f}^\bullet$ of degree 1 by defining it on the generators (and then extending uniquely using the Leibniz rule) as follows:

$$dZ_I = \frac{1}{2} \sum_{I = J \sqcup K} \sigma(J, K)[X_J, X_K]. \tag{1.1.2}$$

Here the summation is over all partitions of I into the union of two disjoint subsets J, K. The number $\sigma(J, K)$ is the sign of the shuffle permutation induced by this partition. For example,

$$dZ_{ij} = [Z_i, Z_j], \quad dZ_{ijp} = [Z_i, Z_{jp}] - [Z_j, Z_{ip}] + [Z_p, Z_{ij}]. \tag{1.1.3}$$

Proposition 1.1.4. *(a) The differential d satisfies $d^2 = 0$ and thus makes $\mathfrak{f}^\bullet$ into a dg-Lie algebra.*
(b) The dg-Lie algebra $\mathfrak{f}^\bullet(V)$ is a resolution of the abelian Lie algebra V, i.e.,

$$H^0(\mathfrak{f}^\bullet(V)) = V, \quad H^i(\mathfrak{f}^\bullet(V)) = 0,\ i \neq 0.$$

Proof: Part (a) can be verified directly on the generators. An alternative way to see the validity of (a) and also to prove (b) is to consider the graded commutative algebra Λ without unit defined as $\Lambda = \Lambda^{\geq 1}(V^*)$, with $\deg(\Lambda^p(V^*)) = p$. Then

$$\mathfrak{f}^\bullet(V) = \mathrm{FL}(\Lambda^*[-1]) = \mathrm{Harr}^\bullet(\Lambda)$$

is nothing but the Harrison cochain complex of Λ. See [3] for background on Harrison (co)homology. To be precise, the standard definition of the Harrison chain complex of a commutative algebra A, is as the space of indecomposable elements of the Hochschild complex $\bigoplus_r A^{\otimes r[r]}$ with respect to the shuffle multiplication [3]. This space is the same as $\mathrm{FL}(A[1])$. The Harrison cochain complex is dual: it is $\mathrm{FL}(A^*[-1])$. The differential (1.1.2) is just the map dual to the multiplication in Λ, so it is indeed the Harrison differential, and the fact that $d^2 = 0$ follows on general grounds. Further, Λ is free as a graded commutative algebra without unit, and V is its space of generators. So part (b) follows from vanishing of the higher Harrison cohomology of a free (graded) commutative algebra. $\square$

Taking the universal enveloping algebra of $\mathfrak{f}^\bullet(V)$, i.e., the free associative dg-algebra on the Z_I, we get a free associative dg-resolution of $S^\bullet(V) = k[Z_1, ..., Z_n]$. Compare with [19].

1.2 The connection.

Let X be a C^∞-manifold and $\mathfrak{g}^\bullet$ a dg-Lie $\mathbb{R}$-algebra. By a *graded connection* on X with values in $\mathfrak{g}^\bullet$ we will simply mean a differential form $A \in (\Omega^\bullet_X \otimes \mathfrak{g}^\bullet)^1$ of total degree 1. We associate to A the operator $\nabla_A = d - A$ in the de Rham complex $\Omega^\bullet_X \otimes \mathfrak{g}^\bullet$, where $d = d_{\mathrm{DR}} + d_{\mathfrak{g}}$ is the sum of the de Rham differential and the differential induced by the one in $\mathfrak{g}^\bullet$. In particular, ∇^2_A is given by multiplication with the *curvature* of A which is the form

$$F_A \ = \ dA - \frac{1}{2}[A, A] \ \in \ (\Omega^\bullet_X \otimes \mathfrak{g}^\bullet)^2. \tag{1.2.1}$$

We say that A is *formally flat*, if $F_A = 0$.

We apply this to the case when $X = V = \mathbb{R}^n$ with basis $Z_1, ..., Z_n$ and the corresponding coordinate system $t_1, ..., t_n$, and $\mathfrak{g}^\bullet = \mathfrak{f}^\bullet = \mathfrak{f}^\bullet(V)$. Introduce a differential form A on V with values in $\mathfrak{f}^\bullet$ as follows:

$$A = \sum_{p=1}^{n} \sum_{1 \leq i_1 < ... < i_p \leq n} Z_{i_1,...,i_p} dt_{i_1} ... dt_{i_p}. \tag{1.2.2}$$

Note that A has total degree 1, so it is a graded connection with values in $\mathfrak{f}^\bullet$.

Proposition 1.2.3. *The graded connection A is formally flat.*

Proof: Let us write $dt_I = dt_{i_1}...dt_{i_p}$. Our form $A = \sum Z_I dt_I$ has constant coefficients. So $d_{\mathrm{DR}}(A) = 0$. On the other hand,

$$\begin{aligned} d_{\mathfrak{f}}(A) \;&=\; \sum_I d(Z_I) dt_I = \frac{1}{2} \sum_{J,K:\; J\cap K = \emptyset} \sigma(J,K)[Z_J, Z_K] dt_{J \sqcup K} \;= \\ &=\; \frac{1}{2} \sum [Z_J, Z_K] dZ_J dZ_K \;=\; \frac{1}{2}[A, A]. \quad \square \end{aligned}$$

It is clear moreover, that requiring A to be formally flat, is equivalent to imposing the differential (1.1.2). Therefore A is the *universal translation equivariant flat graded connection* on V, and so $\mathfrak{f}^\bullet$ has the intepretation as the structure dg-Lie algebra of this universal graded connection.

Remark 1.2.4. It would be very interesting to construct a "curvilinear" version of the dg-Lie algebra $\mathfrak{f}^\bullet(V)$. Recall that in [30], to each smooth manifold X (in C^∞, analytic or algebraic category) we associated a Lie algebroid $\mathcal{P}_X \xrightarrow{\alpha} T_X$ (in the corresponding category) which locally "looks like" $\mathrm{FL}(T_X)$ but with the Lie algebra structure mixing the Lie bracket of vector fields and the formal bracket in the free Lie algebra. The free Lie algebra $\mathrm{FL}(\mathbb{R}^n)$ can be recovered as the algebra of global translation invariant sections of $\mathcal{P}_{\mathbb{R}^n}$. One can view $\mathcal{P}_X$ as the Lie algebroid corresponding to the (infinite-dimensional Lie) groupoid of formal unparametrized paths. In particular, bundles with (not necessarily flat) connections can be seen as modules over $\mathcal{P}_X$.

In this direction, it would be natural to extend the construction of $\mathcal{P}_X$ by associating with each manifold X as above, a dg-Lie algebroid $\mathcal{P}_X^\bullet$ situated in degrees ≤ 0 with $\mathcal{P}_X^0 = \mathcal{P}_X$, so that $\mathfrak{f}^\bullet(\mathbb{R}^n)$ is recovered as the dg-Lie algebra of global translation invariant sections of $\mathcal{P}_{\mathbb{R}^n}^\bullet$. One can possibly define $\mathcal{P}_{\mathbb{R}^n}^\bullet$

by a version of the Tannakian formalism as in [30], i.e., by making it to classify (flat) graded connections. In this way the role of $\mathfrak{f}^\bullet(\mathbb{R}^n)$ in describing translation invariant graded connections would be extended to the curvilinear case. We leave this study for future work.

1.3 Semiabelianization.

Let k be any field of characteristic 0. A dg-Lie k-algebra $\mathfrak{g}^\bullet$ is called abelian, if its bracket vanishes. Thus an abelian dg-Lie algebra is the same as a cochain complex. For any dg-Lie k-algebra $\mathfrak{g}^\bullet$ we denote by $\mathfrak{g}^\bullet_{\mathrm{ab}} = \mathfrak{g}^\bullet/[\mathfrak{g}^\bullet, \mathfrak{g}^\bullet]$ its maximal abelian quotient.

Further, let $\mathfrak{g}^\bullet$ be a dg-Lie k-algebra situated in degrees ≤ 0. We will say that $\mathfrak{g}$ is *semiabelian*, if $[\mathfrak{g}^{\leq -1}, \mathfrak{g}^{\leq -1}] = 0$. Thus the data needed to define a semiabelian dg-Lie algebra are:

(SA1) A Lie algebra structure on $\mathfrak{g}^0$.

(SA2) A $\mathfrak{g}^0$-module structure on each $\mathfrak{g}^i$, taken to be the adjoint action for $i = 0$ and denoted by $[z, x]$, $z \in \mathfrak{g}^0$, $x \in \mathfrak{g}^i$.

(SA3) A differential d of degree $+1$ such that $d^2 = 0$.

Proposition 1.3.1. *In order that the above data define a semiabelian dg-Lie algebra, it is necessary and sufficient that the following hold:*
(a) d is a morphism of $\mathfrak{g}^0$-modules.
(b) For $x, y \in \mathfrak{g}^{-1}$ we have $[dx, y] = [x, dy]$.

Proof: Indeed, the data (SA1) and (SA2) completely define the structure of a graded Lie algebra such that $[\mathfrak{g}^{\leq -1}, \mathfrak{g}^{\leq -1}] = 0$, and all we need is to account for the Leibniz rule for $d[x, y]$. The condition (a) corresponds to the case $x \in \mathfrak{g}^0$, $y \in \mathfrak{g}^i$, $i < 0$, while (b) corresponds to the case $x, y \in \mathfrak{g}^{-1}$, as $d[x, y]$ in this case must be equal to 0. Other cases do not present any restrictions. $\square$

Example 1.3.2 Crossed modules. Suppose that $\mathfrak{g}^\bullet$ is situated in degrees $0, -1$ only:

$$\mathfrak{g}^\bullet \quad = \quad \{\mathfrak{g}^{-1} \xrightarrow{d} \mathfrak{g}^0\}. \tag{1.3.3}$$

Then $\mathfrak{g}^\bullet$ is semiabelian, as for any $x, y \in \mathfrak{g}^{-1}$ we have $[x, y] \in \mathfrak{g}^{-2} = 0$. Recall the concept of a *crossed module of Lie algebras* [32]. By definition, such a crossed module is a diagram as in (1.3.3), except that both $\mathfrak{g}^0$ and $\mathfrak{g}^{-1}$ are assumed to be Lie algebras, d is a homomorphism of Lie algebras and, in addition, $\mathfrak{g}^0$ is acting on $\mathfrak{g}^{-1}$ by derivations, via a homomorphism $\alpha : \mathfrak{g}^0 \to \mathrm{Der}(\mathfrak{g}^{-1})$ which is required to satisfy

$$(1.3.4) \qquad \begin{aligned} [x, y] = \alpha(d(x))(y), \quad [z, d(x)] = d(\alpha(z)(x)), \\ z \in \mathfrak{g}^0, x, y \in \mathfrak{g}^{-1}. \end{aligned}$$

Given a dg-Lie algebra $\mathfrak{g}^\bullet$ situated in degrees $0, -1$, and any two elements $x, y \in \mathfrak{g}^{-1}$, we define their *derived bracket* by

$$(1.3.5) \qquad [x, y]_{-1} := [dx, y] = [x, dy] \ \in \ \mathfrak{g}^{-1},$$

where the second equality follows from $[x, y] = 0$. We leave to the reader the verification of the following fact.

Proposition 1.3.6. *The derived bracket $[-,-]_{-1}$ makes $\mathfrak{g}^{-1}$ into a Lie algebra (in the usual, ungraded sense), and d becomes a homomorphism of Lie algebras. Further, the rule*

$$\alpha(z)(x) = [z, x], \quad z \in \mathfrak{g}^0, \, x \in \mathfrak{g}^{-1},$$

defines an action of $\mathfrak{g}^0$ on $(\mathfrak{g}^{-1}, [-,-]_{-1})$ by derivations, and this action satisfies the axioms of a crossed module of Lie algebras. This correspondence establishes an equivalence of categories between crossed modules of Lie algebras, on one hand, and dg-Lie algebras situated in degrees $0, -1$, on the other hand. □

Similarly, semiabelian dg-Lie algebras of any length can be seen as Lie algebraic analogs of crossed complexes of Brown and Higgins [7], see §3.1 below.

Example 1.3.7. Let $\mathfrak{g}^\bullet$ be a dg-Lie algebra equipped with an increasing filtration

$$\mathfrak{g}_0^\bullet \subset \mathfrak{g}_1^\bullet \subset \cdots$$

by dg-Lie subalgebras. Set $\mathfrak{h}^{-i} = H^{-i}(\mathfrak{g}_i^\bullet / \mathfrak{g}_{i-1}^\bullet)$. These spaces are parts of the term E_1 of the spectral sequence of the filtered complex $\mathfrak{g}^\bullet$. Let us define $d : \mathfrak{h}^{-i} \to \mathfrak{h}^{-i+1}$ to be the given by the differential d_1 in this spectral sequence. Then $(\mathfrak{h}^\bullet, d)$ is naturally a semiabelian dg-Lie algebra, with respect to the induced Lie algebra structure on $\mathfrak{h}^0 = H^0(\mathfrak{g}_0^\bullet)$ and its natural action on each

$\mathfrak{h}^i$. Indeed, both conditions of Proposition 1.3.1 follow easily from the Leibniz rule. For instance, let us show (b).

Let $x, y \in \mathfrak{h}^{-1} = H^{-1}(\mathfrak{g}_1^\bullet/\mathfrak{g}_0^\bullet)$, and choose their representatives $\widetilde{x}, \widetilde{y} \in \mathfrak{g}_1^{-1}$. Then $d\widetilde{x}, d\widetilde{y} \in \mathfrak{g}_0^0$, and their classes in $H^0(\mathfrak{g}_0^\bullet) = \mathfrak{h}^0$ are dx and dy. Now, the element $[dx, y] - [x, dy] \in \mathfrak{h}^{-1}$ is represented by the coboundary $[d\widetilde{x}, \widetilde{y}] - [\widetilde{x}, d\widetilde{y}] = d[\widetilde{x}, \widetilde{y}] \in \mathfrak{g}_1^{-1}$ and therefore is equal to 0.

Semiabelian dg-Lie algebras share some of the properties of abelian dg-Lie algebras, i.e., cochain complexes. In particular, if $\mathfrak{g}^\bullet$ is semiabelian, then the naive truncation

$$\mathfrak{g}^{\geq -m} \quad = \quad \left\{ \mathfrak{g}^{-m} \to \mathfrak{g}^{-m+1} \to \cdots \to \mathfrak{g}^0 \right\}, \tag{1.3.8}$$

is again a (semiabelian) dg-Lie algebra.

Let now $\mathfrak{g}^\bullet$ be any dg-Lie algebra situated in degrees ≤ 0. We call the *semiabelianization* of $\mathfrak{g}^\bullet$ and denote by $\mathfrak{g}^\bullet_{\mathrm{sab}}$ the maximal semiabelian quotient of $\mathfrak{g}^\bullet$:

$$\mathfrak{g}^\bullet_{\mathrm{sab}} = \frac{\mathfrak{g}^\bullet}{[\mathfrak{g}^{\leq -1}, \mathfrak{g}^{\leq -1}] + d([\mathfrak{g}^{\leq -1}, \mathfrak{g}^{\leq -1}])}. \tag{1.3.9}$$

This construction can be reduced to that of Example 1.3.7. Indeed, let us define the *i-skeleton* $\mathfrak{g}_i^\bullet \subset \mathfrak{g}^\bullet$ to be the graded Lie subalgebra generated by the subspace $\mathfrak{g}^{\geq -i}$. This subalgebra is closed under d, so we get a filtration of $\mathfrak{g}^\bullet$ by dg-Lie subalgebras, which we call the *skeleton filtration.*

Proposition 1.3.10. *The dg-Lie algebra $\mathfrak{g}^\bullet_{\mathrm{sab}}$ is isomorphic to the semiabelian Lie dg-algebra $\mathfrak{h}^\bullet = (H^{-i}(\mathfrak{g}_i^\bullet/\mathfrak{g}_{i-1}^\bullet))$ associated as in Example 1.3.7 to the skeleton filtration.*

Proof: By definition, $\mathfrak{g}^{-i}_{\mathrm{sab}}$ is the quotient of $\mathfrak{g}^{-i}$ by the subspace

$$\sum_{p=1}^{i-1} [\mathfrak{g}^{-p}, \mathfrak{g}^{-i+p}] \;+\; \sum_{q=1}^{i} d\,[\mathfrak{g}^{-q}, \mathfrak{g}^{-i-1-q}].$$

On the other hand, $(\mathfrak{g}_i^\bullet/\mathfrak{g}_{i-1}^\bullet)^{-i+1} = 0$, while

$$(\mathfrak{g}_i^\bullet/\mathfrak{g}_{i-1}^\bullet)^{-i} \;=\; \mathfrak{g}^{-i} \Big/ \sum_{p=1}^{i-1} [\mathfrak{g}^{-p}, \mathfrak{g}^{-i+p}],$$

$$(\mathfrak{g}_i^\bullet/\mathfrak{g}_{i-1}^\bullet)^{-i-1} \;=\; \sum_{q=1}^{i} [\mathfrak{g}^{-q}, \mathfrak{g}^{-i-1-q}],$$

so taking the cohomology in degree $(-i)$ gives the same answer. We leave the remaining details to the reader. $\square$

We also denote by $\mathfrak{g}^\bullet_{\mathrm{CM}}$ the maximal crossed module quotient of $\mathfrak{g}^\bullet$, i.e.,

$$(1.3.11) \qquad \mathfrak{g}^\bullet_{\mathrm{CM}} \quad = \quad \left\{ \frac{\mathfrak{g}^{-1}}{d[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]} \longrightarrow \mathfrak{g}^0 \right\} \quad = \quad (\mathfrak{g}^\bullet_{\mathrm{sab}})^{\geq -1}.$$

1.4 Reminder on differential forms and Schur functors.

Let V be a finite-dimensional k-vector space. Consider V as an affine algebraic variety over k. The space $\Omega^p(V)$ of regular p-forms on V has the decomposition

$$(1.4.1) \qquad \Omega^p(V) = \bigoplus_{d=0}^{\infty} \Lambda^p(V^*) \otimes S^d(V^*).$$

The de Rham differential $d : \Omega^p(V) \to \Omega^{p+1}(V)$ makes $\Omega^\bullet(V)$ into a complex exact everywhere except the 0th term. Let us denote by

$$(1.4.2) \qquad \Gamma_p(V) = \bigoplus_{d=0}^{\infty} \Lambda^p(V) \otimes S^d(V)$$

the restricted dual of $\Omega^p(V)$. Geometrically, $\Gamma_p(V)$ can be seen as the space of de Rham currents on V supported at 0. The de Rham differential d on $\Omega^\bullet(V)$ induces differentials $\partial : \Gamma_p(V) \to \Gamma_{p-1}(V)$ by dualization. The dual of

$$\Omega^{p,\mathrm{cl}}(V) = \mathrm{Ker}\big\{ d : \Omega^p(V) \to \Omega^{p+1}(V) \big\}$$

is then

$$(1.4.3) \quad \mathrm{Coker}\big\{ \partial : \Gamma_{p+1}(V) \to \Gamma_p(V) \big\} \; = \; \mathrm{Ker}\big\{ \partial : \Gamma_{p-1}(V) \to \Gamma_{p-2}(V) \big\},$$

which we denote by $\Gamma^{\mathrm{cl}}_{p-1}(V)$.

Let $\mathrm{Vect}^{\mathrm{fd}}_k$ be the category of finite-dimensional k-vector spaces. For any sequence of integers $\alpha = (\alpha_1 \geq ... \geq \alpha_n \geq 0)$ (with arbitrary n) we have the Schur functor $\Sigma^\alpha : \mathrm{Vect}^{\mathrm{fd}}_k \to \mathrm{Vect}^{\mathrm{fd}}_k$, see [36]. For $V = k^n$ the space $\Sigma^\alpha(V)$ is the space of irreducible representation of the algebraic group GL_n (over k) with highest weight α. For example,

$$S^d(V) = \Sigma^{d,0,\dots,0}(V), \quad \Lambda^d(V) = \Sigma^{1,\dots,1,0,\dots,0}(V),$$

where in the last equality there are d occurrences of 1.

The space of closed p-forms has the following well known decomposition as a $GL(V)$-module:

$$\Omega^{p,\mathrm{cl}}(V) \;=\; d(\Omega^{p-1}(V)) \;=\; \bigoplus_{d=1}^{\infty} \Sigma^{d,1,\ldots,1}(V^*) \tag{1.4.4}$$

(with $(p-1)$ occurrences of 1 in the RHS). See, e.g., [20], Proposition 14.2.2.

1.5 The semiabelianization of $\mathfrak{f}^\bullet(V)$.

As before, let $V = k^n$ with basis $Z_1, ..., Z_n$. We denote by $\mathfrak{f}^\bullet_{\mathrm{sab}}(V)$ the semiabelianization of $\mathfrak{f}^\bullet(V)$. Explicitly, it is generated by the elements Z_I as in §1.1 with differential given by (1.1.2), but the Z_I are subject to the relations expressing (1.3.9). The following gives a first sketch of the structure of $\mathfrak{f}^\bullet_{\mathrm{sab}}(V)$.

Proposition 1.5.1. *In $\mathfrak{f}^\bullet_{\mathrm{sab}}(V)$ we have:*
(a) Any Lie monomial containing at least two of the Z_I, $|I| \geq 2$, vanishes.
(b) The Lie monomials $[Z_{i_1}, [Z_{i_2}, ..., [Z_{i_p}, Z_J]...]$, $p \geq 0$, $|J| = m+1$, span $\mathfrak{f}^{-m}_{\mathrm{sab}}(V)$.
(c) If $m \geq 2$, then the Lie monomials in (b) are symmetric in $i_1, ..., i_p$.

Proof: Part (a) is obvious, while (b) follows from (a) and the Jacobi identity. Finally, to see (c), let $M = [Z_{i_{a+1}}, [Z_{i_{a+2}}, ..., Z_{i_p}, Z_J]...]$ for some a. Then the relations (1.3.9) give

$$0 = d([Z_{i_{a-1}, i_a}, M]) = [[Z_{i_{a-1}}, Z_{i_a}], M].$$

Indeed, the term $[Z_{i_{a-1}, i_a}, dM]$ vanishes since both arguments of the bracket lie in $\mathfrak{f}^{\leq -1}_{\mathrm{sab}}(V)$. This equality implies

$$[Z_{i_{a-1}}, [Z_{i_a}, M]] = [Z_{i_a}, [Z_{i_{a-1}}, M]],$$

whence the symmetry. □

Our main result about the structure of $\mathfrak{f}^\bullet_{\mathrm{sab}}$ is as follows.

Theorem 1.5.2. *(a) The dg-Lie algebra $\mathfrak{f}^\bullet_{\mathrm{sab}}(V)$ is a resolution of V, i.e., the projection $\mathfrak{f}^\bullet(V) \to \mathfrak{f}^\bullet_{\mathrm{sab}}(V)$ is a quasiisomorphism.*
(b) For any $m \geq 1$ the space

$$H^{-m}(\mathfrak{f}^{\geq -m}_{\mathrm{sab}}(V)) \quad = \quad \mathrm{Ker}\,\left\{d : \mathfrak{f}^{-m}_{\mathrm{sab}}(V) \longrightarrow \mathfrak{f}^{-m+1}_{\mathrm{sab}}(V)\right\}$$

is identified, as a $GL(V)$-module, with $\Gamma^{\mathrm{cl}}_{m+1}(V)$.

This theorem will be proved by giving an explicit description of $\mathfrak{f}^\bullet_{\mathrm{sab}}(V)$. Let $\widetilde{\mathfrak{f}}^\bullet(V) \subset \mathfrak{f}^\bullet(V)$ be the dg-Lie subalgebra given by

$$(1.5.3)\quad \widetilde{\mathfrak{f}}^0(V) = [\mathrm{FL}(V), \mathrm{FL}(V)] \subset \mathrm{FL}(V) = \mathfrak{f}^0(V); \quad \widetilde{\mathfrak{f}}^i(V) = \mathfrak{f}^i(V),\ i \neq 0.$$

Thus $\widetilde{\mathfrak{f}}^\bullet(V)$ is acyclic by Proposition 1.1.4 (b). We start with describing the abelianization of this subalgebra.

Theorem 1.5.4. *The abelianization $\widetilde{\mathfrak{f}}^\bullet_{\mathrm{ab}}(V)$ is identified, as a cochain complex of $GL(V)$-modules, with the "co-de Rham complex"*

$$\widetilde{\Gamma}^\bullet(V) = \left\{ \Gamma_n(V) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Gamma_2(V) \xrightarrow{\partial} \Gamma_1^{\mathrm{cl}}(V) \right\},$$

with indexing given by

$$\widetilde{\Gamma}^0(V) = \Gamma_1^{\mathrm{cl}}(V), \quad \widetilde{\Gamma}^{-i}(V) = \Gamma_{i+1}(V),\ i > 0.$$

Remark 1.5.5. This generalizes the result of Reutenauer ([41], §8.6.12), which says that

$$(1.5.6)\qquad [\mathrm{FL}(V), \mathrm{FL}(V)]_{\mathrm{ab}} \;=\; \bigoplus_{d=1}^{\infty} \Sigma^{d,1}(V) \;=\; \Gamma_1^{\mathrm{cl}}(V)$$

as a $GL(V)$-module. One can also formulate (1.5.6) in terms of the dual spaces as follows:

$$(1.5.7)\qquad H^1_{\mathrm{Lie}}([\mathrm{FL}(V), \mathrm{FL}(V)], \mathbb{R}) \;=\; \widehat{\Omega}^{2,\mathrm{cl}}_{V,0},$$

where the right hand side stands for the space of formal germs of closed 2-forms on V at 0.

The proof of Theorem 1.5.4, given in the next subsection, provides an explicit homomorphism of dg-Lie algebras (with abelian target)

$$(1.5.8)\qquad \rho_\bullet : \widetilde{\mathfrak{f}}^\bullet(V) \to \widetilde{\Gamma}^\bullet(V),$$

inducing an isomorphism on $\widetilde{\mathfrak{f}}^\bullet_{\mathrm{ab}}(V)$.

For simplicity, let us drop V from the notation, writing Γ_m for $\Gamma_m(V)$, $\mathfrak{f}^\bullet$ for $\mathfrak{f}^\bullet(V)$ etc. Since abelianization factors through semiabelianization, the homomorphism $\rho_\bullet$ gives rise to a commutative diagram:

$$
(1.5.9)\qquad
\begin{array}{ccccccc}
\cdots & \xrightarrow{d} & \mathfrak{f}^{-2}_{\mathrm{sab}} & \xrightarrow{d} & \mathfrak{f}^{-1}_{\mathrm{sab}} & \xrightarrow{d} & [\mathfrak{f}^0,\mathfrak{f}^0] \\
 & & \downarrow \overline{\rho}_{-2} & & \downarrow \overline{\rho}_{-1} & & \downarrow \rho_0 \\
\cdots & \xrightarrow{\partial} & \Gamma_3 & \xrightarrow{\partial} & \Gamma_2 & \xrightarrow{\partial} & \Gamma_1^{\mathrm{cl}}
\end{array},
$$

where $\overline{\rho}_{-m}$, $m \geq 1$, is induced by ρ_{-m}. Theorem 1.5.2 will be a consequence of the following fact.

Theorem 1.5.10. *The rightmost square in* (1.5.9) *is Cartesian. For* $m \geq 2$ *the map* $\overline{\rho}_{-m}$ *is an isomorphism, so* $\mathfrak{f}^{-m}_{sab} = \mathfrak{f}^{-m}_{\mathrm{ab}}$.

1.6 Proofs of Theorems 1.5.4 and 1.5.10.

We start with constructing the homomorphism $\rho_\bullet$ from (1.5.8). The map ρ_0 is constructed as follows, cf [41], §5.3. Recall that $\Gamma_1^{\mathrm{cl}}(V)$ is the restricted dual of $\Omega^{2,\mathrm{cl}}(V)$. Let $t_1, ..., t_n$ be the linear coordinates in V corresponding to the basis $Z_1, ..., Z_n$. Then a (closed) 2-form $\omega \in \Omega^2(V)$ is written in these coordinates as

$$\omega = \sum_{i,j} \omega_{ij} dt_i dt_j, \quad \omega_{ij} \in k[t_1, ..., t_n].$$

Writing $\partial_i = \partial/\partial t_i$, we define, for any $p \geq 2$:

$$(1.6.1)\qquad \rho_0\Big([Z_{i_1}, [Z_{i_2}, ..., [Z_{i_{p-1}}, Z_{i_p}]...]\Big)(\omega) \;=\; \big(\partial_{i_1} ... \partial_{i_{p-2}} \omega_{i_{p-1}, i_p}\big)(0).$$

It follows from Reutenauer's theorem (cf. also Corollary 4.4.5 of [30] for the case of connections on line bundles) that ρ_0 induces an identification as claimed in Theorem 1.5.4 for the 0th level.

Further, let us define $\rho_{-m}, m \geq 1$. We view $\Gamma_{m+1}(V)$ as the restricted dual of $\Omega^{m+1}(V)$, and write any form $\omega \in \Omega^{m+1}(V)$ as $\sum_{|I|=m+1} \omega_I dt_I$ similarly to the above. We then define:

$$(1.6.2)\qquad \begin{array}{r}\rho_{-m}(M)=0, \text{ if } M \text{ is a Lie monomial containing}\\ \text{at least two generators } Z_J \text{ with } |J|>1;\end{array}$$

$$(1.6.3)\quad \rho_{-m}\Big([Z_{i_1},[Z_{i_2},...,[Z_{i_p},Z_I]...]\Big)(\omega)=\big(\partial_{i_1}...\partial_{i_p}\omega_I\big)(0).\quad |I|=m+1.$$

Lemma 1.6.4. *The map $\rho_\bullet$ of graded vector spaces, defined above, is a homomorphism of dg-Lie algebras.*

Proof: To show that $\rho_\bullet$ is a morphism of dg-Lie algebras (with abelian target), it is enough to show that it vanishes on commutators. For any commutator involving at least two of the $Z_J, |J|>1$, it follows from (1.6.2). For a commutator involving none or one of the $Z_J, |J|>1$, it follows from the Jacobi identity and the fact that the partial derivatives in (1.6.1) or (1.6.3) commute with each other. For example,

$$\begin{aligned}\rho_{-m}\big([[Z_a,Z_b],Z_I]\big)(\omega) \quad &= \quad \rho_{-m}\big([Z_a,[Z_b,Z_I]]\big)(\omega)-\rho_{-m}\big([Z_b,[Z_a,Z_I]]\big)(\omega)=\\ &=(\partial_a\partial_b\omega_I)(0)-(\partial_b\partial_a\omega_I)(0)=0.\end{aligned}$$

Next, let us show that $\rho_\bullet$ commutes with the differentials. We denote by $d_{\mathfrak{f}}$ the differential in $\widetilde{\mathfrak{f}}^\bullet(V)$, and by d_{DR} the de Rham differential on forms (dual to ∂). Then on the generators, we have, by (1.1.2):

$$d_{\mathfrak{f}}(Z_{i_0...i_m}) \quad \equiv \quad \sum_{\nu=0}^m(-1)^\nu[Z_{i_\nu},Z_{i_0,...,\widehat{i_\nu},...,i_m}]$$

modulo terns annihulated by ρ_{-m+1}. Therefore

$$\begin{aligned}(\rho_{-m+1}d_{\mathfrak{f}}(Z_{i_0,...,i_m}))(\omega)&=\sum_{\nu=0}^m(-1)^\nu(\partial_{i_\nu}\omega_{i_0,...,\widehat{i_\nu},...,i_m})(0)=\\ &=(d_{\mathrm{DR}}\omega)_{i_0...i_m}(0)=(\partial\rho_{-m}Z_{i_0...i_m})(0).\end{aligned}$$

The statement for more complicated brackets, involving one generator $Z_{i_0...i_m}$ and several degree 0 generators $Z_{j_1},...,Z_{j_p}$, follows in a similar way, by applying the derivatives $\partial_{j_1},...,\partial_{j_p}$ to the above equality. $\square$

Notice further that $\rho_\bullet$ is surjective. So to finish the proof of Theorem 1.5.4, it remains to establish the following fact.

Proposition 1.6.5. *The Lie monomials*

$$[Z_{i_1},[Z_{i_2},...,[Z_{i_{p-1}},Z_{i_p}]...],\quad i_1\geq i_2\geq ...\geq i_{p-1}<i_p,\quad p\geq 2, m=0;$$

$$[Z_{i_1},[Z_{i_2},...,[Z_{i_q},Z_J]...],\quad i_1\geq ...\geq i_q,\quad q\geq 0, |J|=m+1, m\geq 1,$$

span $\widetilde{\mathfrak{f}}^{-m}(V)$ *modulo* $[\widetilde{\mathfrak{f}}^{\bullet}(V),\widetilde{\mathfrak{f}}^{\bullet}(V)]^{-m}$.

Indeed, such monomials correspond to the standard bases in $\Sigma^{p,1}(V)$, resp. $S^q(V)\otimes\Lambda^{m+1}(V)$ which are the direct summands of $\Gamma_1^{\rm cl}(V)$, resp. $\Gamma_m(V)$. So in the case $m\geq 1$, the proposition would imply that $\widetilde{\mathfrak{f}}^{-m}_{\rm sab}$ is a $GL(V)$-module, having an equivariant surjective map $\rho^{\rm ab}_{-m}$ to the space $\bigoplus_q S^q(V)\otimes\Lambda^m(V)$ and having a spanning set mapping bijectively onto the basis of that space. This of course means that $\rho^{\rm ab}_{-m}$ is an isomorphism. Similarly for $m=0$.

Proof of Proposition 1.6.5: The case $m=0$ follows from the cited result of Reutenauer. The case $m\geq 1$ is obvious. Indeed, let us neglect all the iterated brackets containing at least one $[Z_{i_\mu},Z_{i_\nu}]$ and one Z_J, as such brackets belong to $[\widetilde{\mathfrak{f}}^{\bullet}(V),\widetilde{\mathfrak{f}}^{\bullet}(V)]$. Then, modulo such brackets, each Lie monomial $[Z_{i_1},[Z_{i_2},...,[Z_{i_q},Z_J]...]$ depends on $i_1,...,i_q$ in a symmetric way, similarly to the proof of Proposition 1.5.1(c). □

We now prove Theorem 1.5.10. First, we prove that the rightmost square in (1.5.9) is Cartesian. For this, consider the crossed module of Lie algebras

$$\mathfrak{f}^{\geq -1}_{\rm sab}\ =\ \big\{\mathfrak{f}^{-1}_{\rm sab}\xrightarrow{d}\mathfrak{f}^{0}_{\rm sab}\big\},\quad \mathfrak{f}^0_{\rm sab}=\mathfrak{f}^0={\rm FL}(V)$$

(with $\mathfrak{f}^{-1}_{\rm sab}$ equipped with the derived bracket). As $\mathfrak{f}^0$ is free, any Lie subalgebra of it is again free (the Shirshov-Witt theorem [41]). In particular, ${\rm Im}(d)=[\mathfrak{f}^0,\mathfrak{f}^0]$ is free. So the surjection $d:\mathfrak{f}^{-1}_{\rm sab}\to{\rm Im}(d)$ has a section, which is a Lie algebra homomorphism $s:{\rm Im}(d)\to\mathfrak{f}^{-1}_{\rm sab}$ s.t. $ds={\rm Id}$. As for any crossed module of Lie algebras, ${\rm Ker}(d)$ lies in the center of $\mathfrak{f}^{-1}_{\rm sab}$. This implies that

$$\mathfrak{f}^{-1}_{\rm sab}\simeq{\rm Ker}(d)\oplus[\mathfrak{f}^0,\mathfrak{f}^0]\qquad(1.6.6)$$

as a Lie algebra. Now, to prove that the square in question is Cartesian, means to prove that the (surjective) map $\overline{\rho}_{-1}$ maps ${\rm Ker}\{d:\mathfrak{f}^{-1}_{\rm sab}\to[\mathfrak{f}^0,\mathfrak{f}^0]\}$ isomorphically onto ${\rm Ker}\{\partial:\Gamma_2\to\Gamma_1^{\rm cl}\}$. But we know that $\overline{\rho}_{-1}$ is the degree (-1) component of the abelianization map for $\{\mathfrak{f}^{-1}_{\rm sab}\to[\mathfrak{f}^0,\mathfrak{f}^0]\}$ considered as a dg-Lie algebra. This abelianization map consists in quotienting by the action of $[\mathfrak{f}^0, \mathfrak{f}^0]$ (the degree 0 part) on $\mathfrak{f}^{-1}_{\mathrm{sab}}$ (the degree (-1) part). Using the decomposition (1.6.6), we can describe this action directly: it is trivial on the summand $\mathrm{Ker}(d)$ and is the adjoint action on the summand $[\mathfrak{f}^0, \mathfrak{f}^0]$. This implies that $\overline{\rho}_{-1}$ gives an isomorphism

$$\mathrm{Ker}(d) \oplus \frac{[\mathfrak{f}^0, \mathfrak{f}^0]}{[[\mathfrak{f}^0, \mathfrak{f}^0], \mathfrak{f}^0, \mathfrak{f}^0]]} \xrightarrow{\sim} \Gamma_2.$$

and therefore identifies $\mathrm{Ker}(d)$ with $\mathrm{Ker}(\partial)$.

We now prove that $\overline{\rho}_{-m}$ is an isomorphism for $m \geq 2$. Note that by definition (1.6.3) of $\overline{\rho}_{-m}$ the Lie monomials in part (b) of Proposition 1.5.1 are mapped to the standard basis vectors of Γ_{m+1}. By part (c) of the same proposition, these monomials are symmetric in $i_1, ..., i_p$, if $m \geq 1$. This means that the set of distinct monomials in the source of $\overline{\rho}_{-m}$ maps bijectively on the standard basis in the target, i.e., that $\overline{\rho}_{-m}$ is an isomorphism, as claimed. $\square$

2 The crossed module of formal 2-branes.

2.1 Crossed modules of groups.

A. Crossed modules. The following concept is classical, see [5, 40, 43] for more background.

Definition 2.1.1. A *crossed module* (of groups) is a homomorphism of groups

$$G^\bullet \;=\; \bigl\{G^{-1} \xrightarrow{\partial} G^0\bigr\}$$

together with a left action of G^0 on G^{-1} by automorphisms, via a homomorphism $\beta : G^0 \to \mathrm{Aut}(G^{-1})$ which is required to satisfy the axioms:

$$ghg^{-1} \;=\; \beta(\partial(g))(h), \quad u\partial(g)u^{-1} \;=\; \partial(\beta(u)(g)),$$
$$u \in G^0,\ g, h \in G^{-1}.$$

Morphisms of crossed modules are defined in the obvious way. We denote by $\mathcal{CM}$ the resulting category of crossed modules.

It follows from the axioms that $\mathrm{Im}(\partial)$ is a normal subgroup in G^0, so we have the group $H^0(G^\bullet) = \mathrm{Coker}(\partial)$. We also have the group $H^{-1}(G^\bullet) = \mathrm{Ker}(\partial)$ and this group is contained in the center of G^{-1} in virtue of the axioms; in particular, it is abelian. The group $H^0(G^\bullet)$ acts on the abelian group $H^{-1}(G^\bullet)$. In the situation of Definition 2.1.1 we will sometimes say that G^{-1} is a *crossed G^0-module.*

Example 2.1.2. Let Y be a CW-complex, X a subcomplex and $x \in X$ a point. Then

$$G^\bullet \;=\; G^\bullet(Y, X, x) \;=\; \bigl\{\pi_2(Y, X, x) \xrightarrow{\partial} \pi_1(X, x)\bigr\} \tag{2.1.3}$$

is a crossed module.

B. Crossed modules and 2-categories. It is well known [5] (see also [35], Lemma 2.2 and [4], p. 127) that to every crossed module $G^\bullet$ there corresponds a strict monoidal category $\mathrm{Cat}_\otimes(G^\bullet)$. By definition

$$\mathrm{Ob}\bigl(\mathrm{Cat}_\otimes(G^\bullet)\bigr) \;=\; G^0, \quad \mathrm{Mor}\bigl(\mathrm{Cat}_\otimes(G^\bullet)\bigr) \;=\; G^0 \ltimes G^{-1} \tag{2.1.4}$$

(semidirect product with respect to the action β), with the source, target and the composition maps given by

$$\begin{gathered}
s, t : G^0 \ltimes G^{-1} \longrightarrow G^0, \quad s(u,g) = u\partial(g), \ \ t(u,g) = u. \\
\circ : (G^0 \ltimes G^{-1}) \times^{s,t}_{G^0} (G^0 \ltimes G^{-1}) \longrightarrow G^0 \ltimes G^{-1}, \\
(u,g) \circ (v,h) = (u, gh), \quad \text{if} \quad s(u,g) = t(v,h).
\end{gathered} \tag{2.1.5}$$

Thus or any two objects $u, u' \in G^0$ the set $\mathrm{Hom}(u, u')$ is identified with the set of elements $g \in G^{-1}$ such that $\partial(g) = u^{-1}u'$.

Note that all the sets in (2.1.5) are groups and the axioms of a crossed module imply that all the maps are group homomorphisms. Therefore $\mathrm{Cat}_\otimes(G^\bullet)$ is a categorical object in the category of groups (a Cat^1-group, in the terminology of [35]). In particular, it is a monoidal category, with the monoidal operation $\otimes$ on objects, resp. morphisms, given by the group operation in G^0, resp. $G^0 \ltimes G^{-1}$.

We further recall that a (strict, globular, small) *2-category* C consists of sets C_0, C_1, C_2, whose elements are referred to as *objects, 1-morphisms* and *2-morphisms* of C, equipped with maps (called 1- and 0-dimensional *source, target* and *unit maps*)

$$\begin{tikzcd} C_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & C_1 \arrow[l, bend right, "\mathbf{1}_1"'] \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & C_0 \arrow[l, bend right, "\mathbf{1}_0"'] \end{tikzcd}, \quad s_0s_1 = s_0t_1, \ t_0s_1 = t_0t_1, \ s_i\mathbf{1}_i = t_i\mathbf{1}_i \tag{2.1.6}$$

and two compositions

$$*_0 : C_2 \times^{s_0s_1, t_0t_1}_{C_0} C_2 \longrightarrow C_2, \quad *_1 : C_2 \times^{s_1,t_1}_{C_1} C_2 \longrightarrow C_2, \tag{2.1.7}$$

called the *horizontal* and *vertical compositions* of 2-morphisms. These compositions are required to be associative to satisfy the 2-dimensional associativity condition, as well as the unit conditions for the $\mathbf{1}_i$. For more details, see [33]. See also Definition 3.2.7 below for a more general notion of a strict globular n-category.

Any strict monoidal category $(\mathcal{C}, \otimes)$ gives rise to a 2-category $2\,\mathrm{Cat}(\mathcal{C})$ with one object, with 1-morphisms being objects of $\mathcal{C}$ and 2-morphisms being morphisms of $\mathcal{C}$. The horizontal composition $*_0$ of 2-morphisms is given by the monoidal structure $\otimes$ on $\mathcal{C}$. The vertical composition $*_1$ is given by the composition of morphisms in $\mathcal{C}$. Applying this construction to the monoidal category $\mathrm{Cat}_\otimes(G^\bullet)$, we associate to a crossed module $G^\bullet$ a

2-category $2\,\mathrm{Cat}(G^\bullet)$ with one object. See §3.2 for a more general construction, due to Brown-Higgins [7], which associates an n-category to a crossed complex.

The classifying space of the 2-category $2\,\mathrm{Cat}(G^\bullet)$ will be denoted $B(G^\bullet)$. It is a connected topological space with

$$\pi_1(B(G^\bullet)) \;=\; H^0(G^\bullet), \quad \pi_2(B(G^\bullet)) \;=\; H^{-1}(G^\bullet).$$

C. Free crossed modules. Let G be any group and $(g_a)_{a\in A}$ be any family of elements of G. Let C be the group generated by symbols $\sigma(\gamma, a)$ for all $\gamma \in G$ and $a \in A$, which are subject to the relations

$$\sigma(\gamma, a)\sigma(\delta, b)\sigma(\gamma, a)^{-1} \;=\; \sigma(\gamma g_a \gamma^{-1}, b). \tag{2.1.8}$$

We have a homomorphism $\partial : C \to G$, sending $\sigma(\gamma, a)$ to $\gamma g_a \gamma^{-1}$. We also have an action of G on C which on generators is given by

$$\beta(g)(\sigma(\gamma, a)) \;=\; \sigma(g\gamma, a).$$

With these data, $\{C \xrightarrow{\partial} G\}$ is a crossed module, known as the *free crossed G-module* on the set of generators (g_a). It can be characterized by a universal property in the category $\mathcal{CM}$, see [43], Def. 6.5.

At the group-theoretical level, the group $H^0 = \mathrm{Coker}(\partial)$ for this crossed module is the quotient of G by the normal subgroup generated by the g_a, i.e., the result of imposing additional relations $g_a = 1$ in G.

At the 2-categorical level, $2\,\mathrm{Cat}\{C \xrightarrow{\partial} G\}$ is obtained by adding to G (considered as a usual category with one object) new 2-isomorphisms $\sigma(a) : g_a \Rightarrow 1$. The generator $\sigma(\gamma, a)$ of C corresponds to the conjugate 2-morphism $\gamma *_0 \sigma(a) *_0 \gamma^{-1}$.

At the topological level, this construction corresponds to adding 2-cells to a CW-complex. More precisely, we have the following fact, due to J. H. C. Whitehead [47].

Theorem 2.1.9. *Let Y be a connected CW-complex, X a subcomplex such that $Y - X$ consists of 2-dimensional cells only, and $x \in X$. Then $G^\bullet(Y, X, x)$ is isomorphic to the free crossed $\pi_1(X, x)$-module on the set of generators corresponding to the boundaries of the attached 2-cells.* □

The following particular case, also due to Whitehead [47], will be important for us. For a group G we denote $G_{\mathrm{ab}} = G/[G,G]$ the abelianized group. For a CW-complex K we denote by $Z_i(K,\mathbb{Z}) \subset C_i(K,\mathbb{Z})$ the groups of cellular i-cycles and i-chains of K with integer coefficients.

Theorem 2.1.10. *Assume, in addition, that X is equal to the 1-skeleton of Y (so Y is 2-dimensional), and let $G^\bullet = G^\bullet(Y,X,x)$. Denote $\widetilde{Y}$ the universal covering of Y (with the CW-decomposition lifting that of Y) and let $\widetilde{x} \in \widetilde{Y}$ be one of the preimages of x. Then:*

(a) We have $H^0(G^\bullet) = \pi_1(Y,x)$, while

$$H^{-1}(G^\bullet) = \pi_2(Y,x) = \pi_2(\widetilde{Y},\widetilde{x}) = H_2(\widetilde{Y},\mathbb{Z}) = Z_2(\widetilde{Y},\mathbb{Z}).$$

(b) Moreover, we have a commutative diagram

$$\begin{tikzcd} G^{-1} \arrow[r] \arrow[d, "\partial"'] & G^{-1}_{\mathrm{ab}} \arrow[r, "\sim"] \arrow[d, "\partial_{\mathrm{ab}}"'] & C_2(\widetilde{Y},\mathbb{Z}) \arrow[d, "b"] \\ \mathrm{Im}(\partial) \arrow[r] & \mathrm{Im}(\partial)_{\mathrm{ab}} \arrow[r, "\sim"] & C_1(\widetilde{Y},\mathbb{Z}) \end{tikzcd}$$

in which the left square is Cartesian, the two right arrows are isomorphisms, and b is the chain differential.

For convenience of the reader we add a proof that uses modern terminology. The very first equality in (a) follows from the long exact sequence of relative homotopy groups and the fact that $\pi_2(X) = 0$. Subsequent equalities reflect respectively, the invariance of π_2 in coverings, the Hurewitz theorem, and the 2-dimensionality of Y.

To see (b), note that $G^0 = \pi_1(X,x)$ is free and so $F_0 = \mathrm{Im}(\partial)$ is free as a subgroup in a free group. Lifting a system of free generators of F_0 into G^{-1} in an arbitrary way, we realize G^{-1} as a semidirect product of F_0 and $\mathrm{Ker}(\partial)$. Since $\mathrm{Ker}(\partial)$ is contained in the center of G^{-1}, we have that $G^{-1} \simeq F_0 \times \mathrm{Ker}(\partial)$ is a direct product, and ∂ is identified with the projection to the first factor. Since $\mathrm{Ker}(\partial)$ is abelian, this implies that the left square in (b) is Cartesian. Further, $G^0_{\mathrm{ab}} = H_1(X,\mathbb{Z}) = C_1(Y,\mathbb{Z})$ and $\widetilde{Y}_{\leq 1}$, the 1-skeleton of $\widetilde{Y}$, is the covering of X corresponding to the subgroup $\mathrm{Im}(\partial) \subset G^0$. This implies that $H_1(\widetilde{Y}_{\leq 1},\mathbb{Z}) = C_1(\widetilde{Y},\mathbb{Z})$ is identified with $\mathrm{Im}(\partial)_{\mathrm{ab}}$.

Finally, we define a homomorphism $f : G^{-1} \to C_2(\widetilde{Y},\mathbb{Z})$ using the identification of $G^\bullet$ as a free crossed G^0-module on the set of generators corresponding to the boundaries of 2-cells. If $(c_a)_{a\in A}$ are the 2-cells, and $g_a \in G^0$

is the boundary path of c_a, then the generator $\sigma(\gamma, a)$, $\gamma \in G^0$, is sent by f to the cell $(\overline{\gamma})^* c_a$, where $\overline{\gamma} \in \pi_1(Y, x)$ is the image of γ, and $(\overline{\gamma})^*$ means the image under the covering transformation corresponding to $\overline{\gamma}$. One verifies that f is well defined. Since its target is abelian, f gives rise to $f_{\rm ab} : G^{-1}_{\rm ab} \to C_2(\widetilde{Y}, \mathbb{Z})$ making the right square in (b) commutative. Moreover, $f_{\rm ab}$ identifies $\mathrm{Ker}(\partial_{\rm ab})$ with $\mathrm{Ker}(b)$ by the Hurewitz theorem, so we conclude that $f_{\rm ab}$ is an isomorphism as well. $\square$

Example 2.1.11 The crossed module of cubical membranes. Let $F(n)$ be the free group on n generators $X_1, ..., X_n$. Consider the following family of elements of $F(n)$:

$$g_{ij} \;=\; [X_i, X_j] \;=\; X_i X_j X_i^{-1} X_j^{-1}, \quad 1 \le i < j \le n.$$

The free crossed $F(n)$-module on the set of generators (g_{ij}) will be denoted by $\square^\bullet_{\rm CM}(n)$ and called the *crossed module of cubical membranes in* $\mathbb{R}^n$. Thus $\square^0_{\rm CM}(n) = F(n)$, while $\square^{-1}_{\rm CM}(n)$ is generated by the symbols $\sigma_{ij}(\gamma) = \sigma(\gamma, ij)$, $\gamma \in F(n)$ subject to the relations as in (2.1.8).

At the topological level, let $\mathbb{R}^n_\square$ be the CW-decomposition of $\mathbb{R}^n$ into cubes of the standard cubical lattice, and $\mathbb{T}^n = \mathbb{R}^n_\square / \mathbb{Z}^n$ be the n-torus with CW-decomposition obtained by identifying the opposite sides of the unit n-cube. Thus $\mathbb{T}^n$ has $\binom{n}{i}$ cells of dimension i. For $i \ge 0$ denote by $R^n_{\square, \le i}$ and $\mathbb{T}^n_{\le i}$ the i-skeleta of these CW-complexes. In particular, $\mathbb{T}^n_{\le 0}$ consists of a single point which we denote 0. Then Theorem 2.1.9 implies that

$$\square^\bullet_{\rm CM}(n) \;=\; G^\bullet(\mathbb{T}^n_{\le 2}, \mathbb{T}^n_{\le 1}, 0).$$

Since the universal cover of $\mathbb{T}^n_{\le 2}$ is $\mathbb{R}^n_{\square, \le 2}$, Theorem 2.1.10 implies that $\square^{-1}_{\rm CM}(n)$ is the semidirect product of $[F(n), F(n)]$ and the group $Z_2(\mathbb{R}^n_\square, \mathbb{Z})$ of cellular 2-cycles in $\mathbb{R}^n_\square$.

At the level of monoidal categories, $\mathrm{Cat}_\otimes(\square^\bullet_{\rm CM}(n))$ is generated by invertible objects $X_1, ..., X_n$ and invertible morphisms $\sigma_{ij} : X_i \otimes X_j \to X_j \otimes X_i$ which are subject to no relations other than those implied by the definition of a strict monoidal category.

Passing to the 2-category $2\,\mathrm{Cat}(\square^\bullet_{\rm CM}(n))$ with one object, we notice that its 1-morphisms, i.e., elements of $F(n)$, can be seen as a lattice paths in $\mathbb{R}^n_\square$ starting from 0, see, e.g., [29]. The end point of the path corresponding to $\gamma \in F(n)$ is the image of γ in $\mathbb{Z}^n = F(n)_{\rm ab}$. A 2-morphism from γ to γ'

(existing only if the endpoints of γ and γ' coincide) can be visualized as a membrane formed out of 2-dimensional squares of $\mathbb{R}^n_\square$ and connecting γ and γ'. This explains the name "crossed module of cubical membranes". For instance, the element $\sigma_{ij}(\gamma) \in \square^{-1}_{\mathrm{CM}}(n)$ can be seen as a "lasso" formed by the square in the direction (i, j) attached at the end of the path γ. This lasso is a 2-morphism from $[X_i, X_j] *_0 \gamma$ to γ.

The term "lasso" as well as the corresponding "lasso variables" were introduced by L. Gross [23] in connection with quantization of the Yang-Mills theory.

2.2 Crossed modules of Lie groups and Lie algebras.

There are three main constructions relating groups and Lie algebras, and we discuss their effect on crossed modules.

A. From Lie groups to Lie algebras. If G is a Lie group, we denote by $\mathrm{Lie}(G)$ its Lie algebra (over $\mathbb{R}$). The following is then immediate.

Proposition 2.2.1. *Let $G^\bullet$ be a crossed module of Lie groups, and $\mathfrak{g}^i = \mathrm{Lie}(G^i)$. Then $\mathfrak{g}^\bullet$ is a crossed module of Lie $\mathbb{R}$-algebras (Example 1.3.2) and thus, by Proposition 1.3.6, it has a structure of a dg-Lie $\mathbb{R}$-algebra situated in degrees 0 and (-1).* $\square$

B. Lower central series and the Magnus Lie algebra. If G is any group, we have the *lower central series* defined in terms of group commutators:

$$G = \gamma_1(G) \supset \gamma_2(G) \supset \cdots, \quad \gamma_{r+1}(G) = [G, \gamma_r(G)]. \tag{2.2.2}$$

Each $\gamma_{r+1}(G)$ is a normal subgroup in $\gamma_r(G)$ with abelian quotient. Let k be a field of characteristic 0. The k-vector space

$$L(G) \;=\; \bigoplus_{r=1}^{\infty} (\gamma_r(G)/\gamma_{r+1}(G)) \otimes_{\mathbb{Z}} k \tag{2.2.3}$$

with bracket induced by the group commutator in G, is a Lie k-algebra. known as the *Magnus Lie algebra* of G.

More generally, let $G^\bullet$ be a crossed module of groups. We equip G^{-1} by the *lower G^0-central series*

$$G^{-1} = \gamma_1(G^0, G^{-1}) \supset \gamma_2(G^0, G^{-1}) \supset \ldots \tag{2.2.4}$$

where $\gamma_{r+1}(G^0, G^{-1})$ is the normal subgroup in G^{-1}, normally generated by elements of the form

$$\beta(z)(x)\cdot x^{-1}, \quad z \in G^0,\ x \in \gamma_r(G^0, G^{-1}). \tag{2.2.5}$$

See [40], p. 93. We will say that $G^\bullet$ is *nilpotent*, if both series $(\gamma_r(G^0))$ and $(\gamma_r(G^0, G^{-1}))$ terminate.

Proposition 2.2.6. *(a) Both*

$$\big\{\gamma_r(G^0, G^{-1}) \xrightarrow{\partial} \gamma_r(G^0)\big\} \quad \text{and} \quad \big\{G^{-1}/\gamma_r(G^0, G^{-1}) \xrightarrow{\partial} G^0/\gamma_r(G^0)\big\}$$

inherit the structures of crossed modules of groups.

(b) Successive quotients $\gamma_r(G^0, G^{-1})/\gamma_{r+1}(G^0, G^{-1})$ are abelian, and the group commutator in G^{-1} makes

$$L(G^0, G^{-1}) \;=\; \bigoplus_{r=1}^{\infty} \left(\gamma_r(G^0, G^{-1})/\gamma_{r+1}(G^0, G^{-1})\right) \otimes_{\mathbb{Z}} k$$

into a Lie k-algebra. The homomorphism $\partial : G^{-1} \to G^0$ gives rise to a homomorphism of Lie algebras

$$L(\partial) : L(G^0, G^{-1}) \longrightarrow L(G^0)$$

(c) The formula

$$\big(\gamma \in \gamma_p(G^0),\, \delta \in \gamma_q(G^0, G^{-1})\big) \;\longmapsto\; \beta(\gamma)(\delta)\cdot\delta^{-1} \;\in\; \gamma_{p+q}(G^0, G^{-1})$$

defines an action of $L(G^0)$ on $L(G^0, G^{-1})$ by derivations and makes

$$L(G^\bullet) \;=\; \big\{L(G^0, G^{-1}) \xrightarrow{L(\partial)} L(G^0)\big\}$$

into a crossed module of Lie algebras.

Proof: Straightforward, left to the reader. □

C. The Malcev theory. Let k be a field of characteristic 0. If $\mathfrak{g}^\bullet$ is any dg-Lie k-algebra, its *lower central series* is defined in terms of Lie algebra commutators:

$$\mathfrak{g}^\bullet = \gamma_1(\mathfrak{g}^\bullet) \supset \gamma_2(\mathfrak{g}^\bullet) \supset \cdots, \quad \gamma_{r+1}(\mathfrak{g}^\bullet) = [\mathfrak{g}^\bullet, \gamma_r(\mathfrak{g}^\bullet)], \tag{2.2.7}$$

similarly to (2.2.2). As usual, we say that $\mathfrak{g}^\bullet$ is *nilpotent*, if $\gamma_r(\mathfrak{g}^\bullet) = 0$ for some r.

Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie k-algebra (with trivial dg-structure). In this case the Malcev theory produces a unipotent algebraic group $\exp(\mathfrak{g})$ over k. More explicitly, the augmentation ideal $I \subset U(\mathfrak{g})$ is in this case topologically nilpotent: $\bigcap I^d = 0$, and so $U(\mathfrak{g})$ is embedded into the I-adic completion $\widehat{U}(\mathfrak{g}) = \varprojlim U(\mathfrak{g})/I^d$, which is a topological Hopf algebra. The group $\exp(\mathfrak{g})(k)$ of k-points can be identified with the group of group-like elements of $\widehat{U}(\mathfrak{g})$, i.e., of g such that $\Delta(g) = g \otimes g$. As a set, it consists precisely of elements of the form $\exp(x) = \sum_{i=0}^\infty x^i/i!$ for $x \in \mathfrak{g}$. Similarly for points with values in an arbitrary commutative k-algebra. A moprhism $f : \mathfrak{g} \to \mathfrak{g}'$ of finite-dimensional nilpotent Lie algebras gives rise to a morphism $\exp(f) : \exp(\mathfrak{g}) \to \exp(\mathfrak{g}')$ of algebraic groups.

If $\mathfrak{g}$ is any Lie k-algebra, we define its pro-nilpotent completion to be $\widehat{\mathfrak{g}} = \varprojlim \mathfrak{k}$, where $\mathfrak{k}$ runs over finite-dimensional nilpotent quotients of $\mathfrak{g}$. We have then the pro-algebraic group $\exp(\widehat{\mathfrak{g}}) = \varprojlim \exp(\mathfrak{k})$.

Let now $\mathfrak{g}^\bullet$ be a crossed module of Lie algebras. We define the *lower* $\mathfrak{g}^0$*-central series* of $\mathfrak{g}^{-1}$:

$$\begin{aligned} \mathfrak{g}^{-1} = \gamma_1(\mathfrak{g}^0, \mathfrak{g}^{-1}) \supset \gamma_2(\mathfrak{g}^0, \mathfrak{g}^{-1} \supset \cdots \\ \gamma_{r+1}(\mathfrak{g}^0, \mathfrak{g}^{-1}) \;=\; \alpha(\mathfrak{g}^0)\big(\gamma_r(\mathfrak{g}^0, \mathfrak{g}^{-1})\big). \end{aligned} \tag{2.2.8}$$

Proposition 2.2.9. *We have the equality*

$$\gamma_r(\mathfrak{g}^\bullet) \;=\; \big\{\gamma_r(\mathfrak{g}^0, \mathfrak{g}^{-1}) \xrightarrow{d} \gamma_r(\mathfrak{g}^0)\big\}.$$

That is, the diagram on the right is a crossed submodule of Lie algebras in $\mathfrak{g}^\bullet$, and the corresponding dg-Lie algebra coincides with the r-term of the lower central series of the dg-Lie algebra corresponding to $\mathfrak{g}^\bullet$.

Proof: Let $x \in \mathfrak{g}^0, y \in \mathfrak{g}^{-1}$. The commutator $[x, y]$ in the dg-Lie algebra corresponding to $\mathfrak{g}^\bullet$, is equal to the element $\alpha(x)(y)$. Therefore the second line in (2.2.8) can be written, in terms of the dg-Lie algebra structure, as follows:

$$\gamma_{r+1}(\mathfrak{g}^0, \mathfrak{g}^{-1}) \;=\; [\mathfrak{g}^0, \gamma_r(\mathfrak{g}^0, \mathfrak{g}^{-1})].$$

This implies the comparison. □

So we will use the term *nilpotent* for a crossed module $\mathfrak{g}^\bullet$ of Lie algebras to signify that $\gamma_r(\mathfrak{g}^\bullet) = 0$, i.e., $\mathfrak{g}^\bullet$ is nilpotent as a dg-Lie algebra.

Proposition 2.2.10. *Let $\mathfrak{g}^\bullet$ be a crossed module of Lie algebras. Then:*

(a) If $\mathfrak{g}^\bullet$ is nilpotent, then both $\mathfrak{g}^0$ and $\mathfrak{g}^{-1}$ are nilpotent as Lie algebras.

(b) Assume that $\mathfrak{g}^\bullet$ is finite-dimensional. Then $\mathfrak{g}^\bullet$ is nilpotent if and only if $\mathfrak{g}^0$ is a nilpotent Lie algebra.

Proof: (a) Follows from the definition of the bracket $[x, y]_{-1} = [x, d(y)]$ on $\mathfrak{g}^{-1}$ in terms of the dg-Lie algebra structure on $\mathfrak{g}^\bullet$.

(b) The "only if" part is clear. To see the "if" part, note that nilpotence of $\mathfrak{g}^0$ and finite-dimensionality of $\mathfrak{g}^{-1}$ imply that $\gamma_r(\mathfrak{g}^0, \mathfrak{g}^{-1}) = 0$ for some r in virtue of the Engel theorem. □

Proposition 2.2.11. *Let $\mathfrak{g}^\bullet$ be a nilpotent crossed module of finite-dimensional Lie algebras. Then*

$$\exp(\mathfrak{g}^\bullet) \;=\; \left\{\exp(\mathfrak{g}^{-1}) \xrightarrow{\exp(d)} \exp(\mathfrak{g}^0)\right\}$$

has a natural structure of a crossed module of algebraic groups over k. This crossed module is nilpotent. The correspondence $\mathfrak{g}^\bullet \mapsto \exp(\mathfrak{g}^\bullet)$ establishes an equivalence between the categories of nilpotent crossed modules of finite-dimensional Lie k-algebras and of nilpotent crossed modules of algebraic groups over k.

Proof: Follows from the fact that Malcev's construction $\mathfrak{g} \mapsto \exp(\mathfrak{g})$ is an equivalence of categories between finite-dimensional nilpotent Lie k-algebras and unipotent algebraic groups over k. □

Remark 2.2.12. Despite its rather straightforward construction, it is not easy to express the crossed module $\exp(\mathfrak{g}^\bullet)$ in terms of $\mathfrak{g}^\bullet$ as a dg-Lie algebra. This becomes noticeable for example, when $\mathfrak{g}^\bullet$ is itself obtained from some other dg-Lie algebra as the maximal crossed module quotient. In particular, we do not know how to extract all of $\exp(\mathfrak{g}^\bullet)$ out of the universal enveloping dg-algebra $U(\mathfrak{g}^\bullet)$ using some analog of the group-like property.

Note that $\mathfrak{g}^{-1}$, being an abelian ideal in $\mathfrak{g}^\bullet$, contributes an ideal in $U(\mathfrak{g}^\bullet)$ isomorphic to the exterior algebra $\Lambda^\bullet(\mathfrak{g}^{-1})$. On the other hand, Malcev's construction in Proposition 2.2.11 makes use of the (ungraded) enveloping algebra $U(\mathfrak{g}^{-1}, [-,-]_{-1})$ whose size is that of the symmetric algebra $S^\bullet(\mathfrak{g}^{-1})$.

2.3 Unparametrized paths and branes.

In this section we recall the concept of unparametrized paths and branes which based on the technique of thin homotopies introduced in [2, 38].

A. Differentiable spaces. The space of paths. We denote by $\mathcal{M}an$ the category of C^∞-manifolds with corners. The idea of following definition goes back to K.-T. Chen [14]. We present a slightly modified version, closer to that given by R. M. Hain ([24], Def. 4.1).

Definition 2.3.1. A *differentiable space* Y is a contravariant functor $h = h_Y : \mathcal{M}an \to \mathcal{S}et$ satisfying the following *gluing condition*:

(G) Let $(M_i)_{i\in I}$ is any covering of a manifold $M \in \mathcal{M}an$ by open subsets. Call a system $(\phi_i)_{i\in I} \in \prod_{i\in I} h(M_i)$ *compatible*, if for each $i, j \in I$ the images of ϕ_i and ϕ_j in $h(M_i \cap M_j)$ coincide. Then the natural map $h(M) \longrightarrow \prod_{i\in I} h(M_i)$ identifies $h(M)$ with the set of compatible systems.

Elements of $h(M)$ are called *plots* of Y of type M.

For instance, any C^∞-manifold $N \in \mathcal{M}an$ is considered as a differentiable space via the representable functor $M \mapsto C^\infty(M, N)$. For a general differentiable space Y it is convenient to think of $h_Y(M)$ as the set of smooth maps $M \to Y$ and to denote this set $C^\infty(M, Y)$ as above. In particular, the underlying set of Y is recovered as $h_Y(\mathrm{pt})$.

For a differentiable space Y we define the space of smooth differential p-forms on Y (in particular, of smooth functions, for $p = 0$) by

$$\Omega^p_Y \;=\; \varprojlim_{(\phi: M\to Y)} \Omega^p_M, \tag{2.3.2}$$

the limit over all plots. In other words, a p-form on Y is a compatible system of p-forms on all the plots.

Example 2.3.3. Let X be a C^∞-manifold and PX be the space of parametrized smooth paths $\gamma : [0,1] \to X$. The set PX is made into a differentiable space by putting $C^\infty(M, PX)$ to be the space of smooth maps $\phi : M \times [0,1] \to X$. The tangent space $T_\gamma PX$ is understood as the space of smooth sections of the vector bundle $\gamma^* T_X$ on $[0,1]$. We will typically denote such sections by $\delta\gamma$. A differential p-form Φ on PX gives a function $\Phi(\gamma; \delta_1\gamma, \cdots \delta_p\gamma)$ of a point $\gamma \in PX$ and p elements of $T_\gamma PX$, and Φ is uniquely determined by this function.

We define $P_x^y X$ to be the subspace of paths γ as above such that $\gamma(t) \equiv x$ for t sufficiently close to 0, and $\gamma(t) \equiv y$ for t sufficiently close to 1. Thus $T_\gamma P_x^y X$ consists of $\delta\gamma$ vanishing on some neighborhoods of 0 and 1.

Remark 2.3.4. In requiring that paths from $P_x^y X$ are *constant* on some neighborhoods of 0 and 1, we follow [10]. This requirement ensures that the composition of two such paths is again smooth, not just piecewise smooth. On the other hand, any geometric path can be easily parametrized in this way, so this requirement leads to no loss of generality.

Sticking to actual smooth maps becomes particularly important when we pass from paths to membranes which are maps $\Sigma : [0,1]^p \to X$, see Definition 2.3.10 below. The definition of "piecewise" smoothness for such maps is not quite clear (what kind of "pieces" are to be allowed so that the desired constructions go through?). Instead of attempting such a definition, we impose on Σ, as in [2, 38], additional conditions of constancy (in certain directions) near the boundary, so as to ensure the all the necessary compositions of such membranes are again smooth.

B. Thin homotopies. Unparametrized paths. Let X be a C^∞-manifold and $\gamma, \gamma' \in P_x^y X$ be two parametrized paths with common source x and target y.

Definition 2.3.5. A thin homotopy between γ and γ' is a C^∞-map $\Xi = \Xi(a_1, a_2) : [0,1]^2 \to X$ such that:

(1) Ξ is constant in some neighborhoods of te faces $a_1 = 0$, $a_1 = 1$.

(2) Ξ depends only on a_1 in some neighborhoods of the faces $a_2 = 0$, $a_2 = 1$.

(3) $\Xi(a_1, 0) = \gamma(a_1)$, $\Xi(a_1, 1) = \gamma'(a_1)$ for any $a_1 \in [0,1]$.

(4) The rank of the differential $d_a\Xi : T_a[0,1]^2 \to T_{\Xi(a)}X$ at any $a \in [0,1]^2$ is ≤ 1.

The first condition means that all the intermediate paths $\gamma_{a_2} : a_1 \mapsto \Sigma(a_1, a_2)$ lie in $P_x^y X$. The second condition means that γ_{a_2} coincides with γ for a_2 close to 0 and with γ' for a_2 close to 1. It ensures that being thin homotopic is an equivalence relation on $P_x^y X$ which we denote $\gamma \sim \gamma'$. The 4th condition ("thinness") means that the relation $\sim$ includes, in particular, reparametrization of paths as well as cancellation of a segment and the same segment run in the opposite direction immediately after.

We denote by $\Pi_x^y X = P_x^y X/\sim$ the set of this homotopy classes of paths from $P_x^y X$. By the above, we can consider elements of $\Pi_x^y X$ as *unmarametrized paths* from x to y. We define the *groupoid of unparametrized paths* $\Pi_{\leq 1} X$ to be the category with

$$\mathrm{Ob}(\Pi_{\leq 1} X) = X, \quad \mathrm{Hom}_{\Pi_{\leq 1} X}(x, y) = \Pi_x^y X.$$

Composition of paths is given by the standard concatenation (which gives a smooth path, see Remark 2.3.4):

$$(\gamma * \gamma')(a) = \begin{cases} \gamma'(2a), & \text{if} \quad 0 \leq a \leq \frac{1}{2} \\ \gamma(2a-1), & \text{if} \quad \frac{1}{2} \leq a \leq 1. \end{cases} \tag{2.3.6}$$

Because of the thin homotopy relation, this composition is associative, and each path is invertible.

C. Globes and branes. Let $I = [0,1]$ be the unit interval, so I^p is the unit p-cube with coordinates $a_1, ..., a_p \in [0,1]$. The following definition is a reformulation of one from [9]

Definition 2.3.7. Let X be a topological space. A *singular p-globe* in X is a continuous map $\Sigma : I^p \to X$ which for each $i = 1, ..., p$ satisfies the following condition:

(Glob$_i$) The restrictions of Σ to the faces $\{t_i = 0\}$ and $\{t_i = 1\}$ of I^p depend only on the coordinates $t_1, ..., t_{i-1}$ (in particular, are constant, if $i = 1$).

We denote by $\mathrm{Glob}_p(X)$ the set of singular p-globes in X.

The conditions (Glob$_i$) mean that singular p-globes factor through the *universal singular p-globe* $\alpha_p : I^p \to \bigcirc^p$. More precisely, $\bigcirc^p$ is the quotient of I^p by the identifications coming from the (Glob$_i$). As well known $\bigcirc^p$ can be identified with the unit p-ball $D^p = \{x \in \mathbb{R}^p : \|x\| \leq 1\}$, see [9], Def. 2.1 for an explicit map $\beta_p : I^p \to D^p$ establishing this identification. Further, the cell structure on I^p given by the faces, is contracted by β_p into the cell structure on D^p given by the interior open ball and the hemispheres:

$$D^p \;=\; e^p \cup e_{\pm}^{p-1} \cup e_{\pm}^{p-2} \cup \cdots \cup e_{\pm}^0,$$

where

$$e_{\pm}^i \;=\; \big\{x \in D^p : \; \|x\| = 1, \; x_j = 0 \text{ for } j + i < p, \; \pm x_{p-i} > 0.\big\}$$

The conditions (Glob$_i$) implies that for a singular p-globe $\Sigma : I^p \to X$ and $q < p$ the restrictions of Σ to the faces $a_q = 0$ and $a_q = 1$ define singular p-globes which we denote $s_q\Sigma$ and $t_q\Sigma$. These globes are called the *p-dimensional source and target* of Σ. Thus we have maps

$$(2.3.8) \qquad s_q = s_q^p, \, T_q = t_q^p : \; \mathrm{Glob}_p(X) \longrightarrow \mathrm{Glob}_q(X).$$

Example 2.3.9. For the universal globe $\beta_p : I^p \to D^p \simeq \bigcirc^p$ the globe $s_q\beta_p$ takes values in the closed hemisphere e_-^q and $t_q\beta_p$ takes values in e_+^p. So we will denote these hemispheres by $s_q\bigcirc^p$, $t_q\bigcirc^p$.

We now consider the smooth setting and introduce a slight modification of the above concepts, adapted along the lines of Remark 2.3.4.

Definition 2.3.10. Let X be a differentiable space. A *parametrized p-brane* in X is a smooth map $\Sigma : I^p \to X$ which for each $i = 1, ..., p$ satisfies the following condition:

(Br$_i$) There exist neighborhoods of the faces $\{t_i = 0\}$ and $\{t_i = 1\}$ of I^p in which Σ depends only on the coordinates $t_1, ..., t_{i-1}$ (in particular, is constant, if $i = 1$).

We denote by $\widetilde{\Pi}_p(X)$ the set of parametrized p-branes in X. Thus, for a smooth manifold X we have $\widetilde{\Pi}_p(X) \subset \mathrm{Glob}_p(X)$. It is clear that the maps s_q, t_q from (2.3.8) take $\widetilde{\Pi}_p(X)$ to $\widetilde{\Pi}_q(X)$. Note further that a parametrized p-brane in X descends, via α_p, to a *smooth* map $\bigcirc^p = D^p \to X$.

Example 2.3.11. We have $\widetilde{\Pi}_0(X) = X$, while $\widetilde{\Pi}_1(X) \subset PX$ is the space of paths $\gamma : I \to X$ which are constant in some neighborhoods of 0 and 1. Let $\Sigma \in \widetilde{\Pi}_2(X)$. Then by ($\mathrm{Br}_2$) we have that Σ is constant on some neighborhoods of the intervals $\{0\} \times I$ and $\{1\} \times I$. Denoting the images of these neighborhoods by x and y, we can associate to Σ a smooth path

$$\sigma = \tau\Sigma : [0,1] \to P_x^y X, \quad \sigma(a)(b) = \Sigma(b,a),\ a,b \in [0,1],$$

called the *transgression* of Σ.

D. Thin homotopy of branes. Unparametrized branes. Let X be a C^∞-manifold and $\Sigma, \Sigma' : \bigcirc^p \to X$ be two parametrized p-branes.

Definition 2.3.12. A *homotopy* between Σ and Σ' is a smooth map $\Xi : I \times \bigcirc^p \to X$ such that:

(1) For each $b \in I$ the map $\Xi_b = \Xi(b,-) : \bigcirc^p \to X$ is a parametrized p-brane in X.

(2) Ξ_b is independent of b for b in some neighborhoods of 0 and 1 and equals in these neighborhoods to Σ and Σ' respectively.

If $q < p$, then by restricting to $I \times s_q\bigcirc^p$, the homotopy Ξ induces a homotopy Ξ^{s_q} between $s_q\Sigma$ and $s_q\Sigma'$. We similarly obtain a homotopy Ξ^{t_q} between $t_q\Sigma$ and $t_q\Sigma'$.

Definition 2.3.13. A homotopy $\Xi : I \times \bigcirc^p \to X$ between Σ and Σ' is called *thin*, if:

(T_p) The rank of the differential of Ξ at any point of $I \times \bigcirc^p$ is $\leq p$.

($T_{<p}$) For any $q < p$, the homotopies $\Xi^{s_q} : I \times s_q\bigcirc^p \to X$, $\Xi^{t_q} : I \times t_q\bigcirc^p \to X$ are such that their differential at each point has rank $\leq q$.

Remarks 2.3.14. (a) In particular, for a thin homotopy Ξ between Σ and Σ' we have $s_0\Sigma = s_0\Sigma'$ amd $t_0\Sigma = t_0\Sigma'$, as Ξ^{s_0} and Ξ^{t_0} have differentials of rank 0 and so are constant maps.

(b) A requirement equivalent to (T_p) would be tto say that the image of Ξ has Hausdorff dimension $\leq p$, similarly ($T_{<p}$) can be expressed by saying that the images of Ξ^{s_q}, Ξ^{t_q} have Hausdorff dimension $\leq q$.

Note, in particular, that thin homotopies include reparametrizations of branes as well as "p-dimensional cancellations" (preserving the boundary in a compatible way). We also see that being think homotopic is an equivalence relation on $\widetilde{\Pi}_p(X)$, which we denote $\sim$. Define the *set of unparametrized p-branes* in X as the quotient

$$\Pi_p(X) = \widetilde{\Pi}_p(X)/\sim . \tag{2.3.15}$$

Elements of $\Pi_p(X)$ can be thought of as "geometric pieces of p-dimensional surfaces in X", free of the choice of parameters.

E. The 2-groupoid of unparametrized 2-branes. Let X be a C^∞-manifold. The sets of parametrized p-branes in X for $p \leq 2$, organize themselves into a diagram

$$\widetilde{\Pi}_2(X) \underset{t_1}{\overset{s_1}{\rightrightarrows}} \widetilde{\Pi}_1(X) \underset{t_0}{\overset{s_0}{\rightrightarrows}} \widetilde{\Pi}_0(X) = X, \tag{2.3.16}$$

which descents to diagram of sets of unparametrized branes

$$\Pi_2(X) \underset{t_1}{\overset{s_1}{\rightrightarrows}} \Pi_1(X) \underset{t_0}{\overset{s_0}{\rightrightarrows}} \Pi_0(X) = X. \tag{2.3.17}$$

Following [2, 38], we make (2.3.17) into a 2-category, in fact a 2-groupoid $\Pi_{\leq 2}(X)$ with the set of i-morphisms being $\Pi_i(X)$, $i = 0, 1, 2$. For this, we first introduce the composition (2.3.6) on $\widetilde{\Pi}_1(X)$ and two compositions $*_0$ and $*_1$ on $\widetilde{\Pi}_2(X)$, called the *horizontal* and *vertical* composition of 2-branes, defined similarly to (2.3.6) by

(2.3.18)

$$(\Sigma *_0 \Sigma')(a_1, a_2) = \begin{cases} \Sigma'(2a_1, a_2), & \text{if } \ 0 \leq 1 \leq \frac{1}{2} \\ \Sigma(2a_1 - 1, a_2), & \text{if } \ \frac{1}{2} \leq a_1 \leq 1, \end{cases} \qquad \text{when } s_0\Sigma = t_0\Sigma';$$

$$(\Sigma *_1 \Sigma')(a_1, a_2) = \begin{cases} \Sigma'(a_1, 2a_2), & \text{if } \ 0 \leq a_2 \leq \frac{1}{2} \\ \Sigma(a_1, 2a_2 - 1), & \text{if } \ \frac{1}{2} \leq a_2 \leq 1, \end{cases} \qquad \text{when } s_1\Sigma = t_1\Sigma'.$$

These operations do not satisfy the axioms of a 2-category (in particular, they are not associative) but they "descend" to operations on $\Pi_1(X), \Pi_2(X)$ which do. Such descent is completely clear for the composition of paths in $\Pi_1(X)$ and for the $*_0$-composition of thin homotopy classes from $\Pi_2(X)$.

For $*_1$ the descent needs more explanation. Let us denote by $[\Sigma]$ the thin homotopy class of a parametrized p-brane Σ, $p = 1, 2$. Then we need to define $[\Sigma] *_1 [\Sigma']$ for any parametrized 2-branes Σ and Σ' such that the parametrized path $s_1\Sigma$ is thin homotopic (not necessarily equal) to $t_1\Sigma'$. In order to do this, we consider the corresponding thin homotopy as a parametrized 2-brane Ξ with $s_1\Xi = t_1\Sigma'$ and $t_1\Xi = s_1\Sigma$, and define

$$[\Sigma] *_1 [\Sigma'] \;=\; [(\Sigma *_1 \Xi) *_1 \Sigma']. \tag{2.3.19}$$

It is shown in [2, 38] that these composition define a 2-category, in fact a 2-groupoid $\Pi_{\leq 2}(X)$ with the set of i-morphisms being $\Pi_i(X)$, $i = 0, 1, 2$. We will call $\Pi_{\leq 2}(X)$ the *2-groupoid of 2-branes* in X.

2.4 2-dimensonal holonomy.

Here we give a summary of the main points of the theory of connections with values in gerbes (crossed modules) and their 2-dimensional holonomy as developed by Breen and Messing [6] and Baez-Schreiber [2]. Our exposition of the holonomy follows the approach [2] based on the "covariant" generalization of Chen's theory of iterated integrals, as developed in [26] [2]. Additional details of this theory can be found in the Appendix.

A. The Schlessinger formula. Let X be a C^∞-manifold, let G be a Lie group with Lie algebra $\mathfrak{g}$, let Q be a principal G-bundle on X, and ∇ a connection in Q. We denote by $\mathrm{Ad}(Q)$ the bundle of Lie algebras on X associated to Q via the adjoint representation of G. Each fiber $\mathrm{Ad}(Q)_x$ is a Lie algebra is isomorphic (non-canonically) to $\mathfrak{g}$. In a trivialization of Q the connection ∇ is given as $d_{\mathrm{DR}} - A$ for a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_X \otimes \mathfrak{g}$. Denote by $F_\nabla \in \Omega^2_X \otimes \mathrm{Ad}(Q)$ the curvature of ∇. Locally, if ∇ corresponds to A as above, then

$$F_\nabla \;=\; F_A \;=\; dA - \frac{1}{2}[A, A] \;\in\; \Omega^2_X \otimes \mathfrak{g}$$

For any parametrized smooth path $\gamma : [a, b] \to X$ (defined on any interval $[a, b] \subset \mathbb{R}$) we have the holonomy $M_\nabla(\gamma) : Q_{\gamma(a)} \to Q_{\gamma(b)}$. In particular, for any $x, y \in X$ the holonomy gives a smooth function

$$M_\nabla : P^y_x X \longrightarrow \mathrm{Hom}_G(Q_x, Q_y).$$

We denote the logarithmic (group-theoretic) differential of M_∇ by

$$B = M_\nabla^{-1} d_{\mathrm{DR}}(M_\nabla) \ \in \ \Omega^1_{P_x^y X} \otimes \mathrm{Ad}(Q)_x.$$

Then, for any $\delta\gamma$ vanishing near the ends we have the *Schlessinger formula*:

$$(2.4.1) \qquad B(\gamma;\delta\gamma) \ = \ \int_0^1 M_\nabla(\gamma_{\leq t})^{-1} \cdot F_\nabla\big(\gamma(t); \dot{\gamma}(t), \delta\gamma(t)\big) \cdot M_\nabla(\gamma_{\leq t}) dt,$$

where $\gamma_{\leq t} : [0,t] \to X$ is the restriction of γ to $[0,t]$. An equivalent (integrated) formulation of (2.4.1) is that for any smooth path $\sigma : [0,1] \to P_x^y X$ beginning at $\sigma(0) = \gamma$ and ending at $\sigma(1) = \zeta$, the ratio $M_\nabla(\zeta)^{-1} M_\nabla(\gamma)$ can itself be represented as the holonomy of the connection $\frac{d}{ds} - \sigma^*(B)$ on $[0,1]$. This integrated statement was proved by L. Schlessinger in his 1928 paper [42], formula (22), the holonomy of $\frac{d}{ds} - \sigma^*(B)$ being precisely his "gemischtes Doppelintegral".

The Schlessinger formula can be used to give a clear proof of the following well known fact.

Proposition 2.4.2. *Let G be a Lie group, Q a principal G-bundle on X and ∇ a connection in Q. If γ_0 and γ_1 are rank-1 homotopic, then the corresponding holonomies are equal: $M_\nabla(\gamma_0) = M_\nabla(\gamma_1)$. The holonomies of ∇ define then a functor from $\Pi_{\leq 1}(X)$ to the category of G-torsors.*

Proof: The pullback of the curvature 2-form along a rank-1 homotopy $I^2 \to X$ is zero as a 2-form on I^2, so the Schlessinger formula implies the first statement. The second statement is obvious. $\square$

B. Covariant transgression. Let $\beta : G \to \mathrm{Aut}(V)$ be a smooth representation of G in a finite-dimensional $\mathbb{R}$-vector space V, and $V(Q)$ be the vector bundle on X associated to Q. We define the ∇-*transgression* map

(2.4.3)

$$\oint_\nabla : \Omega_X^{m+1} \otimes V(Q) \longrightarrow \Omega^m_{P_x^y X} \otimes V(Q)_x, \quad \left(\oint_\nabla (\Phi)\right)(\gamma; \delta_1\gamma, \cdots, \delta_m\gamma) \ :=$$

$$= \int_0^1 \beta\big(M_\nabla(\gamma_{\leq t})\big)^{-1} \big(\Phi(\gamma(t); \dot{\gamma}(t), \delta_1\gamma(t), \cdots, \delta_m\gamma(t))\big) dt.$$

When Q is the trivial bundle, and $\nabla = d - A$ is given by a 1-form $A \in \Omega^1_X \otimes \mathfrak{g}$, we will write $\oint_A$ for $\oint_\nabla$.

Example 2.4.4. (a) The usual (non-covariant) transgression of (scalar) differential forms

$$\oint : \Omega_X^{m+1} \longrightarrow \Omega^m_{P_x^y X} \tag{2.4.5}$$

is obtained as a particular case, when ∇ is the trivial connection in the trivial bundle and V is the trivial 1-dimensional representation. In this case the action by the holonomy drops out.

(b) As another example, note that Schlessinger's form B above corresponds to the case when $V = \mathfrak{g}$ is the adjoint representation: $B = \oint_\nabla(F_\nabla)$.

When it is important to emphasize the dependence of the covariantly transgressed form on the points $x, y \in X$, we will use the notation

$$\left(\oint_\nabla (\Phi)\right)_x^y \in \Omega^m_{P_x^y X} \otimes V(Q)_x.$$

We will not discuss the relations among such forms for different $x, y \in X$. More precisely, let $x, y, z \in X$ be three points. We then have the composition map

$$\mathrm{com} : P_y^z X \times P_x^y X \longrightarrow P_x^z X, \quad \mathrm{com}(\gamma, \gamma')(t) = \begin{cases} \gamma'(2t), & \text{if } t \in [0, \frac{1}{2}], \\ \gamma(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

For $\gamma \in P_y^X$ we have the *left translation map*

$$l_\gamma : P_x^y X \longrightarrow p_x^z X, \quad r_\gamma(\gamma') = \mathrm{com}(\gamma, \gamma').$$

For $\gamma' \in P_x^y X$ we have the *right translation map*

$$r_{\gamma'} : P_y^x Z \longrightarrow P_x^z X, \quad l_{\gamma'}(\gamma) = \mathrm{com}(\gamma, \gamma').$$

The following "translation invariance" property is a direct consequence of the definitions.

Proposition 2.4.6. *(a) For each* $\gamma \in P_y^z X$ *we have*

$$r_\gamma^* \left(\oint_\nabla (\Phi)\right)_x^z = \left(\oint_\nabla (\Phi)\right)_x^y.$$

(b) For each $\gamma' \in P_x^y X$ *we have*

$$l_\gamma^* \left(\oint_\nabla (\Phi)\right)_x^z = M_\nabla(\gamma)^{-1} \left(\oint_\nabla (\Phi)\right)_y^z$$

(the result of applying the isomorphism $M_\nabla(\gamma)^{-1} : V(Q)_y \to V(Q)_x$ *to an* $V(Q)_y$*-valued* m*-form).* □

C. Connections with values in crossed modules and on the space of paths. Let now $G^\bullet$ be a crossed module of Lie groups and $\mathfrak{g}^\bullet$ the corresponding crossed module of Lie algebras.

We first consider $\mathfrak{g}^\bullet$ as a dg-Lie algebra situated in degrees $-1, 0$. Let X be a C^∞-manifold and and let $A^\bullet \in (\Omega^\bullet_X \otimes \mathfrak{g}^\bullet)^1$ be a graded connection on X with values in $\mathfrak{g}^\bullet$, i.e., a $\mathfrak{g}^\bullet$-valued differential form of total degree 1, see §1.2. Thus $A^\bullet$ has two components:

$$A^1 \in \Omega^1_X \otimes \mathfrak{g}^0, \quad A^2 \in \Omega^2_X \otimes \mathfrak{g}^{-1}.$$

We can consider A^1 as a usual connection in the trivial G^0-bundle on X. Let $F^\bullet$ be the curvature of the graded connection $A^\bullet$, see (1.2.1). Thus $F^\bullet$ has two components:

(2.4.7) $F^2 = F_{A^1} - d_{\mathfrak{g}^\bullet}(A^2) \in \Omega^2_X \otimes \mathfrak{g}^0, \;\; F^3 = d_{\mathrm{DR}}(A^2) - [A^1, A^2] \in \Omega^3_X \otimes \mathfrak{g}^{-1}.$

Now consider $\mathfrak{g}^\bullet$ as a crossed module of Lie algebras, as it was originally defined. From this point of view, $A^\bullet$ is a 2-connection in the trivial $G^\bullet$-gerbe [6] on X. The component F^2 is traditionally called the *fake curvature*, while $F^3 = \nabla_{A^1}(A^2)$ is called the *3-curvature* of $A^\bullet$, see [6]. We say that the 2-connection $A^\bullet$ is *semiflat*, if the fake curvature vanishes: $F^2 = 0$.

Fixing points $x, y \in X$, we associate to $A^\bullet$ a usual connection in the trivial G^{-1}-bundle on the space $P_x^y X$ given by the $\mathfrak{g}^{-1}$-valued 1-form

$$\oint_{A^1} (A^2) \;\in\; \Omega^1_{P_x^y X} \otimes \mathfrak{g}^{-1}.$$

We have the following fundamental fact, see [2], formula (2.57).

Proposition 2.4.8. *If the 2-connection $A^\bullet$ is semiflat, then the curvature of $\oint_{A^1}(A^2)$ is equal to $\oint_{A^1}(F^3)$.*

For convenience of the reader we give here a proof, referring to the Appendix for necessary background. By defiiniton, the curvature in question is equal to

$$d_{\mathrm{DR}} \oint_{A^1} (A^2) \;-\; \frac{1}{2}\left[\oint_{A^1} (A^2), \oint_{A^1} (A^2)\right].$$

Note first that the commutator in the second term vanishes. Indeed, by definition, this is the commutator of two integrals over $[0,1]$, which is expressible as an integral over $[0,1]^2$ of the pointwise commutators of the two integrands. This new integrand is a function on $[0,1]^2$ which is antisymmetric with respect to the interchange of the two variables, so the integral over $[0,1]^2$ vanishes.

Second, we recall Example A.3.5 which implies that we can apply Proposition A.3.6(b) to calculate the exterior derivative of the covariant transgression. It gives that the first term in the above formula for the curvature is equal to

(2.4.9)

$$d_{\mathrm{DR}} \oint_{A^1} (A^2) \;=\; \oint_{A^1} (\nabla_{A^1}(A^2)) \;+\; \oint_{A^1} (F_{A^1}, A^2) \;-\; \oint_{A^1} (A^2, F_{A^1}) \;=$$
$$= \oint_{A^1} (F^3) \;+\; \left[\oint_{A^1} (F_{A^1}),\; \oint_{A^1} (A^2) \right],$$

where the last term is the wedge product of two 1-forms on $P_x^y X$ followed by the commutator, i.e., action of $\alpha : \mathfrak{g}^0 \otimes \mathfrak{g}^{-1} \to \mathfrak{g}^{-1}$. Now, the condition of semiflatness $F_{A^1} = d_{\mathfrak{g}}(A^2)$ implies that the last wedge commutator above is equal to

$$\left[d_{\mathfrak{g}} \oint_{A^1} (A^2),\; \oint_{A^1} (A^2) \right],$$

which is zero since for each $y \in \mathfrak{g}^{-1}$ we have $\alpha(d_{\mathfrak{g}}(y))(y) = d_{\mathfrak{g}}[y,y] = 0$. $\square$

F. 2-dimensional holonomy. Let $G^\bullet = \{G^{-1} \xrightarrow{\partial} G^0\}$ be a crossed module of Lie groups, $\mathfrak{g}^\bullet$ the corresponding crossed module of Lie algebras, and $A^\bullet = (A^1, A^2)$ be a connection on X with coefficients in $\mathfrak{g}^\bullet$, as in §D. Let Σ be a parametrized 2-brane in X. The *2-dimensional holonomy* of $A^\bullet$ along Σ is defined as

$$M_{A^\bullet}(\Sigma) \;=\; M_{\oint_{A^1}(A^2)}(\sigma) \;\in\; G^{-1}. \tag{2.4.10}$$

Here $\sigma : [0,1] \to P_x^y X$ is the path in the space of paths corresponding to Σ, and $\oint_{A^1}(A^2)$ is the connection on $P_x^y X$ obtained as the covariant transgression of A^2 with respect to A^1. The following is the main result of [2] (specialized for the case of 2-connections in trivial 2-bundles).

Theorem 2.4.11. *Assume that $A^\bullet$ is semiflat. Then:*

(a) $M_{A^\bullet}(\Sigma))$ represents a 2-morphism in $2\,\mathrm{Cat}(G^\bullet)$ from $M_{A^1}(\partial_0\Sigma)$ to $M_{A^1}(\partial_1\Sigma)$, i.e.,

$$\partial(M_{A^\bullet}(\Sigma)) \;=\; M_{A^1}(\partial_1\Sigma) \cdot M_{A^1}(\partial_0\Sigma)^{-1} \in G^0.$$

(b) $M_{A^\bullet}(\Sigma)$ is unchanged if Σ is replaced by a thin homotopic 2-brane.
(c) The correspondence

$$x \in X = \Pi_0(X) \longmapsto pt \in \mathrm{Ob}\, 2\,\mathrm{Cat}(G^\bullet),$$
$$\gamma \in \Pi_1(X) \longmapsto M_{A^1}(\gamma) \in G^0 = 1\,\mathrm{Mor}(2\,\mathrm{Cat}(G^\bullet))$$
$$\Sigma \in \Pi_2(X) \longmapsto M_{A^\bullet}(\Sigma) \in 2\,\mathrm{Hom}_{2\,\mathrm{Cat}(G^\bullet))}(M_{A^1}(\partial_0\Sigma), M_{A^1}(\partial_1\Sigma)$$

defines a 2-functor $\mathbb{M}_{A^\bullet} : \Pi_{\leq 2}(X) \to 2\,\mathrm{Cat}(G^\bullet)$.

Proof: (a) The condition of semiflatness, i.e., of vanishing of the fake curvature, means that $d_{\mathfrak{g}^\bullet}(A^2) = F_{A^1}$. Therefore

$$\partial\big(M_{A^\bullet}(\Sigma)\big) = \partial\bigg(M_{\oint_{A^1}(A^2)}(\sigma)\bigg) = M_{\oint_{A^1}(F_{A^1})}(\sigma) = M_B(\sigma),$$

the last equality being the Schlessinger formula. So the statement follows from the fact that B is the logarithmic differential of M_{A^1}.

(b) For a thin homotopy $[0,1]^3 \to S \to X$ between two unparametrized 2-branes the pullback of the 3-curvature F^3 is the zero 3-form on $[0,1]^3$. So the covariant transgression of this pullback with respect to the pullback of A^1 is zero as well. Our statement now follows from Proposition 2.4.8.

(c) We need to prove compatibility of $\mathbb{M}_{A^\bullet}$ with the two compositions $*_0, *_1$. Compatibility with $*_1$ follows from compatibility of the holonomy of usual connections with composition of paths. Indeed, if σ, σ' are the paths in $P_x^y X$ corresponding to parametrized membranes Σ, Σ' s.t. $\partial_0\Sigma = \partial_1\Sigma'$, then $\Sigma *_1 \Sigma'$ corresponds to the composition of σ and σ'.

Let us now prove that $\mathbb{M}_{A^\bullet}(\Sigma *_0 \Sigma') = \mathbb{M}_{A^\bullet}(\Sigma) *_0 \mathbb{M}_{A^\bullet}(\Sigma')$ whenever Σ, Σ' are $*_0$-composable. For this it is enough to assume that either Σ or Σ' is a 1-morphism, i.e., a path considered as a membrane. Indeed, in any 2-category the $*_0$-composition of any two 2-morphisms can be expressed using $*_1$-composition of 2-morphisms as well as $*_0$-composition involving a 2-morphism and a 1-morphism.

We now recall the definition of the $*_0$-composition of a 1-morphism and a 2-morphism in $2\,\mathrm{Cat}(G^\bullet)$. Since $2\,\mathrm{Cat}(G^\bullet)$ has one object, any 1-morphism and 2-morphism can be $*_0$-composed in either order. Now, 1-morphisms are identified with elements of G^0, and 2-morphisms with triples $T = (g_0, g_1, h)$ with $g_i \in G^0, h \in G^{-1}$ such that $\partial(h) = g_0^{-1}g_1$. Such a T is a 2-morphism

$T : g_0 \Rightarrow g_1$. With these conventions, if $u \in G^0$, then

$$
\begin{aligned}
u *_0 T &= (ug_0, ug_1, h) : ug_0 \Rightarrow ug_1, \\
T *_0 u &= \big(g_0 u, g_1 u, \beta(u^{-1})(h)\big) : g_0 u \Rightarrow g_1 u.
\end{aligned}
\tag{2.4.12}
$$

Let now assume that $\Sigma = \gamma \in P_y^z X$ is a path, and Σ' is a parametrized 2-brane corresponding to a path $\sigma' : [0,1] \to P_x^y X$. So we denote by $\sigma'_s = \Sigma(-, s) \in P_x^y X$ the path corresponding to $s \in [0,1]$. Then $\gamma *_0 \Sigma'$ corresponds to a path in $[0,1] \to P_x^z X$ which takes $s \in [0,1]$ into the composition $\gamma * \sigma'_s$, so σ'_s is run first and then followed by γ. This means that $\big(\oint_{A^1}(A^2)\big)_{\gamma * \sigma'_s}$, the covariant transgression evaluated at $\gamma *_0 \sigma'_s$ is equal to $\big(\oint_{A^1}(A^2)\big)_{\sigma'_s}$ the result of evaluation along σ'_s alone. Indeed, the integral in (2.4.3) splits into the sum of two integrals corresponding to σ'_s and γ, and the second integral is equal to 0 because the pullback of the 2-form A^2 to the interval along γ vanishes. So $\mathbb{M}_{A^\bullet}(\gamma *_0 \Sigma')$ as an element of G^{-1} is equal to $M_{A^\bullet}(\Sigma')$, in agreement with the first formula in (2.4.12).

Assume now that $\Sigma' = \gamma' \in P_x^y X$ is a path, and Σ is a parametrized 2-brane corresponding to a path $\sigma : [0,1] \to P_y^z X$. Then $\Sigma *_0 \gamma'$ corresponds to a path in $[0,1] \to P_x^z X$ which takes $s \in [0,1]$ into the composition $\sigma_s * \gamma'$, so γ' is run first and then followed by σ_s. As before, the integral for $\big(\oint_{A^1}(A^2)\big)_{\sigma_s * \gamma'}$ splits into two integrals, one corresponding to σ_s, the other to γ. As before, the integral corresponding to γ' is equal to 0. But since σ_s now follows γ in the order of integration, the integral corresponding to σ_s will be equal to the conjugation of $\big(\oint_{A^1}(A^2)\big)_{\sigma_s}$ by $M_{A^0}(\gamma')$, the holonomy of the connection A^0 along γ'. This agrees exactly with the second formula in (2.4.12). $\square$

2.5 2-dimensional holonomy and the crossed module of formal 2-branes.

Consider the dg-Lie algebra $\mathfrak{f}^\bullet(\mathbb{R}^n)_{\mathrm{CM}}$ defined as in (1.3.11). It is situated in degrees $[-1, 0]$. We will denote this dg-Lie algebra as well as the corresponding crossed module of Lie algebras, by $\mathfrak{g}_n^{\geq -1}$. The notation is chosen to indicate that $\mathfrak{g}_n^{\geq -1}$ is the truncation of a longer (and more fundamental) crossed complex of Lie algebras, see §3.4. Here is a concise summary of the properties of $\mathfrak{g}_n^{\geq -1}$.

Proposition 2.5.1. *(a) As a dg-Lie algebra, $\mathfrak{g}_n^{\geq -1}$ is is generated by symbols $Z_i, i = 1, ..., n$, in degree 0 and Z_{ij}, $1 \leq i < j \leq n$ in degree -1, subject to* *the relations:*

$$d(Z_{ij}) = [Z_i, Z_j], \quad [Z_{ij}, Z_{pq}] = 0, \quad [[Z_i, Z_j], Z_{pq}] = [Z_{ij}, [Z_p, Z_q]].$$

(b) The degree 0 part $\mathfrak{g}_n^0$ is the free Lie algebra $\mathrm{FL}(Z_1, ..., Z_n)$. As a Lie algebra with the bracket $[-,-]_{-1}$, the degree -1 part $\mathfrak{g}_n^{-1}$ is isomorphic to a direct product of a free Lie algebra (isomorphic to $[\mathfrak{g}_n^0, \mathfrak{g}_n^0]$) and an abelian Lie algebra isomorphic to $\Gamma_2^{\mathrm{cl}}(\mathbb{R}^n)$.

(c) In particular, $H^0(\mathfrak{g}_n^{\geq -1}) = \mathbb{R}^n$ (the abelian Lie algebra), while $H^{-1}(\mathfrak{g}_n^{\geq -1}) = \Gamma_2^{\mathrm{cl}}(\mathbb{R}^n)$.

Proof: (a) By definition, see (1.3.11), $\mathfrak{g}_n^{\geq -1}$ is generated by the Z_i, Z_{ij} as stated, with the differential defined as stated, but then quotiented out by all elements of degrees ≤ -2 as well as by l elements of the form $[dx, y] - [x, dy]$ for x, y arbitrary elements of degree -1. Quotienting by all elements of degrees ≤ -2 is equivalent to imposing the second series of relations in (a): that $[Z_{ij}, Z_{pq}] = 0$. Further, the last series of relations in (a) amounts to imposing the conditions $[dx, y] - [x, dy] = 0$ for $x = Z_{ij}$, $y = Z_{pq}$ being the degree -1 generators. Now, the Jacobi identity implies that any other element of the form $[dx, y] - [x, dy]$ will belong to the dg-Lie ideal generated by the above particular elements, so defining $\mathfrak{g}_n^{\geq -1}$ by the relations as in (a), has the same effect as (1.3.11).

(b) This was established in the course of the proof of Theorem 1.5.10, see Eq. (1.6.6).

(c) follows from (b). □

Firther, the crossed module of Lie algebras $\mathfrak{g}_n^{\geq -1}$ carries the universal translation invariant semiflat connection $A^\bullet$ with

$$A^1 = \sum_i Z_i dt_i, \quad A^2 = \sum_{i<j} Z_{ij} dt_i dt_j.$$

We denote by $\mathfrak{g}_{n,d}^{\geq -1}$ the quotient of $\mathfrak{f}^\bullet(\mathbb{R}^n)_{\mathrm{CM}}$ by the $(d+1)$th term of the lower central series. Thus $\mathfrak{g}_{n,d}^0 = \mathrm{FL}(\mathbb{R}^n)/\,\mathrm{FL}_{\geq d+1}(\mathbb{R}^n)$ is the free degree d nilpotent Lie algebra on n generators. We regard $\mathfrak{g}_{n,d}^{\geq -1}$ as a nilpotent crossed module of finite-dimensional Lie algebras.

Example 2.5.2. Let $d = 2$. Then $\mathfrak{g}_{n,2}^0$ is the "universal Heisenberg Lie algebra" generated by $Z_1, ..., Z_n$ such that $\hbar_{ij} := [Z_i, Z_j]$ are linearly independent central elements whose span is naturally identified with $\Lambda^2(\mathbb{R}^n)$. Further, the

Lie algebra $\mathfrak{g}_{n,2}^{-1}$ has a basis formed by the $Z_{ij}, i < j$, spanning $\Lambda^2(\mathbb{R}^n)$, and the $[Z_i, [Z_{pq}]]$, spanning $\mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)$. The bracket $[-,-]_{-1}$ on $\mathfrak{g}_{n,2}^{-1}$ vanishes, while the differential sends Z_{ij} to $\hbar_{ij}$. Thus $\mathrm{Ker}\{d : \mathfrak{g}_{n,2}^{-1} \to \mathfrak{g}_{n,2}^{0}\}$ is identified with $\mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)$.

Note the difference with the full dg-Lie algebra $\mathfrak{g}_n^{\geq -1}$, in which

$$d[Z_i, Z_{pq}] = [Z_i, [Z_p, Z_q]] \neq 0.$$

In this case we need to recall that the space spanned by the $[Z_i, [Z_{pq}]]$, splits, $GL_n(\mathbb{R})$-equivariantly as

$$\mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n) \ = \ \Lambda^3(\mathbb{R}^n) \oplus \Sigma^{2,1}(\mathbb{R}^n)$$

of which the second summand maps by d isomorphically to $\mathrm{FL}_3(\mathbb{R}^n) \subset \mathfrak{g}_n^0$ and the first summand is a part of $\mathrm{Ker}(d) = \Gamma_2^{\mathrm{cl}}(\mathbb{R}^n)$.

Let $G_{n,d}^{\geq -1} = \exp(\mathfrak{g}_{n,d}^{\geq -1})$ be the crossed module of nilpotent Lie group corresponding to $\mathfrak{g}_{n,d}^{\geq -1}$. We have then the 2-dimensional holonomy functor

$$\mathbb{M}_{n,d}^{\leq 2} : \ \Pi_{\leq 2}(\mathbb{R}^n) \longrightarrow 2\,\mathrm{Cat}(G_{n,d}^{\geq -1}) \tag{2.5.3}$$

associated to the tautological semiflat connection with values in $\mathfrak{g}^{\geq -1}$. Consider the projective limit crossed module $\widehat{G}_n^{\geq -1} = \varprojlim G_{n,d}^{\geq -1}$. We call $\widehat{G}_n^{\geq -1}$ the *crossed module of formal 2-branes* in $\mathbb{R}^n$. By construction, the zero part $\widehat{G}_n^0$ is the free prounipotent group over $\mathbb{R}$ with n generators.

Proposition 2.5.4. *We have $\pi_0(\widehat{G}_n^{\geq -1}) = \mathbb{R}^n$, while $\pi_1(\widehat{G}_n^{\geq -1}) = (\Omega^{2,\mathrm{cl}}(\mathbb{R}^n))^*$ is the algebraic dual of the space of closed polynomial 2-forms in $\mathbb{R}^n$.*

Proof: This follows from Proposition 2.5.1(c), as $(\Omega^{2,\mathrm{cl}}(\mathbb{R}^n))^*$ is the pro-finite-dimensional completion of $\Gamma_2^{\mathrm{cl}}(\mathbb{R}^n)$.

$\square$

By passing to the limit we get the functor

$$\widehat{\mathbb{M}}_n^{\leq 2} = \varprojlim_d \mathbb{M}_{n,d}^{\leq 2} : \Pi_{\leq 2}(\mathbb{R}^n) \longrightarrow 2\,\mathrm{Cat}(\widehat{G}_n^{\geq -1}). \tag{2.5.5}$$

This functor is translation invariant: a parallel shift of a membrane does not affect the value of the functor $\widehat{\mathbb{M}}_n^{\leq 2}$. On the level of 1-morphisms, $\widehat{\mathbb{M}}_n^{\leq 2}$ associates to each unparametrized path γ in $\mathbb{R}^n$, the noncommutative power series $E_\gamma(Z_1, ..., Z_n)$ from (0.2), which is a group-like element of $\mathbb{R}\langle\langle Z_1, ..., Z_n\rangle\rangle$ and so is an element of $\widehat{G}_n^0$.

Question 2.5.6. K.-T. Chen proved in [13] that the correspondence $\gamma \mapsto E_\gamma$ is faithful modulo translations. That is, for two piecewise smooth paths γ and γ' the equality $E_\gamma = E_{\gamma'}$ in $\widehat{G}^0_n$ implies that γ' differs from γ only by a rank 1 homotopy (reparametrization + cancellations) and an overall translation in $\mathbb{R}^n$. Can one generalize this to the functor $\widehat{\mathbb{M}}^{\leq 2}_n$?

3 The crossed complex of formal n-branes.

3.1 Crossed complexes of groups and Lie algebras

A. Groups. The following definition is taken from [7]; we restrict to the case of crossed complexes consisting of groups and not groupoids, as in [7].

Definition 3.1.1. A crossed complex (of groups) is a sequence of groups and homomorphisms

$$G^\bullet = \left\{ \cdots \xrightarrow{\partial_{-3}} G^{-2} \xrightarrow{\partial_{-2}} G^{-1} \xrightarrow{\partial_{-1}} G^0 \right\},$$

equipped with an action of G^0 on each G^i, via $\beta_i : G^0 \to \mathrm{Aut}(G^i)$ satisfying the following properties:
(a) The part $\left\{G^{-1} \xrightarrow{\partial_{-1}} G^0\right\}$ is a crossed module of groups; in particular, β_0 is the action of G^0 on itself by conjugation.

(b) For $i \leq -2$ the group G^i is abelian, and the action β_i factors through an action of $H^0(G^\bullet) = \mathrm{Coker}(\partial_{-1})$ on G^i.

(c) The maps ∂_i commute with the action of G^0.

(d) The composition $\partial_i \partial_{i-1}$ is the trivial homomorphism.

Note that for any crossed complex $G^\bullet$ and any $i \leq 0$ the truncation $G^{\geq i}$ is still a crossed complex (a crossed module, if $i = -1$).

Example 3.1.2. (a) Let

$$X_\bullet = \left\{ x_0 \in X_1 \subset X_2 \subset ... \right\}$$

be a filtered topological space, with x_0 being a point. Then setting $G^{-i} = \pi_i(X_i, X_{i-1}, x_0)$ to be the relative homotopy groups of X_i modulo X_{i-1} and ∂_{-i} to be the boundary map on the relative homotopy groups, we get a crossed complex. We denote this crossed complex by $G^\bullet(X_\bullet)$. See [7].

By considering crossed complexes of groupoids [7] one can generalize this example to the case of an arbitrary filtered space.

(b) In particular, let $X = \mathbb{T}^n$ be the n-dimensional torus with the cubical cell decomposition, as in Example 2.1.11. We denote by $\mathbb{T}^n_{\leq i}$ the i-skeleton of this decomposition. This gives a filtered CW-complex which we denote $\mathbb{T}^n_\bullet$. We will call $G^\bullet(\mathbb{T}^n_\bullet)$ the *crossed complex of cubical branes in* $\mathbb{R}^n$. Its truncation in degrees $[-1, 0]$ is the crossed module discussed in Example 2.1.11.

B. Lie algebras. One easily extends the concept of a crossed complex to the Lie algebra case.

Definition 3.1.3. A crossed complex of Lie k-algebras consists of a sequence of Lie k-algebras and homomorphisms

$$\mathfrak{g}^\bullet \quad = \quad \left\{ \cdots \xrightarrow{d_{-3}} \mathfrak{g}^{-2} \xrightarrow{d_{-2}} \mathfrak{g}^{-1} \xrightarrow{d_{-1}} \mathfrak{g}^0 \right\}$$

equipped with an action of $\mathfrak{g}^0$ on each $\mathfrak{g}^i$, via $\alpha_i : \mathfrak{g}^0 \to \mathrm{Der}(\mathfrak{g}^i)$ satisfying the following properties:
(a) The part $\left\{ \mathfrak{g}^{-1} \xrightarrow{d_{-1}} \mathfrak{g}^0 \right\}$ is a crossed module of Lie algebras; in particular, α_0 is the adjoint action of $\mathfrak{g}^0$ on itself.
(b) For $i \leq -2$ the Lie algebra $\mathfrak{g}^i$ is abelian, and the action α_i factors through an action of $H^0(\mathfrak{g}^\bullet) = \mathrm{Coker}(d_{-1})$ on $\mathfrak{g}^i$.
(c) The maps d_i commute with the action of $\mathfrak{g}^0$.
(d) The composition $d_i d_{i-1}$ is the zero homomorphism.

Propositions 1.3.1 and 1.3.6 easily imply the following.

Proposition 3.1.4. *Let $\mathfrak{g}^\bullet$ is a crossed complex of Lie algebras. Then defining the bracket $[x, y] \in \mathfrak{g}^{-i-j}$ for $x \in \mathfrak{g}^{-i}, y \in \mathfrak{g}^{-j}$ by*

$$[x, y] = \alpha_j(x)(y), \ i = 0, \quad [x, y] = -\alpha_i(y)(x), \ j = 0, \quad [x, y] = 0, \ i, j < 0,$$

we make $\mathfrak{g}^\bullet$ into a semiabelian dg-Lie algebra. This correspondence establishes an equivalence between crossed complexes of Lie algebras on the one hand and semiabelian dg-Lie algebras on the other hand. □

In particular, Example 3.1.2 (a) is analogous to the Lie-algebraic Example 1.3.7.

Corollary 3.1.5. *Let $G^\bullet$ be a crossed complex of Lie groups, and $\mathfrak{g}^i = \mathrm{Lie}(G^i)$. Then $\mathfrak{g}^\bullet$ is a crossed complex of Lie $\mathbb{R}$-algebras and thus has a natural structure of a semiabelian dg-Lie $\mathbb{R}$-algebra.* □

C. Lower central series. Let $G^\bullet$ be a crossed complex of groups, with operations written multiplicatively. Its *lower central series* consists of crossed subcomplexes

$$
\begin{gathered}
G^\bullet = \gamma_1(G^\bullet) \supset \gamma_2(G^\bullet) \supset \cdots \\
\gamma_r(G^\bullet) \;=\; \bigl\{\cdots \longrightarrow \gamma_r(G^0, G^{-2}) \longrightarrow \gamma_r(G^0, G^{-1}) \longrightarrow \gamma_r(G^0)\bigr\}.
\end{gathered} \tag{3.1.6}
$$

Here $\gamma_r(G^0, G^i) \subset G^i$ is defined, for $i = -1$, by (2.2.4) and (2.2.5) and for $i \leq -2$ it is defined as the subgroup in the abelian group G^i generated by elements of the form $\beta(z)(x) \cdot x^{-1}$ for $z \in G^0$ and $x \in \gamma_r(G^0, G^i)$. We say that $G^\bullet$ is *nilpotent*, if $\gamma_r(G^\bullet) = 1$ (consists of 1-element groups) for some r.

Similarly, let $\mathfrak{g}^\bullet$ be a crossed complex of Lie k-algebras, Its *lower central series* consists of crossed subcomplexes of Lie algebras

$$
\begin{gathered}
\mathfrak{g}^\bullet = \gamma_1(\mathfrak{g}^\bullet) \supset \gamma_2(\mathfrak{g}^\bullet) \supset \cdots \\
\gamma_r(\mathfrak{g}^\bullet) \;=\; \bigl\{\cdots \longrightarrow \gamma_r(\mathfrak{g}^0, \mathfrak{g}^{-2}) \longrightarrow \gamma_r(\mathfrak{g}^0, \mathfrak{g}^{-1}) \longrightarrow \gamma_r(\mathfrak{g}^0)\bigr\}.
\end{gathered} \tag{3.1.7}
$$

Here the $\gamma_r(\mathfrak{g}^0, \mathfrak{g}^i) \subset \mathfrak{g}^i$ are defined, generalizing (2.2.8) from the case $i = -1$, by

$$
\gamma_{r+1}(\mathfrak{g}^0, \mathfrak{g}^i) \;=\; \alpha(\mathfrak{g}^0)\bigl(\gamma_r(\mathfrak{g}^0, \mathfrak{g}^i)\bigr). \tag{3.1.8}
$$

Proposition 3.1.9. *The dg-Lie algebra corresponding to the crossed complex $\gamma_r(\mathfrak{g}^\bullet)$ is equal to the rth term of the lower central series of the dg-Lie algebra corresponding to $\mathfrak{g}^\bullet$.* □

We say that crossed complex of Lie algebras $\mathfrak{g}^\bullet$ is *nilpotent*, if $\gamma_r(\mathfrak{g}^\bullet) = 0$ for some r, i.e., if $\mathfrak{g}^\bullet$ is nilpotent as a dg-Lie algebra. As in the case of crossed modules (Proposition 2.2.10), the Engel theorem implies the following:

Proposition 3.1.10. *Let $\mathfrak{g}^\bullet$ be a finite crossed complex of finite-dimensional Lie algebras. Then $\mathfrak{g}^\bullet$ is nilpotent, if and only if $\mathfrak{g}^0$ is a nilpotent Lie algebra.* □

D. Malcev theory for crossed complexes. Let $\mathfrak{g}^\bullet$ be a finite crossed complex of finite-dimensional Lie algebras, or, what is the same, a finite-dimensional semiabelian dg-Lie algebra over k. Assume that $\mathfrak{g}^\bullet$ is nilpotent. Applying the Malcev theory to each nilpotent Lie algebra $\mathfrak{g}^i$, we get a sequence of groups and homomorphisms

$$
\exp(\mathfrak{g}^\bullet) \;=\; \bigl\{\cdots \longrightarrow \exp(\mathfrak{g}^{-2}) \xrightarrow{\exp(d_{-2})} \exp(\mathfrak{g}^{-1}) \xrightarrow{\exp(d_{-1})} \exp(\mathfrak{g}^0)\bigr\}. \tag{3.1.11}
$$

Exponentiating the action of $\mathfrak{g}^0$ on $\mathfrak{g}^i$ by derivations, we get an action of $\exp(\mathfrak{g}^0)$ on $\exp(\mathfrak{g}^i)$ by automorphisms, and we see easily:

Proposition 3.1.12. *With the actions described,* $\exp(\mathfrak{g}^\bullet)$ *is a nilpotent crossed complex of algebraic groups over k. We have, therefore, three equivalent categories, with equivalence between (ii) and (iii) given by the functor* exp*:*

(i) Finite-dimensional, nilpotent, semiabelian dg-Lie k-algebras;

(ii) Finite, nilpotent crossed complexes of finite-dimensional Lie k-algebras.

(iii) Finite, nilpotent crossed complexes of algebraic groups over k. □

Remark 3.1.13. In [21], E. Getzler associated a group-like object to any nilpotent dg-Lie algebra situated in degrees ≤ 0. However, in general, his construction is far from being reduced to some data of algebraic groups and their homomorphisms. It is the assumption of being semiabelian which allows for such a reduction.

3.2 Strict n-categories and n-groupoids.

We recall the concept of (small, strict, globular) n-categories, see [44] for more background.

A. Globular sets.

Definition 3.2.1. A globular set $Y_\bullet$ consists of sets $Y_i, i \geq 0$ and maps $s_i, t_i : Y_{i+1} \to Y_i$, $\mathbf{1}_i : Y_i \to Y_{i+1}$ satisfying the identities:

$$s_i s_{i+1} = s_i t_{i+1}, \quad t_i s_{i+1} = t_i t_{i+1}, \quad s_i \mathbf{1}_i = t_i \mathbf{1}_i = \mathrm{id}\,.$$

Elements of Y_i are called *i-cells* of $Y_\bullet$, the maps s_i and t_i are called the *i-dimensional source and target maps* for $(i+1)$-cells, and $\mathbf{1}_i$ is called the *i-dimensional identity map.* For a globular set Y one defines the i-dimensional source and target maps for j-cells, $j \geq i$, by

$$s_i = s_i^j = s_i s_{i+1} \ldots s_{j-1}, \quad t_i = t_i^j = t_i t_{i+1} \ldots t_{j-1} : C_j \to C_i, \tag{3.2.2}$$

and the iterated identity maps

$$\mathbf{1}_i^j = \mathbf{1}_{j-1}\mathbf{1}_{j-2}\ldots\mathbf{1}_i : Y_i \to Y_j, \quad i \leq j. \tag{3.2.3}$$

In particular, for $i = j$ we set $s_i^i = t_i^i = \mathbf{1}_i^i = \mathrm{Id}$.

By an *n-globular set* we will mean a datum as in Definition 3.2.1 but with Y_i defined only for $i \leq n$. An equivalent point of view is to say that an n-globular set is a globular set $Y_\bullet$ with all the $Y_j, j \geq n$, identified with Y_n via $\mathbf{1}_n^j$.

Example 3.2.4. (a) Let $\widetilde{\Pi}_n(X)$ be the set of parametrized n- branes in X, see Definition 2.3.10. The sets $\widetilde{\Pi}_n(X)$, $n \geq 0$, give rise to a globular set $\widetilde{\Pi}_\bullet(X)$ via the maps $s_i, t_i, \mathbf{1}_i$ defined before (2.3.8). That is

$$\begin{aligned} (s_{n-1}\Sigma)(a_1, ..., a_{n-1}) &= \Sigma(a_1, ..., a_{n-1}, 0), \\ (t_{n-1}\Sigma)(a_1, ..., a_{n-1}) &= \Sigma(a_1, ..., a_{n-1}, 1), \\ (\mathbf{1}_n\Sigma)(a_1, ..., a_{n+1}) &= \Sigma(a_1, ..., a_n), \\ \Sigma \in \widetilde{\Pi}_n(X), &\quad a_i \in I = [0, 1]. \end{aligned}$$

In particular, the points given by the images of $\{0\} \times I^{n-1}$ and $\{1\} \times I^{n-1}$ are, in this notation, $s_0\Sigma$ and $t_0\Sigma$, the 0-dimensional source and target of Σ.

Note that a differential n-form on X can be integrated over a parametrized n-brane:

$$\int_\Sigma \omega \;:=\; \int_{I^n} \Sigma^*(\omega). \tag{3.2.5}$$

(b) For $x, y \in X$ let $\widetilde{\Pi}_\bullet(X)_x^y$ be the globular subset in $\widetilde{\Pi}_\bullet(X)$ consisting of Σ such that $s_0\Sigma = x$ and $t_0\Sigma = y$. Each parametrized n-brane $\Sigma \in \widetilde{\Pi}_n(X)_x^y$ gives rise to a parametrized $(n-1)$-brane

$$\tau(\Sigma) \in \widetilde{\Pi}_{n-1}(P_x^y X)$$

in the path space, called the *transgression* of Σ. Explicitly, $\tau(\Sigma)(a_1, ..., a_{n-1})$ is the path $I \to X$ sending a into $\Sigma(a, a_1, ..., a_{n-1})$. This operation is compatible with transgression (2.4.5) of differential forms: if $\omega \in \Omega_X^n$ is an n-form on X, then

$$\int_{\tau\Sigma} \oint(\omega) \;=\; \int_\Sigma \omega. \tag{3.2.6}$$

B. n-categories, n-groupoids and crossed complexes.

Definition 3.2.7. An n-category is an n-globular set C equipped with composition maps

$$*_i : C_j \times_{s_i,t_i} C_j \to C_j.$$

These compositions should satisfy the following properties:
(a) Associativity.
(b) j-morphisms from the image of $\mathbf{1}_i^j$ are units for $*_i$.
(c) The compositions are compatible with the source and target maps:

$$s_i(u *_j v) = s_i(u) *_j s_i(v), \quad t_i(u *_j v) = t_i(u) *_j t_i(v).$$

(d) 2-dimensional associativity:

$$(x *_i y) *_j (z *_i t) = (x *_j z) *_i (y *_j t).$$

Elements of C_i for an n-category C are commonly called *i-morphisms* of C. For $i = 0$ we will call 0-morphisms simply *objects*.

One can also define n-categories inductively, as small categories enriched in the monoidal category of small $(n-1)$-categories, starting with 0-categories which are the same as sets. See [44], Thm. 1.5.

By an *n-groupoid* we mean an n-category with all j-morphisms invertible for all the compositions $*_j$. An *n-group* is, by definition, an n-groupoid with one object. We have the following result of Brown and Higgins [7] which generalizes the case $n = 2$ discussed in §2.1.

Theorem 3.2.8. *Let C be an n-group. Set*

$$G^0 = C_1, \quad G^{-i} = \mathrm{Ker}\{s_i : C_{i+1} \to C_i\} \subset C_{i+1}, \ i = 1, ..., n-1,$$

and define $\partial_{-i} : G^{-i} \to G^{-i+1}$ to be the restriction of t_i. Define the action β_i of G^0 on G^{-i} by

$$\beta_i(g_0)(g_{-i}) = \mathbf{1}_1^i(g_0) *_0 g_{-i} *_0 \mathbf{1}_1^i(g_0^{-1}), \quad g_0 \in C_1, \, g_i \in \mathrm{Ker}(s_i) \subset C_{i+1}.$$

Then $G^\bullet$ with these actions is a crossed complex. This construction establishes an equivalence between n-groups and crossed complexes of groups situated in degrees $[-n+1, 0]$. □

For future reference we recall the inverse equivalence as well, cf. [7], §4. It associates to a crossed complex $G^\bullet$ in degrees $[-n+1, 0]$, an n-group which we denote $n\,\mathrm{Cat}(G^\bullet)$. Set $n\,\mathrm{Cat}(G^\bullet)_0 = \{\mathrm{pt}\}$, and

$$n\,\mathrm{Cat}(G^\bullet)_m = G^{-m+1} \times ... \times G^{-1} \times G^0, \quad m = 1, ..., n. \tag{3.2.9}$$

Define $s_{m-1}, t_{m-1} : n\,\mathrm{Cat}(G^\bullet)_m \to n\,\mathrm{Cat}(G^\bullet)_{m-1}$ by

$$(3.2.10) \qquad \begin{aligned} s_{m-1}(g_{-m+1}, ..., g_0) &= (g_{-m+2}, ..., g_0), \\ t_{m-1}(g_{-m+1}, ..., g_0) &= (g_{-m+2} \cdot \partial(g_{-m+1}), g_{-m+3}, g_{-m+4}, ..., g_0), \end{aligned}$$

and $\mathbf{1}_m : n\,\mathrm{Cat}(G^\bullet)_m \to n\,\mathrm{Cat}(G^\bullet)_{m+1}$ by

$$\mathbf{1}_m(g_{-m+1}, ..., g_0) = (1, g_{-m+1}, ..., g_0).$$

Let now $x = (g_{-m+1}, ..., g_0)$ and $y = (h_{-m+1}, ..., h_0) \in n\,\mathrm{Cat}(G^\bullet)_m$ be given. Then, for a given $i < m$, the condition $s_i(x) = t_i(y)$ means that

$$g_\nu = h_\nu, \ \nu > -i, \quad g_{-i} = h_{-i} \cdot \partial(h_{-i+1}).$$

In this case we set
(3.2.11)

$$x *_i y = \big(g_{-m+1} \cdot h_{-m+1}, ..., g_{-i} \cdot h_{-i}, h_{-i+1}, ..., h_0\big), \quad i > 0,$$

$$x *_0 y = \Big(g_{-m+1} \cdot (\beta_{-m+1}(g_0)(h_{-m+1})), g_{-m+2} \cdot (\beta_{-m+2}(g_0)(h_{-m+2})), ...,$$
$$..., g_{-1} \cdot (\beta_{-1}(g_0)(h_{-1})), g_0 h_0\Big).$$

These structures make $n\,\mathrm{Cat}(G^\bullet)$ into an n-group.

Remark 3.2.12. More generally, one can extend the above to an equivalence between arbitrary n-groupoids and crossed complexes of groupoids [7] of length $n+1$.

C. The fundamental n-groupoid of a filtered space. Let

$$X_\bullet \ = \ \big\{X_0 \subset X_1 \subset X_2 \subset ...\big\}$$

be a filtered topological space. The crossed complex $G^{\geq -n+1}(X_\bullet)$ (described in Example 3.1.2 (a) for the case X_0 is a point) gives, by the equivalence of crossed complexes and n-groupoids (Remark 3.2.12), a certain n-groupoid $\varpi_{\leq n}(X_\bullet)$. We recall here a more direct description of $\varpi_{\leq n}(X_\bullet)$ following [9].

Consider the p-globe $\bigcirc^p \simeq D^p$ with its filtration by skeleta:

$$\mathrm{sk}_q \bigcirc^p \ = \ s_q \bigcirc^p \ \cup t_q \bigcirc^p \ .$$

Definition 3.2.13. (1) A filtered singular p-globe in $X_\bullet$ is a morphism of filtered topological spaces

$$\Sigma : \{\mathrm{sk}_0 \bigcirc^p, \mathrm{sk}_1 \bigcirc^p, \cdots, \bigcirc^p\} \longrightarrow X_\bullet = \{X_0 \subset X_1 \subset X_2 \subset ...\}.$$

(2) A filtered homotopy between two filtered singular p-globes Σ and Σ' in $X_\bullet$, is a map $\Xi : I \times \bigcirc^p \to X$ such that for each $b \in I$ the map $\Xi(b, -)$: is a filtered singular p-globe in X, and

$$\Xi(0, -) = \Sigma,\ \Xi(1, -) = \Sigma'.$$

The set of filtered homotopy classes of filtered simgular p-globes in $X_\bullet$ will be denoted $\varpi_p(X_\bullet)$. The maps $s_i, t_i, \mathbf{1}_i$ from Example 3.2.4 (a) descend on the filtered homotopy classes and make the collection $\varpi_{\leq n}(X_\bullet) = \{\varpi_p(X_\bullet)\}_{p \leq n}$ into an n-globular set.

Theorem 3.2.14. *[9] The operations $*_i$ on $\mathrm{Glob}_p(X)$, $p \leq n$, $i \leq p-1$ given by*

$$(\Sigma *_i \Sigma')(a_1, \cdots, a_p) = \begin{cases} \Sigma(a_1, \cdots, a_i, 2a_{i+1}, a_{i+2}, \cdots, a_p), & \text{if } a_{i+1} \leq 1/2, \\ \Sigma(a_1, \cdots, a_i, 2a_{i+1} - 1, a_{i+2}, \cdots, a_p), & \text{if } a_{i+1} \geq 1/2 \end{cases}$$

whenever $s_i\Sigma = t_i\Sigma'$, descend to well defined operations on $\varpi_{\leq n}(X_\bullet)$ which make it into an n-groupoid with the set of objects X_0. If $X_0 = \{x_0\}$ is a point, then the crossed complex associated to the n-group $\varpi_{\leq n}(X_\bullet)$ by Theorem 3.2.8, is identified with the crossed complex from Example 3.1.2(a).

We refer to [9] for an outline of the proof. This proof, based on [8], uses a certain cubical Kan fibration property which can be seen as a higher-dimensional analog of the construction (2.3.19) connecting the thin homotopic parts of the boundaries of Σ and Σ' to make the composition possible.

An alternative approach would be to first define the crossed complex of groupoids $G^\bullet(X_\bullet)$ associated to $X_\bullet$ (as done in Example 3.1.2(a) for $X_0 = \{x_0\}$) and then use the general (groupoid) form of Theorem 3.2.8 to produce am n-groupoid $\varpi_{\leq n}(X_\bullet)$.

D. The n-groupoid of unparametrized branes in a manifold. Let X be a C^∞-manifold of dimension n with a filtration

$$X_\bullet \;=\; \{X_0 \subset X_1 \subset X_2 \subset ... \subset X\}$$

by closed subsets (not necessarily submanifolds). Then we can form a C^∞-version of the n-groupoid $\varpi_{\leq n}(X_\bullet)$.

Definition 3.2.15. (1) A filtered parametrized p-brane in $X_\bullet$ is a parametrized p-brane $\Sigma : \bigcirc^p \to X$ which take each $\mathrm{sk}_i \bigcirc^p$ to X_i. We denote by $\widetilde{\Pi}_p(X_\bullet)$ the set of filtered parametrized p-branes in $X_\bullet$.

(2) A filtered homotopy of parametrized filtered p-branes Σ, Σ' is a C^∞-homotopy $\Xi : I \times \bigcirc^p \to X$ in the sense of Definition 2.3.12 such that $\Xi(b, \mathrm{sk}_i \bigcirc^p) \subset X_i$ for each $b \in I$ and $i \leq p$. We denote by $\varpi_p^{C^\infty}(X_\bullet)$ the set of filtered homotopy classes of filtered parametrized p-branes in $X_\bullet$.

The following is a natural C^∞-modification of Theorem 3.2.14. The proof may be obtained by the same steps as in [9] for the topological case.

Proposition 3.2.16. *The operations $*_i$ defined as in Theorem 3.2.14, descend to give an n-groupoid structure on $\varpi_{\leq n}^{C^\infty}(X_\bullet) = \{\varpi_p^{C^\infty}(X_\bullet)\}_{p \leq n}$.* $\square$

We now note that a thin homotopy between parametrized p-branes in X can be seen as a version of a filtered homotopy but with respect to an "indetermined" filtration $X_\bullet$ of X such that X_i has Hausdorff dimension $\leq i$, see Remark 2.3.14(b). We can therefore write the set of thin homotopy classes of parametrized p-branes as

$$\Pi_p(X) \;=\; \varinjlim_{(X_\bullet)} \varpi_p^{C^\infty}(X_\bullet),$$

where the inductive limit is taken over the filtering poset formed by filtrations $X_\bullet$ such that the Hausdorff dimension of X_i is $\leq i$. (One can say that considering thin homotopies amounts to a differential-geometric analog of the skeletal filtration.) In this way we obtain:

Proposition 3.2.17. *The operations $*_i$ in the $\varpi_{\leq n}^{C^\infty}(X_\bullet)$ give rise to a well defined n-groupoid structure on $\Pi_{\leq n} X$.* $\square$

We will call $\Pi_{\leq n} X$ the *n-groupod of unparametrized branes* in X.

3.3 Higher holonomy.

A. Connections with values in crossed complexes. Let $G^\bullet$ be a crossed complex of Lie groups situated in degrees $[-n+1, 0]$, and let $\mathfrak{g}^\bullet$ be the corresponding crossed complex of Lie algebras. Let us consider $\mathfrak{g}^\bullet$ as a semiabelian dg-Lie algebra. Let X be a C^∞-manifold and $A^\bullet \in (\Omega_X^\bullet \otimes \mathfrak{g}^\bullet)^1$ be a $\mathfrak{g}^\bullet$-valued differential form of total degree 1. Thus $A^\bullet$ has components $A^i \in \Omega_X^i \otimes \mathfrak{g}^{1-i}$,

$i = 1, ..., n+1$. We want to consider $A^\bullet$ as a connection in the trivial n-bundle with structure n-group $G^\bullet$.

We denote by

$$F^\bullet = d_{\mathrm{DR}} A^\bullet - \frac{1}{2}[A^\bullet, A^\bullet] \;\in\; (\Omega^\bullet_X \otimes \mathfrak{g}^\bullet)^2 \tag{3.3.1}$$

the curvature of $A^\bullet$. Thus $F^\bullet$ has components $F^i \in \Omega^i_X \otimes \mathfrak{g}^{2-i}$, $i = 2, n+2$. We say that $A^\bullet$ is *semiflat*, if all $F^i = 0$ for $i = 2, ..., n+1$. At the level of components, semiflatness means:

$$\begin{aligned} F_{A^1} \;&=\; d_{\mathfrak{g}^\bullet}(A^2), \\ d_{\mathrm{DR}} A^i - [A^1, A^i] \;&=\; d_{\mathfrak{g}^\bullet}(A^{i+1}), \quad i = 2, ..., n. \end{aligned} \tag{3.3.2}$$

Note that $d_{\mathrm{DR}} A^i - [A^1, A^i] = \nabla_{A^1}(A^i)$ is just the covariant differential of A^i with respect to the connection A^1.

B. Forms on the space of paths corresponding to a connection. We fix points $x, y \in X$ and consider the $\mathfrak{g}^{1-i}$-valued differential forms $\oint_{A^1} A^i$ on $P_x^y X$, for $i = 2, ..., n+1$.

Proposition 3.3.3. *If $n \geq 2$ and $A^\bullet$ is semiflat, then:*

(a) The curvature of $\oint_{A^1}(A^2)$ is equal to $\oint_{A^1}(\nabla_{A^1} A^3)$.

(b) For $i = 2, ..., n-1$ we have

$$d_{\mathrm{DR}} \oint_{A^1} (A^i) \;=\; \oint_{A^1} (\nabla_{A^1} A^{i+1}).$$

Proof: Part (a) follows from Proposition 2.4.8 applied to the truncated connection with values in the crossed module $\mathfrak{g}^{\geq -1}$. Indeed, $\nabla_{A^1} A^3$ is the 3-curvature of this truncated connection.

Part (b) follows from Proposition A.3.6(b) in exactly the same way as Proposition 2.4.8. More precisely, we see, similarly to Eq. (2.4.9), that

$$d_{\mathrm{DR}} \oint_{A^1} (A^i) \;=\; \oint_{A^1} (\nabla_{A^1} A^{i+1}) \;+\; \left[d_{\mathfrak{g}} \oint_{A^1} (A^2),\; \oint_{A^1} (A^{i+1}) \right],$$

and the last commutator is equal to 0. Indeed, part (c) of Definition 3.1.3 of crossed complexes of Lie algebras says that $[d_{\mathfrak{g}} x, y] = 0$ for any $x \in \mathfrak{g}^{-1}, y \in \mathfrak{g}^{-i}$. □

C. The p-dimensional holonomy. Let $G^\bullet$ and $\mathfrak{g}^\bullet$ be as before, and $A^\bullet$ be a semiflat form with values in $\mathfrak{g}^\bullet$ of total degree 1. Let $p \geq 1$ and let $\Sigma : I^p \to X$ be a parametrized p-brane in X, with $s_0\Sigma = x$ and $t_0\Sigma = y$. Recall that $\tau(\Sigma)$ denotes the $(p-1)$-brane in $P_x^y X$ obtained as the transgression of Σ, see Example 3.2.4 (b). We now define the *p-dimensional holonomy* of $A^\bullet$ along Σ to be the element

(3.3.4)

$$M_{A^\bullet}(\Sigma) \;=\; \begin{cases} M_{A^0}(\Sigma) \;=\; P\exp\int_\Sigma A^1 \;\in\; G^0, & \text{if} \quad p=1; \\ M_{A^{\leq -1}}(\Sigma) \;=\; P\exp\int_{\tau(\Sigma)}\oint_{A^1}(A^2) \;\in\; G^{-1}, & \text{if} \quad p=2; \\ \exp\int_{\tau(\Sigma)}\oint_{A^1}(A^{p-1}) \;\in\; G^{-p+1}, & \text{if} \quad p\geq 3. \end{cases}$$

Here the first line is the usual holonomy of the connection A^1 along the path Σ, and the second line is the 2-dimensional holonomy of the truncated connection $A^{\leq 2}$ with values in the crossed module $\mathfrak{g}^{\geq -1}$, as defined by Baez-Schreiber [2], see (2.4.10). In the third line, which extends the definitions of [2] to arbitrary crossed complexes, $\exp : \mathfrak{g}^{-p+1} \to G^{-p+1}$ is the exponential map of the abelian Lie group G^{-p+1}. We then define

$$\begin{aligned} \mathbb{M}_{A^\bullet}(\Sigma) \;&=\; \big(M_{A^\bullet}(\Sigma), M_{A^\bullet}(s_{p-1}\Sigma), ..., M_{A^\bullet}(s_1\Sigma)\big) \;\in \\ &\in\; G^{-p+1}\times G^{-p+2}\times ...\times G^0 \;=\; n\,\mathrm{Cat}(G^\bullet)_p. \end{aligned} \tag{3.3.5}$$

Theorem 3.3.6. *(a) The element $\mathbb{M}_{A^\bullet}(\Sigma)$ depends only on the thin homotopy class of Σ. In particular, it is invariant under reparametrizations of Σ identical near the boundary.*

(b) The correspondence $\Sigma \mapsto \mathbb{M}_{A^\bullet}(\Sigma)$ defines a homomorphism (strict n-functor) $\mathbb{M}_{A^\bullet} : \Pi_{\leq n}(X) \to n\,\mathrm{Cat}(G^\bullet)$.

Proof: (a) Follows from Proposition 3.3.3. Let us prove path (b). The first thing to prove is compatibility of $\mathbb{M}_{A^\bullet}$ with the source and target maps. It amounts to the equality, for each p and for each p-brane Σ:

$$\partial(M_{A^\bullet}(\Sigma)) \;=\; M_{A^\bullet}(t_{p-1}\Sigma)\cdot M_A(s_{p-1}\Sigma)^{-1} \;\in\; G^{-p+2}.$$

For $p = 1, 2$ this equality has already been proved in Theorem 2.4.11(a). For $p > 2$ this follows from the equality, in $\mathfrak{g}^{2-p}$, of the logarithms of the both sides which is established using the condition of semiflatness and the Stokes formula:

$$d_{\mathfrak{g}} \int_{\tau(\Sigma} \oint_{A^1} (A^{p-1}) \;=\; \int_{\tau(\Sigma} \oint_{A^1} (d_{\mathfrak{g}} A^{p-1}) \;\overset{(3.3.2)}{=}\; \int_{\tau(\Sigma} \oint_{A^1} \big(d_{\mathrm{DR}} A^{p-2} - [A^1, A^{p-2}]\big) =$$

$$= \int_{\tau(\Sigma)} \oint_{A^1} \big(\nabla_{A^1}(A^{p-2})\big) \;\overset{(3.3.3)(b)}{=}\; \int_{\tau(\Sigma} d_{\mathrm{DR}} \oint_{A^1} (A^{p-2}) \;\overset{\text{Stokes}}{=}\; \int_{\partial\tau(\Sigma)} \oint_{A^1} (A^{p-2}) \;=$$

$$= \;\log M_A(t_{p-1}\Sigma) - \log M_A(s_{p-1}\Sigma).$$

The second thing to prove is that $\mathbb{M}_{A^\bullet}$ commutes with the compositions:

$$\mathbb{M}_{A^\bullet}(\Sigma *_i \Sigma') \;=\; \mathbb{M}_{A^\bullet}(\Sigma) *_i \mathbb{M}_{A^\bullet}(\Sigma'), \tag{3.3.7}$$

whenever Σ and Σ' are $*_i$-composable. Denote

$$p = \dim(\Sigma),\;\; p' = \dim(\Sigma'),\;\; \mu = \min(p, p'),\;\; m = \max(p, p').$$

We use induction on m, the case $m \leq 2$ being already established in Theorem 2.4.11(c). Further, it is enough to prove (3.3.7) under the assumption that $i = \mu - 1$. Indeed, 2-dimensional associativity implies that such "irreducible" types of compositions generate all other compositions of all higher morphisms in any strict n-category. Keeping this assumption, we distinguish two cases and analyze, in each case, the nature of the $(m-1)$-membrane $\tau(\Sigma *_i \Sigma')$ in the path space.

Case 1: $\mu = 1$**.** That is, either $p = 1$ or $p' = 1$, and so $i = 0$. We can suppose $m \geq 3$. Suppose $p = 1$. This means that $\Sigma = \gamma \in P_y^z X$ is a path and $s_0\Sigma' = x, t_0\Sigma' = y$ (so that $\gamma *_0 \Sigma'$ is defined). In this case we have that

$$\tau(\gamma *_0 \Sigma') \;=\; l_\gamma(\tau(\Sigma'))$$

is the left translation of $\tau(\Sigma')$ by γ, and so Proposition 2.4.6(a) implies that

$$M_{A^\bullet}(\gamma *_0 \Sigma') \;=\; M_{A^\bullet}(\Sigma') \;\in G^{-p'+1},$$

which is the highest (degree p') component of the desired equality (3.3.7) with respect to the decompostion (3.3.5). The validity of the lower components follows by induction since

$$s_{p'-1}(\gamma *_0 \Sigma') = \gamma *_0 s_{p'-1}(\Sigma').$$

The case when $p' = 1$ is treated similarly using Proposition 2.4.6(b).

Case 2: $\mu \geq 2$. We can also assume $m \geq 3$. In this case all three branes Σ, Σ' and $\Sigma *_i \Sigma'$, $i = \mu - 1$, have the same $s_0 = x$ and $t_0 = y$. Therefore $\tau(\Sigma)$, $\tau(\Sigma')$ and $\tau(\Sigma *_i \Sigma')$ are branes in the same path space $P_x^y X$. and

$$\tau(\Sigma *_i \Sigma') \;=\; \tau(\Sigma) *_{i-1} \tau(\Sigma'), \tag{3.3.8}$$

is identified with the $*_{i-1}$-composition (which can be seen, geometrically, as the union) of $\tau(\Sigma)$ and $\tau(\Sigma')$ inside $P_x^y X$.

We first concentrate on the highest (degree m) component of (3.3.7). If $p = p'$, then the validity of this compoment follows from additivity of the integral of the $(p-1)$-form $\oint_{A^1}(A^{p-1})$ with respect to the decomposition (3.3.8) of the domain of integration. The validity of lower components follows by induction since, under our assumptions of $p = p'$ and $i = p - 1$ we have

$$s_{p-1}(\Sigma *_i \Sigma') = s_{p-1}(\Sigma) *_{i-1} s_{p-1}(\Sigma').$$

so for the s_{p-1}-parts we have the same situation of equal dimension and "irreducible" composition.

If $p \neq p'$, say $p < p'$, then $m = p'$ and the integral of $\oint_{A^1}(A^{m-1})$ over $\tau(\Sigma *_i \Sigma')$ is equal to the integral over $\tau(\Sigma')$ since the other part has smaller dimension. This equality is precisely the degree m component of (3.3.7). The equality of components of smaller degree follows by induction similarly to the above since

$$s_{p'-1}(\Sigma *_i \Sigma') = \Sigma *_i s_{p'-1}(\Sigma'),$$

so for the s_{p-1}-parts we again have the situation of "irreducible" composition.

The case $p > p$ is treated similarly. $\square$

3.4 The crossed complex of formal branes in $\mathbb{R}^n$ and higher holonomy.

Fix $n \geq 1$ and denote by $\mathfrak{g}_n^\bullet$ the semiabelian dg-Lie algebra $\mathfrak{f}^\bullet(\mathbb{R}^n)_{\mathrm{sab}}$, see §1.5. It is situated in degrees $[-n+1, 0]$. We will also consider $\mathfrak{g}_n^\bullet$ as a crossed complex of Lie algebras, alternating the two points of view.

Let $\mathfrak{g}_{n,d}^\bullet = \mathfrak{g}_n^\bullet / \gamma_{d+1}(\mathfrak{g}_n^\bullet)$ denote the semiabelian dg-Lie algebra (as well as the corresponding crossed complex of Lie algebras) obtained by quotienting $\mathfrak{g}_n^\bullet$ by the $(d+1)$st layer of the lower central series (3.1.7). As $\mathfrak{g}_{n,d}^\bullet$ is nilpotent and finite-dimensional, the Malcev theory for crossed complexes (Proposition 3.1.12) produces a crossed complex of unipotent Lie groups $G_{n,d}^\bullet = \exp(\mathfrak{g}_{n,d}^\bullet)$.

The universal translation invariant connection (1.2.2) descends to a translation invariant connection $A^\bullet_{n,d}$ on $\mathbb{R}^n$ with values in $\mathfrak{g}^\bullet_{n,d}$. This connection gives rise to the higher holonomy n-functor

$$\mathbb{M}_{n,d} = \mathbb{M}_{A^\bullet_{n,d}} : \ \Pi^{\leq n}(\mathbb{R}^n) \longrightarrow n\,\mathrm{Cat}(G^\bullet_{n,d}). \tag{3.4.1}$$

We further consider the pro-unipotent crossed complex $\widehat{G}^\bullet_n = \varprojlim_d G^\bullet_{n,d}$ which we call the *crossed complex of formal branes* in $\mathbb{R}^n$. By passing to the projective limit of the $M_{n,d}$, we get the n-functor

$$\widehat{\mathbb{M}}_n \ = \ \varprojlim_d \mathbb{M}_{n,d} : \ \Pi^{\leq n}(\mathbb{R}^n) \longrightarrow n\,\mathrm{Cat}(\widehat{G}^\bullet_n). \tag{3.4.2}$$

Question 3.4.3. Similarly to Question 2.5.6, it is interesting to understand to what extend is the functor $\widehat{\mathbb{M}}_n$ injective on higher morphisms. Consider in particular the sets

$$\pi_i^{\mathrm{geom}}(\mathbb{R}^n, x) \ := \ \left\{\Sigma \in \Pi^{\leq n}(\mathbb{R}^n)_i \,\middle|\, s_{i-1}(\Sigma) = t_{i-1}(\Sigma) = x\right\}, \quad i \geq 1.$$

As for any strict n-groupoid, these sets are groups, abelian for $i \geq 2$. Translation of membranes in $\mathbb{R}^n$ identifies them for different $x \in \mathbb{R}^n$, so we can assume $x = 0$.

For any i, elements of $\pi_i^{\mathrm{geom}}(\mathbb{R}^n, 0)$ can be understood as geometric unparametrized spheres (maps $\Sigma : (I^i, \partial I^i) \to (\mathbb{R}^n, 0)$ modulo thin homotopies).

The set $\pi_1^{\mathrm{geom}}(\mathbb{R}^n, 0)$ is the group of "paths in $\mathbb{R}^n$ modulo reparemetrization and cancellation" beginning at 0. Chen's theorem [13] implies that $\widehat{\mathbb{M}}_n$ embeds it into the pro-unipotent completion of the free group on n generators.

For $i \geq 2$, the functor $\widehat{\mathbb{M}}_n$ sends any $\Sigma \in \pi_i^{\mathrm{geom}}(\mathbb{R}^n, 0)$ to its associated current (functional on polynomial i-forms)

$$I_\Sigma : \ \omega \ \mapsto \int_\Sigma \omega \ := \ \int_{I^i} \Sigma^*(\omega).$$

and the question is to what extent the thin homotopy class of Σ can be recovered from I_Σ.

A Appendix: forms and connections on the space of paths.

Here we collect the necessary facts about "covariant" generalizations of Chen's iterated integrals, as developed in [26, 2]. We give a direct treatment, reducing these facts to classical results of Chen [14].

A.1 Scalar iterated integrals.

Let X be a C^∞-manifold and $PX = C^\infty([0,1], X)$ be the space of smooth paths in X. Suppose we are given differential forms ω_ν, $\nu = 1, ..., r$ on X of degree $m_\nu + 1$. Chen's *iterated integral* of $\omega_1, ..., \omega_r$ is the form $\oint(\omega_1, ..., \omega_r)$ on PX of degree $m_1 + ... + m_r$, sending tangent vectors $\delta_1\gamma$, ..., $\delta_{m_1+...+m_r}\gamma$ at a point γ to

(A.1.1)

$$\mathrm{ALT} \int_{0\leq t_1 \leq ... \leq t_r \leq 1} \prod_{\nu=1}^{r} \omega_\nu \Big(\gamma(t_\nu); \dot{\gamma}(t_\nu), \delta_{m_1+...+m_{\nu-1}+1}\gamma(t_\nu), ..., \delta_{m_1+...+m_\nu}\gamma(t_\nu) \Big) \times dt_1 ... dt_r.$$

Here ALT stands for alternation in the arguments $\delta_i\gamma$. Put differenty, let

$$\Delta_r = \{0 \leq t_1 \leq ... \leq t_r \leq 1\} \tag{A.1.2}$$

be the integration domain (simplex) in (A.1.1), and consider the diagram

$$PX \xleftarrow{\pi_r} \Delta_r \times PX \xrightarrow{e_r} X^r, \quad e_r(t_1, ..., t_r, \gamma) = (\gamma(t_1), ..., \gamma(t_r)), \tag{A.1.3}$$

with π_r being the projection. Then

$$\oint(\omega_1, ..., \omega_r) = (\pi_r)_* e_r^*(\omega_1 \boxtimes ... \boxtimes \omega_r), \tag{A.1.4}$$

where $(\pi_r)_*$ in the integration along the fibers of π, and $\omega_1 \boxtimes ... \boxtimes \omega_r$ is the wedge product of the pullbacks of the forms ω_ν from the factors. Cf. [22], §2.

One can replace PX in the above definition by the space of Moore paths $C^\infty([a,b], X)$ for any interval $[a,b] \subset \mathbb{R}$. We denote the corresponding forms on this space by the same symbol $\oint(\omega_1, ..., \omega_r)$.

Example A.1.5. (a) Let $r = 1$. Then the correspondence which takes ω to $\oint(\omega)$ restricted to the subspace $P_x^y X \subset PX$, is the transgression (2.4.5) of differential forms.

(b) Suppose that all $m_\nu = 0$. Then $\oint(\omega_1, ..., \omega_r)$ is a function on PX. On the other hand, consider the associative algebra $\mathbb{R}\langle\langle Z\rangle\rangle = \mathbb{R}\langle\langle Z_1, ..., Z_r\rangle\rangle$ of noncommutative formal power series in variables $Z_1, ..., Z_r$. Consider the differential form $A = \sum \omega_i Z_i \in \Omega^1_X \otimes \mathbb{R}\langle\langle Z\rangle\rangle$ and the corresponding connection $\nabla = d_{\mathrm{DR}} - A$ on X with values in $\mathbb{R}\langle\langle Z\rangle\rangle$. Since $\mathbb{R}\langle\langle Z\rangle\rangle$ is a projective limit of finite-dimensional associative algebras, this connection has a well defined holonomy which is a function $M_A = M_\nabla : PX \to \mathbb{R}\langle\langle Z\rangle\rangle^\times$. The following fundamental formula of Chen can be used as an alternative definition of iterated integrals of 1-forms:

$$M_A = \sum_{d=0}^{\infty} \sum_{1\leq i_1,...,i_d \leq r} Z_{i_1}...Z_{i_p} \oint(\omega_{i_1}, ..., \omega_{i_d}). \tag{A.1.6}$$

(c) More generally, let m_ν be arbitrary. Following [14], §2, we consider first the algebra $\mathbb{R}\langle Z\rangle = \mathbb{R}\langle Z_1, ..., Z_r\rangle$ of noncommutative polynomials in the Z_ν, equipped with grading $\deg(Z_\nu) = -m_\nu$. Then denote by $\mathbb{R}\langle\langle Z\rangle\rangle$ the completion of $\mathbb{R}\langle Z\rangle$ allowing power series only in those generators which have degree 0. Then $\mathbb{R}\langle\langle Z\rangle\rangle$ inherits the grading, and

$$A = \sum \omega_i Z_i \in (\Omega^\bullet_X \otimes \mathbb{R}\langle Z\rangle)^1$$

is an element of total degree 1 called the *formal power series connection* associated to the forms $\omega_1, ..., \omega_r$ of arbitrary degrees. Now, defining M_A by the infinite sum as in (A.1.6), we find that M_A is an invertible element of $\Omega^\bullet_{PX} \otimes \mathbb{R}\langle\langle Z\rangle\rangle$ of total degree 0.

(d) Similarly, for any interval $[a, b] \subset \mathbb{R}$, the sum as in (A.1.6) defines an invertible element

$$M_{A,[a,b]} \in (\Omega^\bullet_{C^\infty([a,b],X)} \otimes \mathbb{R}\langle\langle Z\rangle\rangle)^0.$$

We now recall the well known result of Chen, see [14], §4, describing the products and differentials of iterated integrals. As usual, by an (r, s)-*shuffle* we mean a permutation $w \in S_{r+s}$ such that $w(i) < w(j)$ for all $i < j$ such that ether both $i, j \leq r$ or both $i, j \geq r$. The set of (r, s)-shuffles will be denote by $\mathrm{Sh}(r, s)$.

Proposition A.1.7. *Let $x, y \in X$ and consider the iterated integrals as differential forms on the subspace $P_x^y X \subset PX$. Then:*

(a) The wedge product of two iterated integrals is given by

$$\oint(\omega_1, ..., \omega_r) \ \wedge \ \oint(\omega_{r+1}, ...\omega_{r+s}) \ = \sum_{w \in \mathrm{Sh}(r,s)} (-1)^{\delta(w)} \oint(\omega_{w(1)}, ..., \omega_{w(r+s)}),$$

$$\text{where} \quad \delta(w) \ = \sum_{\substack{i<j \\ w(i)>w(j)}} m_i m_j.$$

(b) The exterior differential of an iterated integral is given by

$$d\oint(\omega_1, ..., \omega_r) \ = \ -\sum_{\nu=1}^{r} (-1)^{\sum_{\nu'<\nu} m'_\nu} \oint(\omega_1, ..., d\omega_\nu, ..., \omega_r) -$$
$$-\sum_{\nu=1}^{r-1} (-1)^{\sum_{\nu' \leq \nu}(m'_\nu+1)} \oint(\omega_1, ..., \omega_\nu \wedge \omega_{\nu+1}, ..., \omega_r).$$

Note that part (a) is obvious: it follows by decomposing the product of two simplices $\Delta_r \times \Delta_s$ into $\binom{r+s}{r}$ simplices labelled by shuffles. The integral over each of these simplices is the iterated integral as stated.

On the contrary, the standard proofs of part (b) are usually much less transparent. For future purposes, let us indicate a simple proof in the case when all $m_\nu = 0$. In this case we use Chen's formula (A.1.6) for the holonomy of the connection A. The curvature of A is then

$$\text{(A.1.8)} \qquad F \ = \ dA - \frac{1}{2}[A, A] \ = \ \sum d\omega_\nu \cdot Z_\nu \ - \sum_{\nu'<\nu} (\omega_{\nu'} \wedge \omega_\nu) \cdot [Z_{\nu'}, Z_\nu].$$

After that part (b) is obtained by applying the Schlessinger formula (2.4.1) to A, writing $dM_A = M_A \cdot B$, using part (a) to evaluate the product and comparing coefficients at the monomial $Z_1 ... Z_r$.

We now formulate a generalization of the Schlessinger formula to higher degree forms, found by Chen. This is achieved by expressing the original formula in a more diagrammatic fashion and then replacing functions by differential forms. We use the conventions of Example A.1.5(c) and consider A as a graded connection with values in the associative algebra $\mathbb{R}\langle Z \rangle$ (considered as a Lie algebra). The curvature

$$F \in (\Omega^\bullet_{P_x^y X} \otimes \mathbb{R}\langle Z \rangle)^2$$

of A is then given by the same formula (A.1.8), with the commutators understood in the graded sense. For $t_0 \in I$ consider the diagram

$$P_x^y X \xrightarrow{j_{t_0}} I \times P_x^y X \xrightarrow{e_1} X, \quad j_{t_0}(\gamma) = (t_0, \gamma),\, e_1(t,\gamma) = \gamma(t),$$

and define

$$\text{(A.1.9)} \qquad \mathbf{i}(F, t_0) \;=\; j_{t_0}^*\big(i_{\frac{\partial}{\partial t}} e_1^* F\big) \;\in\; (\Omega^\bullet_{P_x^y X} \otimes \mathbb{R}\langle Z\rangle)^1.$$

Here $i_{\frac{\partial}{\partial t}}$ is the contraction of differential forms with the vector field $\frac{\partial}{\partial t}$ on $I \times P_x^y X$. Further, for $t \in I$ consider the map

$$\pi_t : P_x^y X \longrightarrow C^\infty([0,t], X)$$

given by restricting paths $\gamma : I \to X$ to the subinterval $[0,t]$, and let

$$\text{(A.1.10)} \qquad M_{A,\leq t} \;=\; \pi_t^* M_{A,[0,t]} \;\in\; (\Omega^\bullet_{P_x^y X} \otimes \mathbb{R}\langle\langle Z\rangle\rangle)^0,$$

see Example A.1.5(d).

Theorem A.1.11 ([14], Thm. 2.3.2)**.** *In this situation we have*

$$M_A \wedge d_{\mathrm{DR}} M_A \;=\; \int_0^1 M_{A,\leq t}^{-1} \wedge \mathbf{i}(F,t) \wedge M_{A,\leq t} \cdot dt. \quad \square$$

One can deduce from this the general case of Proposition A.1.7(b) in the same way as indicated above for the case when all $m_\nu = 0$.

Remarks A.1.12. (a) The notation $\oint$, different from Chen's but compatible with [26, 2], is chosen to emphasize that we work with forms on $P_x^y X$, the subspace of paths having source and target fixed. In particular, if $x = y$, then the paths are closed.

(b) Let $(\mathcal{A}^\bullet, d)$ be a commutative dg-algebra. Its *bar-construction* is a new commutative dg-algebra $\mathrm{Bar}^\bullet(\mathcal{A}^\bullet, d)$ which, as a graded vector space, is $\bigoplus_r \mathcal{A}[-1]^{\otimes r}$, the tensor algebra of the graded space $\mathcal{A}^\bullet$ with degree shifted by (-1). This space is equipped with the *shuffle multiplication*

(A.1.13)

$$(\omega_1 \otimes \cdots \otimes \omega_r) \cdot (\omega_{r+1} \otimes \cdots \otimes \omega_{r+s}) \;=\; \sum_{w \in \mathrm{Sh}(r,s)} (-1)^{\delta(w)}\; \omega_{w(1)} \otimes \cdots \otimes \omega_{w(r+s)}$$

$$\delta(w) \;=\; \sum_{\substack{i<j \\ w(i)>w(j)}} m_i m_j, \quad \omega_i \in \mathcal{A}^{m_i+1},$$

and with the differential

$$
\begin{aligned}
d(\omega_1 \otimes \cdots \otimes \omega_r) \;&=\; -\sum_{\nu=1}^{r} (-1)^{\sum_{\nu'<\nu} m'_\nu}\, \omega_1 \otimes \ldots \otimes d\omega_\nu \otimes \ldots \otimes \omega_r - \\
&\quad - \sum_{\nu=1}^{r-1} (-1)^{\sum_{\nu'\leq\nu} (m'_\nu+1)}\, \omega_1 \otimes \ldots \otimes (\omega_\nu \cdot \omega_{\nu+1}) \otimes \ldots \otimes \omega_r.
\end{aligned} \tag{A.1.14}
$$

Proposition A.1.7 can be formulated by saying that iterated integrals define a morphism of commutative dg-algebras

$$
\oint : \mathrm{Bar}^\bullet(\Omega^\bullet_X, d_{DR}) \longrightarrow (\Omega^\bullet_{P^y_x X}, d_{DR}). \tag{A.1.15}
$$

The main result of Chen [14] is that $\oint$ is a quasi-isomorphism, and so $\mathrm{Bar}^\bullet(\Omega^\bullet_X, d_{DR})$ calculates the coholomogy of $P^y_x X$.

Remark A.1.16. Another approach to Theorem A.1.11 would be to deduce it from the classical Schlesinger formula (2.4.1) but applied to *supermanifolds*, see [37] for background. More precisely, for any C^∞-manifold X we have the supermanifold

$$
\mathcal{S}X \;=\; \mathrm{Spec}(\Omega^\bullet_X) \;=\; \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, X),
$$

see [34, 31]. Thus differential forms on X can be seen as functions on $\mathcal{S}X$. We can then extend the concept of a differentiable space and the functor $\mathcal{S}$ to the super-situation and, in particular, have the identification

$$
\mathcal{S}P^y_x X \;=\; P^y_x \mathcal{S}X \quad\subset\quad \mathcal{S}PX = \underline{\mathrm{Hom}}(\mathbb{R}^{0|1} \times [0,1], X) = P\mathcal{S}X.
$$

Further, by analyzing differential forms on $\mathcal{S}X$, i.e., functions on $\mathcal{S}\mathcal{S}X$, as in [31], one can interpret p-forms on X, $p \geq 2$, as certain 1-forms on $\mathcal{S}X$. In this way iterated integrals of forms of higher degree on X can be expressed through iterated integrals of 1-forms on $\mathcal{S}X$ so the Schlesinger formula applied to $\mathcal{S}X$, implies Theorem A.1.11. These remarks also apply to more sophisticated iterated integrals considered further in this Appendix.

A.2 Iterated integrals with values in an associative algebra.

Let $V_0, \ldots, V_r$ be finite-dimensional $\mathbb{R}$-vector spaces, and let $\mu : V_1 \otimes \cdots \otimes V_r \to V_0$ be a multilinear map. Given differential forms $\omega_\nu \in \Omega^{m_\nu+1}_X \otimes V_\nu$, $i = 1, \ldots, r$, we have the iterated integral

$$\oint(\omega_1, ..., \omega_r) \;=\; \oint^{\mu}(\omega_1, ..., \omega_r) \;\in\; \Omega_{PX}^{\sum m_\nu} \otimes V_0. \tag{A.2.1}$$

It is obtained by tensoring the usual (scalar) iterated integral map

$$\oint : \bigotimes_{n=1}^{r} \Omega_X^{m_\nu+1} \longrightarrow \Omega_{PX}^{\sum m_\nu}$$

(tensor product over $\mathbb{R}$), with μ.

Next, let $R^\bullet = \bigoplus_{d\leq 0} R^d$ be a $\mathbb{Z}_-$-graded associative $\mathbb{R}$-algebra, with each R^d finite-dimensional. Let $A^\bullet \in (\Omega_X^\bullet \otimes R^\bullet)^1$ be a graded connection on X with values in $R^\bullet$ (considered as a graded Lie algebra in a standard way). As usual, we write $A^p \in \Omega_X^p \otimes R^{1-p}$ for the component of $A^\bullet$ which is a p-form. Let $F^\bullet$ be the curvature of $A^\bullet$. Let $\mu_r : (R^\bullet)^{\otimes r} \to R^\bullet$ be the $(r-1)$-fold product map. Consider the *Picard series*

$$M_A \;=\; \sum_{r=0}^{\infty} \oint^{\mu_r} \underbrace{(A^\bullet, \cdots, A^\bullet)}_{r} \;\in\; (\Omega_{PX}^\bullet \otimes R^\bullet)^0. \tag{A.2.2}$$

Unlike the series (A.1.6) which is purely formal, the Picard series involves actual infinite summation of real numbers because the component A^1 can enter the iterated integrals arbitrarily many times without raising the degree of the resulting form on PX.

Proposition A.2.3. *The Picard series converges to an invertible element of* $(\Omega_{PX}^\bullet \otimes R^\bullet)^0$.

Proof: The degree 0 component $\sum_r \oint^{\mu_r}(A^1, ..., A^1)$ represents the standard Picard (ordered exponential) series for the holonomy of the usual connection A^1, so it converges to an invertible element of $\Omega_{PX}^0 \otimes R^0$. As for higher degree components, the question reduces to the convergence of series like

$$\sum_{N_1,...,N_s=0}^{\infty} \oint^{\mu_r} \big(A^{p_1}, (A^1)^{N_1}, A^{p_2}, (A^1)^{N_2}, ..., A^{p_s}, (A^1)^{N_s}\big), \quad r = s + \sum N_i,$$

where $p_1, ..., p_s \geq 2$ are fixed and the notation $(A^1)^N$ stands for the fragment $A^1, ..., A^1$ (N times). This convergence follows from that of the series for the holonomies of the connections in the trivial bundles with fibers R^{1-p_i} induced by A^1 via the left and right actions of R^0 on R^{1-p_i}. $\square$

As in §A.1, we denote by $M_{A,[a,b]}$ the form on the Moore path space $C^\infty([a,b],X)$ defined in the same way but with iterated integrals taken over $[a,b]$. We define

$$\mathbf{i}(F^\bullet, t) \in (\Omega^\bullet_{PX} \otimes R^\bullet)^1, \quad M_{A,\le t} \in (\Omega^\bullet_{PX} \otimes R^\bullet)^0, \quad t \in I,$$

using the same formulas as in (A.1.9), (A.1.10).

Proposition A.2.4. *Fix $x, y \in X$. Then we have the equality of forms on $P_x^y X$:*

$$M_A \wedge d_{\mathrm{DR}} M_A \;=\; \int_0^1 M_{A,\le t}^{-1} \wedge \mathbf{i}(F,t) \wedge M_{A,\le t} \cdot dt \;\in\; (\Omega^\bullet_{P_x^y X} \otimes R^\bullet)^1.$$

Proof: Taking bases in R^{1-p}, $1 \le p \le \dim(X)$, we can reduce the statement to Theorem A.1.11. That is, we can find forms $\omega_\nu \in \Omega_X^{m_\nu+1}$, $\nu = 1, ..., r$ with the corresponding grading on the free algebra $\mathbb{R}\langle Z \rangle$ and the corresponding graded connection $\widetilde{A}^\bullet = \sum Z_\nu \cdot \omega_\nu$, together with a homomorphism $\phi : \mathbb{R}\langle Z \rangle \to R^\bullet$ such that $\phi_*(\widetilde{A}^\bullet) = A^\bullet$ and therefore $\phi_*(F_{\widetilde{A}}) = F^\bullet$. Further, applying ϕ to convergent infinite sums, we see that $\phi_*(M_{\widetilde{A}}) = M_A$ etc. so the statement follows from the cited theorem. Alternatively, one can repeat, in the present context, the proof of Theorem A.1.11 given by Chen, by using the differential equation of $M_{A,\le t}$ with respect to t. $\square$

A.3 Covariant iterated integrals.

Let $E_0, ..., E_r$ be C^∞ real vector bundles on X, with connections $\nabla_0, ..., \nabla_r$, and let $\mu : E_1 \otimes ... \otimes E_r \to E_0$ be a morphism of vector bundles preserving the connections. We write ∇ for the system $(\nabla_\nu)_{\nu=0}^r$. Let $x, y \in X$ be two points. Given differential forms $\omega_\nu \in \Omega_X^{m_\nu+1} \otimes E_\nu$, $\nu = 1, ..., r$, we define, following [26, 2], the *(left) covariant iterated integral*, which is a form

$$\text{(A.3.1)} \qquad \oint_\nabla (\omega_1, ..., \omega_r) \;=\; \oint_\nabla^\mu (\omega_1, ..., \omega_r) \;\in\; \Omega^{m_1+...+m_r}_{P_x^y X} \otimes E_{0,x}.$$

The value of this form at a path γ and tangent vectors $\delta_i\gamma$, $i = 1, ..., m_1 + ... + m_r$, is defined to be

$$\text{(A.3.2)} \qquad \begin{aligned} &\mathrm{ALT} \int_{0 \le t_1 \le ... \le t_r \le 1} dt_1 ... dt_r \cdot \mu \Bigg\{ \bigotimes_{\nu=1}^{r} \Bigg[M_{\nabla_\nu}(\gamma_{\le t_\nu})^{-1} \Bigg(\omega_\nu \Big(\gamma(t_\nu); \\ &\qquad \dot{\gamma}(t_\nu), \delta_{m_1+...+m_{\nu-1}+1}\gamma(t_\nu), \cdots, \delta_{m_1+...+m_{\nu+1}}\gamma(t_\nu) \Big) \Bigg) \Bigg] \Bigg\}. \end{aligned}$$

The idea is simple: in order to be able to form the tensor product, we transport the value of each ω_ν from the fiber of E_ν over $\gamma(t_\nu)$, to the fiber over $x = \gamma(0)$. In the case when all the bundles are trivial and $\nabla_\nu = d - A_\nu$ we write A for the system $(A_0, ..., A_r)$ are $\overleftarrow{\oint}_A$ for $\overleftarrow{\oint}_\nabla$.

We can also transport the values to the fibers over $y = \gamma(1)$, to define the *right covariant iterated integral* to be the form

$$\text{(A.3.3)} \qquad \overrightarrow{\oint}_\nabla^\mu (\omega_1, ..., \omega_r) \;:=\; M_{\nabla_0} \overleftarrow{\oint}_\nabla^\mu (\omega_1, ..., \omega_r) \;\in\; \Omega^{m_1+...+m_r}_{P_x^y X} \otimes E_{0,y}.$$

(Note the difference between the symbols $\overleftarrow{\oint}$ and $\overrightarrow{\oint}$ for the left and right covariant iterated integrals.) We will use the notation

$$\text{(A.3.4)} \qquad \overleftarrow{\oint}_\nabla ((\omega_1, ..., \omega_r)) \;:=\; \overleftarrow{\oint}_\nabla^{\mathrm{Id}} (\omega_1, ..., \omega_r), \quad E_0 = E_1 \otimes ... \otimes E_r$$

for the covariant iterated integral corresponding to $\mu = \mathrm{Id}$, and similarly for $\overrightarrow{\oint}_\nabla((\omega_1, ..., \omega_r))$.

Example A.3.5. The covariant transgression introduced in §2.4 C, is a particular case of the construction (A.3.4) for $r = 1$. More precisely, let G be a Lie group, Q a principal G-bundle on X and $\alpha : G \to GL(V)$ a representation of G. A connection ∇ in Q induces a connection ∇_V in the vector bundle $V(Q)$. In this notation, for each $\Phi \in \Omega_X^{m+1} \otimes V(Q)$ we have

$$\overleftarrow{\oint}_\nabla (\Phi) \;=\; \overleftarrow{\oint}_{\nabla_Q} ((\Phi)).$$

We now formulate the main properties of covariant iterated integrals, cf. [26], formula (15) and [2], Prop.2.7.

Proposition A.3.6. *(a) The tensor/wedge product of two covariant iterated integrals with* $\mu = \mathrm{Id}$ *is found by:*

$$\oint_{\nabla}((\omega_1, ..., \omega_r)) \wedge \oint_{\nabla}((\omega_{r+1}, ..., \omega_{r+s})) = \sum_{w \in \mathrm{Sh}(r,s)} (-1)^{\delta(w)} w^* \oint_{\nabla}((\omega_{w(1)}, ..., \omega_{w(r+s)})),$$

where $\delta(w)$ *is as in Proposition A.1.7(a), and*

$$w^* : E_{w(1),x} \otimes ... \otimes E_{w(r+s),x} \longrightarrow \big(E_{1,x} \otimes ... \otimes E_{r,x}\big) \otimes \big(E_{r+1,x} \otimes ... \otimes E_{r+s,x}\big)$$

is the permutation map corresponding to w.

(b) For any $\mu : E_1 \otimes ... \otimes E_r \to E_0$ *as before, the de Rham differential of a covariant iterated integral (as a form on* $P_x^y X$ *with values in* $E_{0,x}$*) is found by*

$$\begin{aligned} d_{\mathrm{DR}} \oint_{\nabla}^{\mu} (\omega_1, ..., \omega_n) \; &= \; - \sum_{\nu=1}^{r} (-1)^{\sum_{\nu' < \nu} m'_\nu} \oint_{\nabla}^{\mu} (\omega_1, ..., \nabla_\nu \omega_\nu, ..., \omega_r) - \\ & - \sum_{\nu=1}^{r-1} (-1)^{\sum_{\nu' \leq \nu} (m_{\nu'}+1)} \oint_{\nabla}^{\mu'} (\omega_1, ..., \omega_\nu \wedge \omega_{\nu+1}, ..., \omega_r) \\ & + \sum_{\nu=1}^{r} (-1)^{\sum_{\nu' < \nu} (m_{\nu'}+1)} \oint_{\nabla}^{\mu''_\nu} (\omega_1, ..., \omega_{\nu-1}, F_{\nabla_\nu}, \omega_\nu, ..., \omega_n). \end{aligned}$$

Here $F_{\nabla_\nu} \in \Omega^2_X \otimes \mathrm{End}(E_\nu)$ *is the curvature of* ∇, *while*

$$\mu'_\nu : E_1 \otimes ... \otimes (E_\nu \otimes E_{\nu+1}) \otimes ... \otimes E_r \longrightarrow E_0$$

is obtained from μ *using the associativity of tensor product, and*

$$\mu''_\nu : E_1 \otimes ... \otimes E_{\nu-1} \otimes \mathrm{End}(E_\nu) \otimes E_\nu \otimes ... \otimes E_r \longrightarrow E_0$$

is obtained by composing μ *with the action map* $\mathrm{End}(E_\nu) \otimes E_\nu \to E_\nu$.

Proof: (a) This follows, just like the corresponding part of Proposition A.1.7, by decomposing the product of two simplices $\Delta_r \times \Delta_s$ into $\binom{r+s}{r}$ simplices labelled by shuffles.

(b) First, it is enough to prove the statement under the assumption that $E_0 = E_1 \otimes ... \otimes E_r$ and $\mu = \mathrm{Id}$. Indeed, the general statement follows from that one by applying a given $\mu : E_1 \otimes ... \otimes E_r \to E_0$ to the above "universal" statement. So the rest of the proof will be under the assumption that $\mu = \mathrm{Id}$.

Second, it is enough to assume that each bundle E_ν is trivial. Indeed, we can embed each E_ν a direct sum of a trivial bundle: $E_\nu \oplus E'_\nu = X \times V_\nu$, where V_ν is a finite-dimensional $\mathbb{R}$-vector space. We can choose a connection on E'_ν in an arbitrary way thus getting a connection on $X \times V_\nu$. After this, each form ω_ν can be considered as a form with values in V_ν, and the statement follows if we know it for the trivial bundles $X \times V_\nu$, by applying the projection $V_0 = E_{0,x} \oplus E'_{0,x} \to E_{0,x}$. So the rest of the proof will be under the additional assumption that $E_\nu = X \times V_\nu$ for each ν.

Consider the tensor algebra

$$\text{(A.3.7)}\quad T \;=\; \bigoplus_{d=0}^{\infty} (V_1 \oplus ... \oplus V_r)^{\otimes d} \;=\; \bigoplus_{d=0}^{\infty} \bigoplus_{\nu_1,...,\nu_d=1}^{r} Z_{\nu_1} ... Z_{\nu_d} \cdot (V_{\nu_1} \otimes ... \otimes V_{\nu_d}).$$

Here the monomials in noncommuting variables $Z_1, ..., Z_r$ are added as a bookkeeping device, and to emphasize the analogy with Example A.1.5(b-c). Thus an element of T can be seen as a noncommutative polynomial in the Z_i, but with the coefficient at each monomial lying in the corresponding tensor product of vector spaces as specified. We introduce a grading on T by putting $\deg(Z_\nu) = 1 - m_\nu$. We also consider the completion $\widehat{T}$ of T with respect to the powers of the ideal generated by the Z_ν of degree 0. We view $\widehat{T}$ as the algebra of formal series in the Z_ν, with coefficient at each monomial being as above. For instance, if each $V_\nu = \mathbb{R}$, then $T = \mathbb{R}\langle Z \rangle$ is the algebra of noncommutative polynomials, and $\widehat{T} = \mathbb{R}\langle\langle Z \rangle\rangle$, as in Example A.1.5(c). Let $\mathrm{End}(T)$ be the algebra of endomorphisms of T as an $\mathbb{R}$-vector space, and $\mathrm{End}(\widehat{T})$ be the algebra of continuous endomorphisms of $\widehat{T}$ as a topological vector space.

We have a natural connection in the trivial bundle $X \times T$ on X, induced by the ∇_ν on $E_\nu = X \times V_\nu$, and denoted by ∇'. It is compatible with the direct sum decomposition in the RHS of (A.3.7) as well as with the tensor product. Let write the connection ∇_ν as $\nabla_\nu = d_{\mathrm{DR}} - A'_\nu$, with $A'_\nu \in \Omega^1_X \otimes \mathrm{End}(V_\nu)$ and write accordingly $\nabla' = d_{\mathrm{DR}} - A'$. Using the terminology of §1.2, we view A' as a graded connection with values in the graded associative algebra $\mathrm{End}(T)$ (considered as a graded Lie algebra in the usual way).

In addition, we have a 1-form

$$\text{(A.3.8)}\qquad A'' \;=\; \sum_{\nu=1}^{r} \omega_\nu \cdot Z_\nu \;\in\; (\Omega^\bullet_X \otimes T)^1 \;\subset\; (\Omega^\bullet_X \otimes \mathrm{End}(T))^1,$$

where the embedding on the right is via action of T on itself by left multiplication. We use it to construct a new graded connection $A^\bullet = A' + A''^\bullet$ on X with values in T, which is no longer compatible with the product. Let $\mathrm{End}(\widehat{T})^{\leq 0} \subset \mathrm{End}(\widehat{T})$ be the subalgebra of endomorphisms which take any homogeneous element into a (possibly infinite) sum of elements of equal or lesser degrees. Then $\mathrm{End}(\widehat{T})^{\leq 0}$ can be represented as a projective limit of $\mathbb{Z}_-$-graded associative algebras $R^\bullet$ with all graded components finite-dimensional. So applying the Picard series (A.2.2) to such algebras $R^\bullet$ and taking the projective limit, we associate to $A^\bullet$ an invertible element $M_A \in (\Omega^\bullet_{P_x^y X} \otimes \mathrm{End}(\widehat{T}))^0$.

Lemma A.3.9. *Denoting by $1_T \in T$ the unit element, we have:*

$$M_A(1_T) \;=\; \sum_{d=0}^{\infty} \sum_{\nu_1,\dots,\nu_d=1}^{r} Z_{\nu_1}\dots Z_{\nu_d} \cdot \ointclockwise_{\nabla} ((\omega_{\nu_1},\dots,\omega_{\nu_d})) \;\in\; \Omega^\bullet_{P_x^y T} \otimes \widehat{T}.$$

Here in the RHS we use right covariant iterated integrals.

Proof: This follows directly from comparison with the Picard series. $\square$

Notice that the curvature of the graded connection A has the form

$$F_A \;=\; F_{\nabla'} \;-\; \sum_{\nu'<\nu} [Z_{\nu'}, Z_\nu] \cdot \omega_{\nu'} \wedge \omega_\nu, \tag{A.3.10}$$

Notice further that the Schlessinger-Chen formula (Proposition A.2.4) is applicable to our situation since it is a projective limit of situations considered in that proposition. So we write

$$d_{\mathrm{DR}} M_A = M_A \wedge B, \tag{A.3.11}$$

where B is given by the integral in the Schlessinger-Chen formula. We then compare coefficients in (A.3.11) at $Z_1 \dots Z_r$ and use use the (already proved) part (a) of Proposition A.3.6 to express the products of iterated integrals appearing on the right. Note that the multiplication by M_A in the RHS of (A.3.11) accounts for the difference between $\ointctrclockwise$ and $\ointclockwise$. This established part (b) of the Proposition.

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