ON CYCLOTOMIC NUMBERS AND REDUCTION MAP FOR K-THEORY OF THE INTEGERS

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Abstract. We apply recently proven compatibility of Beilinson and Soulé elements in K-theory to investigate density of rational primes p, for which the reduction map

$$K_{2n+1}(\mathbb{Z}) \to K_{2n+1}(\mathbb{F}_p)$$

is nontrivial. Here n is an even, positive integer and $\mathbb{F}_p$ denotes the field of p elements. In the proof we use arithmetic of the cyclotomic numbers

$$u_k^{(n)} = \prod_{(a,l)=1;\, 1\leq a<l^k} (1-\xi_{l^k}^a)^{a^n},$$

where l is an odd prime, $k \geq 1$ and ξ_{l^k} is a primitive root of unity of order l^k. We investigate divisibility of the number $u_k^{(n)}$ in the group of certain S-units of the cyclotomic field $\mathbb{Q}(\xi_{l^k})$. It turns out that the divisibility of $u_k^{(n)}$ is bounded from above by the maximal power of l dividing the index of the subgroup of $K_{2n+1}(\mathbb{Z})$ generated by a nontorsion element which maps to the Beilinson element. We reformulate the Vandiver conjecture on the class group of the cyclotomic field $\mathbb{Q}(\xi_l)$ in terms of divisibility properties of $u_k^{(n)}$. Using the density computation on the reduction map we describe an infinite family of elements of the Brauer group of the field $\mathbb{Q}(\xi_{l^k})$.

1. Introduction

It is known that the problem of computing K-groups of the rational integers is closely related to deep questions in number theory, such as the Vandiver conjecture and the Iwasawa cyclicity conjecture (cf. [K], section 3, [BG1], section 5 and [BG2]). On the other hand, investigation of the K-groups of $\mathbb{Z}$ can provide some valuable information about class groups of cyclotomic fields. Let A and $A^{[i]}$ denote the Sylow l-subgroup of the ideal class group of $\mathbb{Q}(\xi_l)$ and the ith eigensubmodule of A under

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2000 *Mathematics Subject Classification.* Primary 19F27; Secondary 19Fxx.

Part of this work was done while the author was visiting the SFB 343 ,,Diskrete Strukturen in der Mathematik" in Bielefeld and the Max-Planck-Institute für Mathematik in Bonn. He would like to thank both institutions for hospitality and support. In Poznań, the author was partially supported by the KBN grant 2 P03A 023 17.

the action of the Galois group $G(\mathbb{Q}(\xi_l)/\mathbb{Q})$, respectively. Kurihara observed in [K], Cor. 3.8 that the vanishing of $K_4(\mathbb{Z})$ (proven by Rognes, Soulé and Voevodsky), implies that the eigenspace $A^{[l-3]}$ is trivial. Soulé using K-theory and Voronoi's reduction theory proved in [So3] that for a fixed number n the eigenspace $A^{[l-n]}$ vanishes for primes l bigger than a certain large but explicitly computed number. We refer the reader to [BG1] and to [BG2], where the subgroup of infinitely divisible elements in $K_{2m}(\mathbb{Q})$ was investigated in connection with the classical conjectures mentioned above.

Our first objective in this paper is to use K-theory to investigate certain l-units $u_k^{(n)}$, $u_k^{(n)+} \in \mathbb{Z}[\xi_{l^k}]$ (see Theorem B and Definition 2.1, below) which emerge naturally when one explores the structure of the group $K_{2n+1}(\mathbb{Z})$ for n even and positive. The group has a nontorsion element b_{2n+1} which is provided by the compatibility of the Beilinson element in $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Q}$ with the Soulé elements in $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l$ for all odd primes l. The compatibility follows by results of Beilinson and Deligne, [BD], and also Huber and Wildeshaus, [HW] who proved Conjecture 6.2, [BK] of Bloch and Kato. Using standard techniques of K-theory we show that the element b_{2n+1} maps onto the element $u_k^{(n)} \otimes \xi_{l^k}^{\otimes n}$ of the group of twisted S-units of the cyclotomic field $E = \mathbb{Q}(\xi_{l^k})$ (see Lemma 2.9, below). This implies the following result about the numbers $u_k^{(n)}$ and $u_k^{(n)+}$.

Theorem A. *[Th. 2.14]*
Assume that the l-torsion part of the group $K_{2n+1}(\mathbb{Z})$ is trivial. Let l^{k_0} be the largest power of l which divides the index of the subgroup of $K_{2n+1}(\mathbb{Z})$ generated by the element b_{2n+1}. Let k be a natural number not smaller than k_0.

(1) *If $u_k^{(n)} = \eta^{l^{k_1}}$, for some $\eta \in \mathbb{Z}[\xi_{l^k}]$, then $k_1 \leq k_0$.*
(2) *The same property holds true for ${u_k}^{(n)+}$, if we assume in addition that l does not divide $(n+1)C_n$, where C_n denotes the product of denominators of the divided Bernoulli numbers $\frac{B_2}{2}, \frac{B_4}{4}, \ldots, \frac{B_n}{n}$ (written in the lowest terms).*

It is well known that divisibility properties of cyclotomic units are related to the Vandiver conjecture on the ideal class group of the field $\mathbb{Q}(\xi_l)$ (see Section 8.3 of Washington's book [W]). Such a relation exists also in the case of the divisibilty property considered in Theorem A. This is indicated by the following proposition which is probably known to experts.

Theorem B. *[Prop. 2.15]*
Let l be an odd prime and let n be an even interger such that $2 \leq n \leq l-3$. Let k be a positive integer. The following statements are equivalent.

(1) *The number*
$$u_k^{(n)} = \prod_{(a,l)=1;\, 1 \leq a < l^k} (1 - \xi_{l^k}^a)^{a^n}$$
is not l-divisible in the group $\mathcal{O}_{E,S_k}^{\times}$. Here, S_k is any finite set of primes containing primes over l and such that the class group of $\mathcal{O}_{E,S_k}$ vanishes.

(2) *The eigensubmodule $A^{[l-1-n]}$ is trivial.*

We deduce Theorem B from the formula for the index of the subgroup of the group $H^1_{cts}(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(n+1))$ generated by the Soulé cyclotomic element, which was proven by Bloch and Kato in [BK] using the main conjecture in Iwasawa theory. Note that Thaine shoved (see [Th], Prop. 2, p. 305), without using main conjecture, that $A^+ = \bigoplus_{even} A^{[i]} = 0$, (i.e., the Vandiver conjecture holds true for l), if and only if none of $u_1^{(n)}$ is an lth power in $\mathbb{Z}[\xi_l]$, for all n even and $2 \leq n \leq l-3$.

Our second goal in this paper is to apply the arithmetic of the numbers $u_k^{(n)}$ to investigate $K_*(\mathbb{Z})$. We consider the reduction map in K-theory of $\mathbb{Z}$

$$K_{2n+1}(\mathbb{Z}) \longrightarrow K_{2n+1}(\mathbb{F}_p)_l,$$

where l is a fixed odd prime and p is another prime such that $l|(p^{n+1}-1)$. It follows by [AB], Theorem 1, p. 68 that the reduction map is nontrivial for infinitely many primes p, if n is an even, positive integer. Using the arithmetic of the Kummer fields $E(\sqrt[l^k]{u_k})$ and the theorem of Chebotarev we refine the result of [AB] in the case of the group $K_{2n+1}(\mathbb{Z})$.

Theorem C. *[Th. 3.6]*
Let l be an odd prime, n an even, positive integer such that l is relatively prime to $n+1$. Assume that l does not divide the index of the subgroup of $K_{2n+1}(\mathbb{Z})$ generated by the element b_{2n+1}. Let k be a fixed positive integer. Denote by M_k the set of rational primes p which satisfy the following two conditions:

(1) *the l-torsion part of the group $K_{2n+1}(\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}/l^k$*
(2) *the reduction map $K_{2n+1}(\mathbb{Z}) \longrightarrow K_{2n+1}(\mathbb{F}_p)_l$ is nontrivial.*

Then the set M_k has positive density, which equals $\frac{l^k-1}{l^{2k}}$.

Observe that the set $M = \cup_{k=1}^{\infty} M_k$ consists of primes p for which the reduction map $K_{2n+1}(\mathbb{Z}) \to K_{2n+1}(\mathbb{F}_p)_l$ has nontrivial image. Since the sets M_k for $k \geq 1$ are pairwise disjoint, it is natural to expect that the set M has Dirchlet density and that it is equal to $\sum_{k=1}^{\infty} \frac{l^k-1}{l^{2k}} = \frac{l}{l^2-1}$. By definition, the density of M should be

$$\lim_{x\to\infty} \frac{\#\{p \leq x: \quad p \in M\}}{\#\{p \leq x\}}.$$

It does not seem to be known whether the limit exists.

In the last part of the paper we prove the following consequence of Theorem C.

Theorem D. *[Prop 3.15]*
Let n, l and k be as in the assumptions of Theorem C. Then the following statements hold.

(1) *The set of primes p for which the l-torsion part of the group $K_{2n+1}(\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}/l^k$ and such that the class of the central simple algebra*

$A_{\xi_{l^k}}(u_k^{(n)}, p)$ is a nontrivial element of the Brauer group $Br(E)$ has positive density which equals $\frac{l^k-1}{l^{2k}}$.

(2) *Assume in addition that l does not divide the number $(n+1)C_n$. Then the set of primes for which the l-torsion part of the group $K_{2n+1}(\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}/l^k$ and such that the class of the algebra $A_{\xi_{l^k}}(u_k^{(n)+}, p)$ in $Br(E^+)$ is nontrivial has positive density which equals $\frac{l^k-1}{l^{2k}}$.*

It is clear that the numbers $u_k^{(n)}$ considered in this paper are implicitly present in the work of Soulé on cyclotomic elements in K-theory and étale cohomology (cf. [So2] and proof of Lemma 2.9 given below). For some other applications of $u_k^{(n)}$ (where $k \geq 1$) to the arithmetic geometry, we refer the reader to papers by Ihara, see for example [I].

Acknowledgements

During the work on this paper I benefited from discussions or from correspondence with Grzegorz Banaszak, Radan Kućera and B.Z. Moroz. I am greatful to all of them. I would like to thank Don Zagier for suggesting Remark 2.17.

Notation

n is a positive, even integer

l is a rational prime which is assumed to be odd, if not stated otherwise

l^k is a fixed power of l which will be specified later

$\xi_{l^k} = e^{\frac{2\pi i}{l^k}}$ is the fixed primitive root of 1 of order l^k.

$E = \mathbb{Q}(\xi_{l^k})$ is the cyclotomic field

$\mathbb{Q}(\xi_{l^k})^+$ denotes the maximal real subfield of E

A denote the Sylow l-subgroup of the ideal class group of $\mathbb{Q}(\xi_l)$

$A^{[i]}$ for $0 \leq i \leq l-2$ is the ith component of A cf. [W], p. 101

$l^k || m$ means that l^k is the largest power of l which divides the natural number m

B_l denotes the lth torsion part of an abelian group B

$B[l^k]$ denotes the subgroup of the abelian group B consisting of elements od order dividing l^k

$\mathcal{O}_F$ denotes the ring of integers of the number field F

$\mathcal{O}_{F,S}$ is the ring of S-integers of the number field F, defined for a set of primes S

$K_m(R)$ denotes the mth algebraic K-theory group of the commutative ring R with identity, as was defined by Quillen.

$H^i_{cts}(-,-)$ denotes the ith continuous étale cohomology group, as was defined by Jannsen in [J].

2. Some computations with cyclotomic elements

Definition 2.1. Let k be a positive integer.

(1) We put

$$u_k^{(n)} = \prod_{1\le a<l^k,\,(a,l)=1} (1-\xi_{l^k}^a)^{a^n}$$

$$u_k^{(n)+} = \prod_{1\le a<\frac{l^k}{2},\,(a,l)=1} \left[-\xi_{l^k}^{-a}\left(\frac{1-\xi_{l^k}^a}{1-\xi_{l^k}}\right)^2\right]^{a^n}.$$

¿From now on, in order to simplify notation we will not indicate that $u_k^{(n)}$ and $u_k^{(n)+}$ depend on n and denote them simply by u_k and u_k^+, respectively. This should not lead to any confusion, since most of the time the number n will be fixed.

(2) We denote by C_n the product of denominators of the numbers $\frac{B_2}{2}$, $\frac{B_4}{4}$, $\dots$, $\frac{B_n}{n}$ (put in the lowest terms), where B_m denotes the mth Bernoulli number.

Our basic reference for the theory of cyclotomic fields is the book of Washington, [W]. For some more information on the numbers u_k we refer the reader to [Th], where the case $k=1$ is treated and also to [I], where the case of arbitrary k is considered. Note that the number C_n defined above depends only on n. In order to check this, let us write $B_{2i} = \frac{N_{2i}}{D_{2i}}$ and $\frac{B_{2i}}{2i} = \frac{N'_{2i}}{D'_{2i}}$ with $D_{2i}>0$, $D'_{2i}>0$ and $(D_{2i}, N_{2i}) = 1$, $(D'_{2i}, N'_{2i}) = 1$, for $1\le i\le \frac{n}{2}$. It follows by the theorem proven by Glaisher [G] that a prime l divides the number D_{2i} if and only if it divides D'_{2i}. The theorem of Clausen-von Staudt ([W], Theorem 5.10. p. 56) implies that D_{2i} is a square free integer such that its prime factors are primes l for which $l-1$ divides $2i$.

Lemma 2.2.

(1) *The number u_k is an l-unit and the number u_k^+ is a unit in the ring $\mathbb{Z}[\xi_{l^k}]$.*

(2) *Assume that l is relatively prime to the number $(n+1)C_n$. Then $u_k^+ \in \mathbb{Z}[\xi_{l^k}+\xi_{l^k}^{-1}]$, i.e., the unit u_k^+ is real.*

Proof. (1) See [W], Chapter 8.
(2) Let σ_{-1} denote the complex conjugation. It is easy to check that

$$\sigma_{-1}(u_k^+) = \xi_{l^k}^{2\sum' a^n} u_k^+, \tag{2.3}$$

where the summation is over a such that $1\le a<\frac{l^k}{2}$ and $(a,l)=1$. Observe that

$$2{\sum}' a^n \equiv \sum_{1\le a<l^k,\,(a,l)=1} a^n \mod l^k,$$

hence it suffices to show that the last sum is divisible by l^k. Using the classical formula due to Jacob Bernoulli we obtain

$$\begin{aligned}\sum_{1\le a<l^k,\,(a,l)=1} a^n &= \sum_{1\le a\le l^k} a^n - \sum_{1\le a\le l^{k-1}} l^n a^n \\ &= \frac{1}{n+1}[B_{n+1}(l^k) - B_{n+1}] - \frac{l^n}{n+1}[B_{n+1}(l^{k-1}) - B_{n+1}],\end{aligned}$$

where

$$B_m(x) = \sum_{i=0}^{m} \binom{m}{i} B_i x^{m-i}.$$

By a simple calculation we obtain

$$\begin{aligned}\sum_{1\le a<l^k,\,(a,l)=1} a^n &= \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i} B_i(l^{k(n+1-i)} - l^{(k-1)(n+1-i)+n}) \\ &= \frac{(l-1)l^{k(n+1)-1}}{n+1} + \sum_{i=1}^{n}\binom{n}{i-1} l^{k(n+1-i)}(1-l^{i-1})\frac{B_i}{i}. \qquad (2.4)\end{aligned}$$

The claim (2) follows, since by assumption l^k divides the number on right hand side of the equality (2.4). $\square$

Lemma 2.5. *Let $v_l(-)$ denote the l-adic valuation. Then we have*

$$v_l(p^{n+1} - 1) = v_l(n+1) + v_l(p-1).$$

Proof. We use the l-adic logarithm. Let us put $m = n+1$. Then we have

$$log_l(p^m) = m\, log_l(1+(p-1)) = m\, l^{v_l(p-1)}\epsilon_1$$

$$log_l(p^m) = log_l(1+(p^m-1)) = l^{v_l(p^m-1)}\epsilon_2,$$

where ϵ_1 and ϵ_2 are l-adic units. $\square$

Consider the following sequence of maps.

$$
\begin{array}{c}
K_{2n+1}(\mathbb{Z}) \\
\downarrow{\scriptstyle \alpha_0} \\
(K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l)/l^k \\
\downarrow{\scriptstyle \alpha_1} \\
H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))/l^k \\
\downarrow{\scriptstyle \alpha_2} \\
H^1_{cts}(\mathcal{O}_{E,S_k};\, \mathbb{Z}_l(n+1))/l^k \\
\downarrow{\scriptstyle \alpha_3} \\
H^1_{\acute{e}t}(\mathcal{O}_{E,S_k};\, \mathbb{Z}/l^k(n+1)) \\
\downarrow{\scriptstyle \alpha_4}\ \cong \\
\mathcal{O}^{\times}_{E,S_k}/\mathcal{O}^{\times l^k}_{E,S_k} \otimes \mathbb{Z}/l^k(n)
\end{array}
\tag{2.6}
$$

We use étale cohomology and the continuous étale cohomology $H^*_{cts}(-)$ in the sense of Jannsen, [J]. Note that

$$
\begin{aligned}
H^1_{cts}(\mathbb{Z}[\tfrac{1}{l}];\, \mathbb{Z}_l(n+1)) &= \varprojlim H^1_{\acute{e}t}(\mathbb{Z}[\tfrac{1}{l}];\, \mathbb{Z}/l^m(n+1)) \\
H^1_{cts}(\mathcal{O}_{E,S_k};\, \mathbb{Z}_l(n+1)) &= \varprojlim H^1_{\acute{e}t}(\mathcal{O}_{E,S_k};\, \mathbb{Z}/l^k(n+1)),
\end{aligned}
$$

since the $\lim^1$-term vanishes, where the inverse system is over reduction of coefficients maps, [J], p.216, (3.1). Now we will describe maps in the diagram (2.6).

(i) The number k is a positive integer. It will be specified in Theorem 2.14 and further in the beginning of section 3.

(ii) The map labeled α_0 is induced by the map

$$K_{2n+1}(\mathbb{Z}) \to K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l$$

which sends x to $x \otimes 1$.

(iii) The map α_1 is induced by the Dwyer-Friedlander map from K-theory to continous étale cohomology, [DF], Proposition 4.4 and Remark 8.8. It is a surjection by [DF], Theorem 8.7 and Remark 8.8. The map α_1 is an isomorphism for primes l such that $K_{2n+1}(\mathbb{Z})_l = 0$.

(iv) We put $E = \mathbb{Q}(\mu_{l^k})$ and we choose S_k to be a finite set of primes of E such that it contains primes over l and that the class group of the ring of S_k-integers $\mathcal{O}_{E,S_k}$ vanishes.

(v) The arrow α_2 is an imbedding which comes from the Hochschild-Serre spectral sequence of the injection $\mathbb{Z}[\frac{1}{l}] \to \mathcal{O}_{E,S_k}$.

(vi) The map α_3 is an imbedding which comes from the long exact sequence in cohomology induced by the exact sequence of l-adic sheaves

$$0 \longrightarrow \mathbb{Z}_l(n{+}1) \xrightarrow{\times l^k} \mathbb{Z}_l(n{+}1) \longrightarrow \mathbb{Z}/l^k(n{+}1) \longrightarrow 0.$$

(vii) Finally, the map α_4 is an isomorphism provided by the Kummer exact sequence, since by (iv) $Pic\,\mathcal{O}_{E,S_k} = 0$.

Consider the cyclotomic element

$$c_l = (c_{l^m}) \in H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1)) = \varprojlim H^1_{\acute{e}t}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}/l^m(n+1)) \tag{2.7}$$

constructed in cohomology by Soulé [So2], (4.1)-(4.3), for l odd. It is defined as the norm compatible system of elements

$$c_{l^m} = Tr_{\mathbb{Q}(\mu_{l^m})/\mathbb{Q}}((1-\xi_{l^m}) \cup \xi_{l^m}^{\otimes n}), \tag{2.8}$$

where

(1) $\xi_{l^m} = e^{\frac{2\pi i}{l^m}}$, $1-\xi_{l^m} \in H^1(\mathbb{Z}[\xi_{l^m}, \frac{1}{l}];\, \mathbb{Z}/l^m(1))$

(2) $\xi_{l^m}^{\otimes n} \in H^0(\mathbb{Z}[\xi_{l^m}, \frac{1}{l}];\, \mathbb{Z}/l^m(n))$

(3) $\cup$ is the cup product and $Tr_{\mathbb{Q}(\mu_{l^m})/\mathbb{Q}}$ is the transfer map in étale cohomology

$$Tr_{\mathbb{Q}(\mu_{l^m})/\mathbb{Q}}\colon \quad H^1_{\acute{e}t}(\mathbb{Z}[\xi_{l^m}, \frac{1}{l}];\, \mathbb{Z}/l^m(n+1)) \quad \longrightarrow \quad H^1_{\acute{e}t}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}/l^m(n+1)).$$

In a similar way, using produtcs, Bott elements and transfer maps in K-theory with finite coefficients, Soulé constructed cyclotomic elements in $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l$, for l odd, [So4], p. 384. Note that the cyclotomic elements of Soulé exist also when $l = 2$ in which case one has to modify the construction at the level $m = 1$, see [ABG], Def. 2.2 and Def. 1.5. The group $K_{2n+1}(\mathbb{Z})$ has rank one by results of Borel, [Bo]. There is a nontrivial element in $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Q}$ which was constructed by Beilinson [Be], (7.1.5). Corollary 9.8, [HW] implies that there is a nontorsion element $b_{2n+1} \in K_{2n+1}(\mathbb{Z})$ which maps to the Beilinson element in $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Q}$ and such that for l odd, the image of b_{2n+1} by

$$K_{2n+1}(\mathbb{Z}) \quad \longrightarrow \quad K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l \quad \longrightarrow \quad H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))$$

is the cyclotomic element c_l. The second map above is the Dwyer-Friedlander map. Note that the element $b_{2n+1} \otimes 1$ is uniquely determined in the group $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}]$. By some abuse of terminology, in the sequel we call $b_{2n+1} \in K_{2n+1}(\mathbb{Z})$ the Beilinson element. In order to simplify notation we will use the same symbol for element of the group $\mathcal{O}^{\times}_{E,S_k}$ and its image in the quotient group $\mathcal{O}^{\times}_{E,S_k}/\mathcal{O}^{\times l^k}_{E,S_k}$; what is meant will be clear from the context.

Lemma 2.9.

(1) *The composition of maps in the diagram (2.6) sends the Beilinson element* b_{2n+1} *onto the element* $u_k \otimes \xi_{l^k}^{\otimes n}$ *of the group* $\mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n)$.

(2) *Assume that* l *does not divide* $(n+1)C_n$, *where* C_n *is the number from Definition 2.1 (3). Then we have* $u_k \otimes \xi_{l^k}^{\otimes n} = u_k^+ \otimes \xi_{l^k}^{\otimes n}$ *in the group* $\mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n)$.

Proof. (1) This is a strightforward calculation. We give details of the proof for the convenience of the reader. By the naturality of the long exact sequence in cohomology, the imbedding $\mathbb{Z}[\frac{1}{l}] \to \mathcal{O}_{E,S_k}$ induces the following commutative diagram.

$$\begin{array}{ccc} H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))/l^k & \xrightarrow{\alpha_3'} & H^1_{\acute{e}t}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}/l^k(n+1)) \\ {\scriptstyle \alpha_2}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_2'} \\ H^1_{cts}(\mathcal{O}_{E,S_k};\, \mathbb{Z}_l(n+1))/l^k & \xrightarrow{\alpha_3} & H^1_{\acute{e}t}(\mathcal{O}_{E,S_k};\, \mathbb{Z}/l^k(n+1)) \end{array} \tag{2.10}$$

It follows that the image of the element b_{2n+1} under the composition (2.6) is the same as if we replace the map α_2 by α_3' and the map α_3 by α_2', respectively. This and the definition of c_l gives

$$\alpha_3\alpha_2\alpha_1\alpha_0(b_{2n+1}) = \alpha_2'(Tr_{\mathbb{Q}(\mu_{l^k})/\mathbb{Q}}((1-\xi_{l^k}) \cup \xi_{l^k}^{\otimes n}).$$

By the basic property of the transfer map, [So1], III.1.5, the last element equals

$$\sum_{\sigma_a \in G} (1-\xi_{l^k})^{\sigma_a} \cup \xi_{l^k}^{\otimes n\sigma_a} \in H^1_{\acute{e}t}(\mathcal{O}_{E,S_k};\, \mathbb{Z}/l^k(n+1)). \tag{2.11}$$

Here, we have put $G = Gal(E/\mathbb{Q})$. By the bilinearity of the cup product the element (2.11) can be identified with

$$\sum_{1 \le a < l^k,\, (a,l)=1} (1-\xi_{l^k}^a)^{a^n} \cup \xi_{l^k}^{\otimes n}. \tag{2.12}$$

Applying the isomorphism α_4 to (2.12) gives

$$\prod_{1 \le a < l^k,\, (a,l)=1} (1-\xi_{l^k}^a)^{a^n} \otimes \xi_{l^k}^{\otimes n} \quad \in \quad \mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n). \tag{2.13}$$

To prove (2) observe that in the group $\mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k}$

$$\begin{aligned} \prod_{1 \le a < l^k,\, (a,l)=1} (1-\xi_{l^k}^a)^{a^n} &\equiv (1-\xi_{l^k})^{\sum a^n} \prod_{1 \le a < l^k,\, (a,l)=1} \left[\frac{1-\xi_{l^k}^a}{1-\xi_{l^k}}\right]^{a^n} \\ &\equiv (1-\xi_{l^k})^{\sum a^n} \prod_{1 \le a < \frac{l^k}{2},\, (a,l)=1} \left[\frac{1-\xi_{l^k}^a}{1-\xi_{l^k}}\right]^{a^n} \left[\frac{1-\xi_{l^k}^{(l^k-a)}}{1-\xi_{l^k}}\right]^{(l^k-a)^n} \\ &\equiv (1-\xi_{l^k})^{\sum a^n} \prod_{1 \le a < \frac{l^k}{2},\, (a,l)=1} \left[-\xi_{l^k}^{-a}\left(\frac{1-\xi_{l^k}^a}{1-\xi_{l^k}}\right)^2\right]^{a^n}, \end{aligned}$$

where the sum is over a such that $1 \leq a < l^k$ and $(a,l) = 1$. The claim (2) follows, since the assumption on l implies that l^k divides $\sum a^n$, as was checked in the proof of Lemma 2.2 (2). $\square$

As a consequence of the compatibility of the Beilinson and the Soulé elements we obtain the following divisibility property of the cyclotomic numbers u_k and $u_k{}^+$. It would be interesting to know whether Theorem 2.14 can be proven by an argument which does not rely on K-theory.

Theorem 2.14. *Assume that the l-torsion part of the group $K_{2n+1}(\mathbb{Z})$ is trivial. Let l^{k_0} be the largest power of l which divides the index of the subgroup of $K_{2n+1}(\mathbb{Z})$ generated by the Beilinson element. Let k be a natural number which is not smaller than k_0.*

(1) *If $u_k = \eta^{l^{k_1}}$, for some $\eta \in \mathbb{Z}[\xi_{l^k}]$, then $k_1 \leq k_0$.*
(2) *The same property holds true for the real unit u_k^+, if we assume in addition that l does not divide $(n+1)C_n$.*

Proof. The map α_0 from (2.6) is a surjection because the group $K_{2n+1}(\mathbb{Z})$ is finitely generated. Since b_{2n+1} is nontorsion and $l^k \geq l^{k_0}$, the element $\alpha_0(b_{2n+1})$ is divisible by at most l^{k_0} in $(K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l)/l^k$. Let x be the generator of $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l$. Observe that the last group equals $\mathbb{Z}_l$, since by assumption $K_{2n+1}(\mathbb{Z})_l = 0$. Then $b_{2n+1} \otimes 1 = l^{k'}x$, for some k' and it is clear that $l^{k_0} \geq l^{k'}$. The map α_1 is an isomorphism, by [DF], Theorem 8.7 and Remark 8.8, and since $K_{2n+1}(\mathbb{Z})_l = 0$.

We prove (1). The order of the images of x by compositions of the imbeddings α_1, α_2, α_3 and α_4 is l^k, because all the groups in (2.6), except for the top one have exponent l^k. Hence, the image of the Beilinson element by (2.6) is divisible in the group

$$\mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n)$$

by at most l^{k_0}. However, by Lemma 2.9 (1) $u_k \otimes \xi_{l^k}^{\otimes n}$ is the image of the Beilinson element by the map (2.6). This implies (1). The second claim follows by (1) and Lemma 2.9 (2). $\square$

In some sense the following proposition can be thought of as an analogue for groups of units of the corollary to the theorem of Furtwängler ([W], Cor. 10.6 and Cor. 10.5, p. 187) which shows that the Vandiver conjecture for the prime l is equivalent to vanishing of the l-Sylow subgroup of the class group of the field $\mathbb{Q}(\xi_{l^k})^+$, for all $k \geq 1$.

Proposition 2.15. *Let l be an odd prime and let n be an even interger such that $1 \leq n \leq l-3$. Let k be a positive integer. The following statements are equivalent.*

(1) *The number*
$$u_k = \prod_{(a,l)=1;\, 1\leq a<l^k} (1-\xi_{l^k}^a)^{a^n}$$
is not l-divisible in the group $\mathcal{O}_{E,S_k}^{\times}$.

(2) *The eigensubmodule $A^{[l-1-n]}$ is trivial.*

Proof. It follows by [K], Corollary 1.5, p. 226 that for an even n as in the assumptions above we have $A^{[l-1-n]} = 0$, if and only if, $H^2_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1)) = 0$. Let C denote the subgroup of $H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))$ generated by the Soulé element c_l. Using the main conjecture of Iwasawa theory Bloch and Kato proved that the number of elements of the group $H^2_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))$ equals the index $[H^1_{cts}(\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}_l(n+1))\ ;\ C]$, ([BK], Theorem 6.1 (i), see formulas (6.8) and (6.9), p. 388). We prove that (1) implies (2). If $A^{[l-1-n]} \neq 0$, then by results of Kurihara and Bloch and Kato *loc. cit.* we obtain that $c_l = lx$ for some nonzero $x \in H^1_{cts}(\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}_l(n+1))$. It was shown in the proof of Lemma 2.9 (1) that the map (2.16)
$$H^1_{cts}(\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}_l(n+1)) \longrightarrow H^1_{cts}(\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}_l(n+1))/l^k \longrightarrow \mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n),$$

(where the second map is the composition of maps α_2, α_3 and α_4 from (2.6)) sends the element c_l to $u_k \otimes \xi_{l^k}^{\otimes n}$. This gives $u_k = d^l y^{l^k} = (dy^{l^{k-1}})^l$ for some $d,\, y \in \mathcal{O}_{E,S_k}^{\times}$, where $d \otimes \xi_{l^k}^{\otimes n} \in \mathcal{O}_{E,S_k}^{\times}/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n)$ is the image of x by the map (2.16). This contradicts (1).

We prove that (2) implies (1). Assume that $A^{[l-1-n]} = 0$. It follows by the results of Kurihara and Bloch and Kato that c_l generates the group $H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))$. At this point we can repeat the proof of Theorem 2.14 (1) in which we replace: the group $K_{2n+1}(\mathbb{Z})$ by the group $H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))$, the element b_{2n+1} by the element c_l and the map α_1 by the identity. Since c_l generates $H^1_{cts}(\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}_l(n+1))$, and since here $k > k_0 = 0$, the same argument as in the proof of Theorem 2.14 (1) shows that $u_k \notin \mathcal{O}_{E,S_k}^{\times l}$. $\square$

Remark 2.17. I am indebted to Don Zagier for the following observation. There is a direct way of showing that for every $k > 1$ the following three statements are eequivalent

(1) u_1 is an lth power in the group $\mathbb{Q}(\xi_l)^{\times}$
(2) u_1 is an lth power in the group $\mathbb{Q}(\xi_{l^k})^{\times}$
(3) u_k is an lth power in the group $\mathbb{Q}(\xi_{l^k})^{\times}$.

In order to see this, observe that for every n and k we have the following equalities in the group $\mathbb{Q}(\xi_{l^k})^\times/\mathbb{Q}(\xi_{l^k})^{\times l}$.

$$\begin{aligned} u_k &\equiv \prod_{1\le b<l} [\prod_{a\equiv b \mod l;\, 1\le a<l^k} (1-\xi_{l^k}^a)]^{b^n} \\ &\equiv \prod_{1\le b<l} [\prod_{1\le j\le l^{k-1}} (1-\xi_{l^k}^{b+lj})]^{b^n} \\ &\equiv \prod_{1\le b<l} [\prod_{\lambda^{l^{k-1}}=1} (1-\lambda\xi_{l^k}^b)]^{b^n} \\ &\equiv \prod_{1\le b<l} [(1-(\xi_{l^k}^b)^{l^{k-1}})]^{b^n} \\ &\equiv \prod_{1\le b<l} (1-\xi_l^b)^{b^n} = u_1 \end{aligned}$$

This shows that (2) and (3) are equivalent. Also, (1) clearly implies (2). To check that (2) implies (1), let us assume that $u_1 \notin \mathbb{Q}(\xi_l)^{\times l}$. Consider the Kummer extension $L = \mathbb{Q}(\xi_l, \sqrt[l]{u_1})$. By assumption it has degree l over $\mathbb{Q}(\xi_l)$. Also, the Galois group $G(L/\mathbb{Q})$ is not abelian (use Proposition 3.2, below for $k = 1$.) Since $G(\mathbb{Q}(\xi_{l^k})/\mathbb{Q})$ is obviously abelian, we see that $L \not\subset \mathbb{Q}(\xi_{l^k})$, hence $\sqrt[l]{u_1} \notin \mathbb{Q}(\xi_{l^k})$.

Remark 2.18. According to the Quillen-Lichtenbaum conjecture the Dwyer-Friedlander map

$$K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l \longrightarrow H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n{+}1))$$

should be an isomorphism for l odd. If we assume the conjecture, then by the result of Bloch and Kato *loc. cit.*, the number l^{k_0} from Theorem 2.14 equals $\#H^2_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n{+}1))$. Hence, using again [K], Corollary 1.5, p. 226 we conclude that the following two statements are equivalent, if the Quillen-Lichtenbaum conjecture is true for $\mathbb{Z}$ and all odd primes.

(1) The Vandiver conjecture holds true for all odd primes l.
(2) The element $b_{2n+1} \otimes 1$ generates the group $K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}]$ for all even, positive n.

Remark 2.19. Recall that G. Banaszak and the author showed that the following statements are equivalent for a fixed odd prime l (see [BG1], Prop. 8 and Prop. 9. pp. 248).

(1) The Vandiver conjecture holds true for l.
(2) The groups of infinitely l-divisible elements $D_{n+1}(\mathbb{Q}) \subset K_{2n}(\mathbb{Q})_l$ vanish for all n, even and positive.
(3) The étale K-theory group $K_4^{et}(\mathbb{Z}[\frac{1}{l}, \xi_l{+}\xi_l^{-1}])$ is trivial.

We refer the reader to [BG2], [K], [K-NQD] and to [So3] for other results relating the conjectures on the class groups of the cyclotomic fields to the arithmetic of the groups $K_*(\mathbb{Z})$.

Remark 2.20. Note that Ichimura and Sakaguchi in [IS], Cor., p. 62 proposed another reformulation of the Vandiver conjecture in terms of Kummer character which is defined using the cyclotomic numbers $u_k^{(n)}$ (see also [K-NQD], Cor. 4.7 (ii), p. 444).

3. Reduction map and densities of primes

As in the previous sections n is an even, positive integer. ¿From now on we assume that the odd prime l does not divide the number $n+1$ and that l is relatively prime to the index of the subgroup of $K_{2n+1}(\mathbb{Z})$ generated by the Beilinson element b_{2n+1}. The last assumption means that the number k_0 from Theorem 2.14 equals 0. Hence, in particular $K_{2n+1}(\mathbb{Z})_l = 0$. We consider the image of the reduction map

$$\phi_{p,l}\colon \quad K_{2n+1}(\mathbb{Z}) \longrightarrow K_{2n+1}(\mathbb{F}_p)_l$$

onto the l-torsion part of the K-theory of the finite field $\mathbb{F}_p$, where p is an odd prime.

Consider the diagram of field extensions

$$\begin{array}{c} L = E(\sqrt[l^k]{u_k}) \\ \Big| \\ E = \mathbb{Q}(\xi_{l^k}) \\ \Big| \\ \mathbb{Q} \end{array}$$

where L denotes the splitting field of the polynomial $x^{l^k} - u_k$. By the assumption on l we have $l^{k_0} = 1$. It follows by Theorem 2.14 (1) that u_k is not an lth power, hence the polynomial $x^{l^k} - u_k$ is irreducible over E. We have the exact sequence of Galois groups

$$0 \longrightarrow G(L/E) \longrightarrow G(L/\mathbb{Q}) \longrightarrow G(E/\mathbb{Q}) \longrightarrow 0. \tag{3.1}$$

For $1 \leq a < l^k$ and $(a, l) = 1$ we denote by σ_a the element of $G(E/\mathbb{Q}) \cong (\mathbb{Z}/l^k)^\times$ such that

$$\sigma_a(\xi_{l^k}) = \xi_{l^k}^a.$$

Every element γ of $G(L/E) \cong \mathbb{Z}/l^k$ is determined by its action on $\sqrt[l^k]{u_k}$. Let

$$\gamma(\sqrt[l^k]{u_k}) = \xi(\gamma)\sqrt[l^k]{u_k},$$

for an l^kth root of unity $\xi(\gamma)$. It is clear that the order of γ in the group $G(L/E)$ equals the order of the root $\xi(\gamma)$. We consider the action of $G(E/\mathbb{Q})$ on $G(L/E)$ by conjugation. With this action the group $G(L/\mathbb{Q})$ becomes a semidirect product $G(L/E) \rtimes G(E/\mathbb{Q})$.

Proposition 3.2.

(1) *For* $\gamma \in G(L/E)$ *and* $\sigma_a \in G(E/\mathbb{Q})$ *as above, we have* $\sigma_a \gamma \sigma_a^{-1} = \gamma^{a^{n+1}}$.

(2) *Let* $\gamma\sigma_a$ *be an element of* $G(L/\mathbb{Q})$, *where* σ_a *and* γ *are as in (1). Then for any integer* m *we have*

$$(\gamma\sigma_a)^m = \begin{cases} \gamma^{x_m}\sigma_{a^m} & \textit{if} \quad a^{n+1} \not\equiv 1 \mod l^k \\ \gamma^m \sigma_{a^m} & \textit{if} \quad a^{n+1} \equiv 1 \mod l^k, \end{cases}$$

where $x_m = \frac{a^{m(n+1)}-1}{a^{(n+1)}-1}$.

Proof. (1) To simplify notation, we put $\xi_0 = \xi(\gamma)$. Lift σ_a to $\widetilde{\sigma}_a \in G(L/\mathbb{Q})$. Let b be such that $ab \equiv 1 \mod l^k$. Then in the group $\mathcal{O}_E^{\times}/\mathcal{O}_E^{\times l^k}$ we have

$$\widetilde{\sigma}_a^{-1}(u_k) \equiv \widetilde{\sigma}_b(u_k) \equiv \prod_{1 \le a < l^k,\, (a,l)=1} (1-\xi_{l^k}^{ab})^{(ab)^n a^n} \equiv u_k^{a^n}.$$

Hence, there exists $z \in \mathcal{O}_E[\frac{1}{l}]^{\times}$ such that equality

$$\widetilde{\sigma}_a^{-1}(u_k) = {u_k}^{a^n} z^{l^k}$$

holds in the group $\mathcal{O}_E[\frac{1}{l}]^{\times}$. This gives the following equality in $\mathcal{O}_L[\frac{1}{l}]^{\times}$

$$\widetilde{\sigma}_a^{-1}(\sqrt[l^k]{u_k}) = (\sqrt[l^k]{u_k})^{a^n} z\xi_1,$$

where $\xi_1 \in \mathcal{O}_E^{\times}$ is a root of unity which depends on a. Now we compute

$$\begin{aligned} \widetilde{\sigma}_a \gamma \widetilde{\sigma}_a^{-1}(\sqrt[l^k]{u_k}) &= \widetilde{\sigma}_a\gamma((\sqrt[l^k]{u_k})^{a^n} z\xi_1) \\ &= \widetilde{\sigma}_a(\xi_0^{a^n}(\sqrt[l^k]{u_k})^{a^n} z\xi_1) \\ &= \widetilde{\sigma}_a(\xi_0^{a^n})\,\widetilde{\sigma}_a((\sqrt[l^k]{u_k})^{a^n} z\xi_1) \\ &= \xi_0^{a^{n+1}} \sqrt[l^k]{u_k} \end{aligned}$$

(2) follows by (1), since by the multiplication rule in the semidirect product $G(L/\mathbb{Q}) = G(L/E) \rtimes G(E/\mathbb{Q})$ we have $(\gamma\sigma_a)^m = \gamma^{1+a^{n+1}+\ldots+a^{(n+1)(m-1)}}\sigma_{a^m}$. □

Propostion 3.3. *Let* n *and* l *be as specified above. Let* l^k *be the largest power of* l *which divides* $p-1$. *Let* S_k *be the finite set of primes of* $\mathcal{O}_E$ *from (2.6) (cf. condition (iv) below (2.6)), which we construct for* p *and* l^k. *Let* $w \notin S_k$ *be a prime of* E *over* p *and let* $\tilde{w}$ *be a prime of* $L = E(\sqrt[l^k]{u_k})$ *lying over* w. *We denote by* Fr_w *the Frobenius element which generates the subgroup* $G(\kappa_{\tilde{w}}/\kappa_w)$ *of* $G(L/E)$. *Then the number of elements of the image of the reduction map*

$$\phi_{p,l}\colon \quad K_{2n+1}(\mathbb{Z}) \to K_{2n+1}(\mathbb{F}_p)_l$$

equals the order of the Frobenius element Fr_w.

Proof. The proof is based on an idea of Rubin [Ru], p. 319, see also [AB], p. 67 or [BG1], section 4.4. Because $v_l(n{+}1) = 0$, it follows by Lemma 2.5 and by results of Quillen [Q2] that $K_{2n+1}(\mathbb{F}_p)_l = \mathbb{Z}/l^k$. Expanding the diagram (3.1) by adding reduction maps as horizontal arrows we obtain the following commutative diagram.

$$
\begin{array}{ccc}
K_{2n+1}(\mathbb{Z}) & \xrightarrow{\phi_{p,l}} & K_{2n+1}(\mathbb{F}_p)_l \\
\alpha_0 \downarrow & & \downarrow = \\
(K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l)/l^k & \longrightarrow & (K_{2n+1}(\mathbb{F}_p) \otimes \mathbb{Z}_l)/l^k \\
\alpha_1 \downarrow \cong & & \downarrow \cong \\
H^1_{cts}(\mathbb{Z}[\frac{1}{l}];\, \mathbb{Z}_l(n+1))/l^k & \longrightarrow & H^1(\mathbb{F}_p;\, \mathbb{Z}_l(n+1))/l^k \\
\alpha_2 \downarrow & & \downarrow \\
H^1_{cts}(\mathcal{O}_{E,S_k};\, \mathbb{Z}_l(n+1))/l^k & \longrightarrow & H^1(\kappa_w;\, \mathbb{Z}_l(n+1))/l^k \\
\alpha_3 \downarrow & & \downarrow \cong \\
H^1_{\acute{e}t}(\mathcal{O}_{E,S_k};\, \mathbb{Z}/l^k(n+1)) & \longrightarrow & H^1(\kappa_w;\, \mathbb{Z}/l^k(n+1)) \\
\alpha_4 \downarrow \cong & & \downarrow \cong \\
\mathcal{O}_{E,S_k}^\times/\mathcal{O}_{E,S_k}^{\times l^k} \otimes \mathbb{Z}/l^k(n) & \xrightarrow{\psi_{p,l}} & \kappa_w^\times/\kappa_w^{\times l^k} \otimes \mathbb{Z}/l^k(n)
\end{array}
\tag{3.4}
$$

In the diagram (3.4) $w \notin S_k$ is a prime which lies over p and κ_w denotes its residue field. Let $\pi'_w\colon \mathcal{O}_{E,S_k} \longrightarrow \kappa_w$ be the reduction at the ideal $w \notin S_k$. Let

$$
\pi_w\colon \quad \mathcal{O}_{E,S_k}^\times/\mathcal{O}_{E,S_k}^{\times l^k} \longrightarrow \kappa_w^\times/\kappa_w^{\times l^k} \tag{3.5}
$$

denote the map induced by π'_w. For $x \in \mathcal{O}_{E,S_k}^\times/\mathcal{O}_{E,S_k}^{\times l^k}$ we put $\psi_{p,l}(x \otimes \xi_{l^k}^{\otimes n}) = \pi_w(x) \otimes \xi_{l^k}^{\otimes n}$. Remaining horizontal arrows in (3.4) are the maps induced on K-theory and on cohomology by reduction at p and at w. The right vertical arrows are imbeddings by the choices of l^k, S_k and since the cohomological dimension of a finite field is one. Let H denote the subgroup of $\mathcal{O}_{E,S_k}^\times/\mathcal{O}_{E,S_k}^{\times l^k}$ generated by the image of the number u_k. By assumptions on l and k the element $\alpha_0(b_{2n+1})$ is a generator of $(K_{2n+1}(\mathbb{Z}) \otimes \mathbb{Z}_l)/l^k$. It follows by Lemma 2.9 (1) that the image of $K_{2n+1}(\mathbb{Z})$ by the map in the left column of (3.4) equals $H \otimes \mathbb{Z}/l^k(n)$. Since all vertical maps in the diagram (3.4), except α_0 are injections, the image of the reduction map $\phi_{p,l}$ is isomorphic to the group $\pi_w(H)$. Note that $\kappa_{\tilde{w}} = \kappa_w(\sqrt[l^k]{u_k})$, because for local rings we have $\mathcal{O}_{L,\tilde{w}} = \mathcal{O}_{E,w}[\sqrt[l^k]{u_k}]$ what follows by a simple argument using the different (cf. [A], p. 85), since u_k is an l-unit and its minimal polynomial over $\mathcal{O}_{E,w}$ divides $x^{l^k} - u_k$. The Frobenius element Fr_w generates the group $G(\kappa_{\tilde{w}}/\kappa_w) \subset G(L/E) \cong \mathbb{Z}/l^k$. On the other hand, by the Kummer theory, [L], Chapter VIII.8, Theorem 13, we have

$$\pi_w(H) \cong Hom(G(\kappa_{\tilde{w}}/\kappa_w); \mathbb{Z}/l^k(1)).$$

This establishes the claim. $\square$

Theorem 3.6. *Let n and l be as specified in the beginning of this section. Let k be a fixed natural number. Denote by M_k the set of rational primes p which satisfy the following two conditions:*

1. *the l-torsion part of the group $K_{2n+1}(\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}/l^k$*
2. *the reduction map $K_{2n+1}(\mathbb{Z}) \longrightarrow K_{2n+1}(\mathbb{F}_p)_l$ is nontrivial.*

Then the set M_k has positive density, which equals $\frac{l^k-1}{l^{2k}}$.

In the proof of the theorem we will need some notation and a lemma. Consider the following field extensions

$$\begin{tikzcd}[column sep=small]
 & K \arrow[dl, dash] \arrow[dr, dash] & & & & \beta \arrow[dr, dash] & \\
E' \arrow[dr, dash] & & L \arrow[dl, dash] & & & & \tilde{w} \arrow[dl, dash] \\
 & E \arrow[d, dash] & & & & w \arrow[d, dash] & \\
 & \mathbb{Q} & & & & p &
\end{tikzcd}$$

where $E = \mathbb{Q}(\xi_{l^k})$, $E' = \mathbb{Q}(\xi_{l^{k+1}})$ and $L = E(\sqrt[l^k]{u_k})$. We denote by K the composite of the fields E' and L. For a prime p we choose a prime w in E lying over p and primes $\tilde{w}$ in L, and β in K such that $\beta|\tilde{w}|w$ as is shown at the picture above.

Lemma 3.7. *Let n be an even, positive integer and l be an odd prime. Assume that l^k is the largest power of l which divides $p-1$. Then*

1. $E' \cap L = E$
2. $G(K/E) = G(E'/E) \oplus G(L/E)$.
3. $[K : \mathbb{Q}] = l^{2k}(l-1)$.

Proof of Lemma 3.7. Observe that the group $G(K/\mathbb{Q})$ is a semidirect product of $G(K/E)$ by $G(E/\mathbb{Q})$, where the latter group acts on the former by conjugation. It is clear that (1) implies (2) and (3). We sketch the proof of (1) which is a standard exercise in Galois theory. Since $[E' : E] = l$, it is enough to show that $E' \not\subset L$. If $E' \subset L$, then the group $G(L/\mathbb{Q})$ would contain an element of order $\phi(l^{k+1})$, where $\phi(-)$ is the Euler function. However, an easy check based on the formula from Proposition 3.2 (2) shows that orders of elements in $G(L/\mathbb{Q})$ divide either $\phi(l^k)$ or l^k. The assumptions on numbers l and n are essential for the last conclusion.

Proof of Theorem 3.6. We use the Chebotarev density theorem, [Se1], Theorem 1, p. 331. Let us fix k' such that $1 \leq k' \leq k$. It follows by Proposition 3.3 that the image of reduction map on the l^k-torsion of $K_{2n+1}(\mathbb{F}_p)$ has order $l^{k'}$, if and only if the following conditions hold

1. p splits completely in E, i.e., restriction of the Frobenius element $Fr_\beta \in G(K/\mathbb{Q})$ to the field E is trivial
2. $Fr_\beta|_L$ has order $l^{k'}$
3. restriction of Fr_β to E' is nontrivial. Since $G(E'/E) = \mathbb{Z}/l$, the last condition means that the order of $Fr_\beta|_{E'}$ is l.

We count the number of elements of $G(K/\mathbb{Q})$ which satisfy conditions (1)-(3). By Lemma 3.7 and Proposition 3.2 (1) the group $G(K/\mathbb{Q})$ is a semidirect product of $G(K/E) = G(E'/E) \oplus G(L/E)$ by $G(E/\mathbb{Q})$, where the action is given by

$$\sigma_a(\tau,\, \gamma)\sigma_a^{-1} = (\tau,\, \gamma^{a^{n+1}})$$

for $\tau \in G(E'/E)$, $\gamma \in G(L/E)$ and $\sigma_a \in G(E/\mathbb{Q})$. Let $\gamma \in G(L/E)$ have order $l^{k'}$. The conjugacy class of (τ, γ) in $G(K/\mathbb{Q})$ has

$$\#\{a^{n+1}\colon \quad a \in (\mathbb{Z}/l^{k'})^\times\}$$

elements. From this and Lemma 3.7 it follows easily that in $G(K/\mathbb{Q})$ there are $\phi(l^{k'})(l-1)$ elements g such that

(a) $g|_E = 1$,
(c) $g|_L$ has order $l^{k'}$ and
(b) $g|_{E'}$ has order l.

Using this and the Chebotarev density theorem for the extension $K/\mathbb{Q}$ we obtain that the set of rational primes p for which the Frobenius Fr_β meets conditions (1)-(3) for $1 \leq k' \leq k$, has Dirichlet density which equals

$$\frac{\sum_{k'=1}^{k} \phi(l^{k'})(l-1)}{l^{2k}(l-1)} = \frac{l^k - 1}{l^{2k}}. \qquad \square$$

Remark 3.8. Assume that we fix k' such that $1 \leq k' \leq k$. The proof of Theorem 3.6 shows that for l, p and n as in assumptions, the density of the primes p for which the image of the reduction map

$$K_{2n+1}(\mathbb{Z}) \longrightarrow K_{2n+1}(\mathbb{F}_p)_l$$

has order exactly $l^{k'}$, equals $\frac{\phi(l^{k'})(l-1)}{l^{2k}}$.

Remark 3.9. The set

$$M = \cup_{k=1}^{\infty} M_k$$

consists of primes p for which the reduction map $K_{2n+1}(\mathbb{Z}) \to K_{2n+1}(\mathbb{F}_p)_l$ has nontrivial image. Since the sets M_k for $k \geq 1$ are pairwise disjoint, it seemes natural to expect that the set M has density

$$\sum_{k=1}^{\infty} \frac{l^k - 1}{l^{2k}} = \frac{l}{l^2 - 1}.$$

By definition, the density of M should be

$$\lim_{x \to \infty} \frac{\#\{p \leq x: \quad p \in M\}}{\#\{p \leq x\}}.$$

It is an interesting question whether the above limit exists.

Theorem 3.6 has the following consequence for K_2 of the cyclotomic field $E = \mathbb{Q}(\xi_{l^k})$. Let l and p be two odd primes. Assume that n and l meet conditions formulated in the beginning of this section. If $l^k || p^{n+1} - 1$ for $k > 0$, then $l^k || p - 1$ by Lemma 2.5 and under the assumptions we have $K_{2n+1}(\mathbb{F}_p)_l = \mathbb{Z}/l^k$ by [Q2]. We consider the Galois symbol map

$$A_{\xi_{l^k}}: \quad K_2(E) \longrightarrow Br(E)$$

which to a symbol $\{a, b\}$ associates the class of the central simple algebra $A_{\xi_{l^k}}(a, b) \in Br(E)$ cf. [M], Chapter 15. It follows by the main result of [MS] that the map $A_{\xi_{l^k}}$ induces an isomorphism

$$K_2(E)/l^k \xrightarrow{\cong} Br(E)[l^k]. \tag{3.10}$$

Let S_k be the same set of primes as in condition (iv) below the diagram (2.6). Choose a prime $w \notin S_k$ in E which lies over p. Consider the symbol $\{u_k, p\}$, where u_k is the l-unit from Definition 2.1. Let $v_w(-)$ denote the valuation at the prime w. For l and p as above $v_w(u_k) = 0$ and $v_w(p) = 1$. Hence, the value of the tame symbol map ∂'_w at $\{u_k, p\}$ is

$$\partial'_w\{u_k, p\} = (-1)^{v_w(u_k)v_w(p)}\pi'_w\Big(\frac{u_k^{v_w(p)}}{p^{v_w(u_k)}}\Big) = \pi'_w(u_k), \tag{3.11}$$

where $\pi'_w\colon \mathcal{O}^{\times}_{E,S_k} \to \kappa^{\times}_w$ denotes the reduction map at the ideal w. Clearly,

$$\partial_w\{u_k, p\} = \pi_w(u_k),$$

where $\partial_w\colon K_2(E)/l^k \longrightarrow \kappa^{\times}_w/\kappa^{\times l^k}_w$ and $\pi_w\colon \mathcal{O}^{\times}_{E,S_k}/\mathcal{O}^{\times l^k}_{E,S_k} \longrightarrow \kappa^{\times}_w/\kappa^{\times l^k}_w$ are the maps induced by ∂'_w and π'_w, respectively.

Lemma 3.12. *The following conditions are equivalent.*

(1) $A_{\xi_{l^k}}(u_k, p) \neq 0$ *in the group* $Br(E)$.

(2) *There exists a prime* w *of* E *such that* $w|p$ *and* $\pi_w(u_k) \neq 0$ *in* $\kappa^{\times}_w/\kappa^{\times l^k}_w$.

Proof. By the calculation of the tame symbol above, it remains to show that (1) implies (2). Consider the commutative diagram.

$$\begin{array}{ccc} K_2(E)/l^k & \longrightarrow & \bigoplus_v K_2(E_v)/l^k \\ \cong\big\downarrow & & \cong\big\downarrow \\ Br(E)[l^k] & \longrightarrow & \bigoplus_v Br(E_v)[l^k] \end{array} \tag{3.13}$$

The direct sums in (3.13) are over finite places of E. The vertical arrows are isomorphisms of [MS]. The upper horizontal arrow is given by the power residue symbols $(\,,\,)_v$ cf. [M], Theorem 16.1. The lower horizontal map in (3.13) is induced by the localization map. It is an injection because, by the class field theory we have the short exact sequence

$$0 \longrightarrow Br(E)[l^k] \longrightarrow \bigoplus_v Br(E_v)[l^k] \xrightarrow{\ \Sigma\ } \mathbb{Z}/l^k \longrightarrow 0. \tag{3.14}$$

The direct sum in (3.14) is over finite places, since the field E is totally imaginary. Assume that $A_{\xi_{l^k}}(u_k, p) \neq 0$ and denote by A_v its image in the Brauer group $Br(E_v)$ by the lower horizontal map of (3.13). Observe that the symbol $(u_k, p)_v$ may be non trivial only at $v|p$ or $v|l$. This follows easily by the formula (3.11) and [Se2], Corollary 8, p. 211. Since there is exactly one prime over l in E, the exactness of (3.14) implies that there is at least one prime $w|p$ such that $(u_k, p)_w$ is nontrivial. By commutativity of (3.13) we obtain $A_w \neq 0$. Using again (3.11) and [Se2] *loc. cit.* we conclude that $\partial_w\{u_k, p\} = \pi_w(u_k) \neq 0$. $\square$

Theorem 3.6 gives the density of primes p for which $\pi_w(u_k)$ is nontrivial. Thus Lemma 3.11 and Theorem 3.6 imply the following result. Claim (2) of Proposition 3.15 below follows by an easy transfer argument, if we replace u_k by u_k^+ in the calculation and apply Lemma 2.9 (2).

Proposition 3.15.

(1) *The set of primes p for which the l-torsion part of the group $K_{2n+1}(\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}/l^k$ and such that the class of the algebra $A_{\xi_{l^k}}(u_k, p)$ is a nontrivial element of the Brauer group $Br(E)$ has positive density which equals $\frac{l^k-1}{l^{2k}}$.*

(2) *Assume in addition that l does not divide the number $(n+1)C_n$. Then the set of primes for which the l-torsion part of the group $K_{2n+1}(\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}/l^k$ and such that the class of the algebra $A_{\xi_{l^k}}(u_k^+, p)$ in $Br(E^+)$ is nontrivial has positive density which equals $\frac{l^k-1}{l^{2k}}$.*

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