Four manifolds with tree like ends

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Introduction

In this paper we study anti self dual connections over the Casson handles by equipping complete Riemannian metrics on them. Casson handles are effectively used by M.Freedman, to develop four manifold theory. For example they can be used to construct topological connected sums for closed four manifolds. This leads to the fact that simply connected oriented topological 4 manifolds are determined by their intersection forms and the Kirby Siebenmann classes.

Let $Q_1 \oplus Q_2$ be a representation of the intersection form of a simply connected smooth closed 4 manifold M. Then as above M decomposes as $W_1 \# W_2$ topologically and compatibly with respect to the decomposition of intersection forms. By the classification of the intersection forms plus Donaldson's theorem for definite forms on M ([Do1]), at least one of Q_1 or Q_2, say Q_1, may be representable by the intersection form of a smooth closed 4 manifold (e.g. $Q_1 = lH = l\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $k(1) \oplus l(-1)$). We denote the manifold by $|Q_1|$. H is called the hyperbolic matrix. Then one finds that on a complement of a topological disk $D \subset W_1$, $W_1 \backslash D$ may be regarded as a smooth submanifold in $|Q_1|$, and the end of it admits the handle body with infinitely many 1 handles in $W_1 \# W_2 = M$. They are called Casson handles.

In this paper we study open 4 manifold which corresponds to $S \equiv W_1 \backslash D$ part in the above. Recall that any Casson handle is expressed by an infinite tree (embedded in $\mathbf{R}^2$) with one end point, and with signs $\pm$ on every edge of it. We say that such a tree is a signed tree. In this paper we will define a subclass of infinite trees which is called homogeneous trees of finite type. These are constructed by iterating to attatch infinitely many another half periodic trees. A connected infinite subtree of a homogeneous tree of finite type is called a tree of finite type. Any homogeneous tree can be approximated by a family of homogeneous trees of finite type.

We say that an open four manifold S has *tree-like end* if there are finite number of signed trees $T_1, \ldots, T_l$, such that S is diffeomorphic to $D^4 \cup_j CH(T_j)$, where $CH(T_j)$ is the Casson handle corresponding T_j, and every $CH(T_j)$ is attatched to the 4 ball D^4 along the attatching S^1 of the first kinky handle (which corresponds to the end point in T_j). Notice that there is natural smooth structure on it induced from the handle body.

Let M be an oriented closed smooth four manifold with $\pi_1(M) = 1$. Let us take a marking of the standard form (M, φ):

$$\varphi : (H_2(M; \mathbf{Z}), <,>) \cong (\oplus \mathbf{Z}, \quad k(-E_8) \oplus lH, \text{ or } k(-1) \oplus l(1)).$$

Then by Freedman, one finds an open four manifold S with tree-like end, and finds a smooth embedding $S \hookrightarrow M$ which induces an embedding of the form:

$$(H^2(S : \mathbf{Z}), < \quad , \quad >) \cong (\oplus \mathbf{Z}, lH \text{ or } l(1)) \hookrightarrow (H^2(M : \mathbf{Z}), < \quad , \quad >).$$

In order to construct the moduli space of $SO(3)$ ASD connections over M, one takes $\hat{\alpha} \in H^2(M; \mathbf{Z}_2)$ which is mod 2 reduction of $\alpha \in H^2(M; \mathbf{Z})$. Let us take $\hat{\alpha} \in H^2(M; \mathbf{Z}_2)$ and a positive integer k. Suppose the intersection form of M is isomorphic to $L = (-E_8) \oplus (k-1)(-E_8) \oplus lH$ or $(-1) \oplus (k-1)(-1) \oplus l(1)$ for $k \geq 2$ and $l \geq 1$. Then one has the corresponding decomposition $\hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2 \oplus \hat{\alpha}_3$. Let $\Sigma_1, \ldots, \Sigma_d \subset S$ be embedded surfaces which represents homology classes $[\Sigma_i] \in H_2(S; \mathbf{Z}) \hookrightarrow H_2(M; \mathbf{Z})$.

Theorem 0.1 *Suppose $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3 \neq 0$, and $S \subset M$ consists of signed homogeneous trees. Then if the $SO(3)$ Donaldson's invariant $Q_{k,\hat{\alpha}}(M; [\Sigma_1], \ldots, [\Sigma_d])$ does not vanish, then (1) at least one of the trees is of infinite type, (2) or L^2 ASD moduli space over S is generically w-separated.*

Here w is a weight function over S. Our main construction, when S is of finite type, is to equip a complete Riemannian metric on S and construct Atiyah, Hitchin and Singer complex as a Fredholm operator between w weighted Sobolev spaces. Then one can use any L^2 ASD connection over S as a base point, and can construct L^2_w ASD moduli space. w separated property is some discontinuity of L^2_w ASD moduli space in the larger space, L^2 ASD moduli space over S, under generic deformation of metrics (see $1.E$). If one gets an L^2 ASD connection as a limit of elements in $\mathfrak{M}(P, (M, h_i); [\Sigma_1], \ldots, [\Sigma_m])$, then it cannot be a base of any generic L^2_w ASD moduli space over S. Here h_i are Riemannian metrics on M which approach to a specified complete Riemannian metric on S on each compact subset.

Vanishing of Q is well known when M decomposes as a connected sum $M = W_1 \sharp W_2$ diffeomorphically with respect to the decomposition of the intersection form $L = L_1 \oplus L_2$ ([Do2]). S is homeomorphic to the interior of $l(S^2 \times S^2) \backslash D^4$, or $l(\mathbf{C}P^2) \backslash D^4$. In particular one has a topological connected sum $M \cong (M \backslash S) \sharp S$. Our theorem states that the vanishing result holds if the decomposition is not so complicated smoothly around the end.

Let us outline the proof of it as follows. Suppose the invariant is non zero. Let $L_2 = lH$ which is the intersection form of $l(S^2 \times S^2)$. Let us remove a small disk D^4 from this, and denote $S' = l(S^2 \times S^2) \backslash D^4$. Then one has a smooth immersion $S' \to M$ which is compatible with the intersection form. By Casson and Freedman's tower construction, one finds an open subset $S'' \subset S'$ and a smooth embedding $S'' \hookrightarrow M$ which realizes the form L_2. S'' has tree-like end. Notice that one may assume that all Σ_i are embedded in S''. Our main construction is to admit a complete Riemannian metric g on S'' so that one can do gauge theory over (S'', g), when S'' is of finite type. Let us choose a family of Riemannian metrics g_i on M so that $g_i | S''$ converges to g on every compact subsets of S''. Let $M_{k,\hat{\alpha}}((M, g_i); [\Sigma_1], \dots, [\Sigma_d])$ be the oriented 0 dimensional moduli space of ASD connections. Its number is invariant for any i, since it is invariant under generic deformation of Riemannian metrics. In particular for all i, the above spaces are all non empty. Let us take a sequence $[A_i] \in M_{k,\hat{\alpha}}((M, g_i); [\Sigma_1], \dots, [\Sigma_d])$. Then one finds a subsequence such that its restriction on S'' converges on every compact subset, to $[A_\infty] \in M_{k',\hat{\alpha}'}((S'', g); [\Sigma_1], \dots, [\Sigma_{d'}])$ which should be a smooth manifold. Since $\hat{\alpha}_i \neq 0$, this space has a negative formal dimension. This is a contradiction.

One has an application of the main theorem (of the proof). Let $T_1, \dots, T_k$ be signed homogeneous trees defining open four manifolds S with tree-like end, not necassarily of finite type. In the general case, one doesnot get moduli spaces as smooth manifolds over S. However in the situation of the main theorem, one has some estimate on the decay of the ASD connections $\{A_i\}$ over S. This can be obtained by approximating S by a family of another open manifolds such that ASD moduli spaces are constructed over the approximations.

Let S be an open four manifold with tree-like end, smoothly embedded into a compact manifold M. Suppose the conditions in the above theorem are satisfied except on trees. Notice that any tree with one end point admits a natural distance from the fixed point. Let us use e to denote the edges in T_j.

Let $F(\{T_j\}) : \mathbf{N} \to \mathbf{N}$ be a function:

$$F(\{T_j\})(n) = \text{ Sup}_{v,j} \#\{e| \text{ one end of } e \text{ is } v, \text{ and } v \in T_j, |v| = n\}.$$

For any tree T, let us put $T(n) = \{x \in T : |x| \leq n\}$. For any infinite homogeneous tree, one gets a family of another homogeneous trees of finite type $\{T^n\}_n$ such that $T(n)$ and $T^n(n)$ are isomorphic. Let us put:

$$Y_n = D^4 \cup_{j=1}^k CH(T_j^n), \quad W_n = D^4 \cup_{j=1}^k CH(T_j(n)).$$

Each Y_n admits a compatible Riemannian metric g_n, $g_{n+1}|W_n = g_n$, and a weight function w_n such that one can do gauge theory over (Y_n, g_n, w_n).

Let us choose a sequence $A_i \in \mathfrak{M}((M, g_i), [\Sigma_1], \ldots, [\Sigma_m]) \neq \phi$. Then a subsequence $\{A_i\}$ converges to a L^2 ASD connection A_∞ over S on each compact subset. Let us define $G(\{A_i\}) : \mathbf{N} \times \mathbf{N} \mapsto (0, C]$ and λ_i by:

$$G(\{A_i\})(n, n') = |F_{A_n}|L^2_{w_n}(W_n \backslash W_{n-n'}, g; \Lambda^2_+), \quad \lambda_n = \text{ inf} \frac{|(d^+_{A_n})^*_{w_n}(u)|L^2_{w_n}(W_n)}{|u|L^2_{w_n}(W_n)}.$$

Let $\mathbf{F} = \{F : \mathbf{N} \mapsto \mathbf{N}\}$, $\mathbf{G} = \{h : \mathbf{N} \times \mathbf{N} \mapsto (0, C(h)]\}$ and $\mathbf{H} = \{h : \mathbf{N} \mapsto (0, C(h)]\}$ be the sets of functions.

Suppose the restriction of the Donaldson's invariant $Q|H_2(S : \mathbf{Z})$ is non zero for a principal bundle P with $\hat{\alpha}_j \neq 0$ as before. Then one can choose a sequence of ASD connections $A_i \in \mathfrak{M}((M, h_i), P)$ which converges to A_∞ as before. The following estimate is independent of any data on $M \backslash S$:

Theorem 0.2 *There exists a universally defined map $\mathfrak{F} : \mathbf{F} \times \mathbf{H} \mapsto \mathbf{G}$ and a constant C such that one has the pointwise estimate at (i, j) with $j \geq C\lambda_i^{-1}$:*

$$\mathfrak{F}(F(\{T_j\}), \{\lambda_i\}) \leq G(\{A_i\}).$$

Notice that all Y_n, Riemannian metrics and weight functions are determined by the asymptotics of $\{T_j\}_j$. Thus the above statement saids that all the constants to choose $\mathfrak{F}$ above is estimated by the growth function $F(\{T_j\}_j)$ and $\{\lambda_i\}$.

The estimate is independent of $M \backslash S$ and embedded surfaces. One will use the fact that the invariant is polynomial. In order to verify this, one can use the above argument. If $G(\{A_i\})(n, n')$ is sufficiently small, then one gets A'_n on Y_n by cutting off A_n around $W_n \backslash W_{n-n'}$. Since A'_n has a sufficiently small $L^2_{w_n}$ norm on its curvature, one can find an ASD connection A''_n on Y_n around A'_n. Since Y_n has tree-like end of finite type, one can construct ASD moduli space $\mathfrak{M}(Y_n, [\Sigma_1], \ldots, [\Sigma_{m'}])$. Its formal dimension is negative. This is a contradiction.

One has an application of the vanishing results above from some relationship between Kahler forms and Picard lattice over algebraic surfaces. Let M be a simply connected complex algebraic surface with $p_g > 0$. Let us take a marking:

$$\psi : H^2(M; \mathbf{Z}) \cong \oplus \mathbf{Z} \oplus (\oplus \mathbf{Z}) \oplus (\oplus \mathbf{Z})$$

with respect to the decomposition of the form as $(-E_8) \oplus_{k-1} (-E_8) \oplus^l H$ or $(-1) \oplus (k-1)(-1) \oplus l(1)$, $k \geq 2$. Then one has an open manifold with tree-like end $S \subset M$, which corresponds to lH or $l(1)$. Let h be a hyperplane class of M, and decompose $h = h_1 \oplus h_2 \oplus h_3$ with respect to above. Suppose there is an integral $(1,1)$ form $\alpha = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \in H^{1,1}(M; \mathbf{R}) \cap H^2(M; \mathbf{Z})$, such that the mod 2 reduction $\hat{\alpha_1}, \hat{\alpha_2}, \hat{\alpha_3} \neq 0 \in H^2(M; \mathbf{Z}_2)$.

Corollary 0.1 *Let M be as above, and α satisfy the above condition. Then if there is a hyperplane class $h \in H^2(M; \mathbf{Z})$ with $h_1 = h_2 = 0$, then at least one of T_j for S have infinite type, or L^2 ASD moduli space over S is generically w-separated.*

M can be holomorphically embeddable into a projective space $\mathbf{C}P^N$. Let us take such embedding $i : M \hookrightarrow \mathbf{C}P^N$. Then one has a natural homology class $h \in H_2(M; \mathbf{Z})$, the hyperplane class. This is defined as the intersection of M with the hyperplane $\mathbf{C}P^{N-1} \subset \mathbf{C}P^N$. Let ω be the Fubini Study form over $\mathbf{C}P^N$. Then the Poincaré dual to h is the restriction of $\omega = \omega|M$. This form gives a Kahler (Hodge) structure on M. There are some restriction of ω on the position in $H^2(M; \mathbf{R})$. For example, let y be a closed analytic subvariety in M. Then one has a positivity $< \omega, y >> 0$. The above theorem is a generalization of this fact. It follows from the first theorem before, and the non vanishing result:

Theorem 0.3 (Do) *Let M be as before and take any hyperplane class h. Then there is $k_0 = k(M, h)$ such that for any $k \geq k_0$, and any $\hat{\alpha}$ which is mod 2 reduction of an integral $(1,1)$ class, the Donaldson's invariant $Q_{\hat{\alpha},k}(h)$ is positive.*

The non realizable result above, gives some restrictions of the growth of trees, from relation between the Kahler cone and the Picard lattice.

Recall that the Picard number is the rank of the image of the Picard group $\text{Pic}(M)$ in $H^2(M : \mathbf{Z})$. Let M be a complex algebraic surface with even type of the form, $k(-E_8) \oplus lH$. Let us say that the Picard group of M is large if the Picard number is grater than $\max\{8k + 1, 2l + 1\}$. Thus the Picard numbers of all projective Kummer surfaces are large (more than 17).

Let M be as above, and suppose the Picard group of M is large.

Corollary 0.2 *Suppose a complex algebraic surface M of even type of the form has large Picard group. Let $T_1, \ldots, T_{2l}$ be signed homogeneous trees which represents $S \subset M$. Then at least one of them must be of infinite type, or L^2 ASD moduli space over S is generically w-separated.*

In order to see this, one uses the following fact. Let M be an oriented smooth closed 4 manifold with $\pi_1(M) = 1$. Then $H^2(M; \mathbf{Z})$ is a free abelian group. Let us take a marking $\varphi : (H^2(M; \mathbf{Z}, < \quad , \quad >) \mapsto (\oplus \mathbf{Z}, L)$ where L is a unimodular form on a free $\mathbf{Z}$ module, and $< \quad , \quad >$ is the intersection form. In general there are many choices of markings. For example suppose L is of even type which contains at least two hyperbolic matrices. Then for any primitive sublattices $l, l' \subset L$ of rank 1, there is an automorphism on L which sends l to l'. In particular there exists a marking ϕ as above, such that $\phi(h) \in \oplus_{2l} \mathbf{Z}$, and for some $\alpha = \alpha_1 + \alpha_2 + \alpha_3 \in \text{im Pic}(M)$, one has $[\alpha_1]_2, [\alpha_2]_2, [\alpha_3]_2 \neq 0 \in \oplus \mathbf{Z}_2$.

The main construction in this paper is to obtain ASD moduli spaces over Casson handles of finite type. In order to compute the indices of the Atiyah, Hitchin and Singer (AHS) complex between weighted Sobolev spaces, one will use the analytic tool, developped by [T1]. In the paper, it is shown that one can make gauge theory over periodic open 4 manifolds, using Fourier Laplace transform between spaces. Here in order to get Fredholmness for AHS complex over Casson handles, one uses Fourier Laplace transform iteratedly. As a generalization of the result by [T1], one gets the following. Notice that it is known that any open 4 manifold admits some smooth structure ([Q]).

Proposition 0.1 *Any smoothing on $|kE_8| \backslash$ pt cannot admit tree-like end of finite type.*

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I Yang Mills moduli space

1 Moduli space

1.A Admissible four manifolds: In this section, one will construct Yang Mills moduli spaces over non compact four manifolds. We treat open four manifolds which can admit Riemannian metrics and weight functions such that they are able to construct Fredholm complex.

Let Y be a non compact smooth four manifold. Throughout this section, one always assumes that Y is simply connected, and simply connected at infinity.

Let g be a complete Riemannian manifold of bounded geometry and $w : Y \mapsto [0, \infty)$ be a smooth function. Recall that a complete Riemannian metric is of bounded geometry, if (1) the injectivity radious is more than a positive constant $\epsilon > 0$ at any point, and (2) for any $l \geq 0$, $\sup_{x \in Y} |\nabla^l R| < \infty$, where ∇ and R are with respect to g. Throughout this paper, we always assume that g is of bounded geometry, and $|\nabla^l(w)|C^0(Y) < \infty$ for all $l \geq 1$.

Let $\Lambda^i(Y)$ and $\Lambda^2_+(Y)$ be the exterior differentials on i forms and self dual 2 forms with respect to g, $i = 0, 1$. Then we have the Atiyah Hitchin Singer complex (without coefficient) as:

$$0 \ \to C_c^\infty(Y; \Lambda^0) \ \xrightarrow{d} \ C_c^\infty(Y; \Lambda^1) \ \xrightarrow{d^+} \ C_c^\infty(Y; \Lambda^2_+) \ \to 0$$

where $d^+ = (1 + *) \circ d$. Using L^2 adjoint operator (here we do not use any weight functions), we get the next elliptic operator:

$$P = d^* \oplus d^+ : C_c^\infty(Y; \Lambda^1) \mapsto C_c^\infty(Y; \Lambda^0 \oplus \Lambda^2_+).$$

Let us introduce weighted Sobolev k norms on Y by:

$$|u|W^k_w = (\Sigma_{l \leq k} \int exp(w) |\nabla^l u|^2 vol)^{\frac{1}{2}}.$$

We denote these Sobolev weighted spaces by W^k_w or L^2_w (for $k = 0$). Then one has a complex of bounded maps:

$$0 \ \to W^{k+2}_w(Y; \Lambda^0) \ \xrightarrow{d} \ W^{k+1}_w(Y; \Lambda^1) \ \xrightarrow{d_+} W^k_w(Y; \Lambda^2_+) \ \to 0$$

Now let us call that the pair (g, w) is admissible, if the following conditions are satisfied; (1) the above consists of a Fredholm complex, namely each differential has closed range, and its cohomology group is of finite dimension as a vector space over $\mathbf{R}$. We denote their cohomology groups by $H^*(AHS)$, $* = 0, 1, 2$. Notice that by the condition, one has L^2_w adjoint operators d^*_w and $(d_+)^*_w$.

(2) Y admits a partition $Y = K \cup Y_1 \cup \ldots Y_l$ such that (a) K is a compact subset and each Y_j is an open subset of Y, (b) let us put $D = d^*_w \oplus d_+$ or $= d \oplus (d_+)^*_w$. Then there is a positive constant C_k such that for any $u \in C_c^\infty(Y_j; \Lambda^*)$, $* = 1, 2$, one has the bound $|u|W^{k+1}_w \leq C|D(u)|W^k_w$.

(3) For any $f \in C^1(Y)$ with $|df|L^2_w(Y) < \infty$, there is $\bar{f} \in \mathbf{R}$ with $f - \bar{f} \in L^2_w(Y)$.

(4) There is a compact subset $K \subset Y$ and a homeomorphism $Y \backslash K \cong S^3 \times [0, \infty)$.

In part II, we will construct admissible pairs for all Casson handles constructed from homogeneous signed trees of finite type.

Proposition 1.1 *Let (g, w) be an admissible pair on Y. Then $P_w = d_w^* \oplus d_+ : W_w^1(Y; \Lambda^1) \mapsto L_w^2(Y; \Lambda^0 \oplus \Lambda_+^2)$ has closed range with $H^0(AHS) = H^1(AHS) = 0$ and $H^2(AHS) \geq k = H_+^2(Y : \mathbf{R})$.*

Proof : Clearly $H^0(AHS) = H^1(AHS) = 0$. We show $\dim H^2(AHS) \geq k$. For this, one takes two steps.

Let $H^2_{\text{cp}}(Y : \mathbf{R})^+$ be a subspace of $H^2_{\text{cp}}(Y : \mathbf{R})$ consisting of vectors with $\int_Y u \wedge u > 0$. Then $H^2_{\text{cp}}(Y : \mathbf{R})^+$ is a linear subspace of $H^2(Y : \mathbf{R})$ of dimension k, since the natural map $I : H^2_{\text{cp}}(Y : \mathbf{R}) \mapsto H^2(Y : \mathbf{R})$ gives an isomorphism.

Now by the above, one has $\dim H^2_{\text{cp}}(Y : \mathbf{R})^+ = k$. For any element $u \in C^\infty(Y; \Lambda^2)$, let us denote by u^+ the projection to Λ_+^2 part. Then one defines:

$$Pr : H^2_{\text{cp}}(Y : \mathbf{R})^+ \mapsto H^2(AHS)$$

by assigning $[u] \to [u^+]$. This map is well defined. We show that Pr is an injection.

Suppose $[u^+] = 0 \in H^2(AHS)$. Thus one has $\alpha \in W_w^1(Y; \Lambda^1)$ such that $u^+ = d^+(\alpha)$, or $(u - d(\alpha))^+ = 0$. Let us put $v = u - d(\alpha)$, and take the cup product $< v, v >= \int_Y v \wedge v$. Notice that by the Stokes theorem, $\int_Y u \wedge d(\alpha) = 0$, since u is a closed form. Thus one has the equality $\int_Y v \wedge v = \int_Y u \wedge u \geq 0$. On the other hand since v does not have self dual part, one has $\int_Y v \wedge v \leq 0$. This shows $u = d(\alpha)$. This implies $[u] = 0 \in H^2_{\text{cp}}(Y : \mathbf{R})^+$ since I gives an isomorphism. Thus one concludes $\dim H^2 \geq k$.

Example 1.1 *In general, $\dim H^2_{cp}(Y; \mathbf{R})^+$ and $\dim H^2(AHS) = Ker(d_+)^*_w$ do not coincide. One takes $Y = \mathbf{R}^2 \times \mathbf{R}^2$, where one equips the standard metric on $\mathbf{R}^2$, and the product one on Y. Let us choose a weight function w on Y. We show $\dim H^2(AHS)$ is non zero. Let u_1 and u_2 be 2 forms $\mathbf{R}^2$ respectively, such that $|u_i| \equiv 1$, $i = 1, 2$, pointwisely. Then clearly $du_i = 0$, $i = 1, 2$, and $u_1 + u_2$ gives a self dual 2 form with bounded pointwise norm. Then one puts $v = exp(-w)(u_1 + u_2) \in L_w^2(Y; \Lambda_+^2)$. Then v satisfies the equation $d(exp(w)v) = 0$. Thus v is a non trivial element in $H^2(AHS)$.*

1.B ASD moduli space: Let $E \to Y$ be a G ($= SO(3)$ or $U(1)$) vector bundle such that except some compact subset $K \subset Y$, $E|Y \backslash K$ is trivial. One denotes the corresponding principal G bundle by P. For $SO(3)$ bundles, E is determined by $w_2(E) \in H^2(Y : \mathbf{Z}_2)$. Let A be a smooth connection over E such that except some compact subset on Y, it satisfies the ASD equation:

$$F_A + *F_A = 0$$

where F_A is the curvature form of A. Let us denote by $\tilde{R}(Y)$ the set of smooth connections as above satisfying $|F_A|L^2(Y) < \infty$. In particular, if A is the trivial connection except some compact subset, it is an element in this.

Lemma 1.1 *For $A \in \tilde{R}(Y)$, $p_1(A) = -\frac{1}{4\pi^2}\int_Y tr(F_A \wedge F_A)$ is an integer.*

In order to show this, one uses the following:

Sublemma 1.1 (U) *Let $D \subset Y$ be δ ball with any point as the center, where $\delta > 0$ is suficiently small. Then there exists another small $\epsilon > 0$ with the following property; suppose $|F_A|L^2(D) < \epsilon$. Then there exists a gauge transformation g over D, such that:*

$$Sup_{x \in D}|\nabla^l(g^*(A) - d)|(x) \leq C_l|F_A|L^2(D)$$

where C_l are constants, independent of A.

Proof of lemma: Let us take any small $\epsilon > 0$. Then there exists a compact subset $K \subset Y$, and a smooth gauge transformation $g \in C^\infty(Y; Aut(E))$ such that $|g^*A - d|W^1(Y \backslash K) < \epsilon$, where d is the trivial connection (Y is simply connected at infinity). One may assume that K is a compact sumanifold with boundary. Let $D(K)$ be a double of K. Then using a cut off function φ around ∂K, one may extend $A' \equiv \varphi(g^*A)$ over $D(K)$, by putting d over the other side of $K \subset D(K)$. By the Chern Weil theory, $p_1(A')$ is an integer. On the other hand, since $p_1(A) - p_1(A')$ may be arbitrarily small as above, one concludes that $p_1(A)$ is an integer. This completes the proof.

Let us put $Ad(P) = P \times_G \mathfrak{G}$, where $\mathfrak{G}$ is the Lie algebra of G. ($P \times_G \mathbf{R}^3 = E$). We also denote by $\mathfrak{G}$ as the trivial Lie$G$ bundle. Then we have the Atiyah Hitchin Singer complex (AHS complex) as:

$$0 \to C_c^\infty(Y; Ad(P)) \xrightarrow{d_A} C_c^\infty(Y; Ad(P) \otimes \Lambda^1) \xrightarrow{d_A^+} C_c^\infty(Y; Ad(P) \otimes \Lambda_+^2) \to 0$$

where $d_A^+ = (1 + *) \circ d_A$. Using L_w^2 adjoint operator, we get the next elliptic operator:

$$P(A)_w = (d_A)_w^* \oplus d_A^+ : W_w^{k+1}(Y; Ad(P) \otimes \Lambda^1) \mapsto W_w^k(Y; Ad(P) \otimes (\Lambda^0 \oplus \Lambda_+^2)).$$

Let us fix $A_0 \in \tilde{R}(Y)$. One defines an affine Hilbert space as:

$$\mathfrak{A}_k(P)_0 = \{A_0 + a | a \in W^k_w(Y; AdP \otimes T^*Y)\}, \quad k \geq 3.$$

Let us take $g \in C^1_{\mathrm{loc}}(Y; AutE)$. By embedding as $g \in C^1_{\mathrm{loc}}(Y; Hom(E,E))$, one may consider $\nabla_{A_0} g \in C^0(Y; Hom(E,E) \otimes T^*Y)$. Notice that if g is locally W^4, then it is of C^1 class. Now one defines the weighted Sobolev gauge group:

$$\mathfrak{G}_l(P) = \{h \in W^l_{loc}(Y; Aut(E)) | \nabla_{A_0} h \in W^{l-1}_w(Y; Hom(E,E) \otimes T^*Y)\}, \quad l \geq 4.$$
$$\mathfrak{G}_l(P)_0 = \{h \in W^l_{loc}(Y; Aut(E)) | h - id \in W^{l-1}_w(Y; Hom(E,E) \otimes T^*Y)\}.$$

First of all one has the following property of $\mathfrak{G}_l(P)$. Suppose one of the following two conditions; (1) $G = U(1)$, or (2) $G = SO(3)$ and $A|Y \backslash K \in W^k_w(Y \backslash K; Ad(P) \otimes \Lambda^1)$ where $E|Y\backslash K$ admits a trivialization. Then:

Lemma 1.2 *For each $h \in \mathfrak{G}_{k+1}(P)$, there exists $\bar{h} \in G$ such that $h - \bar{h} \in W^k_w(Y \backslash K; Aut(E))$.*

Proof : This follows from the Kato (toshio)'s inequality, $|\nabla_{A_0} h| \geq d|h|$ a.e., and admissibility condition (3).

The Lie algebras of $\mathfrak{G}_l(P)$ and $\mathfrak{G}_l(P)_0$ are correspondingly as follows:

$$\mathfrak{g}_l(P) = \{h \in W^l_{loc}(Y; Ad(P)) | \nabla_{A_0} h \in W^{l-1}_w(Y; Ad(P) \otimes T^*Y)\},$$
$$\mathfrak{g}_l(P)_0 = W^l_w(Y; Ad(P)).$$

Lemma 1.3 *(1) $\mathfrak{G}_{k+1}(P)_0$ acts on $\mathfrak{A}_k(P)_0$ by $g^*(A_0 + a) = g^{-1}\nabla_{A_0} g + g^{-1} a g$.*

(2) Suppose P is a $U(1)$ bundle with $G = U(1)$. Then $\mathfrak{G}_{k+1}(P)$ acts on $\mathfrak{A}_k(P)$.

The proof is standard, and we omit it.

Notice that $\mathfrak{G}_{k+1}(P)$ may not act on $\mathfrak{A}_k(P)_0$, since A_0 may live only in $L^2(Y)$.

Let us define:

$$\hat{\mathfrak{M}}_k(P) = \{A \in \mathfrak{A}_k(P)_0 : F_A + *F_A = 0\}.$$

Then $\mathfrak{G}_{k+1}(P)_0$ acts on $\hat{\mathfrak{M}}_k(P)$. Thus one gets the quotient space:

$$\mathfrak{M}_k(P) = \hat{\mathfrak{M}}_k(P) / \mathfrak{G}_{k+1}(P)_0.$$

Let us find a linear space which corresponds to the tangent space of $\mathfrak{M}(P)_k$.

Lemma 1.4 *There exist natural continuous maps:*

$$exp : \mathfrak{g}_{k+1}(P), \quad (W^{k+1}_w(Y; Ad(P))) \mapsto \mathfrak{G}_{k+1}(P) \quad (\mathfrak{G}_{k+1}(P)_0).$$

Moreover let us put $\mathfrak{G}'_{k+1}(P)(\epsilon) = \{g \in \mathfrak{G}_{k+1}(P) || g-1|W^{k+1}_w(Y) \leq \epsilon\}$. Then for sufficiently small $\epsilon > 0$, there exists $Log : \mathfrak{G}_{k+1}(P)_0(\epsilon) \mapsto W^{k+1}_w(Y; AdP)$ which inverts exp. One has a similar statement for $exp|W^{k+1}_w(Y; Ad(P))$.

We also omit the proof.

Let us take any $A \in \mathfrak{A}_k(P)_0$, and consider the following continuous map:

$$d_A : \mathfrak{g}_{k+1}(P) \mapsto W^k_w(Y; AdP \otimes T^*Y)$$

by $u \to \frac{d}{dt}(exp(tu)A)|_{t=0}$. Locally this map is expressed as $u \to du + [u, a]$.

Lemma 1.5 *Suppose an admissible pair (g, w) is given. Then for the AHS complex, d_A and d^+_A are closed operators.*

Proof : It is enough to show that $d_A|W^{k+1}_w(Y; Ad(P))$ has closed range.

Let $A'_0 \in \mathfrak{A}_k(P)$ be a smooth connection which coincides with A on some compact subset K, and is trivial except another compact subset. Then one has $A = A'_0 + a$, $a \in W^k(Y; AdP \otimes T^*Y)$. Let φ be a cut off function such that for some compact subsets $K \subset K' \subset K'' \subset Y$, one has $\varphi|K' \equiv 1$ and $\varphi|(K'')^c \equiv 0$. By the assumption, $d : W^{k+1}_w(Y; \Lambda^0) \mapsto W^k_w(Y; \Lambda^1)$ is invertible (satisfies $|d(u)|W^k_w \geq C|u|W^{k+1}_w$). Then one sees $d_{(1-\varphi)a}$ is invertible by choosing sufficiently large $K' \subset K''$, using the Sobolev embedding, $W^k_{\text{loc}} \hookrightarrow C^0$. Then one has the following estimate:

$$\begin{aligned}
|u|W^{k+1}_w &\leq C(|d_A(u)|W^k_w + |u|W^k_w) \\
&\leq C(|d_A(u)|W^k_w + |\varphi u|W^k_w + |(1-\varphi)u|W^k_w) \\
&\leq C(|d_A(u)|W^k_w + |\varphi u|W^k_w + |d_{(1-\varphi)a}((1-\varphi)u)|W^k_w) \\
&\leq C(|d_A(u)|W^k_w + |\varphi u|W^k_w + |(1-\varphi)d_{(1-\varphi)a}(u)|W^k_w + \\
&\qquad |[d_{(1-\varphi)a}, (1-\varphi)]u|W^k_w) \\
&\leq C(|d_A(u)|W^k_w + |\varphi u|W^k_w + |(1-\varphi)d_A(u)|W^k_w + \\
&\qquad |[d_{(1-\varphi)a}, (1-\varphi)]u|W^k_w + |(1-\varphi)[\varphi a + A'_0, u]|W^k_w) \\
&\leq C(|d_A(u)|W^k_w + |\psi u|W^k_w)
\end{aligned}$$

where ψ is another compactly supported cut off function with $\psi|$ Supp$\varphi \equiv 1$. The last inequality shows that d_A is a closed operator (see lemma 6.1).

Next let us take $u \in W^{k+1}_w(Y; Ad(P) \otimes \Lambda^1)$ with $(d_A)^*_w(u) = 0$. Then by admissibility condition, one has a similar inequality:

$$|u|W^{k+1}_w \leq C(|d^+_A(u)|W^k_w + |\psi u|W^k_w)$$

where a cut off function ψ has compact support. This completes the proof.

Corollary 1.1 *Let $A \in \mathfrak{A}_k(P)_0$. Then the AHS complex:*

$$0 \to W_w^{k+1}(Y; Ad(P)) \xrightarrow{d_A} W_w^k(Y; Ad(P) \otimes \Lambda^1)$$
$$\xrightarrow{d_A^+} W_w^{k-1}(Y; Ad(P) \otimes \Lambda_+^2) \to 0$$

is a Fredholm complex of index $-2p_1(P) + 3(dimH^1(AHS) - dimH^2(AHS))$.

The index computation uses the excision principle, or relative index theorem ([GL]). One can also use [APS] and the method in section 8 here. One denotes their cohomology groups by $H_A^*(AHS)$, $* = 0, 1, 2$.

Let us consider the restriction:

$$d_A : W_w^{k+1}(Y; AdP) \mapsto W_w^k(Y; AdP \otimes T^*Y).$$

By the above corollary, one may consider the adjoint operator:

$$(d_A)_w^* : W_w^k(Y; AdP \otimes T^*Y) \mapsto W_w^{k-1}(Y; AdP).$$

One understands $(d_A)_w^*$ in the geometric way as follows. Let us take $A \in \mathfrak{A}_k(P)_0$. Then one has the 'tangent space' of $\mathfrak{A}_k(P)_0/\mathfrak{G}_{k+1}(P)_0$ at $[A]$:

$$\frac{W_w^k(Y; AdP \otimes T^*Y)}{d_A(W_w^{k+1}(Y; AdP))} \cong \text{ Ker}(d_A)_w^* : W_w^k(Y; AdP \otimes T^*Y) \mapsto W_w^{k-1}(Y; AdP).$$

Let us take a smooth family of connections $A(t) \in \mathfrak{A}_k(P)_0$, $t \in [0, 1]$. Then one has:

$$\frac{\partial F_{A(t)}^+}{\partial t}|_{t=0} = (d_{A(0)} + *d_{A(0)})B \equiv d_{A(0)}^+ B$$

where $B = \frac{dA(t)}{dt}|_{t=0} \in W_w^k(Y; AdP \otimes T^*Y)$. Notice that near infinity, $F_{A(t)}$ can be written as $dA(t) + A(t) \wedge A(t)$. Each $A(t)$ can be written as $A(t) = A_0 + a_t$, where $a_t \in W_w^k(Y; AdP \otimes \Lambda^1)$. Since $F_{A_0}^+ = 0$, one sees that $F_{A_t}^+$ is a smooth family in $W_w^{k-1}(Y; AdP \otimes \Lambda_+^2)$. Now one gets the 'tangent space' of $\mathfrak{M}_k(P)$ at $[A]$ as:

$$\text{Ker } (d_A)_w^* \oplus d_A^+ : W_w^k(Y; AdP \otimes T^*Y) \mapsto W_w^{k-1}(Y; AdP \otimes (\Lambda^0 \oplus \Lambda_+^2)).$$

In particular this space is isomorphic to $H_A^1(AHS)$ which is of finite dimension.

Lemma 1.6 *Ker $d_A : W_w^{k+1}(Y; Ad(P)) \mapsto W_w^k(Y; Ad(P) \otimes \Lambda^1)$ is zero.*

Proof : This follows from the Kato's inequality, $|\nabla_A u| \geq |d|u||$ a.e. Notice that there are also no L^2 functions u with $d_A(u) = 0$. This is also seen as follows.

Suppose $u \in L^2(Y; Ad(P))$, $|u|L^2 = 1$, satisfies $d_A(u) = 0$. Then for $g = exp(u) \in \mathfrak{G}$, one has $g^*(A) = A$. Then it is known that for any path l between p and q in Y, one has $P_l \circ g(p) = g(q) \circ P_l$, where P_l is the parallel translation along l. Let $q \to \infty$. Then $g(q)$ approaches to the identity. On the other hand since P_l holds the inner product, one has $< P_l(g(p)(u)), g(q)(P_l(u)) >_q \sim < g(p)(u), u >_p$. This shows that $g(p)$ is near the identity. This impossible. This completes the proof.

Let us consider $\mathfrak{M}_k(P) = \hat{\mathfrak{M}}_k(P)_0 / \mathfrak{G}_{k+1}(P)_0$, and show that under some assumptions, this space is a finite dimensional smooth manifold. For $A \in \mathfrak{A}_k(P)_0$, let us put:

$$CG^k_A = \{A' \in \mathfrak{A}_k(P)_0 | (d_A)^*_w (A' - A) = 0\}.$$

$$I : CG^k_A \times \mathfrak{G}_{k+1}(P)_0 \mapsto \mathfrak{A}_k(P)_0 : (A, g) \to g^* A.$$

Let us calculate dI at (A, id). For $u \in W^{k+1}_w(Y; Ad(P))$ and $v \in W^k_w(Y; Ad(P) \otimes \Lambda^1)$, one has:

$$\frac{d}{dt}(\exp\, tu)^*(A + tv)|_{t=0} = d_A u + v.$$

Thus dI is given by:

$$\begin{aligned} dI = 1 \oplus d_A : \ \mathrm{Ker}(d_A)^*_w \cap W^k_w(Y; AdP \otimes T^*Y) \times W^{k+1}_w(Y; AdP) \\ \mapsto W^k_w(Y; AdP \otimes T^*Y). \end{aligned}$$

Since d_A has closed range, it is clear that the above map is an isomorphism at every (A, id).

Corollary 1.2 *There are neighbourhoods $U \subset \mathfrak{A}_k(P)_0$ of A and $V \subset \mathfrak{G}_{k+1}(P)_0$ of id such that:*

$$I : U \cap CG^k_A \times V \mapsto \mathfrak{A}_k(P)_0$$

is a homeomorphism into its image. By restriction,

$$I : U \cap CG^k_A(P) \cap \hat{\mathfrak{M}}_k(P)_0 \times V \mapsto \hat{\mathfrak{M}}_k(P)_0$$

is also a homeomorphism into $\hat{\mathfrak{M}}_k(P)_0$.

Now one has the following:

Proposition 1.2 *Suppose $d^+_A : W^k_w(Y; AdP \otimes T^*Y) \mapsto W^{k-1}_w(Y; AdP \otimes \Lambda^2_+)$ is surjection for $A \in \hat{\mathfrak{M}}_k(P)_0$. Then $\mathfrak{M}_k(P) = \hat{\mathfrak{M}}_k(P)_0 / \mathfrak{G}_{k+1}(P)_0$ is a finite dimensional smooth manifold at $[A]$. Its tangent space is naturally isomorphic to $Ker d^+_A /\ im d_A$.*

Proof : This follows from the general facts on group action ([FU] p48). From the above corollary, it is enough to verify that the above I gives a slice for the projection:

$$\hat{\mathfrak{M}}_k(P)_0 \mapsto \mathfrak{M}_k(P) = \hat{\mathfrak{M}}_k(P)_0/\mathfrak{G}_{k+1}(P)_0.$$

These can be verified by boostrapping as [FU]. We omit to write it.

1.C Perturbation of Riemannian metrics: In order to construct smooth moduli spaces, one uses K.Uhlenbeck's generic metric theorem. Let (g, w) be an admissible pair for Y. Let us choose a smooth map $h : Y \mapsto [0, \infty)$ with $h(x) \geq \frac{w}{2}(x)$. Then one introduces the following Banach manifold:

$$\mathfrak{C} = \{\phi \in C^l(Gl(TM)) : limsup_K(\Sigma_{j=0}^l e^h|\nabla^j(\phi^*g - g)|K = 0, K : \text{ compact}\}.$$

Let us take $\phi \in \mathfrak{C}$, and put $g' = \phi^* g$. Then one has the AHS complex with respect to g':

$$0 \rightarrow W_w^{k+1}((Y, g'); \Lambda^0) \xrightarrow{d} W_w^k((Y, g'); \Lambda^1) \xrightarrow{d^+} W_w^{k-1}((Y, g'); \Lambda_+^2) \rightarrow 0.$$

One sees that this is a Fredholm complex with the same index as the unperturbed one. Our aim in this section, is to show the following:

Proposition 1.3 *Suppose Y is indefinite. Then by a small perturbation of the Riemannian metric, one has no orbit of reducible connections in $\mathfrak{M}_k(P)$. In particular $\mathfrak{M}_k(P)$ is a smooth manifold.*

In order to verify this, we follows [FU]. Let us take any L^2 ASD connection A_0 over Y, and consider $\mathfrak{A}_k(P)_0 = \{A_0 + a | a \in W_w^k(Y; Ad(P) \otimes \Lambda^1)\}$. Then one introduces a $\mathfrak{G}_{k+1}(P)_0$ equivariant map:

$$P_+ : \mathfrak{A}_k(P)_0 \times \mathfrak{C} \mapsto W_w^{k-1}((Y, g); \Lambda_+^2)$$

by $P_+(A, \phi) = P_+(g)(\phi^{-1}(F_A))$, where $P_+(g)$ is the projection to the self dual part with respect to g. Notice that this map is well defined since A_0 is L^2 ASD with respect to g. Let us put $\bar{\mathfrak{M}}_k(P) = P_+^{-1}(0)$.

Proposition 1.4 $\bar{\mathfrak{M}}_k(P) \cap \mathfrak{A}_k^*(P) \times \mathfrak{C}/\mathfrak{G}_{k+1}(P)_0$ *is a smooth Banach manifold.*

We sketch its proof. First we see that dP_+ is surjective at any (A, φ) with $P_+(A, \varphi) = 0$. Then it follows that $\bar{\mathfrak{M}}_k(P)$ is a Banach manifold on which $\mathfrak{G}_{k+1}(P)_0$ acts. Then as before by making slice for the action, one gets the result.

Let $\mathfrak{c}$ be the Lie algebra of $\mathfrak{C}$. Then dP_+ splits as:

$$dP_+ = d_1P_+ \oplus d_2P_+ : W^k_w(Y; Ad(P) \otimes \Lambda^1) \oplus \mathfrak{c} \mapsto W^{k-1}_w(Y; Ad(P) \otimes \Lambda^2_+),$$

where $d_1P_+(\alpha) = P_+(g)(\varphi^{-1}(d_A(\alpha)))|(u, \varphi)$, and $d_2P_+(r) = P_+((\varphi^{-1})^*(r^*F))$. We show that the differential of P_+ is surjective (notice $\mathrm{Ker} d_A \cap W^{k+1}_w(Y; Ad(P)) = 0$).

Let us consider the AHS complex:

$$0 \to W^{k+1}_w(Ad(P)) \xrightarrow{d_A} W^k_w(Ad(P) \otimes \Lambda^1) \xrightarrow{d_A^+} W^{k-1}_w(Ad(P) \otimes \Lambda^2_+) \to 0.$$

Since this is Fredholm, one sees dP_+ has finite codimension. Let us take a representative $u \in \mathrm{Coker} dP_+$ with $(d^+_A)^*_w(u) = 0$. Then one has $(d^+_A)^*(e^w u) = 0$. Then one has the equations, $d_A(F_A) = d^*_A(F_A) = d_A(v) = d^*_A(v) = 0$, where $v = \varphi^*(e^w u)$. Then the same argument as [FU] p 56, shows that on open dense subset of Y, F_A can be expressed as $\alpha \otimes a \in \Lambda^2_+ \otimes Ad(P)$, with $|a| = 1$ (pointwise norm) and $d_A(a) = 0$. By the irreducibility, it follows $a = 0$, which contradicts to non triviality of $p_1(P)$. This completes the proof.

Proposition 1.5 *Let us fix A_0, L^2 ASD connection with respect to g. Then for a Baire set of $\phi \in \mathfrak{C}$, there are no reducible connection in $\mathfrak{A}_k(P)_0$ with respect to $\phi^*(g)$.*

Proof : Suppose A_0 is reducible, and denote the corresponding $U(1)$ connection by the same A_0. Let P be a $G = U(1)$ bundle. Then similarly as before one puts:

$$\mathfrak{A}_k(P)_0 = \{A_0 + a | a \in W^k_w(Y; \Lambda^1)\},$$
$$\mathfrak{G}_{k+1}(P)_0 = \{h \in W^{k+1}_{loc}(Y; Aut(E)) | h - id \in W^{k+1}_w(Y; T^*Y \otimes \mathbf{C})\}.$$

Then $\mathfrak{G}_{k+1}(P)_0$ acts on $\mathfrak{A}_k(P)_0$. Now as above, one considers $P_+ : \mathfrak{A}_k(P)_0 \times \mathfrak{C} \mapsto W^{k-1}_w((Y, g'); \Lambda^2_+)$. As [FU], if dP_+ is not surjective, then it follows $F_{A_0} = 0$. This shows that dP_+ is surjective.

Now one has a smooth Banach manifold $\bar{\mathfrak{M}}_k(P) = P_+^{-1}(0)/\mathfrak{G}_{k+1}(P)_0$. Let us consider a Fredholm map between Banach manifolds $\bar{\pi} : \bar{\mathfrak{M}}_k(P) \mapsto \mathfrak{C}$. Its Fredholm index is the one of the following:

$$0 \to W^{k+1}_w(Y; \Lambda^0) \xrightarrow{d} W^k_w(Y; \Lambda^1) \xrightarrow{d^+} W^{k-1}_w(Y; \Lambda^2_+) \to 0.$$

One has $H^0(AHS) = H^1(AHS) = 0$, and $H^2(AHS) \geq b^2_+$. This is also the case when one purturbes the Riemannian metrics slightly. In particular, if $b^2_+ > 0$, then one has dimH^1- dim$H^2 < 0$. Then the Sard Smale theorem shows that for a Baire set of $\mathfrak{C}$, ther are no L^2 ASD connections over non trivial line bundles. This shows that there are no orbits of reducible ASD connections in $\mathfrak{M}_k(P)$, for a Baire set of $\mathfrak{C}$. This completes the proof.

Let us put the projection as $\pi : \bar{\mathfrak{M}}_k(P) \mapsto \mathfrak{C}$. Then the direct application of [FU] to this case, shows:

Corollary 1.3 *Suppose $b_+^2 > 0$. Then a Baire set of $\phi \in \mathfrak{C}$ exists such that $\bar{\mathfrak{M}}_k(\phi) \equiv \pi^{-1}(\phi)$ are smooth finite dimensional manifolds.*

1.D Large deformation of metrics: Let M be an oriented smooth closed four manifold, and $Y \subset M$ be its connected open subset. Suppose Y can admit (g, w), an admissible pair. Here one considers to obtain ASD connections over Y, from sequences of another ASD connections over M.

Let us take a G bundle $P \mapsto M$. Let us take an exaustion $K_0 \subset K_1 \subset \dots Y$ by compact subsets. Let us take a family of Riemannian metrics $\{h_i\}_i$ on M such that $|h_i|K_i - g|K_i|C^l(K_i) \to 0$ as $i \to \infty$. Let us take a sequence of ASD connections A_i over P with respect to (M, h_i).

Proposition 1.6 (U1,2) *After gauge transformation $g_i^*(A_i)$ and taking a subsequnce (which we also denote A_i), $\{A_i\}$ converges to an ASD connection A_∞ with respect to (Y, g), on every compact subsets of Y except finite set of points.*

In order to construct an ASD moduli space $\mathfrak{M}_k(Y, g)$, one wants to choose A_∞ as the base connection. Notice that one knows only that F_{A_∞} lies in $L^2((Y, g); \Lambda_+^2)$. In order to remove the case when A_∞ is a flat connection, one will require some conditions on $w_2(P) \in H^2(M : \mathbf{Z})$. This will be considered in the next section.

1.E Discontinuity of L^2 ASD connection under a small deformation of metrics: In order to get a moduli space as a smooth manifold, one needs to make a small deformation of Riemannian metrics. Suppose for any generic metric h on M, ASD moduli space $\mathfrak{M}_k(M, h)$ is non empty. Let us take a family of irreducible ASD connections $\{A_i \in \mathfrak{M}_k(M, h_i)\}$ such that $h_i|K_i$ converges to g, and A_i converges to A_∞ as above. Let us choose a generic small perturbation $g' = \varphi^*(g)$ of g on Y, where $\varphi \in \mathfrak{C}$. Then one addresses the following continuity problem for the limit ASD connection:

For any sequence $\{A_i\}$ as above, there exist generic small perturbations h'_i of h_i with $h'_i \to \varphi^*(g)$ on compact subsets, and another family $\{A'_i \in \mathfrak{M}_k(M, h'_i)\}$

such that A'_i converges to $A'_\infty \in \mathfrak{M}_k(Y, g')$ on compact subset $\setminus$ pts, where:

$$A'_\infty \in \mathfrak{A}(P)_0 = A_\infty + W^k_w(Y; AdP \otimes \Lambda^1). \quad (**)$$

Let us formulate w-separation property which states some discontinuity, contrary to $(**)$. Let (Y, g, w) be an admissible triple, and take an $SO(3)$ bundle $P \to Y$. Then one defines the L^2 ASD moduli space $\mathfrak{M}(g)$ as the set of smooth ASD connections A on P with $\int_Y |F_A|^2 < \infty$. By taking a trivialization near infinity, one assumes $A \in W^k(Y; \Lambda^1)$ (after gauge transforming).

In $2.C$,one will introduce the Donaldson's invariant $Q(M, P; [\Sigma_1], \ldots, [\Sigma_m]) \in \mathbf{Z}$. For this, one will choose generic sections of line bundles $\Lambda_1, \ldots, \Lambda^m$ over spaces. Notice that these spaces are constructed only from embedded surfaces $\Sigma_j \subset S$.

Now let us take any $A \in \mathfrak{M}(g)$. We say that A is generically w-separated under small perturbation, if there exists a Baire set $\mathfrak{B} \subset \mathfrak{C}$ such that for any $\varphi \in \mathfrak{B}$ and any generic sections $s_1, \ldots, s_m$, one has the following property; let us choose another generic sections $s^i_1, \ldots, s^i_m$ and h_i as $1.D$ so that (1) each $s^i_j \to s_j$ as $i \to \infty$, and (2) $\mathfrak{M}(P, (M, h_i); s^i_1, \ldots, s^i_m)$ is generically defined, and (3) $h_i \to \varphi^*(g)$.

Then for any convergent subsequence of $A_i \in \mathfrak{M}(P, (M, h_i); s^i_1, \ldots, s^i_m)$ as $1.D$ with the limit A_∞, A and A_∞ has w-separated property:

$$A - A_\infty \neq L^2_w(Y; \Lambda^1).$$

If each $A \in \mathfrak{M}(g)$ is generically w-separated, then the L^2 moduli space $\mathfrak{M}(g)$ is called generically w-separated (with respect to $Q(M, P; [\Sigma_1], \ldots, [\Sigma_m])$).

1.F Asymptotics of ASD connections: Here one formulates asymptotics of L^2 norms of F_A for ASD connections over a complete Riemannian manifold (Y, g). In the next section, one will get some lower bound of the sequence, if A is obtained as a limit under deformation of Riemannian metrics on closed manifold M. The estimate is independent of any data on $M \backslash Y$.

Let $Y \subset M$ be an open connected subset. Let us equip a complete Riemannanian metrtic g of bounded geometry. An admissible approximation of (M, Y, g) is a sequence of admissible triples $\{(Y_i, g_i, w_i)\}$ with the following property; there exists an exaustion of Y by compact subsets $L_0 \subset L_1 \subset \ldots Y$ and diffeomorphisms $\varphi_i | L_i$ from L_i into Y_i such that (1) $g|L_i = g_i|L_i$,

(2) $d(\partial L_{i+1}, L_i) \geq 1$, and

(3) $\pi_1(L_{l+1} \backslash L_l) \mapsto \pi_1(L_{l+2} \backslash L_l)$ has zero image.

We always assume that the addmissible approximation consists of uniformly bounded geometry.

Let us take a generic family of Riemannian metrics on M, $\{h_i\}$ such that $|h_i - g|C^l(L_i) < \epsilon_i \to 0$. Then one chooses a sequence $\{A_i \in \mathfrak{M}_k(M, h_i)\}_i$. By taking a subsequence, one may assume that $\{A_i\}$ converges on each compact subsets, to A_∞, an ASD connection on (Y, g). One defines an asymptotics of convergent sequence $\{A_i\}$ and a partial lowest spectrum:

$$\{a_{i,j} = |F_{A_i}|L^2_{w_i}(L_i \backslash L_{i-j}; \Lambda^2_+)\}_{i,j}, \quad \lambda_i = \lambda(L_i, A_i) = \inf \frac{|(d^+_{A_i})^*_{w_i}(u)|L^2_{w_i}(L_i)}{|u|L^2_{w_i}(L_i)}.$$

Let us take an admissible approximation $\{(Y_i, g_i, w_i)\}$ of (M, Y, g) as above. Let $\{A_i\}$, $\{\lambda_i\}$ and $\{a_{i,j}\}$ be as above. Let us take cut off functions $\varphi_{i,j}$ for each i, j on M with $\varphi_{i,j}|M \backslash L_i \equiv 0$ and $\varphi_l|L_{i-j} \equiv 1$.

Proposition 1.7 *There is a function $G : \mathbf{N} \times \mathbf{N} \times (0, 1] \mapsto (0, 1]$ which only depends on the addmissible approximation with the following property; if one has bound $G(i, j, \lambda_i) > a_{i,j}$ for (i, j) with $j \geq C\lambda_i^{-1}$, then one gets an ASD connection A'_i over Y_i with resepct to g_i such that (1) $|A'_i|W^k_{w_i}(Y_i, g_i) < \infty$, and (2) its norm satisfies $|A'_i - \varphi_{i,j}A_i|W^k_{w_i} < C(Y_i)a_{i,j}$.*

The proof follows by applying the continuous method over each (Y_i, g_i). Notice the inequality:

$$\begin{aligned}
|u|L^2_{w_i}(Y_i) \leq & |(1 - \varphi_{i,j})u|L^2_{w_i} + |\varphi_{i,j}u|L^2_{w_i} \\
\leq & \lambda_i^{-1}(|d(1 - \varphi_{i,j})u|L^2_{w_i} + |d_{A_i}\varphi_{i,j}u|L^2_{w_i}) \\
\leq & \lambda_i^{-1}(|(1 - \varphi_{i,j})d_{A'_i}u|L^2_{w_i} + |\varphi_{i,j}d_{A'_i}u|L^2_{w_i}) + \frac{C}{2}j^{-1}\lambda_i^{-1}|u|L^2_{w_i}(L_i \backslash L_{i-j}).
\end{aligned}$$

2 Casson handles and Donaldson's invariant

2.A Classification: In this section, we recall several known facts and constructions on simply connected four manifolds. The realization problem of unimodular forms has been obtained completely by Freedman ([F]). Let λ be a non degenerate quadratic form over a free $\mathbf{Z}$ module Λ. We denote by $|\lambda|$, a topological closed oriented four manifold with the intersection form λ. More precisely this means that there exists an isomorphism:

$$(H^2(|\lambda|; \mathbf{Z}), (\quad,\quad)) \cong (\Lambda, \lambda).$$

When Λ is a direct sum of $\mathbf{Z}$, $\Lambda = \oplus\mathbf{Z}$, then one saids that this isomorphism is a marking of $|\lambda|$.

Theorem 2.1 (F) *Let λ be as above. If λ is of even type, we assume $k \equiv \frac{\sigma(\lambda)}{8} \mod 2$. Then there is a closed oriented simply connected topological four manifold $|\lambda|$ with the form λ, and its Kirby Siebennman invariant is equal to $ks(|\lambda|) = k$. Moreover $|\lambda|$ satisfying above is unique up to homeomorphisms.*

On the other hand, one of the striking result for smooth four manifolds is as follows:

Theorem 2.2 (Do) *Let M be an oriented simply connected smooth closed four manifold with definite form λ. Then $\lambda = \pm kI$.*

For example $|(-E_8) \oplus (-E_8)|$ cannot admit any smooth structure. Notice that $K3$ surface (Kummer surface) M is homeomorphic to $|2(-E_8)|\sharp 3(S^2 \times S^2)$.

Finally one has the following classification of indefinite unimodular forms:

Theorem 2.3 *Let λ be of indefinite odd type. Then λ is isomorphic to $k(-1) \oplus l(1)$.*

If λ is of indefinite even type, then it is isomorphic to $kE_8 \oplus l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv H$.

2.B Decomposition of M: By the Donaldson's theorem, one knows all types of intersection forms of simply connected smooth closed oriented four manifolds. Here we review the way to construct a homeomorphism $M \cong |k(-E_8)| \oplus l(S^2 \times S^2)$ for an even type M. For simplicity of the notation, one restricts to the $K3$ surface M. However the process will be applied to the general case. Let us fix a marking:

$$(H^2(M; \mathbf{Z}), (\quad , \quad)) \mapsto (\oplus_{22}\mathbf{Z}, 2(-E_8) \oplus 3H).$$

Let D be a closed topological disk in $3(S^2 \times S^2)$. Then from the above homeomorphism, one finds that there exists a smooth embedding $S = 3(S^2 \times S^2) \backslash D \hookrightarrow M$ which is compatible with the forms respectively. In particular S admits a smooth structure on $3(S^2 \times S^2) \backslash D$, induced from M. This smoothing on the end is far from the standard product $S^3 \times [0, \infty)$.

In the Appendix, we recall the basic construction to find $3(S^2 \times S^2) \backslash D$, due to Casson and Freedman's tower construction. Here we have an outline of it. In general it is known that if one removes a point from a simply connected closed four manifold M, then it is homotopy equivalent to the wedges of S^2 ([M]). Recall that $S^2 \times S^2$ admits a handle body, with one 0 and 4 handles, and two 2 handles. The two 2 handles is expressed by a link, Hopf link with 0 framing.

Let D^4 be a smooth disk. Then $3(S^2 \times S^2)\backslash D^4$ has the handle body:

$$D^4 \cup h_1 \cup \cdots \cup h_6$$

where h_i are 2 handles with framing 0. Each pair (h_{2i-1}, h_{2i}) represents a Hopf link with 0 framing. Only h_{2i-1} and h_{2i} links from each other.

Now by the Hurwitz surjection, there is a smooth map:

$$h : 3(S^2 \times S^2)\backslash D^4 \mapsto M$$

which induces an injection $\pi_2(3(S^2 \times S^2)\backslash D^4) \hookrightarrow \pi_2(M)$. This corresponds to the decomposition $\pi_2(M) \cong \pi_2(3(S^2 \times S^2)) \oplus \pi_2(|2(-E_8)|)$. Thus this map preserves the intersection form. By the general position argument, one may assume the following:

(1) h is a smooth immersion, (2) $h|D^4$ is an embbeding.

The main construction by Casson and Freedman, is to find $\tilde{h}_i \subset h_i$ with the following properties:

(1) $\tilde{h}_i$ contains $\partial_0 \bar{h}_i$, a small neighbourhood of $\partial_0 h_i$, where $\partial_0 h_i = S^1 \times D^2$ is the attaching region of $h_i = D^2 \times D^2$,

(2) $\tilde{h}_i$ is homeomorphic to h_i itself,

(3) $\tilde{h}_i \cup \partial_0 \bar{h}_i$ has a tree like end,

(4) there are regular homotopies of $h|\tilde{h}_i$ rel $h|\partial_0 h_i$ such that the result $h'|\tilde{h}_i$ is a topological embedding. Moreover $h'|\ \text{int}\tilde{h}_i \cup \partial_0 \bar{h}_i$ is a smooth embedding,

(5) Let us put $D = 3(S^2 \times S^2)\backslash[D^4 \cup_i \tilde{h}_i]$. Then D has a topological color neighbourhood. Notice that D inherits a smooth structure from the inclusion $D \hookrightarrow 3(S^2 \times S^2)$.

Proposition 2.1 (F) *D is homeomorphic to* $\mathbf{R}^4$.

Each pair $(\tilde{h}_i, S^1 \times D^2)$ is called a Casson handle. It is denoted by CH. In the Appendix, one will recall some basic construction of the Casson handles. Smooth type of each CH can be expressed by an infinite tree T with a base point, such that every edge is assigned one of $\pm$. Later one calls such a tree as a signed tree.

Let us put $Y = D^4 \cup_i \tilde{h}_i \subset M$. One may express Y by $D^4 \cup_i CH(T_i)$, where each T_i is a signed tree. Induced from the embedding, Y admits a smooth structure.

In the above case, one can consider the pair $(M, Y = D^4 \cup_i CH(T_i) \subset M)$. In part II, one will verify the following:

Proposition 2.2 *Suppose Y is constructed from a finite number of signed trees of finite type. Then Y contains another Y' with the property from (1) to (5) above, and Y' has an admissible pair (g,w).*

2.C Donaldson's invariant: Let M be an oriented simply connected smooth closed four manifold. Let h be any element in $H_2(M;\mathbf{Z})$. One may represent h by a smoothly embedded Riemann surface. In fact by Hurwitz surjection, h has a representative by a smooth immersion $f: S^2 \mapsto M$. Let $p_1, p_2 \in S^2$ be a self intersection pair. Remove small neighbourhoods $B_\epsilon(p_i)$, and add a small cylinder $S^1 \times [0,1]$ along $\partial B_\epsilon(p_i)$ which gives an embedded Riemann surface which represents the same homology class as h (notice that $H_2(M;\mathbf{Z})$ is torsion free).

Let $\Sigma_1, \dots, \Sigma_m \subset M$ be embedded surfaces representing homology classes $[\Sigma_i] \in H_2(M;\mathbf{Z})$. Let $\phi : H_2(M;\mathbf{Z}) \cong \oplus \mathbf{Z}$ be a marking. Thus the form of M is isomorphic to $(-E_8) \oplus (k-1)(-E_8) \oplus lH$, or $(-1) \oplus (k-1)(-1) \oplus l(1)$, $k \geq 2$ and $l \geq 1$. Associated to the decomposition of this form, one may express:

$$\oplus \mathbf{Z} = \oplus_{n_1} \mathbf{Z} \oplus_{n_2} \mathbf{Z} \oplus_{n_3} \mathbf{Z}, \quad (n_1, n_2, n_3) = (8, 8(k-1), 2l) \text{ or } (1, k-1, l).$$

Through ϕ and this decomposition, one has decompositions of $[\Sigma_i]$ as $[\Sigma_i] = h_i^1 \oplus h_i^2 \oplus h_i^3$, where $h_i^1 \in \oplus_{n_1} \mathbf{Z}$, $h_i^2 \in \oplus_{n_2} \mathbf{Z}$, and $h_i^3 \in \oplus_{n_3} \mathbf{Z}$.

Now suppose $Y \subset M$ is homeomorphic to $l(S^2 \times S^2) \backslash D^4$, or $l\mathbf{C}P^2 \backslash D^4$ which induces an isomorphism:

$$i : H^2(Y;\mathbf{Z}) (\hookrightarrow H^2(M:\mathbf{Z})) \quad \cong \oplus \mathbf{Z}$$

through ϕ. Moreover suppose the induced smooth structure on Y admits a tree like end of finite type. Thus one has an admissible pair (g,w) on Y.

Under the above assumption, one can construct a Donaldson's invariant $Q(P; (Y, g, w); h_1, \dots, h_m)$. One can construct Q for all oriented closed smooth four manifold. The constructions for Y are parallel to the closed case. Here one reviews for the former.

We follow [FS2]. Let E be an $SO(3)$ bundle over M, and take any embedded Riemann surface $\Sigma \subset M$. Let $\mathfrak{B}_k^*(\Sigma; E|\Sigma) = \mathfrak{A}(E|\Sigma)_k^* / \mathfrak{G}_{k+1}(E|\Sigma)$ be the gauge equivalence classes of irreducible Sobolev k connections. Similarly one has $\tilde{\mathfrak{B}}_k^*(\Sigma; E|\Sigma) = \mathfrak{A}(E|\Sigma)_k^* / \mathfrak{G}_{k+1}(E|\Sigma)_0$, where $\mathfrak{G}_{k+1}(E|\Sigma)_0 \subset \mathfrak{G}_{k+1}(E|\Sigma)$ is a subgroup with $g(x_0)$ is the identity for a fixed point $x_0 \in \Sigma$. Thus one has a fibration:

$$SO(3) \hookrightarrow \tilde{\mathfrak{B}}_k^*(\Sigma; E|\Sigma) \mapsto \mathfrak{B}_k^*(\Sigma; E|\Sigma)$$

which gives an $SO(3)$ principal bundle β over $\mathfrak{B}_k^*(\Sigma; E|\Sigma)$. Now one has the sublemma:

Sublemma 2.1 (FS2) *(1) If $< w_2(E), \Sigma > \neq 0$, β lifts to a $U(2)$ bundle $\tilde{\beta}$.*

(2) For Σ, there is another embedded surface $\Sigma' \subset M$ such that restriction of any irreducible connection A over E to $E|\Sigma$ is also irreducible.

For our application, it is enough to assume the condition (1) in the above sublemma. Thus let us choose a lift $\tilde{\beta}$ over $\mathfrak{B}_k^*(\Sigma; E|\Sigma)$. One has $c_1(\tilde{\beta}) = w_2(\beta)$ mod 2. Let us choose a complex line bundle Λ over $\mathfrak{B}_k^*(\Sigma; E|\Sigma)$ with $c_1(\Lambda) = c_1(\tilde{\beta})$.

Now by the above sublemma (2), one has the restriction map:

$$r : \mathfrak{M}_k^*(P) \mapsto \mathfrak{B}_k^*(\Sigma; E|\Sigma).$$

One chooses a generic section s of Λ. Then by the transversality argument ([DK] p192), one has a smooth manifold $\mathfrak{M}_k^*(P) \cap r^{-1}(\text{ Ker} s)$ of codimension 2. One may generalize this construction. Suppose $\dim \mathfrak{M}_*(P) = 2m$, and choose m embedded surfaces $\Sigma_1, \dots, \Sigma_m \subset M$. As before one has determinant bundles over $\mathfrak{B}_k^*(\Sigma_j; E|\Sigma_j)$. Then for generic sections, $s_1, \dots, s_m$, one has a smooth submanifold of dimension 0 as:

$$\mathfrak{M}_*(P) \cap r_1^{-1}(\text{ Ker} s_1) \cap \cdots \cap r_m^{-1}(\text{ Ker} s_m).$$

By choosing these embeddings with general positions, one may assume that this 0 dimensional manifold consists of finite oriented points. Thus one may count the number of points so that one obtains an integer. This number is, by definition, $Q(M, P; [\Sigma_1], \dots, [\Sigma_m])$ (modulo positive constant).

2.D The main theorem: Our main theorem is as follows:

Theorem 2.4 *Let $\pi_1(M) = 1$, all $h_i^1 = h_i^2 = 0$, $1 \leq i \leq l$ and $\hat{\alpha}_i \neq 0$, $i = 1, 2, 3$. Then if the Donaldson's invariant does not vanish $Q(M, P; h_1, \dots, h_m) \neq 0$, then the end of Y is tree like of infinite type, or L^2 ASD moduli space over (Y, g) is generically w-separated.*

Proof : Let us fix a marking of $H^2(M; \mathbf{Z})$, and take a smooth embedding $Y = l(S^2 \times S^2) \backslash D \hookrightarrow M$, where D is a topologically closed disk, and this embedding is compatible with the forms. On the other hand, there exists a topological disk $(D^3, S^2) \subset (M \backslash Y, \partial Y)$ such that (1) $M \backslash Y \cup D^3$ consists of two components $M_1 \cup M_2$,

(2) for even case, M_1 and M_2 are homeomorphic to $|(-E_8)| \backslash$ pt and $|(k-1)(-E_8)| \backslash$ pt respectively,

(3) for odd case, these are homeomorphic to $\bar{\mathbf{C}P}^2 \backslash$ pt and $(k-1)\bar{\mathbf{C}P}^2 \backslash$ pt respectively.

By taking any integral lift $\alpha \in H^2(M; \mathbf{Z})$, it gives a lift of the $SO(3)$ bundle to another $U(2)$ bundle $\bar{P}$. Then for any $h \in H_2(M : \mathbf{Z})$ and its mod 2 reduction $[h]_2$, one has the equality $< w_2(P), [h]_2 >=< c_1(\bar{P}), h >$ mod 2.

Now Y is an open manifold on which one can equip a complete Riemannian metric g of bounded geometry. Moreover one has a smooth function $w : Y \mapsto [0, \infty)$ so that one gets weighted Sobolev spaces $W^l_w(Y; \Lambda^*)$. Let us choose any complete Riemannian metrics of bounded geometry g_1 and g_2 on the interiors of M_1 and M_2.

Let $E \to Y$ be an $SO(3)$ bundle with $\frac{1}{4}p_1(E) = -k$ and $w_2(E) = \hat{\alpha}$, which satisfies the conditions in the theorem. Let us take an L^2 ASD connection A_0 over (E, Y, g). Then by a generic choice of g, one sees that for all A_0, d_{A_0} is an injection and $d^+_{A_0}$ is a surjection. Thus one has a smooth moduli space $\mathfrak{M}_*(E)$ of ASD connections over E. Let $\Sigma_1, \dots, \Sigma_{m'} \subset Y$ be embedded surfaces as above. Then one has a smooth manifold $\mathfrak{M}(E; [\Sigma_1], \dots, [\Sigma_{m'}])$ defined similarly as before.

Now let $\Sigma_1, \dots, \Sigma_m \subset M$ be as the theorem. By the assumption, one may assume that all these surfaces are included in Y. Moreover one may assume the general position of these, for any $\Sigma_i, \Sigma_j, \Sigma_k$, one has empty intersection $\Sigma_i \cap \Sigma_j \cap \Sigma_k = \phi$.

Suppose one has $Q(P, M; [\Sigma_1], \dots, [\Sigma_m]) \neq 0$. This shows that for a generic choice of Riemannian metric g on M, and also generic choice of these surfaces, one has non empty 0 dimensional moduli space $\mathfrak{M}(P, (M, g); [\Sigma_1], \dots, [\Sigma_m])$ whose algebraic number is equal to the above invariant Q.

Recall that one has n signed trees $T_1, \dots, T_n$ Then one obtains Y by the tower construction as $Y = D^4 \cup CH(T_1) \cup \cdots \cup CH(T_n)$.

Let us choose an exausting families of compact subsets as:

$$\begin{aligned} &K_0 \subset K_1 \subset \cdots \subset Y, \\ &K'_0 \subset K'_1 \subset \cdots \subset M_1, \\ &K''_0 \subset K''_1 \subset \cdots \subset M_2. \end{aligned}$$

Let us choose a family of metrics g'_i on M with $g'_i|K_i = g|K_i$, $g'_i|K'_i = g_1|K'_i$ and $g'_i|K''_i = g_2|K''_i$. Let us choose a family of small constants $\epsilon_i > 0$, $i = 0, 1, \dots$, so that $\epsilon_i \to 0$. As section 1.E, in order to construct a smooth moduli space $\mathfrak{M}(E, M; [\Sigma_1], \ldots, [\Sigma_m])$, one needs to perturbe g_i' slightly. Let us choose a family of generic Riemannian metrics g_i on M with $|g - g_i|W^l_w(K_i \cup K_i' \cup k_i'', g \cup g_1 \cup g_2) \leq \epsilon_i$. One may assume that the family $\{g_i\}$ consists of uniformly bounded geometry. Then one has a family of smooth manifolds:

$$\{\mathfrak{M}(P, (M, g_i); [\Sigma_1], \ldots, [\Sigma_m])\}_{i \geq 0}.$$

Let us choose any sequence $[A_i] \in \mathfrak{M}(P, (M, g_i); [\Sigma_1], \ldots, [\Sigma_m])$, $i = 0, 1, \ldots$. Then there is a subsequence which we also denote by $[A_i]$, and gauge transformations $h_i \in Aut(P)$ so that $h_i^*(A_i)$ converges to $[A_\infty]$, L^2 ASD connection with respect to (Y, g) in C^∞ topology on every compact subset of $Y\backslash$ pts. More precisely for any compact subset $K_n \subset Y\backslash$ pts and any small $\epsilon > 0$, there exists $i_0 >> 0$ so that for any $i \geq i_0$, one has $|h_i^*(A_i) - A_\infty|C^l(K_n) < \epsilon$.

Similarly for $l = 1, 2$, $h_i^*(A_i)|M_l\backslash$ pts converges to L^2 ASD connections A^l_∞ over (M_l, g_l) respectively.

Now let $\{p_1, \ldots, p_l\} \subset Y$ be the bubbling points. Then at most $2l$ surfaces can contain these points. Thus by reordering surfaces, one has:

$$A_\infty \in \mathfrak{M}(E, (Y, g); [\Sigma_1], \ldots, [\Sigma_{m-2l}])$$

Let us see that A_∞ defines an $SO(3)$ bundle E' on Y with strictly smaller $-p_1(E')$ than $-p_1(E)$ on M. For this, one will see $|F_{A^l_\infty}|L^2 > 0$, $l = 1, 2$. One has chosen E so that each $\hat{\alpha}_j \neq 0$ where $w_2(E) = \hat{\alpha} = \Sigma_{j=1}^3 \hat{\alpha}_j \in \mathbf{Z}_2^8 \oplus_{k-1} \mathbf{Z}_2^8 \oplus \oplus_{2l} \mathbf{Z}_2 \cong H^2(M; \mathbf{Z}_2)$.

Let $\alpha_j \in H^2(M; \mathbf{Z})$ be the corresponding lift. Let us take homology classes $h = h_1 + h_2 + h_3 \in H_2(M; \mathbf{Z})$ such that the mod 2 reduction of $< \alpha_j, h_j >_2 \in \mathbf{Z}_2$ are all non zero ($= 1$). First one sees $w_2(E') = \hat{\alpha}_3 \in H^2(Y; \mathbf{Z}_2)$. Let $\Sigma_j \subset M$ be embedded surfaces which represent h_j. As below one has $w_2(E') \neq 0$.

If the family $\{F_{A_i}\}_i$ concentrates at some points $p \in M_1$ and $p' \in M_2$, then clearly one has $-p_1(E') \leq -p_1(E) - 4$. Now suppose it does not concentrate on M_1. Let $\Sigma_1 \subset M_1$ as above. Then one has the equality $< w_2(E), \Sigma_1 > = < c_1(\bar{P}), \Sigma_1 > = < c_1(A_i), \Sigma > \neq 0 \bmod 2$. Since there are no concentrating points, one has also $< c_1(A^1_\infty), \Sigma > \neq 0$. Then clearly $\lim \sup_i |F_{A_i}|L^2(M_1, g_i|M_1) \geq 4$. Thus in any situation, one has $-p_1(E') \leq -p_1(E) - 4$.

Then using A_∞ as the base point, one has non empty moduli space of ASD connections $\mathfrak{M}(E', (Y, g); [\Sigma_1], \ldots, [\Sigma_{m-2l}])$. One may assume that this space contains no reducible orbits. Thus one has $\dim \mathfrak{M} = \dim \operatorname{Ker} d^+_{A_\infty} / \operatorname{im} d_{A_\infty} - 2(m - 2l)$. Since $\hat{\alpha}_i \neq 0$, $i = 1, 2$, one has $-p_1(P) > -p_1(E)$. Thus one has dim

$\mathrm{Ker}d^+_{A_\infty}/\ \mathrm{im}d_{A_\infty} + 4 - 3 - 2(m-2l) \leq 0$. In particular the formal dimension of $\mathfrak{M}$ is negative. This is a contradiction.

2.E Asymptotics of ASD connections: Let T be any infinite tree and $|\quad|$ be the norm which is the distance from the base point. Every integer value corresponds to some vertices of T. Let us use notations e to denote some edge, and v to some vertex. If v is one end point of e, then one denotes $v \in e$. One defines a finite subtree $E(v) = \{e \in T | v \in e\}$. Let us define a growth function of T by $F(T) : \mathbf{N} \mapsto \mathbf{N}$:

$$F(T)(n) = \ \sup_v \sharp \{e \in T | v \in e, |v| = n\}.$$

For a family of trees $\{T_j\}_j$, one puts $F(\{T_j\}_j)(n) = \max_j F(T_j)(n)$. Let us denote $T(n) = \{x \in T : |x| \leq n\}$. Any homogeneous tree T admits a family of homogeneous trees of finite type $\{T^n\}$ such that there are isomorphisms $\varphi_n : T(n) \subset T \cong T^n(n) \subset T^n$. These are called an asymptitics of T.

Let M be a simply connected oriented smooth closed manifold with an embedding $Y \subset M$ as before. Here the end of $Y = D^4 \cup \cup_{j=1}^l CH(T_j)$ may be of infinite type. For this case also, one can admit a complete Riemannian metric g and a weight function w on Y as before. Suppose the defining signed trees $T_1, \dots, T_l$ are homogeneous trees of infinite type. By taking asymptotics $\{T_j^n\}$ for all T_j, one gets a family of open manifolds by $Y_n = D^4 \cup \cup_{j=1}^l CH(T_j^n)$. Let us put $W_n' = D^4 \cup \cup_{j=1}^l CH(T_j^n(n))$. Y_n has end $Y_n([0,\infty))$ on vertical direction, where $Y_n([0,\infty))$ is isometric to $Y_n(0) \times [0,\infty)$ and $Y_n(0)$ is a complete Riemannian three manifold (see $5.A$). Similarly W_n' has vertical end isometric to $W_n(0)' \times [0,\infty)$. Let us choose an increasing sequence of large numbers $T_0 << T_1 << \dots$. Then one puts $W_i = W_i' \backslash W_i'([T_i,\infty))$.

Recall that each kinky handle of $CH(T)$ is determined by each $E(v) \subset T$. Thus the family $\{Y_n\}_n$ can be regarded as an approximation of $CH(T)$. Let (g_n, w_n) be a family of admissible pairs over Y_n. Let g be a complete Riemannian metric of bounded geometry on Y with $g_n | W_n = g | W_n$. Then $\{(Y_n, g_n, w_n)\}$ gives an admissible approximation of (Y, g, w) with an increasing sequence $W_0 \subset W_1 \subset \dots$.

Let us choose a sequence $A_i \in \mathfrak{M}((M, g_i), [\Sigma_1], \dots, [\Sigma_m])$. Then a subsequence $\{A_i\}$ converges to a L^2 ASD connection A_∞ over (Y, g) on each compact subset except finite set of points. Then one can assign an asymptotics of $\{A_i\}$ as a proper function $G(\{A_i\}) : \mathbf{N} \times \mathbf{N} \mapsto (0, C]$ by:

$$G(\{A_i\})(i,j) = |F_{A_i}| L^2_{w_i}((W_i \backslash W_{i-j}\}), g).$$

Let $\mathbf{F} = \{F : \mathbf{N} \mapsto \mathbf{N}\}$, $\mathbf{G} = \{h : \mathbf{N} \times \mathbf{N} \mapsto (0, C(h)]\}$ and $\mathbf{H} = \{h : \mathbf{N} \mapsto (0, C(h)]\}$ be the sets of functions.

Let P be an $SO(3)$ bundle over M with $\hat{\alpha}_i \neq 0$, $i = 1, 2, 3$ as before. Suppose the restriction of the Donaldson's invariant $Q|H_2(Y : \mathbf{Z})$ is non zero. Then there exists a family of ASD connections A_i over P which converges to A_∞ with the following property. Let $\{\lambda_i\}$ be the family of partial lower spectrums with respect to $\{W_i\}$ (see 1.F).

Theorem 2.5 *There exists a universally defined map $\mathfrak{F} : \mathbf{F} \times \mathbf{H} \mapsto \mathbf{G}$ and a constant C such that one has the pointwise estimate at (i, j) with $j \geq C\lambda_i^{-1}$:*

$$\mathfrak{F}(F(\{T_j\}_j), \{\lambda_i\}) \leq G(\{A_i\}).$$

Proof : Let us take an exaustion of Y by $W_1 \subset \dots \subset W_n \subset \dots Y$. Recall the properties $g|W_n = g_n|W_n$.

We show that for a small $\epsilon = \epsilon(F(\{T_j\}_j)(n), \lambda_n)$, if $G(\{A_i\})(n, n') < \epsilon$, then one can find $A'_n \in \mathfrak{M}((Y_n, g_n), [\Sigma_1], \dots, [\Sigma_{m'}])$, $m' \leq m$, which should be a smooth manifold of negative formal dimension. This is enough for the proof.

Let $H_2(Y : \mathbf{Z}) \cong \oplus \mathbf{Z}$ be an isomorphism with a generating set $a_1, \dots, a_v$. Since Q is polynomial, there exist some choice of $a_{i(1)}, \dots, a_{i(m)}$ from the generating set with $Q(M, P; a_{i(1)}, \dots, a_{i(m)}) \neq 0$. Then one chooses the embedded surfaces $\Sigma_1 = a_{i(1)}, \dots, \Sigma_m = a_{i(m)} \subset Y$ and $A_i \in \mathfrak{M}((M, h_i); a_{i(1)}, \dots, a_{i(m)})$ so that the above ϵ is independent of Σ_i.

Let us take a subsequence of $\{A_i\}$ which converges to $A_\infty = lim_i A_i$ on each compact subsets. If the above condition is satisfied, then one finds an ASD connection A'_n over (Y_n, g_n) such that $|F^+_{A'_n}|L^2_{w_n}(W_n) < C(g_n)G(A_\infty)(n, n')$, where C is determined by g_n. ϵ and C are determined by $\{T^n_j\}$ and $\{\lambda_i\}$. Thus one may assume ϵ and C are determined by $F(\{T_j\}_j)$ and $\{\lambda_i\}$. Let us take a cut off function φ on M with $\varphi|W_{n-n'} \equiv 1$ and $\varphi|(W_n)^c \equiv 0$. Then if $G(\{A_i\})(n, n')$ is sufficiently small, one can get a smooth connection A'_n over (Y_n, g_n) such that $A'_n|W_{n-n'}$ is ASD and $|F^+_{A'_n}|L^2_{w_n}((Y_n, g_n), \Lambda^2_+)$ is sufficiently small. Then by the continuous method, one gets $A''_n \in \mathfrak{M}((Y_n, g_n), [\Sigma_1], \dots, [\Sigma_{m'}])$. The argument that this space has negative formal dimension, is the same as the proof of the previous theorem. This completes the proof.

3 The position of Kahler forms and Picard lattice

One has an application of the vanshing result above from some relationship between Kahler forms and Picard lattice over algebraic surfaces. Let M be a simply connected complex algebraic surface with $p_g > 0$. Suppose M is of even type, and take a marking $\psi : H^2(M; \mathbf{Z}) \cong \oplus_{8k} \mathbf{Z} \oplus_{2l} \mathbf{Z}$ with respect to the decomposition of the form as $k(-E_8) \oplus lH$. Then one has an open manifold $S \subset M$ which corresponds to lH. The diffeomorphism type of the end of S is determined by $T_1, \ldots, T_{2l}$, where each T_j is an infinite signed tree.

M can be holomorphically embeddable into a projective space $\mathbf{C}P^N$. Let us take such embedding $i : M \hookrightarrow \mathbf{C}P^N$. Then one has a natural homology class $h \in H_2(M; \mathbf{Z})$, the hyperplane class. This is defined as the intersection of M with the hyperplane $\mathbf{C}P^{N-1} \subset \mathbf{C}P^N$. Let ω be the Fubini Study form over $\mathbf{C}P^N$. Then the Poincaré dual to h is the restriction of $\omega = \omega|M$. This form gives a Kahler (Hodge) structure on M.

Let h be a hyperplane class of M, and decompose $h = h_1 \oplus h_2$ with respect to above. Suppose there is an integral $(1,1)$ form $\alpha = \alpha_1 \oplus \alpha_2 \in H^{1,1}(M; \mathbf{R}) \cap H^2(M; \mathbf{Z})$, such that the mod 2 reduction $\hat{\alpha}_i \neq 0 \in H^2(M; \mathbf{Z}_2)$, $i = 1, 2$.

Theorem 3.1 *Let M be as above and α satisfy the above condition. Then if there exists a hyperplane class $h \in H^2(M; \mathbf{Z})$ with $h_1 = 0$, then one of the following holds; (1) at least one of T_j is of infinte type, or (2) L^2 ASD moduli space over (S, g) is generically w-separated.*

This theorem follows from our main theorem before, and the non vanishing result as follows:

Theorem 3.2 (Do) *Let M be as before and take any hyperplane class h. Then there is $k_0 = k(S, h)$ such that for any $k \geq k_0$, and any $\hat{\alpha}$ which is mod 2 reduction of an integral $(1,1)$ class, the Donaldson's invariant $Q_{\hat{\alpha},k}(h)$ is positive.*

Theorem 3.1 above gives some condition on the smooth structure on the end of S. Recall that the Picard number is the rank of the image of the Picard group $\mathrm{Pic}(M)$ in $H^2(M : \mathbf{Z})$. Let M be a complex algebraic surface with even type of the form, $k(-E_8) \oplus lH$. Let us say that the Picard group of M is large if the Picard number is grater than $\max\{8k+1, 2l+1\}$. Thus the Picard numbers of all projective Kummer surfaces are large (more than 17).

Lemma 3.1 *Let M be a complex algebraic surface of even type of the form. Let ω be a Hodge form. Then there exists a marking $\phi : H^2(M : \mathbf{Z}) \cong \oplus_{8k} \mathbf{Z} \oplus_{2l} \mathbf{Z}$ corresponding to the decomposition $k(-E_8) \oplus lH$, such that $\phi(\omega) \in \oplus_{2l} \mathbf{Z}$.*

This follows easily from the result in [J] for unimodular forms over $\mathbf{Z}$.

Then one has the following statement; let M be as above, and suppose the Picard group of M is large. Then there exists a marking ϕ as above, such that $\phi(\omega) \in \oplus_{2l}\mathbf{Z}$, and for some $\alpha = \alpha_1 + \alpha_2 + \alpha_3 \in$ im Pic(M), one has $[\alpha_j]_2 \neq 0 \in \oplus \mathbf{Z}_2$.

Corollary 3.1 *Let M be as above. Let S be an open sub manifold in M with tree like end as before. Then for the defining signed trees $T_1, \dots, T_{2l}$, at least one of them is of infinite type with respect to ϕ above, or L^2 ASD moduli space over (S, g) is generically w-separated.*

For $K3$ surface M, the Kahler cone is described as $\mathfrak{C} = \{y \in H^{1,1}(M; \mathbf{C}) | < y, y >> 0, < y, d >> 0, d \in \Delta\}$, where Δ consists of the effective classes with self intersection -2 in the Picard lattice.

For $K3$ surface M, there is a 1 dimensional holomorphic section w of holomorphic 2 forms. w satisfies (1) $< w, w >= 0$, $< w, \bar{w} >> 0$. Then $\omega \in H^2(S; \mathbf{Z})$ can be a Kahler form on some K3 surface M if and only if the following conditions are satisfied:

(2) $< \omega, w >= 0$, $< \omega, \omega >> 0$,

(3) $< \omega, d > \neq 0$ for any $d \in H^2(M; \mathbf{Z})$ with $< d, d >= -2$ and $< w, d >= 0$.

In the case when M is a Kummer surface, then one has the natural marking. With respect to this particular marking, one has the following non vanishing result.

Lemma 3.2 *Let M be a projective Kummer surface. Then there exists a particular marking φ such that for any Hodge form ω, $\varphi(\omega) = \varphi(\omega)_1 \oplus \varphi(\omega)_2$ satisfies $\varphi(\omega)_1 \neq 0$.*

Proof : Notice that for this marking, -2 curves d spans $\oplus^{16}\mathbf{Z} \subset \oplus^{16}\mathbf{Z} \oplus^{6} \mathbf{Z}$ with respect to the decomposition. Then one takes any $d \in \oplus^{16}\mathbf{Z}$ with $< w, d >= 0$. For this d, one must have $< \omega, d > \neq 0$ which implies $\omega_1 \in \oplus^{16}\mathbf{Z}$ does not vanish.

For another $K3$ surfaces, one knows the following.

Proposition 3.1 *Let (M, ϕ) be a marked Kahler K3 surface with Kahler form ω, and $w \neq 0 \in H^{2,0}(M; \mathbf{C})$. Let $E(w) \subset H^2(M; \mathbf{R})$ be a linear 2 plane spanned by Rew and Imw. Suppose $H^2(M; \mathbf{Z}) \cap E(w)$ consists of a primitive rank 2 lattice in $H^2(M; \mathbf{Z})$. Then there is an automorphism τ on $H^2(M; \mathbf{Z})$ such that for all Hodge form ω, one has $(\tau \circ \phi(\omega))_1 \neq 0 \in \oplus^{16}\mathbf{Z}$.*

Recall that a sublattice $L' \subset L$ is primitive if L/L' is free as a $\mathbf{Z}$ module.

Proof of the proposition: Let us put $L = H^2(M;\mathbf{Z}) \cap E(w)$. Then there is an automorphism τ of $H^2(M;\mathbf{Z})$ such that $\tau(L)_1 = 0 \in \oplus^{16}\mathbf{Z}$ (see [BPV] p15 2.9(2)). Since $\tau(L)$ is primitive, one has an orthogonal decomposition $H^2(M;\mathbf{Z}) = (\tau(L))^\perp \oplus \tau(L)$, Let us take $d \in (\tau(L))^\perp$ with $< d, d >= -2$. Then by the condition of ω above, one has $< \tau(\omega), d > \neq 0$. This shows the result.

4 Appendix

Here we recall basic construction of Casson and Freedman's tower construction.

Casson handles: Before entering the tower construction, one recalls about the spin structure on four manifolds. Let M be a simply connected closed smooth four manifold. Then one has the second Stiffel Whitney class $w_2(M) \in H^2(M;\mathbf{Z}_2)$.

Lemma 4.1 *Let M be as above. Then M admits a spin structure (i.e. $w_2(M) = 0$) if and only if the intersection form of M is of even type.*

Suppose M has form $\lambda = \lambda_1 \oplus \lambda_2$, such that $||\lambda_1||$ is realized by a smooth N. Later when we construct some smooth embedding $N\backslash D \equiv Y \hookrightarrow M$, one assumes one of the following conditions:

(1) $w_2(M) = 0$, or

(2) $w_2|H_2(M : \mathbf{Z}_2) \cap (H_2(N : \mathbf{Z}_2))^\perp \neq 0$.

For the Kummer surface, (1) is satisfied. For the treatment of topological manifolds M which does not satisfy (1) or (2), we refer to [FQ].

In general, N will contain 1 and 3 handles for a handle body. However one has the following ([FQ] p176):

Lemma 4.2 *For any handle body on N, one may change 1 and 3 handles into 2 handles by taking appropreate number of connected sums with $S^2 \times S^2$.*

Suppose N admits a handle body with one 0 and 4 handles. All the other handles consist of 2 handles. From the handle body theory, one may assume that all 2 handles are disjoint. Let $Y = N\backslash D^4 \mapsto M$ be a smooth immersion as $2.B$:

$$h : N\backslash D^4 = D^4 \cup h_1 \cup \cdots \cup h_k \mapsto M.$$

Let $D_i = 0 \times D^2 \subset h_i$ be the cores of the 2 handles.

Lemma 4.3 (see FQ) *Suppose $w_2(M)$ satisfies the above condition. Then:*

(1) By changing h by a regular homotopy, one may assume that the (self) intersection numbers of $h|D_i$ are all trivial.

(2) For each D_i, there are $\alpha_i \in \pi_2(M)$ so that the intersection numbers are given by $\lambda(h(D_i), \alpha_j) = \delta_{i,j}$.

Tower construction: Let D^2 be a 2 dimensional disk, and consider an immersion $f : D^2 \mapsto M$ into a 4 manifold. When one considers such immersion, it is always assumed that f is an embedding on the boundary, $S^1 = \partial D^2$. Let us regard D^2 as the core, $D^2 \times \{0\} \subset D^2 \times D^2$. We say that f is a framed immersion if f has an extension to $f|D^2 \times D^2$, and for every double points $p, p' \in D^2$, $f(p) = f(p')$, one has the identification $(z_1, w_1) \cong (w_2, z_2)$, where (z_1, w_1) and (z_2, w_2) are local charts around $p = (0, 0)$ and $p' = (0, 0)$ respectively.

Let M be a simply connected 4 manifold, and consider a framed immersion $f : S^2 \mapsto M$ and an immersion $h : D^2 \mapsto M$. The image $f(S^2) \equiv (D^2)^t$ is called a transverse sphere for h if $h(D^2) \cap (D^2)^t$ consists of one point. $(D^2)^t$ is called an algebraically transverse sphere if the intersection number of these is equal to 1. When h is a map from a disjoint union of D^2, $h : \cup_i (D^2)_i \mapsto M$, then a transverse sphere for h consist of transverse spheres for every component $(D^2)_i$. The main theorem coming from the tower construction, is as follows:

Theorem 4.1 (F) *Let $h : \cup D^2 \mapsto M$ be an immersion whose algebraic intersection numbers are all 0, with algebraic transverse spheres. Then h is regularly homotopic rel ∂ to a topologically union of embedded disks with transverse spheres.*

One may replace the condition, that algebraic intersection numbers $h((D^2)_i) \cap h((D^2)_j)$ are all 0, by that such numbers for $(D^2_i)^t \cap (D^2_j)^t$ are all 0.

One finds proofs of this theorem in [F], [FQ]. Roughly speaking, the process of the proof is as follows. Suppose one has an immersion as above. One may extend h, and thick the domain as $h| \cup D^2 \times D^2$. The image of every component $h(D^2 \times D^2)$ is diffeomorphic to $S^1 \times D^3 \natural_{D^3} S^1 \times \natural_{D^3} \dots S^1 \times D^3$ (n times). It is called a kinky handle. The number of times to take end connected sums, is equal to the one of double points. Now one tries to remove intersections in the image, by adding another kinky handles. Let $\gamma_i : S^1 \hookrightarrow h(D^2 \times D^2)$, $1 \leq i \leq n$, be a set of embeddings, representing a set of generator of $\pi_1(h(D^2 \times D^2))$. Then one may construct another immersions, $h_i : D^2 \mapsto M$ whose restriction $h_i|\partial D^2$ are identical to γ_i. One may extend it to a framed immersion $h_i|D^2 \times D^2$. Then the images of h_i produces another kinky handles. One may iterate this process infinitely many times. The above theorem saids that the result of union of kinky handles, are homeomorphic to $D^2 \times D^2$ rel boundary.

Here we will recall the process more concretely. Let us denote a 2 handle by $(D^2 \times D^2, \partial D^2 \times D^2)$, where $\partial D^2 \times D^2$ is the attatching region.

A 2 handle is expressed by a knot in S^3. A knot corresponds to the attatching circle $S^1 \times \{0\} \subset \partial D^2 \times D^2$. For example the trivial knot implies $B^4 \cup D^2 \times D^2$, diffeomorphic to $D^2 \times S^2$. In general one needs to determine a framing for a knot which is an integer. For example a trivial knot with 1 framing implies $\mathbf{C}P^2 \backslash D^4$.

Let $h : D^2 \times D^2 \mapsto M$ be a framed immersion, such that $h|\partial D^2 \times D^2$ is embedding. Then the image is diffeomorphic to $\natural_n S^1 \times D^2$. For each double point of the image, or each generator $\gamma_i \in \pi_1(\natural_n(S^1 \times D^3))$, one associates $\pm$ sign. We call the pair $(\natural_n S^1 \times D^3, (\pm, \dots, \pm))$, a kinky handle with n kinks. Thus there are at most 2^n kinky handles with n kinks.

Kinky handles are conveniently described by link diagrams. Usually a 1 handle is denoted a circle with a dot. This is understood as follows. Let us choose a standard embedding $i : S^1 = \partial D^2 \hookrightarrow S^3 = \partial B^4$, and extend this to the embedding of pairs, $i : (D^2, S^1) \hookrightarrow (B^4, S^3)$. We remove a tublar neighbourhood of the image of i in B^4. Then the result, $B^4 \backslash D^2 \times D^2$ is diffeomorphic to $S^1 \times D^3$, union of one 0 handle and one 1 handle.

Notice that a generator γ of $\pi_1(B^4 \backslash D^2 \times D^2)$ is given as the following loop in $S^3 \subset B^4$:

Figure

Let k be a kinky handle with 1 kink. Thus it is diffeomorphic to $S^1 \times D^3$. Let us consider $\partial k = S^1 \times S^2$, and regard it as a union of two solid tori, $\partial k = \cup_{\pm}(S^1 \times D^2)_{\pm}$. Notice that when k is an image of an immersion $h : D^2 \times D^2 \mapsto M$, It is important to consider the position of the attatching circle, $h(S^1 \times \{0\}) \subset (S^1 \times D^2)_+ \subset \partial k$. For 1 kinks, there are two possible positions as the following pictures:

Figure

Thus if one writes a 1 kink with specified attatching region, it is expressed using link picture as follows:

Figure

When one has n kinks, where n is more than 1, one has the following link picture, to denote the kinky handle with the attaching region:

Figure

Let k be a n kinky handle. Then one has a disjoint n loops in k, $\gamma_1, \ldots, \gamma_n$, which represents a generator of $\pi_1(k)$. For every γ_i, let us assign another kinky handle k_i, with the attatching region γ_i. It is clear that γ_i is null homotopic in k_i. In particular one has the zero map $\pi_1(k) \to \pi_1(k \cup_i k_i)$ induced from the embedding. The link diagram for $k \cup_{i=1}^{n} k_i$ is as follows:

Figure

Now we look for the atatching regions in more detail. Let k be a n kinky handle. Let $\mu \subset \partial k$ be the atatching circles. Let us regard ∂k as:

$$\sharp_n S^1 \times S^2 \cong \cup_{\pm}(\natural_n S^1 \times D^2)_{\pm}.$$

Then one may assume that μ has the image in $(\natural_n S^1 \times D^2)_+$. The image of μ is described as follows. Let k_+ or k_- be 1 kinky handles. Then $k_\pm$ can be denoted by an embedding of a circle $\mu_\pm$ into $\partial k_\pm = S^1 \times S^2$, $k_\pm = (S^1 \times D^3, \mu_\pm)$. Associated to the end connected sum of two $S^1 \times S^2$, one may take a bund connected sum of two $\mu_\pm$'s. The result becomes $(\natural^2 S^1 \times D^3, \mu_\pm \natural \mu_\pm)$, or shortly $\mu_\pm \natural \mu_\pm$. Let $\mu_1, \dots, \mu_n$ be the one of μ_+ or μ_- respectively. Then $\natural_{i=1}^n \mu_i$ is exactly representing the previous μ in $(\natural_n S^1 \times D^2)_+$ as above, for some choice of $\pm$.

Let us consider the previous situation, $k \cup_{i=1}^n k_i$. From the above description, one may represent any kinky handle k with n kink by a tree $T(k)$ with n edges $e_1, \dots, e_n$, and all the edges has the common one end point $v(k)$. Moreover every edge is assigned with a sign $\pm$. Recall that $\pi_1(k)$ has a generator by $\gamma_1, \dots, \gamma_n$. Then every edge e_i corresponds to γ_i. Now suppose another kinky handle k_i is attatched along γ_i. Let us represent k_i by another tree $T(k_i)$. Then in order to express $k \cup k_i$, one connects $T(k)$ and $T(k_i)$ by identifying the end point of e_i which is different from $v(k)$, and $v(k_i)$. The result is another tree $T(k \cup k_i)$. By making this attatching for every i, one gets a tree $T(k \cup_{i=1}^n k_i)$.

Figure

Now let us consider $k \cup k_i$. Associated to every loop $\gamma_{i,l}$, in k_i which represent an element in $\pi_1(k_i)$ as before, one may attatch another kinky handle $k_{i,l}$. By adding $T(k_{i,l})$ with the vertex corresponding to $\gamma_{i,l}$, one may similarly construct another tree, $T(k \cup k_i \cup k_{i,l})$. One may iterate this process infinitely many times. The result, $k \cup_i k_i \cup_{i,l} k_{i,l} \cup_{i,l,m} k_{i,l,m} \cup \dots$, is called a Casson handle, CH. Thus every CH may be expressed by an infinite signed tree T with one end point. Let $S^1 \times D^2 \subset CH$ be the attatching region of the first kinky handle k.

Theorem 4.2 (F) *Any Cassson handle pair $(CH, S^1 \times D^2)$ is pairwisely homeomorphic to $(D^2 \times D^2, S^1 \times D^2)$.*

II Spectrum on complete Riemannian manifolds

5 A complete Riemannian metric on int W_0

Roughly speaking the set of the anti self dual connections modulo equivalence forms a finite dimansional manifold, in the case when the base four manifold is closed. Then the dimension is equal to the Fredholm index of the elliptic complex, the Atiyah Hitchin Singer complex. One may make a Fredholm operator for four manifold with boundary, using suitable complete Riemannian metrics on its interior. In this case one uses the weighted Sobolev spaces on the elliptic operator. For our purpose this is a basic construction, and later we will use infinitely many such open Riemannian manifolds. In fact by glueing out all of them iteratedly, we will get another complete Riemannian manifold such that the Atiyah Hitchin Singer complex becomes Fredholm between the weighted Sobolev spaces.

5.A A specified embedding of W_0 into $\mathbf{R}^N$: Let W_0 be a compact manifold with boundary of arbitrary dimension n. Let us choose compact submanifolds with boundary $M_1, N_0 \subset \partial W_0$ of the same dimension, with empty intersection, $M_1 \cap N_0 = \phi$. Suppose M_1 and N_0 are diffeomorphic each other, and denote $\tilde{W}_0 = \text{int}W_0 \cup M_1 \cup N_0$. Let us take a countable number of $\tilde{W}_0$ and make indices them by $\tilde{W}_0^i = \tilde{W}_0$, $i \in \mathbf{Z}$. Then one constructs an open manifold Y by $\cup \tilde{W}_0^i$ identified M_1^i with N_0^{i+1}:

$$Y = \dots \tilde{W}_0^i \cup_{M_1^i \sim N_0^{i+1}} \tilde{W}_0^{i+1} \cup \dots$$

There is a natural smooth $\mathbf{Z}$ action on Y, and we call Y a periodic open manifold. We have the obvious notion, periodic, for vector bundles over Y, or for differential operators over E.

In order to define (weighted) Sobolev spaces over Y, we need first of all, complete Riemannian metric on int W_0 which extends to the one of Y. We recall an explicit picture of end connected sums along M_1 and N_0. Let us choose smooth embeddings:

$$\gamma(\gamma') : M_1(N_0) \times [0, \infty) \hookrightarrow \partial W_0 \times [0, \infty) \subset \text{ int } W_0.$$

Then we have the following equality (at this stage it is C^∞ sense, by smoothing corners):

$$\text{int } W_0 \natural \text{ int } W_0 = \text{ int } W_0 \backslash \text{ im } \gamma \cup \text{ int } W_0 \backslash \text{ im } \gamma'$$

where we identify $\partial M_1 \times [0,\infty) \cup M_1 \times \{0\}$ with $\partial N_0 \times [0,\infty) \cup N_0 \times \{0\}$. In order to make this construction compatible with the metric on int W_0, we choose a particular style of Riemannian metric on int W_0 as follows. Let us choose an embedding:

$$I : W_0 \hookrightarrow \mathbf{R}^{N+2}$$

with the following properties:

(1) Let us choose a small neighbourhood of ∂W_0 in W_0 and identify it with $\partial W_0 \times [0,\epsilon] \equiv W_0(\epsilon)$, $\partial W_0 = \partial W_0 \times \{0\}$. Then for any $\tau \in [0,\epsilon]$, we have the following:

$$W_0 \cap \mathbf{R}^{N+1} \times [0,\epsilon] = W_0(\epsilon), \quad W_0(\epsilon) \cap \partial W_0 \times \tau \subset \mathbf{R}^{N+1} \times \tau.$$

(2) For a smaller $\epsilon' < \epsilon$, $\mathrm{pr}\{W_0(\epsilon') \cap \partial W_0 \times \tau\} \subset \mathbf{R}^{N+1}$ is independent of $\tau \in [0,\epsilon']$, where pr: $\mathbf{R}^{N+2} \mapsto \mathbf{R}^{N+1}$ is the projection.

Figure

(3) $I|W_0 \backslash W_0(\epsilon)$ is C^∞ embedding (i.e. without corners).

(4) Let $\bar{M}_1$ and $\bar{N}_0$ be tublar neighbourhoods of M_1 and N_0 in ∂W_0 respectively. As before we identify small neighbourhoods of $\partial \bar{M}_1$ and $\partial \bar{N}_0$ with $\partial M_1 \times (0,\delta] \equiv M_1(\delta)$ and $\partial N_0 \times (0,\delta] \equiv N_0(\delta)$ respectively. Then we have the following:

(a) $\bar{M}_1$ and $\bar{N}_0$ are diffeomorphic to M_1 and N_0 respectively,

(b) $\partial W_0 \cap \mathbf{R}^N \times \{0\} = M_1, \quad \partial W_0 \cap \mathbf{R}^N \times \{1\} = N_0$,

(c) for $0 < \tau < \delta$, $M_1(\delta) \cap \mathbf{R}^N \times \tau$ and $N_0(\delta) \cap \mathbf{R}^N \times \{1-\tau\}$ are diffeomorphic to ∂M_1 and ∂N_0 respectively,

Figure

(4) $W_0(\epsilon) \cap \mathbf{R}^N \times \tau \times [0,\epsilon]$ and $W_0(\epsilon) \cap \mathbf{R}^N \times \{1-\tau\} \times [0,\epsilon]$, $\tau \in (0,\delta]$, are diffeomorphic to $\partial M_1 \times [0,\epsilon] \cup M_1$ and $\partial N_0 \times [0,\epsilon] \cup N_0$ respectively.

Figure

Moreover $W_0(\epsilon) \cap \mathbf{R}^N \times \tau \times [0,\epsilon] \subset \mathbf{R}^N \times \{0\} \times [0,\epsilon]$ and $W_0(\epsilon) \cap \mathbf{R}^N \times \{1-\tau\} \times [0,\epsilon] \subset \mathbf{R}^N \times \{0\} \times [0,\epsilon]$ are independent of τ.

(5) Let us denote $\bar{M}_1(\tau) = W_0(\epsilon) \cap \mathbf{R}^N \times \{\tau\} \times [0,\epsilon] = \{(a,b,c) \in \bar{M}_0(\delta) \cap \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}\}$. We have similar notations for N_0. Then we have $\bar{N}_0(\tau) = \bar{M}_1^\sharp(\tau) \equiv \{(a,1-b,c) | (a,b,c) \in \bar{M}_1(\tau)\}$, $\tau \in (0,\delta]$.

Figure

Let us denote $\tilde{W}_0' = W_0 \backslash W_0 \cap \mathbf{R}^N \times \{0,1\} \times [0,\epsilon]$. Then $I|\tilde{W}_0'$ is a smooth embedding with corners. Let us denote by g_0, the Riemannian metric on $\tilde{W}_0'$ induced from the standard one on $\mathbf{R}^{N+2}$ through I. Notice important properties of g_0; for $\epsilon' < \epsilon$:

$$g_0|W_0(\epsilon') = (g_0|\partial W_0) \times dy^2, \quad g_0|\bar{M}_1((0,\delta]) \cap W_0(\epsilon') = (g_0|\partial M_1) \times dx^2 \times dy^2$$

where we have the coordinates $(t,x,y) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}$. We have a similar expression for $g_0|\bar{N}_0(\delta)$.

Now we define the desired Riemannian metric on int W_0 as follows. Let us put:

$$\tilde{W}_0 = \tilde{W}_0' \cup [\partial W_0 \backslash \{M_1 \cup N_0\}] \times (-\infty, 0] \cup$$
$$[\partial M_1 \times (-\infty, \epsilon] \cup M_1 \times \{\epsilon\}] \times (-\infty, 0] \cup [\partial N_0 \times (-\infty, \epsilon] \cup N_0 \times \{\epsilon\}] \times [1, \infty).$$

Figure

Clearly $\tilde{W}_0$ is diffeomorphic to int W_0. On the other hand, by the above remark, there is a natural extension of g_0 to $g|\tilde{W}_0$. This is the desired complete Rimannian metric. Later we will use the following notations. Let us choose large $t >> 0$ and $s >> 0$. Then we denote:

$$\tilde{W}_0(t) = [\partial W_0 \backslash \{M_1 \cup N_0\}] \times \{-t\} \cup \partial M_1 \times \{-t\} \times (-\infty, 0]$$
$$\cup\, \partial N_0 \times \{-t\} \times [1, \infty),$$

$$M_1(s) = \{\partial M_1 \times (-\infty, \epsilon] \cup M_1 \times \{\epsilon\}\} \times \{-s\}.$$

We denote $M_1([S_0, S_1]) = \cup_{s \in [S_0, S_1]} M_1(s)$. We have a similar notation $N_0(s)$.

5.B Casson handles as Riemannian manifolds: Let us generalize the previous construction over the triple (W_0, M_1, N_0) as follows. Suppose we have $n+1$ submanifolds $M_0, M_1, \ldots, M_n$ in ∂W_0 such that they are disjoint from each other, $M_i \cap M_j = \phi$. Then by choosing an appropreate embedding $I : W_0 \hookrightarrow \mathbf{R}^N$, one can construct a complete Riemannian metric on intW_0, by the same way as $5.A$. Recall the notation $\tilde{W}_0'$ in $5.A$. Then one expresses this as $\tilde{W}_n$:

$$\tilde{W}_n = \tilde{W}_0' \cup [\partial W_0 \backslash \cup_i M_i] \times (-\infty, 0] \cup_i [\partial M_i \times (-\infty, \epsilon] \cup M_i \times \{\epsilon\}] \times (-\infty, 0].$$

Let Σ be a Riemann surface of genus $g \geq 1$, and $\bar{\Sigma}$ be its solid Riemann surface. Thus $\bar{\Sigma}$ is a three manifold with boundary Σ. Let us put $W_g = \bar{\Sigma} \times [0, 1]$, and take g embeddings of $S^1 \times D^2$ into ∂W_g which represents a generating set of $\pi_1(W_g)$. We denote their images by $M_1, \ldots, M_g$. One may assume that all M_i and M_j are disjoint. A kinky handle with k kinks, are expressed as

$(W_k, N_0, M_1, \dots, M_k)$, where N_0 is another embedding of $S^1 \times D^2$ into ∂W_k. The isotopy type of N_0 is determined by the sign $(\pm, \cdots \pm)$ (k times). Thus there are exactly 2^k types of kinky handles with k kinks.

Casson handle is constructed by taking end connected sums of kinky handles infinitely many times. It is expressed by an infinite tree (embedded in $\mathbf{R}^2$) with the additional information. Let us introduce the following:

Definition 5.1 *Let T be an infinite tree with only one end point $*$. T is said to be a signed tree, if every edge v is assigned one of $+$ or $-$.*

Let $n(v)+1$ be an integer which is the number of edges of T with common end point v. A signed tree T is said to be of bounded multiplicity if there exists a constant C with $C \geq n(v)$ for any $v \in T$.

Any tree with one end point $*$, admits a natural distance from $*$. One will denote it by $| \quad |$. Let T be a signed tree with one end point $*$. Let us take any vertex $v \in T$. Then there exists a unique edge e on T such that one of the end point of e is v, and the other v' satisfies $|v'| < |v|$. To specify this e, one denotes it as $e(v)$. Let us denote finite subtrees by $T_v = \cup_{j=0}^{n(v)} e_j$, where all the edges e_j has v as one end point, and $e_0 = e(v)$. Recall that each e_j is assigned with $\pm$. Since e_j, $1 \leq j \leq n(v)$, are all assigned one of $\pm$, the set $\{e_1, \dots, e_{n(v)}\}$ has one to one correspondence with the set $(W_{n(v)}, N_0, M_1, \dots, M_{n(v)})$, where $W_{n(v)}$ is diffeomorphic to $\bar{\Sigma}_{n(v)} \times [0,1]$, and all N_0 and M_j are diffeomorphic to $S^1 \times D^2$ with specified embeddings into $\partial W_{n(v)}$. Thus every T_v corresponds to a kinky handle with $n(v)$ kinks. In particular one may assign a complete Riemannian manifold $\tilde{W}_v$ for every vertex $v \in T$.

Let us take an edge e with end points v and v', and assign a large number $S >> 0$. Then one can make end connected sums of $\tilde{W}_v$ with $\tilde{W}_{v'}$ as:

$$\tilde{W}_{v'} \natural_{e,S} \tilde{W}_v \equiv \tilde{W}_{v'} \backslash M_e([S,\infty)) \natural \tilde{W}_v \backslash N_v([S,\infty)).$$

One will also denote the set (T, S) by T, and also call it a signed tree. For every e, one may connect $\tilde{W}_v$ and $\tilde{W}_{v'}$ as above. By iterating these end connected sums for every edge, one gets a complete Riemannian four manifold $CH(T)$. The diffeomorphism type of this space, is the Casson handle corresponding to T.

We denote (W_0, N_0, M_1) as the simplest kinky handles. Then $\tilde{Y}(S)$ is a periodic simplest Riemannian Casson handle:

$$CH(\mathbf{R}) = \tilde{Y}(S) = \cup_{j \in \mathbf{Z}} \tilde{W}_0^j([-S,S]), \quad CH(\mathbf{R}_+) = \tilde{Y}(S)_0 = \cup_{j \in \mathbf{N}} \tilde{W}_0^j([-S,S])$$

where $\tilde{W}_0^j([-S,S]) = \tilde{W}_0^j \backslash N_0([S,\infty)) \cup M_1([S,\infty))$, and one numerates the same $\tilde{W}_0$ by $\tilde{W}_0^j$, $j \in \mathbf{Z}$.

Let $(W_l, N_0, M_1, \ldots, M_l)$ be a kinky handle with l kinks. Then one defines the following another Riemannian manifolds. Let us choose any $1 \le j \le l$, say $j = 1$. Now one defines:

$$
\begin{aligned}
&V(S,l)_0 \equiv \tilde{W}_l \backslash \cup N_0([S,-\infty)) \cup_{k=1}^{l} M_k([S,\infty)),\\
&Y(S,l,1)_0 = V(S,l)_0 \cup_{j=2}^{l} \cup_{M_j(S)\sim N_0(S)} \tilde{Y}(S)_0 \backslash N_0([S,-\infty)),\\
&Y(S,l,1) = Y(S,l,1)_0 / N_0(S) \sim M_1(S),\\
&\tilde{Y}(S,l,1)_0 = \cup_{j\in\mathbf{N}} \cup_{M_1^j(S)\sim N_0^{j+1}(S)} Y(S,l,1)_0^j,\\
&\tilde{Y}(S,l,1) = \cup_{j\in\mathbf{Z}} \cup_{M_1^j(S)\sim N_0^{j+1}(S)} Y(S,l,1)_0^j.
\end{aligned}
$$

One expresses the complete Riemannian manifold $\tilde{Y}(S,l,1)$ by $CH(T_{l,1})$. Similarly there is an infinite signed tree with a base point $(T_{l,1})_0$ such that $\tilde{Y}(S,l,1)_0$ is diffeomorphic to $CH((T_{l,1})_0)$. Let us take l half lines $\mathbf{R}_+, \ldots \mathbf{R}_+$ (l times). Then one identifies all $\mathbf{R}_+$ at 0. The result is a connected infinite tree with a base point. One denotes it by T_+^l. $T_{l,1}$ can be expressed as $\mathbf{R} \cup_{j\in\mathbf{Z}} (T_+^{l-1})^j$, where $(T_+^{l-1})^j$ are the same (T_+^{l-1}) indexed by $j \in \mathbf{Z}$. Similarly one has $(T_{l,1})_0 = \mathbf{R}_+ \cup_{j\in\mathbf{N}} (T_+^{l-1})^j$.

Let us define:

$$
\begin{aligned}
&Y(S,l,m,1)_0 \equiv V(S,l)_0 \cup_{j=2}^{l} \cup_{M_1^j(S)\sim N_0^1(S)} \tilde{Y}(S,m,1)_0,\\
&Y(S,l,m,1) = Y(S,l,m,1)_0 / N_0(S) \sim M_1(S),\\
&\tilde{Y}(S,l,m,1)_0 = \cup_{j\in\mathbf{N}} \cup_{M_1^j(S)\sim N_0^{j+1}(S)} Y(S,l,m,1)_0^j,\\
&\tilde{Y}(S,l,m,1) = \cup_{j\in\mathbf{Z}} \cup_{M_1^j(S)\sim N_0^{j+1}(S)} Y(S,l,m,1)_0^j.
\end{aligned}
$$

Similarly one has $\tilde{Y}(S,l,m,1) = CH(T_{l,m,1})$, $\tilde{Y}(S,l,m,1)_0 = CH((T_{l,m,1})_0)$. Let us take $(T_{m,1})_0, \ldots (T_{m,1})_0$ (l times). Then one identifies all $(T_{m,1})_0$ at 0. The result is a connected infinite tree with a base point. One denotes it by $(T_{m,1})_0^l$. As above, $T_{l,m,1}$ can be expressed as $\mathbf{R} \cup_{j\in\mathbf{Z}} [(T_{m,1})_0^{l-1}]^j$, where $[(T_{m,1})_0^{l-1}]^j$ are the same $(T_{m,1})_0^{l-1}$ indexed by $j \in \mathbf{Z}$. Similarly one uses a notation $(T_{l,m,1})_0^l$, which one identifies l $(T_{l,m,1})_0$ at 0. Then using this, one can obtain $T_{l,m,k,1}$ by a similar method.

By iteration, one obtains $Y(S,l,m,n,1)$ and $\tilde{Y}(S,l,m,n,1) = CH(T_{l,m,n,1})$ as above. Thus one can obtain $CH(T_{n_1,\ldots,n_k,1})$ by iteration. One calls $T_{n_1,\ldots,n_k,1}$ a homogeneous tree of finite type.

Let $\bar{n} = \{n_1, n_2, \dots, n_l, \dots\}$ be an infinite set of positive integers. One can iterate the previous construction infinitely many times. Then one gets a complete Riemannian manifold:

$$CH(T_{\bar{n}}) = CH(T_{n_1,n_2,\dots})$$

where we call $T_{\bar{n}}$ a homogeneous tree. If there is a bound C with $n_j \leq C$ for all j, then we call the homogeneous tree of bounded multiplicity.

6 Some properties of elliptic operators over $\tilde{W}_0$

6.A Spectral decomposition: Throughout this paper, one uses the following lemmas. Let P be an order 1 elliptic differential operator over a complete Riemannian manifold W of bounded geometry. Let w be an weight function on W, and weighted Sobolev k norms on W by $|u|W^k_w = (\Sigma_{l \leq k} \int exp(w)|\nabla^l u|^2 vol)^{\frac{1}{2}}$. Thus one has a bounded map $P : W^{k+1}_w(W) \mapsto W^k_w(W)$. Let us denote P^*_w, the formal adjoint operator with respect to the weighted L^2_w inner product.

Lemma 6.1 *Suppose* $Spec\Delta = P^*_w \circ P$ *is contained in* $[\lambda^2, \infty)$ *for some* $\lambda > 0$. *Then* $P : W^{k+1}_w \mapsto W^k_w$ *has closed range.*

Similarly $P^*_w : W^{k+1}_w \mapsto W^k_w$ *also has closed range.*

Proof : Let us consider the first statement. One verifies this by induction. Let $k = 0$. Let $\{u_i\}$ be a sequence in W^1_w with $|P(u_i)|L^2_w \mapsto 0$. Since one has the estimate $|P(u)|L^2_w \geq \lambda|u|L^2_w$, one sees $|u_i|L^2_w \to 0$. Moreover by the elliptic estimate:

$$|u|W^{k+1}_w \leq C_k(|P(u)|W^k_w + |u|W^k_w), \quad k \geq 0$$

for some constants C_k, one concludes $|u_i|W^1_w \to 0$. This shows the result for $k = 0$.

We show by induction that if a sequence $\{u_i\}$ in W^{k+1}_w satisfies $|P(u_i)|W^k_w \to 0$, then $|u_i|W^k_w \to 0$. Then by the elliptic estimate, one gets the conclusion. Suppose the result holds for $k \leq k_0$, and take a sequence $\{u_i\}$ in $W^{k_0+2}_w$ with $|P(u_i)|W^{k_0+1}_w \to 0$. Then since $|P(u_i)|W^{k_0}_w$ also converges to 0, one knows that $|u_i|W^{k_0}_w$ converges to 0 by the induction hypothesis. By the elliptic estimate, one gets the result for k_0+1. Thus one gets the result for k_0+1. This completes the proof of the first statement.

Next let us consider the second statement. Let $H^\perp \subset L^2_w$ be a closed subspace which consists of the orthogonal complement of $H = P(W^1_w) \subset L^2_w$. Notice the following relation:

$$\Delta(W_w^{k+2}) = P_w^*(H \cap W_w^{k+1}) \subset W_w^k.$$

Now let $\{v_i\}$ be a sequence in $H \cap W_w^1$ with $|P_w^*(v_i)|L_w^2 \mapsto 0$. Since one may express $v_i = P(u_i)$ for $u_i \in W_w^2$, one has $|\Delta(u_i)|L_w^2 \to 0$. Thus one has $|u_i|L_w^2 \to 0$. Then by the Cauchy Schwartz, one has the estimate $|P(u_i)|^2 L_w^2 \leq |\Delta(u_i)|L_w^2 |u_i|L_w^2$. This shows $|v_i|L_w^2 \to 0$. By the elliptic estimate, one sees $|v_i|W_w^1 \to 0$. Then by proceeding similarly as above, one verifies the conclusion by induction. This completes the proof.

Let W and w be as above, and consider an elliptic complex:

$$0 \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \to 0$$

over vector bundles on W.

Lemma 6.2 *(1) Suppose $d_0 : W_w^{k+1}(E_0) \mapsto W_w^k(E_1)$ has closed range, and $Kerd_0 = 0$. Then there exists a constant C such that:*

$$|(d_0)_w^* \circ d_0(u)|W_w^k(E_0) \geq C|u|W_w^{k+2}(E_0)$$

for any u. In particular $(d_0)_w^* \circ d_0$ gives an isomorphism.

(2) Suppose the corresponding bounded complex:

$$0 \to W_w^{k+2}(E_0) \xrightarrow{d_0} W_w^{k+1}(E_1) \xrightarrow{d_1} W_w^k(E_2) \to 0$$

satisfies the following; (1) d_0 and d_1 have closed range, (2) $1st$ cohomology group $Kerd_1/\ imd_0 = 0$, and (3) $Kerd_0 = 0$.

Then for $P = (d_0)_w^* \oplus d_1$ and $\Delta = P_w^* \circ P$, one has $Spec\Delta \subset [\lambda, \infty)$ for some positive constant $\lambda > 0$. Moreover $(d_1)_w^*$ has closed range.

Proof : (1) Suppose contrary, and take a sequence $\{u_i\} \subset W_w^2$ with $|(d_0)_w^* \circ d_0(u_i)|L_w^2 \to 0$. Then by Cauchy Schwartz, one has a convergent sequence $|d_0(u_i)|L_w^2 \to 0$. Then by the elliptic estimate, one has $|d_0(u_i)|W_w^1 \to 0$. By closedness and the open mapping theorem, this shows existence of non trivial kernel. This is a contradiction. The rest of the proof follows by induction.

(2) follows from (1) and lemma 6.1. This completes the proof.

Let us use the above notations. Let us put $H = L_w^2(E_1) \cap (\text{ Ker}d_1)^\perp$. Let us consider restriction of $\Delta = (d_1)_w^* \circ d_1|H$.

Corollary 6.1 *Suppose d_0 and d_1 has closed range as above. Then $Spec\Delta|H$ is contained in $[\lambda, \infty)$ for some positive $\lambda > 0$.*

6.B Analysis over $\tilde{W}_0$: Let us take the complete Riemannian manifold $\tilde{W}_0$ in 5.A. Let us take a smooth function $w : \tilde{W}_0 \mapsto [0, \infty)$. Then as before one has weighted Sobolev k norms on $\tilde{W}_0$ by $|u|W^k_w = (\Sigma_{l \leq k} \int exp(w)|\nabla^l u|^2 vol)^{\frac{1}{2}}$. Then we have the Atiyah Hitchin Singer complex (AHS complex) as:

$$0 \ \to W^{k+2}_w(\tilde{W}_0; \Lambda^0) \ \stackrel{d}{\to} \ W^{k+1}_w(\tilde{W}_0; \Lambda^1) \ \stackrel{d_+}{\longrightarrow} W^k_w(\tilde{W}_0; \Lambda^2_+) \ \to 0$$

where $d^+ = (1 + *) \circ d$.

Let M be a complete Riemannian 3 manifold without boundary, but possibly be non compact. We have in mind $M = \tilde{W}_0(t)$ or $N_0(s)$, $M_1(s)$. Let g be a Riemannian metric on M, and denote also by g as the product one on $M \times \mathbf{R}$. Let $\Lambda^1(M \times \mathbf{R})$ and $\Lambda^2_+(M \times \mathbf{R})$ be the exterior differentials on 1 forms and self dual 2 forms with respect to g. Then we have the natural identification:

$$\Lambda^1(M \times \mathbf{R}) = p^*(\Lambda^1(M)) \oplus p^*(\Lambda^0(M)),$$

$$\Lambda^2_+(M \times \mathbf{R}) = p^*(\Lambda^1(M))$$

where $p : M \times \mathbf{R} \mapsto M$ is the projection. The isomorphisms are given by:

$$u + vdt \leftrightarrow (u, v), \quad *_M u + u \wedge dt \leftrightarrow u.$$

Let us put $X = M \times \mathbf{R}$. Using L^2 adjoint operator (here we do not use any weight functions), we get the elliptic operator $P = d^* \oplus d^+ : \Lambda^1(X) \mapsto \Lambda^0(X) \oplus \Lambda^2_+(X)$. Using the above identification, this is expressed as:

$$P = d^* \oplus d^+ : p^*(\Lambda^0(M) \oplus \Lambda^1(M)) \mapsto p^*(\Lambda^0(M) \oplus \Lambda^1(M))$$

where $p : M \times \mathbf{R} \mapsto M$ is the projection. Let us use t as the coordinate on $\mathbf{R}$. Then by a straightforward calculation, one has the following expression:

$$P = -\frac{d}{dt} + \begin{pmatrix} *_M d & d \\ d^* & 0 \end{pmatrix} \equiv -\frac{d}{dt} + Q.$$

Notice that Q is an elliptic self adjoint differential operator on $L^2(M; \Lambda^0(M) \oplus \Lambda^1(M))$. Notice that if M is closed, then Q is not invertible (one has constant functions). When M is non compact, 0 may be contained in the spectrum of Q, even if there are no kernels for Q.

Let us introduce a weight function τ on X as follows. Let M be possibly non compact as above. Let us choose a positive number $\delta > 0$ which will be specified later, and choose a smooth proper function $\tau' : \mathbf{R} \mapsto [0, \infty)$ with

$\tau'|\mathbf{R}\backslash[-1,1](t) = \delta|t|$. Then we define $\tau : M \times \mathbf{R} \mapsto [0,\infty)$ by $\tau(m,t) \equiv \tau'(t)$. Let us introduce weighted Sobolev k norms on X by:

$$|u|W^k_\tau = (\Sigma_{l\leq k}\int exp(\tau)|\nabla^l u|^2 vol)^{\frac{1}{2}}.$$

We denote these Sobolev weighted spaces by W^k_τ or L^2_τ (for $k=0$). Then one has an isometry:

$$I : L^2(X, p^*(\Lambda^0 \oplus \Lambda^1)) \mapsto L^2_\tau(X, p^*(\Lambda^0 \oplus \Lambda^1))$$

by $I(u) = exp(-\frac{\tau}{2})u$. Then one has the equality:

$$I^{-1}PI = P + \frac{1}{2}\frac{d\tau}{dt} : W^{k+1}(X; p^*(\Lambda^0 \oplus \Lambda^1)) \mapsto W^k(X; p^*(\Lambda^0 \oplus \Lambda^1)).$$

Let d^*_τ be the L^2_τ adjoint operator, i.e. $< u, d(v) > |L^2_\tau =< d^*_\tau(u), v > |L^2_\tau$, and put $P_\tau = d^*_\tau \oplus d^+$. Then we have the following expression:

$$I^{-1}P_\tau I = -\frac{d}{dt} + \begin{pmatrix} *_M d & d \\ d^* & -\frac{d\tau}{dt} \end{pmatrix} + \frac{1}{2}\frac{d\tau}{dt} \equiv -\frac{d}{dt} + Q_\tau.$$

We recall the definition of $\tilde{W}_0(t)$ or $M_1(s), N_0(s)$ in the last paragraph in $5.A$. Practically we will apply the next lemma for $M = \tilde{W}_0(t)$. Let us consider the operator:

$$Q_\delta = \begin{pmatrix} *_M d + \frac{\delta}{2} & d \\ d^* & -\frac{\delta}{2} \end{pmatrix} : W^{k+1}(M, \Lambda^1 \oplus \Lambda^0) \mapsto W^k(M, \Lambda^1 \oplus \Lambda^0).$$

Notice that Q_δ coincides with Q_τ above for $|t| \geq 1$.

Lemma 6.3 *Suppose M is a closed Riemannian manifold. Then for a small $\delta > 0$ and $k \geq 0$, Q_δ gives an isomorphism.*

Proof : Let us take $(u,v) \in W^{k+1}(M, \Lambda^1 \oplus \Lambda^0)$, and put $Q_\delta(u,v) = (x,y)$. Then we have:

$$(*d + \frac{\delta}{2})u + dv = x, \quad d^*u - \frac{\delta}{2}v = y.$$

For convenience, we recall $*^2 u = (-1)^{p(n-p)}u$, $d^*u = (-1)^{np+n+1} * d * u$ for p form u on n dimensional manifold. Then from the above, we have:

$$d^* * du + \frac{\delta}{2}d^*u + d^*dv = d^*x = ((\frac{\delta}{2})^2 + d^*d)v + \frac{\delta}{2}y.$$

Thus we can solve v from (x,y) as follows:

$$v = ((\frac{\delta}{2})^2 + d^*d)^{-1}(d^*x + \frac{\delta}{2}y).$$

Next we have the following equality:

$$*^2 du + \frac{\delta}{2} * u + *dv = *x = du + \frac{\delta}{2} * u + *dv.$$

$$\begin{aligned} d^* du + \frac{\delta}{2} d^* * u + & d^* * dv = d^* * x \\ & = d^* du + \frac{\delta}{2} * du = d^* du + \frac{\delta}{2}(x - \frac{\delta}{2} u - dv). \end{aligned}$$

Then from this, we have the following equation:

$$\begin{aligned} (d^* d - (\frac{\delta}{2})^2) u = & - \frac{\delta}{2}(x - dv) + d^* * x, \\ dd^* u = & \frac{\delta}{2} dv + dy. \end{aligned}$$

Thus one can again solve u as:

$$u = (\Delta - (\frac{\delta}{2})^2)^{-1}(dy + d^* * x - \frac{\delta}{2} x + \delta dv).$$

It follows that there is a constant $C = C(M)$ such that $\mathrm{Spec}|Q_\delta| \geq C(\frac{\delta}{2})^4$. Notice the following inequalities:

$$\begin{aligned} |(u, v)| W^{k+1} \leq & C(\frac{\delta}{2})^{-2} |(\Delta \pm (\frac{\delta}{2})^2)(u, v)| W^{k-1} \\ \leq & C(\frac{\delta}{2})^{-4}(|x| W^k + |y| W^k) \leq C(\frac{\delta}{2})^{-4} |Q_\delta(u, v)| W^k). \end{aligned}$$

In fact one may assume that C only depends on $\mathrm{Sup}_{0 \leq l \leq 2} \mathrm{Sup}_{x \in M} |\nabla^l R|(x)$, where R is the curvature form with respect to the Levi Chivita connection.

This completes the proof.

Let $(P_\tau)^*_\tau$ be the formal adjoint of the Sobolev weighted spaces. Then one considers the same analysis for $(P_\tau)^*_\tau$. Notice the equality:

$$(I^{-1} \circ P_\tau \circ I)^* = I^{-1} \circ (P_\tau)^*_\tau \circ I.$$

Thus one can analysis an operator of the form, $P^*_\tau = \frac{d}{dt} + Q_\delta$, since Q_δ is self adjoint over M.

Let us take a smooth (non proper) function:

$$\tau : \tilde{W}_0 \mapsto [0, \infty)$$

which is horizontalwisely constant, namely, for some positive constant $\delta > 0$, we have $\tau | \tilde{W}_0(t) = \delta t$ for $t \geq 0$. Using τ, one defines weighted Sobolev spaces $W^k_\tau((\tilde{W}_0, g), \Lambda_*)$. One has the expression $P_\tau | M_1([0,\infty)) = -\frac{d}{ds} + Q_{\tau'}$ where:

$$Q_{\tau'} = \begin{pmatrix} *d & d \\ d^*_{\tau'} & 0 \end{pmatrix} : W^{k+1}_{\tau'}(M_1(s); \Lambda^1 \oplus \Lambda^0) \mapsto W^k_{\tau'}(M_1(s); \Lambda^1 \oplus \Lambda^0)$$

is an elliptic operator over $M_1(s)$, and $\tau' = \tau | M_1(s)$ is its restriction. It follows that $Q_{\tau'}$ has closed range with empty kernel. We do not use this fact later. In this case one can not see whether P_τ defined above is Fredholm or not, since one has no simple statement as above for the weighted adjoint operator $(Q_{\tau'})^*_{\tau'} | W^k_{\tau'}(M_1(s); \Lambda^1)$.

Let us take another weight function $\mu : \tilde{W}_0 \mapsto [0,\infty)$, with $\mu | M_1(s), N_0(s) \equiv \delta' s$, where we choose a sufficiently small $0 < \delta' << \delta$. Let us put:

$$w = \tau + \mu : \tilde{W}_0 \mapsto [0,\infty).$$

We call w as a weight function with weight (δ, δ'). Let us use w as the weight function. Then one obtains a bounded operator:

$$P_w = d^*_w \oplus d^+ : W^{k+1}_w(\tilde{W}_0; \Lambda^1) \mapsto W^k_w(\tilde{W}_0; \Lambda^0 \oplus \Lambda^2_+).$$

Passing through the isometry $I : L^2 \to L^2_w$, $I(u) = exp(-\frac{w}{2})u$, P_w has the following expression on $M_1([0,\infty))$ and $N_0([0,\infty))$:

$$I^{-1} \circ P_w \circ I = -\frac{\partial}{\partial s} + \begin{pmatrix} *d & d \\ d^*_{\tau'} & 0 \end{pmatrix} + \frac{\delta'}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv -\frac{\partial}{\partial s} + Q^v_w.$$

On the other hand one has another expression on $\tilde{W}_0([0,\infty))$:

$$I^{-1} \circ P_w \circ I = -\frac{\partial}{\partial t} + \begin{pmatrix} *d + \frac{\delta}{2} & d \\ d^*_\mu & -\frac{\delta}{2} \end{pmatrix} = -\frac{\partial}{\partial t} + Q^h_w.$$

6.C Decay estimate: Let M be a possibly non compact n dimensional Riemannian manifold of bounded geometry. In practice M stands for $\tilde{W}_0(t)$ (or $N_0(s), M_1(s)$). Let us take a smooth function $\mu : M \mapsto [0,\infty)$. Let E and F be vector bundles over M, and take an elliptic operator:

$$Q : W^{k+1}_\mu(M; E) \mapsto W^k_\mu(M; F).$$

Let us choose $\tau : M \times \mathbf{R} \mapsto \mathbf{R}_+$ by $\tau(m,t) = \delta |t|$ for $|t| \geq 1$, and put $w = \tau + \mu$. Then one considers the following operator:

$$P = -\frac{d}{dt} + Q : W^{k+1}_w(M \times \mathbf{R}, p^* E) \mapsto W^k_w(M \times \mathbf{R}; p^* F).$$

Let P_w^* be the formal adjoint operator with respect to the weighted inner product. Then P_w^*P admits a spectral decomposition. For $u \in W_w^k(M \times \mathbf{R}, p^*(E))$, we denote $\text{Spec}u \in [0,\epsilon]$ if u lies in the image of the spectral projection $\mathfrak{P}([0,\epsilon])$.

Suppose the above Q is a Fredholm operator, and is self adjoint with respect to the weighted L^2_μ inner product.

Proposition 6.1 (LM, F,T) *For sufficiently small $\delta > 0$, the above P is Fredholm. Moreover if Q is invertible, then one may choose $\delta = 0$. In particular P is also invertible.*

Suppose $Q : W_\mu^{k+1}(M;E) \mapsto W_\mu^k(M;F)$ has closed range with $KerQ = 0$. Let us put $\Delta = Q_\mu^* \circ Q$, and choose a positive constant $\lambda_0 > 0$ with $Spec\Delta \in [\lambda_0^2, \infty)$. Then under the above condition, one has the following; for any small $\epsilon << \lambda_0$, there exist positive constants $C, \delta_0 > 0$ such that for any $u \in W^{k+l}(M \times \mathbf{R}, p^*E)$ with $Specu \in [0,\epsilon]$ for $P_w^* \circ P$, we have the following decay estimate:

$$|u(\quad,t)|L^2_\mu(M) \leq Cexp(-\delta_0|t|)|u|L^2_\mu(M \times [0,\infty)), \quad t \geq 0.$$

Proof : The first statement follows from the decay estimate. Next let us take u as above, and denote its slice on $M \times s$ by u_s. Let us put a smooth function $f : [0,\infty)$ by $f(s) = \int_{M_s} exp(\mu)|u_s|^2$. Then by differentiating, one has the inequalities:

$$\begin{aligned} f''(s) &= 2[\int_{M_s} exp(\mu) < u_s', u_s' > + \int_{M_s} exp(\mu) < u_s'', u_s >] \\ &\geq 2[\int_{M_s} exp(\mu) < u_s', u_s' > + \int_{M_s} exp(\mu) < (\Delta - \epsilon)u_s, u_s >] \\ &\geq 2\int_{M_s} exp(\mu) < (\Delta - \epsilon)u_s, u_s > \geq 2(\lambda_0 - \epsilon)f(s). \end{aligned}$$

Thus one has differential inequalities $f'' \geq \lambda_0 f$. From this, one can get the desired estimate. This completes the proof.

Let Y be a complete Riemannian four manifold such that except compact subset $K \subset Y$, $Y \backslash K$ is isometric to $Y(0) \times [0,\infty)$ where $Y(0)$ is a closed manifold. Let us equip a weight function $\tau : Y \mapsto [0,\infty)$ by $\tau | Y(0) \times t = \delta t$. Let $I : L^2(Y) \cong L^2_\tau(Y)$ be the isometry.

Corollary 6.2 (LM) *The corresponding*

$$P = I^{-1}P_\tau I : W^{k+1}(Y;\Lambda^1) \mapsto W^k(Y;\Lambda^0 \oplus \Lambda^2_+)$$

gives a Fredholm map.

Proof : This is well known, but for convenience we recall the proof. Let us take a cut off function $\varphi|Y$ with $\varphi|Y(0) \times [1, \infty) \equiv 1$, and $\varphi|K \equiv 0$. Let us choose any $u \in W^{k+1}(Y; \Lambda^1)$. Then one may regard $\varphi u \in W^{k+1}(Y(0) \times \mathbf{R}; \Lambda^1)$. Thus one has the estimate:

$$\begin{aligned} |u|W^{k+1} &\leq |\varphi u|W^{k+1} + |(1-\varphi)u|W^{k+1} \\ &\leq C\{|\varphi P(u)|W^k + |[\varphi, P]u|W^k + |(1-\varphi)Pu|W^k \\ &\qquad + |[(1-\varphi), P]u|W^k + |(1-\varphi)u|W^k\}. \end{aligned}$$

This shows that P has closed range with finite dimensional kernel. One has a similar estimate for P^*. This shows the result.

6.D 1st cohomology: Let $w = \tau + \mu$ be the last paragraph in 6.B.

Lemma 6.4 *For a sufficiently small choice of $\delta' << \delta$, $d : W^{k+1}_w(\tilde{W}_0; \Lambda^0) \mapsto W^k_w(\tilde{W}_0; \Lambda^1)$ has closed range with Ker $d = 0$.*

Proof : Clearly Ker $d = 0$.

As lemma 6.3, passing through the isometry I, $I^{-1} \circ d \circ I|\tilde{W}_0([0, \infty))$ can be expressed as:

$$\frac{d}{dt} + (d, -\frac{\delta}{2}) : W^{k+1}(\tilde{W}_0([0, \infty)) \mapsto W^k(\tilde{W}_0([0, \infty); \Lambda^0 \oplus \Lambda^1).$$

Clearly this has closed range. One has the same property on $N_0([0, \infty))$ and $M_1([0, \infty))$.

Now let $H_i \subset W^{k+1}(\tilde{W}_0)$, $i = 0, 1, 2, 3$ be closed subsets whose supports lye in $N_0([0, \infty))$, $M_1([0, \infty))$, $\tilde{W}_0([0, \infty))$ and K respectively, where $K \subset \tilde{W}_0$ is some compact subset. Then one may assume $\cup_i H_i = W^{k+1}(\tilde{W}_0)$. Now by the open mapping theorem and above, every image $d(H_i)$ has closed range. This completes the proof.

Let us consider $\tilde{W}_0$ and the weight function w on it with the weight constants (δ, δ'). Let us recall that we have a bounded complex:

$$0 \longrightarrow W^{k+2}_w(\tilde{W}_0) \xrightarrow{d} W^{k+1}_w(\tilde{W}_0; \Lambda^1) \xrightarrow{d_+} W^k_w(\tilde{W}_0; \Lambda^2_+) \longrightarrow 0$$

where the weight function is $w = \tau + \mu$.

It is clear Ker$d_0 = 0$. Suppose $\omega \in W^1_w(\tilde{W}_0; \Lambda^1)$ with $d_+(\omega) = 0$. Then by integration by parts, one has $0 = \int_{\tilde{W}_0} |d_+\omega|^2 = \int_{\tilde{W}_0} |d\omega|^2$. In particular we have $d\omega = 0$. The following sublemma shows $H^1(AHS) = 0$.

Let $W = \tilde{W}_0$ or $\tilde{Y}(S)$ in 5.B. Recall that one has a weight function τ over $\tilde{W}_0$. There is a natural extension of τ on $\tilde{Y}(S)$. Let $\mu : \tilde{Y}(S) \mapsto [0, \infty)$ be another weight function defined as $\mu|\tilde{W}_0^n([-S,S])(x) = \delta'(|n| + t(x))$, where $t : \tilde{W}_0^0([-S,S]) \mapsto [0,1]$ satisfies $t|N_0(S) \equiv 0$ and $t|M_1(S) \equiv 1$.

Let $w = \tau + \mu$ be a weight function on W In order to make explicit the weight constants, sometimes one uses notations $\tau(\delta)$ and $\mu(\delta')$. Later one will show that $d : W_w^{k+1}(\tilde{Y}(S)) \mapsto W_w^k(\tilde{Y}(S);\Lambda^1)$ has closed range. Assuming this, one has the following:

Sublemma 6.1 *Let $f \in C^\infty(W)$ satisfies $|df|L_w^2(W) < \infty$. Then there are constants C and $k = k(f)$ such that:*

$$|f - k|L_w^2 < \infty.$$

Proof: The situation diffres from [T1] lemma 5.2, in that on the end, we have non compact slices, $\tilde{W}_0(t)$ and $M_1(s), N_0(s)$. In order to verify this, one takes two steps. First one sees a weaker version of the sublemma.

Step 1: Let us choose any $\delta_0 < \delta$ and $\delta_0' < \delta'$ so that the pair (δ_0, δ_0') is sufficiently near (δ, δ'). Let us put $w = \tau(\delta) + \mu(\delta')$, $w' = \tau(\delta_0) + \mu(\delta_0')$. Then we claim that $f - k \in L_{w'}^2$.

By lemma 6.2(1) and lemma 6.4, one has an estimate:

$$\int_W exp(w)|u|^2 \leq C \int_W exp(w)|du|^2.$$

This shows that one has a Hilbert space, which is a completion of $C_{\mathrm{cp}}^\infty(W)$ by the norm $|u|^2 = \int_W exp(w)|du|^2$. Let us take any $f \in C^\infty(W)$ with $df \in L_w^2$. Using a functional $s : H \to \mathbf{R}$, by $s(v) = \int_{\tilde{W}_0} exp(w)(|dv|^2 - 2 < dv, df >)$, one gets a unique $u = (d_w^* d)^{-1}(d_w^* df) \in H$ with $d_w^* d(f - u) = 0$. We show that $g = f - u$ is in $L_{w'}^2$. From $dg \in L_w^2$, one has estimates $\int_{W([t,t+1])} |dg|^2 \leq exp(-\delta t)\epsilon_t$, where $\epsilon_t \to 0$. Notice that dg satisfies the elliptic equation, $(d_w^* \oplus d)dg = 0$. Thus by the local Sobolev embedding and the above estimate, one gets:

$$\mathrm{Sup}_{\tilde{W}_0(t)}|dg| \leq C exp(-\frac{\delta t}{2})\epsilon_t.$$

In particular, one has the next estimate,

$$|g(m,t)| \leq C_1 + \int_0^t C_2 exp(-\frac{\delta s}{2})\epsilon_s ds,$$

where $C_1 = |g(m,0)|$. This shows that there is a constant C_m such that $lim_{t\to\infty}$ $\mathrm{Sup}_{W(m,t)}|g - C_m| = 0$. It is clear that C_m is independent of m, and we write the same constant by C.

Let us make a similar procedure on s direction along $M_1(s)$ and $N_0(s)$. Then one finds also another constant C' (C'') such that $g - C'(C'')|M_1(m,s), (N_0(m,s))$ vanishes at infinity for every m. Again it is clear that $C = C' = C''$.

Now one finds a constant c with $|dg|(x) \leq cexp(-\frac{w(x)}{2})$. Then by integration, one has the inequality:

$$|g - C|(x) \leq cexp(-\frac{w(x)}{2}).$$

Notice that for any $\alpha > 0$, w^α is integrable over W. Combining with this and the above inequality, one gets the first claim.

Step 2: For any sufficiently small pair (δ, δ'), one has a well defined 1st cohomology group $H^1(\delta, \delta') = \text{Ker} d \cap L^2_w(W; \Lambda^1)/d(W^1_w(W))$. Then one has a natural map:

$$i : H^1(\delta, \delta') \mapsto H^1(\delta_0, \delta'_0).$$

We claim that i is injection. This is enough for the proof of the sublemma.

Let us put $\Delta(\delta, \delta') = (d^*_w \oplus d)^*_w (d^*_w \oplus d)$ on $L^2_w(W; \Lambda^1)$, $w = w(\delta, \delta')$. Then the spectrum of $\Delta(\delta, \delta')$ is discrete near 0. Thus for (δ_0, δ'_0) sufficiently near (δ, δ'), one has $\dim H^1(\delta, \delta') \geq \dim H^1(\delta_0, \delta'_0)$.

Suppose these dimensions are different. Then one chooses another $\delta_1 < \delta_0$ and $\delta'_1 < \delta'_0$. If $\dim H^1(\delta_0, \delta'_0) = \dim H^1(\delta_1, \delta'_1)$, then one puts $\delta_0 = \delta$ and $\delta' = \delta'_0$, and gets the conclusion. If these dimensions are different, one has $\dim H^1(\delta_0, \delta'_0) > \dim H^1(\delta_1, \delta'_1)$. If these are different, one must have $\dim H^1(\delta, \delta') \geq 2$. Let us take another $\delta_2 < \delta_1 < \delta_0$ and δ'_2 similarly.

One itarates this process until one gets the equality of dimensions. If this is not the case, one has an infinite sequence (δ_i, δ'_i). By choice, one may assume that this sequence converges to $(\delta_\infty, \delta'_\infty)$ with $2\delta_0(') - \delta(') < \delta_\infty(') < \delta(')$. Then $\Delta(\delta_\infty, \delta'_\infty)$ must have a continuous spectrum near 0. This is a contradiction. This completes the proof.

Remark 6.1: When $\delta' = 0$ (no weight in the horizontal direction), one can still find some constant C such that $g - C$ vanishes at infinity. Let us take any $(m', t) \in \tilde{W}^n_0([-S, S]) \subset \tilde{Y}(S)$. Then we show that if n is sufficiently large, then $|g(m', t) - C_m|$ is sufficiently small for C_m as above. With respect to n, let us choose a sufficiently large T. Then since one has the estimate $|dg|\tilde{Y}(S)(T) \leq Cexp(-\frac{\delta T}{2})$, one may assume $|g(m, T) - g(m', T)|$ is sufficiently small. On the other hand, $|g(m', t) - g(m', T)|$ is less than ϵ_n, where $\epsilon_n \to 0$ as $n \to \infty$. Thus $|g(m', t) - C_m|$ is sufficiently small.

Lemma 6.5 *Let $W = \tilde{Y}(S)$. $H^1(\delta, 0)$ is naturally isomorphic to $H^1(\delta, \delta')$.*

Proof : Let $\alpha \in H^1(\delta, 0)$ be a harmonic representative. Then from corollary 7.1 below, and by the same method as [T1], proof of lemma 5.3, one can verify $\alpha \in L^2_{w(\delta,\delta')}(\tilde{Y}(S); \Lambda^1)$. This completes the proof.

7 Fourier Laplace transform between open manifolds

Our main application is the analysis over (W_0, N_0, M_1) = kinky handles. Recall that for the simplest kinky handle, W_0 is diffeomorphic to $S^1 \times D^3$. In this section, one considers $(W_0 = S^1 \times D^3, M_1, N_0)$, where M_1, N_0 are all diffeomorphic to $S^1 \times D^2$, and the embeddings are given as follows. Let us regard $\partial W_0 = S^1 \times D^2_+ \cup S^1 \times D^2_-$, and consider two knots $\mu \subset S^1 \times D^2_+$, $\gamma \in S^1 \times D^2_-$ as follows:

Figure

Notice that μ is nullhomotopic, and γ represents a generator of $\pi_1(W_0)$. Let us take tublar neighbourhoods of μ and γ in $S^1 \times D^2_\pm$ respectively. We denote them as $\bar{\mu}$ and $\bar{\gamma}$. Then we choose $\bar{\gamma} = M_1$ and $\bar{\mu} = N_0$. Let us denote the quotient space by $Y = W_0/M_1 \sim N_0$. By a simple calculation, one knows the following:

Sublemma 7.1 $H^1(Y : \mathbf{R}) = \mathbf{R}$ *and* $H^2(Y : \mathbf{R}) = H^3(Y : \mathbf{R}) = 0$.

In this section, one introduces the Fourier-Laplace transform over Y, which is an open analogue of the one in [T1]. In order to do this, one will only require the above topological conditions. In particular one can apply the method in this section for cylindrical manifolds Y which includes the simplest Casson handles.

Now by the previous construction, one has a complete Riemannian manifold $\tilde{W}_0$. Let us take infinitely many $\tilde{W}_0$, and for convenience, we numerate the same

$\tilde{W}_0$ as $\tilde{W}_0^i$, $i \in \mathbf{Z}$. Let us choose a sufficiently large S, and introduce notations:

$$\tilde{W}_0([-S,S]) = \tilde{W}_0 \backslash \{M_1([S,\infty)) \cup N_0([S,\infty))\},$$
$$Y(S) = \tilde{W}_0([-S,S])/N_0(S) \sim M_1(S).$$

One has a natural extension of notation $Y(S)(t) = \tilde{W}_0(t)/N_0(S,t) \sim M_1(S,t)$, $t \in [0,\infty)$. Recall that one has a weight function $\tau : \tilde{W}_0 \mapsto [0,\infty)$ by $\tau | \tilde{W}_0(t) = \delta t$. One can make a natural extnsion of τ on $Y(S)$. We will use the same τ for this.

Now one has a periodic Riemannian maifold as:

$$\tilde{Y}(S) = \cdots \cup_{M_1^{-n}(S) \sim N_0^{-n+1}(S)} \tilde{W}_0^{-n+1}([-S,S])$$
$$\cup_{M_1^{-n+1}(S) \sim N_0^{-n+2}(S)} \tilde{W}_0^{-n+2}([-S,S]) \dots$$

There is a free isometric $\mathbf{Z}$ action on $\tilde{Y}(S)$, and we denote the action $1 \in \mathbf{Z}$ by T.

Let us take vector bundles $E_i \to Y(S)$, $i = 1,2$ and a differential operator D between them. Then it lifts to the following one:

$$D : C^\infty_{\rm cp}(\tilde{E}_1) \mapsto C^\infty_{\rm cp}(\tilde{E}_2)$$

where $\tilde{E}_i \to \tilde{Y}(S)$ are the natural lifts.

Let us take any $\psi \in C^\infty_{\rm cp}(\tilde{E}_1)$ and $z \in \mathbf{C}^*$. Then we define the Fourier Laplace transform of ψ by:

$$\hat{\psi}_z(\quad) = \Sigma_{n=-\infty}^{\infty} z^n (T^n \psi)(\quad).$$

Let us define another vector bundle over $Y(S) \times \mathbf{C}^*$ as:

$$E_1' \equiv [\tilde{E}_1 \otimes_{\mathbf{R}} \mathbf{C}]/\mathbf{Z} \mapsto Y(S) \times \mathbf{C}^*$$

where $1 \in \mathbf{Z}$ sends $(\rho, \lambda) \in \tilde{E}_1 \otimes_{\mathbf{R}} \mathbf{C}$ to $(T\rho, z\lambda)$. One may regard E_1' as a family of vector bundles $\{E_1'(z)\}$ over $Y(S)$, parametrized by $z \in \mathbf{C}^*$.

Now by restriction $\hat{\psi} | \tilde{W}_0^0([-S,S])$, $\hat{\psi}_z$ defines a smooth section over $Y(S)$ of $E_1'(z)$, where $\tilde{W}_0^0([-S,S]) \subset \tilde{Y}(S)$. Thus one may regard $\hat{\psi}_z \in C^\infty_{\rm cp}(E_1'(z))$.

The Fourier Laplace inversion formula is as follows. Let us take a smooth section $\hat{\eta} \in C^\infty_{\rm cp}(E_1')$ with $\hat{\eta}_z \in C^\infty_{\rm cp}(E_1'(z))$. Then for any $s \in (0,\infty)$, the following:

$$T^n \eta(x) \equiv \frac{1}{2\pi i} \int_{|z|=s} z^{-n} \hat{\eta}_z(\pi(x)) \frac{dz}{z}$$

defines a smooth section over $\tilde{E}_1 \to \tilde{Y}(S)$, where $x \in \tilde{W}_0^0([-S,S])$ and π : $\tilde{Y}(S) \to Y(S)$ is the projection. By the Cauchy's formula, these are inverses each other, independently of s.

Let D be as above. Passing through the Fourier Laplace transform, one has another differential operator between $(E_0)'$ and $(E_1)'$ by:

$$\hat{D}_z \hat{\psi}_z \equiv (\hat{D\psi})_z.$$

Notice that at $z = 1$, $(E_i)'$, $i = 0,1$, are isomorphic to $E_i \to Y(S)$ respectively. In fact every $(E_i)'(z)$ is isomorphic to E_i as follows. Let us take $t : \tilde{W}_0([-S,S]) \mapsto [0,1]$, a smooth map such that $t \equiv 0$ near $N_0(S)$, and $t \equiv 1$ near $M_1(S)$. By taking $[0,\infty) \subset \mathbf{C}$ as a branched line, one may define $logz$. Let us put $z^t \equiv exp(tlogz)$, and consider $z^t \hat{\psi}_z$. This gives a $\mathbf{C}$ valued section over E'_0. One calculates $\hat{D}_z$ in terms of this identification. The result is as:

$$\hat{D}(z) = D + z^t[D, z^{-t}] : C^\infty_{\mathrm{cp}}(E_0) \mapsto C^\infty_{\mathrm{cp}}(E_1).$$

Let $\{E_i, D_i\}_{i=0,1,2}$ be an elliptic complex over $Y(S)$. It is clear that it gives a bounded one as:

$$0 \to W^{k+2}_\tau(\tilde{E}_0) \xrightarrow{D_0} W^{k+1}_\tau(\tilde{E}_1) \xrightarrow{D_1} W^k_\tau(\tilde{E}_2) \to 0.$$

Proposition 7.1 *The above D_i has closed range, if for all $z \in C = \{w \in \mathbf{C}^* ; |w| = 1\}$, $\{\hat{D}^i_z : W^{k+i}_\tau \to W^{k+i-1}_\tau\}$ is acyclic Fredholm complex, i.e. all cohomologies H^i vanish.*

Proof: For any $u \in W^k_\tau(\tilde{Y}(S); \tilde{E}_i)$, $\hat{u}_z$ has the following property; it lies in $W^k_\tau((E_i)_z; Y(S)_z)$ for all z with $0 < |z| \leq 1$.

When $|z| = 1$, the inner product on E_i has a natural extension of a hermitian metric on $E_i(z)$. Recall that $D_i : W^{k+1}_\tau(Y(S); E_i(z)) \to W^k_\tau(Y(S); E_{i+1}(z))$ has closed range. Let $(D_i)^*_\tau$ be the formal adjoint operator. This differential operator is defined independently of z, with $|z| = 1$ over $E_i(z)$. By the assumption, one has an isomorphism:

$$D = D_0 \oplus (D_1)^*_\tau : W^{k+1}_\tau(Y(S); E_0(z) \oplus E_2(z)) \mapsto W^k_\tau(Y(S); E_1(z)).$$

We denote by D^{-1} as its inverse. On the other hand one has a natural lift of D as:

$$D : W^{k+1}_w(\tilde{Y}(S); \tilde{E}_0 \oplus \tilde{E}_2) \mapsto W^k_w(\tilde{Y}(S); \tilde{E}_1).$$

Here we show that there is another bounded operator:

$$R : W^k_w(\tilde{Y}(S); \tilde{E}_1) \mapsto W^{k+1}_w(\tilde{Y}(S); \tilde{E}_0 \oplus \tilde{E}_2)$$

with $R \circ D$ is the identity. From this, one knows that D has closed range. In particular D_0 has closed range.

Next one proceeds similarly for $D_\tau^* = (D_0)_\tau^* \oplus D_1$. Then one also knows that its lift D_τ^* has closed range. Clearly it follows that D_1 has also closed range.

Now let us take $\psi \in W_w^{k+1}(\tilde{Y}(S); \tilde{E}_0 \oplus \tilde{E}_2)$. Then one considers $b(\psi) = D^{-1}(\hat{\psi}) \in W_\tau^{k+1}(Y(S); E_1(z))$. This element also has the property in the first paragraph of the proof.

Then one defines $R(\psi) \in W_w^{k+1}(\tilde{Y}(S); \tilde{E}_1)$ by:

$$T^n R(\psi)|\tilde{W}_0^0([-S,S])(x) = \frac{1}{2\pi i}\int_C z^{-n}(D^{-1}(\hat{\psi})(\pi(x)))\frac{dz}{z}.$$

We claim that this assignment:

$$R : W_w^k(\tilde{Y}(S); \tilde{E}_1) \to W_w^{k+1}(\tilde{Y}(S); \tilde{E}_0 \oplus \tilde{E}_2)$$

gives a bounded operator. Then by definition of D_i over $Y(S)$, it satisfies $R \circ D = 1$.

Let us put $C = \{z = exp(i\theta) \in \mathbf{C} : |z| = 1\}$, $\psi_n|\tilde{W}_0^0([-S,S]) = T^n(\psi)$. By regarging $b(\psi)_z \in W_\tau^{k+1}(\tilde{W}_0^0([-S,S]))$, one has a Fourier expansion $b(\psi) = \Sigma_n z^n \varphi_n$, $\varphi_{n+1}|N_0(S) = \varphi_n|M_1(S)$. Notice the equality $D(\varphi_n) = \psi_n$. One can regard the value of the inner product $< b(\psi), b(\psi) > |W_\tau^k(Y(S); E_*(z))$ as a real valued function of $\theta \in [0, 2\pi]$. Then one has the equality:

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta |b(\psi)|^2 W_\tau^{k+1}(Y(S); E_*(z)) =$$
$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \Sigma_{n,m} exp(i\theta(n-m)) < \varphi_n, \varphi_m > |W_\tau^{k+1}(\tilde{W}_0^0([-S,S])) = \Sigma_n |\varphi_n|^2 W_\tau^{k+1}.$$

Combining with the inequality (D^{-1} are bounded operators for all $z \in C$) $|b(\psi)|W_\tau^{k+1}(Y(S); E_*(z)) \leq C|\hat{\psi}|W_\tau^k$ for every θ, one gets the estimate:

$$\Sigma_n |\varphi_n| W_\tau^{k+1}(\tilde{W}_0^0([-S,S])) \leq C\Sigma_n |\psi_n| W_\tau^k(\tilde{W}_0^0([-S,S])).$$

From the equality, $T^n(\psi) = \psi_n$, one has the desired estimate:

$$\begin{aligned} |R(\psi)|^2 W_\tau^{k+1}(\tilde{Y}(S); \tilde{E}_*) &= \Sigma_n |\varphi_n|^2 W_\tau^{k+1}(\tilde{W}_0^0([-S,S])) \leq C\Sigma_n |D(\varphi_n)|^2 W_\tau^k \\ &= C\Sigma_n |T^n(\psi)|^2 W_\tau^k(\tilde{W}_0^0([-S,S])) = C|\psi|^2 W_\tau^k. \end{aligned}$$

This completes the proof.

One needs to have some circle $C \subset \mathbf{C}^*$ which satisfies the above conditions. For this, one needs the following:

Sublemma 7.2 *$D_i : W^{k+1}_\tau(Y(S); E_i(z)) \mapsto W^k_\tau(Y(S); E_{i+1}(z))$ has closed range.*

Proof : It is true for $z = 1$. Recall $Y(S) = \tilde{W}_0([-S, S])/N_0(S) \sim M_1(S)$. Let $H_i \subset W^{k+1}_\tau(Y(S); E_i(z))$ be closed subsets satisfying

(1) $H_1 \cup H_2 = W^{k+1}_\tau(Y(S); E_i(z))$ and

(2) Supp$H_1 \subset \tilde{W}_0([-(S-1), S-1])$ and Supp$H_2 \subset N_0([-S, -(S-1)]) \cup M_1([S-1, S])$.

By regarding $H_1 \subset W^{k+1}_\tau(Y(S); E_i(1))$, it follows that $D_i(H_1)$ is a closed subspace of $W^k_\tau(Y(S); E_{i+1}(z))$. Let $u \in H_2$. Then one may associate $\tilde{u} \in W^{k+1}_\tau(Y(S); E_i(1))$ by $\tilde{u}|N_0([-S, -(S-1)]) = u$ and $\tilde{u}|M_1([S-1, S]) = z^{-1}u$. Clearly $D_i(\tilde{H_2}) = D_i(\tilde{H}_2)$ is a closed subspace of $W^k_\tau(Y(S); E_{i+1}(1))$. Since this assignment is isometric, it follows that H_2 is also a closed subspace of $W^k_\tau(Y(S); E_{i+1}(z))$.

This completes the proof.

Proposition 7.2 *Let $\{D_*\}$ be the AHS complex. Suppose it gives a Fredholm complex and all the cohomology groups vanish, $H^i(W^k_\tau(Y(S); \Lambda^*); D_*) = 0$, $i = 0, 1, 2$. Then for $C(1) = \{z : |z| = 1\}$, one has $H^*(W^k_\tau(Y(S); \Lambda^*); (D_*)_z) = 0$ for all $z \in C(1)$ and $i = 0, 1, 2$.*

Proof : The first part of the proof of is essentially [T1] p390. Let $u \in W^k_\tau(Y(S); \Lambda^*_z)$ with $d_0(u) = 0$. Then passing through the identification $z^t : \Lambda^0_z \cong \Lambda^0$, one has $d(z^{-t}u) = 0$ where $z^{-t}u \in W^k_\tau(Y(S); \Lambda^0)$. Clearly this shows $u = 0$.

Next let us take $\alpha \in W^k_\tau(Y(S); \Lambda^1_z)$ with $d_+(\alpha) = 0, d^*_\tau(\alpha) = 0$. Then by integration by parts, one has $d(\alpha) = 0$. α is zero in $H^1(\tilde{W}_0 : \mathbf{R})$ (see the proof of lemma 8.1). Thus one may express $\alpha = d(f)$ on $\tilde{W}_0$ for $f \in C^\infty(\tilde{W}_0([-S, S]))$ and $d^*_\tau d(f) = 0$. Let $i : N_0(S) \cong M_1(S)$ be the identification. Then one has $i^*(f) = zf + const$. If one takes f so that it vanishes at infinity, then $const = 0$. Thus one has the equality:

$$0 = \int_{\tilde{W}_0([-S,S])} exp(\tau) < f, d^*_\tau d(f) > vol = -\int exp(\tau)|df|^2.$$

This shows $df = 0$. Thus $H^*(W^k_\tau(Y(S); \Lambda^*); (D_*)_z) = 0$, $* = 0, 1$ for all $z \in C(1)$.

Now let us put $D = (D_0)^*_\tau \oplus D_1$, and consider $\Delta = D^*_\tau D$ on $L^2_\tau(Y(S); \Lambda^1(z))$. Then there is a constant $\lambda > 0$ with $\lambda_z > \lambda$, where λ_z is the bottom of Δ. Suppose $H^2 \neq 0$ at $z = exp(i\theta_0)$, but $H^2 = 0$ for all $z = exp(i\theta)$, $0 \leq \theta < \theta_0$.

Then by choosing some θ, it follows that there exists $u \in W^1_\tau(Y(S); \Lambda^2_z)$, $z = exp(i\theta)$ with (1) $|u|W^1_\tau = 1$ and (2) $|(D_1)^*_\tau(u)|L^2_\tau < \epsilon$.

Since D_1 is surjective, one finds $\alpha \in W^2_\tau(Y(S); \Lambda^1_z)$ with $D_1(\alpha) = u$ and $(D_0)^*_\tau(\alpha) = 0$. Then there is a constant C independent of z such that:

$$|u|W^1_\tau \leq C|\alpha|W^2_\tau \leq C|(D_1)^*_\tau D_1(\alpha)|L^2_\tau.$$

This is a contradiction. This shows that the second cohomology also vanishes for all $z \in C(1)$. This completes the proof.

Now we compute the cohomology $H^i(W^k_\tau(Y(S); \Lambda^*); D_*)$ of the AHS complex for the simplest kinky handles (W_0, N_0, M_1). Notice that the end of $Y(S)$ is isometric to $M \times [0, \infty)$ for a compact Riemannian three manifold M. We will denote $M_t = M \times \{t\} \subset Y(S)$.

Proposition 7.3 *Let $Y(S)$ be as above. Then for all $i = 0, 1, 2$, the cohomologies $H^i(W^k_\tau(Y(S); \Lambda^*); D_*)$ vanish.*

Proof : Clearly $H^0(AHS) = 0$. Let us consider $H^1(AHS)$. Let us take any representative u. By integration by parts, one finds $du = 0$. One may express $u = w + fdt$ around the end $Y(S) \cong M \times [0, \infty)$, where w does not contain dt component. Then through this isometry, one considers $w_t \in \Lambda^1(M)$ for $t \in [0, \infty)$. Let d_3 be the differential over M. Then for every t, one has $d_3 w_t = 0$. Thus one may express $w_t = \Sigma_i g_i(t)\alpha_i + d\mu_t$, where $\{\alpha_i\}_i$ consists of the orthogonal basis of $H^1(M; \mathbf{R})$. Here one may assume $d_3^*(\alpha_i) = 0$. By taking into account of dt component of du, one has the following equation:

$$\Sigma_i g_i(t)'\alpha_i + d_3\mu(t)' = d_3 f_t.$$

This shows $g_i(t)' \equiv 0$. Since $u \in L^2_\tau$, one concludes $g_i(t) \equiv 0$. Then one has another equality, $f_t = \mu(t)' + c(t)$, where $c(t)$ are constants depending on t. Thus one gets $u = d_3\mu_t + (\mu'_t + c(t))dt$.

Sublemma 7.3 *If $d_3\mu_t \in W^k_\tau$, then for a smooth family of constants $d(t)$, one has $\mu_t - d(t) \in W^k_\tau$.*

Proof : Let us put $d(t) = (vol_M)^{-1} \int_M \mu_t vol$. Then we show that this family is the desired one. Notice that one has the following bound: $|d_3\mu_t|L^2_\tau \geq C|\mu_t - d(t)|L^2_\tau$ for some positive constant C. In particular one has $\mu_t - d(t) \in L^2_\tau$. Notice that one has the equality $\mu_t - d(t) = (\Delta_0)^{-1}d^*d(\mu_t - d(t))$, and $(\Delta_0)^{-1}d^*$ is a translation invariant bounded operator. Thus we get:

$$\frac{\mu_t - d(t)}{dt} = \frac{d}{dt}[(\Delta_0)^{-1}d_3^*d_3(\mu_t - d(t))] = (\Delta_0)^{-1}d^*\frac{d}{dt}[d_3\mu_t].$$

From the last term, one sees that $\frac{\mu_t - d(t)}{dt} \in L^2_\tau$. By a similar consideration, one gets the result. This completes the proof.

Proof of Proposition: Then by replacing μ_t by $\mu_t - d(t)$, one may assume $\mu_t \in W^k_\tau$. Thus $d\mu_t = d_3\mu_t + \mu'_t dt \in W^{k-1}_\tau$. On the other hand, let us consider the equality $f_t = \mu'_t + c(t)$. Then it follows $c(t) \in W^{k-1}_\tau$. Let us put $C(t) = \int_0^t c(s)ds - \int_0^\infty c(t)$. Then by [T1] lemma 5.2 (see sublemma 6.1), one finds that $C(t) \in W^k_\tau$, and $dC(t) = c(t)$. In particular we have $u|\ \mathrm{end}Y(S) = df$, $f \in W^k_\tau$. Then using a cut off function, one may represent u by a compactly supported smooth 1 form, which is itself exact by a compactly supported smooth function, since $H^1_{\mathrm{cp}}(Y(S); \mathbf{R}) = 0$. This shows $H^1(AHS) = 0$.

Next we consider $H^2(AHS)$. Let us take a representative $u \in H^2(AHS)$. One may choose u so that it satisfies $d(e^\tau u) = e^{-\tau} d(ue^\tau) = 0$. Since $H^2(Y(S); \mathbf{R}) = 0$, one may express $e^\tau u = d\mu$, $\mu \in C^\infty(\Lambda^1)$. Let us denote $\mu|\ \mathrm{end}Y(S) = \beta + f dt$, where β does not contain dt component. Then we have the following relation:

$$d\mu|\ \mathrm{end}Y(S) = d_3\beta_t + (d_3 f - \beta'_t)dt = d_3\beta_t + *_3 d_3\beta_t \wedge dt$$

where $*_3$ is the star operator over M. Let us decompose $\beta_t = \beta^1_t + \beta^2_t$, where $\beta^1_t(\beta^2_t)$ does (not) consists of a closed form over M. Then from the last two terms, one finds $d_3 f_t = (\beta^1_t)'$. In particular one may represent:

$$e^w u|\ \mathrm{end}Y(S) = d\beta^2_t = d_3\beta^2_t - (\beta^2_t)' \wedge dt = d_3\beta^2_t + *_3 d_3\beta^2_t \wedge dt.$$

By the decomposition, one finds a positive constant C such that:

$$|d_3\beta^2_t|W^{k-1}(M_t) \geq C|\beta^2_t|W^k(M_t).$$

Now we have the next relations (put $\mu = \beta^2_t$ on the end):

$$e^\tau u = d\mu, \quad |\mu_t|W^k(M_t) \leq C|e^\tau u|W^{k-1}(M_t).$$

Sublemma 7.4 *If $\delta > 0$ is sufficiently small with respect to S, then $e^\tau u \in L^2(Y(S); \Lambda^2_+)$.*

Proof : Let us consider $e^\tau u|\mathrm{end} = d\mu$. One may assume that for every t, $\mu_t \in C^\infty(\Lambda^1(M_t))$ lies on the orthogonal complement of Ker$d$. Then $*_3 d$ is invertible on $(\mathrm{Ker}d)^\perp$ by corollary 6.1. Moreover it is self adjoint with respect to L^2 inner product. Since μ satisfies the equation, $(\frac{\partial}{\partial t} + *_3 d_3)\mu = 0$, one has the exponential decay estimate for μ. More precisely there exist constants $C > 0, \lambda_0 > 0$ which are independent of δ, such that:

$$|\mu|L^2(M_t) \leq exp(-\lambda_0 t) \sup\{|\mu|L^2(M_s); 0 \leq s \leq 2t\}.$$

Notice that a priori, μ satisfies the following bound of its growth $|\mu|L^2(M_t) \leq Cexp(t\frac{\delta}{2})|u|L^2(M_t)$. Combining these estimates, one gets the exponential estimate for μ. Then one has also the exponential decay estimate for $d\mu$ on the end. This completes the proof.

Proof of proposition: Let $\varphi_t : Y(S) \mapsto [0,1]$ be a cut off function such that $\varphi|M \times [t+1,\infty) \equiv 0$, $\varphi|(M \times [t,\infty))^c \equiv 1$. Let us fix a large S, and let $\delta > 0$ satisfy the above condition. By the above estimate on W^k norm, one finds $\varphi_t\mu \in L^2(Y(S);\Lambda^2_+)$ for all t. Then for any small $\epsilon > 0$, one finds a large t so that:

$$\int |e^w u|^2 = |\int d(\varphi_t\mu) \wedge d(\varphi_t\mu) + \int |e^w u|^2| < \epsilon.$$

This shows $u \equiv 0$, and we have shown $H^2(AHS) = 0$ for this pair (δ, S). This completes the proof.

Remark 7.1 Suppose for all $\delta' \in [\delta, \delta_0]$, the differential of AHS complex has closed range, where δ is sufficiently small. Then $H^*(AHS)$ also vanish for all δ'. This is seen as follows. Notice that for any choice of the constants, one knows $H^0(AHS) = H^1(AHS) = 0$. We want to calculate $H^2(AHS)$ when we vary $\delta' > 0$. Let us take an isomorphism:

$$I : W^*_{\tau(\delta)}(Y(S);\Lambda^*) \cong W^*_{\tau(\delta')}(Y(S);\Lambda^*)$$

where we denote $\tau(\delta)$ to express the weight constant. Passing through I, one has $\{W^*_{\tau(\delta)}(\Lambda^* : Y(S)); I^{-1} \circ d_* \circ I\}$, a continuous family of Fredholm complexes. In particular, the indices of these complexes are invariant. Since for all cases, one has $H^0(AHS) = H^1(AHS) = 0$, one concludes $H^2(AHS) = 0$.

Remark 7.2 Suppose a cylindrical four manifold Y has non zero $H^2(Y : \mathbf{R})$. Then in this case, one has a bound:

$$\dim H^2(AHS) \leq 2 \dim H^2(Y : \mathbf{R}).$$

This directly follows from the proof of proposition 7.3.

Corollary 7.1 *Let us choose a sufficiently small $\delta > 0$. Then one has an invertible operator:*

$$P_w : W^{k+1}_w(\tilde{Y}(S);\Lambda^1) \cong W^k_w(\tilde{Y}(S);\Lambda^0 \oplus \Lambda^2_+).$$

8 An asymptotic method to compute cohomology

8.A P_w over $Y(S,2)$: Let (W_0, N_0, M_1) be a simplest kinky handle, and $\tilde{W}_0$ be the Riemannian manifold constructed before. In the previous section, one has the invertible operator:

$$P_\tau : W_\tau^{k+1}(Y(S);\Lambda^1) \mapsto W_\tau^k(Y(S);\Lambda^0 \oplus \Lambda^2_+).$$

By the Fourier Laplace transform, one gets an invertible Fredholm P_τ over $\tilde{Y}(S)$ where:

$$\tilde{Y}(S) = \cdots \cup \tilde{W}_0^{i-1}([-S,S]) \cup \tilde{W}_0^{i}([-S,S]) \cup \ldots \quad i \in \mathbf{Z}.$$

Let T_2 be the periodic tree as $\mathbf{R} \cup_{n\in\mathbf{Z}} \mathbf{R}_0$, where $\mathbf{R}_0$ is the half real line. The aim here is to show that the corresponding P_τ over $CH(T_2)$ is invertible. Let (W_2, N_0, M_1, M_2) be a kinky handle with 2 kinks, and $\tilde{W}_2$ be the corresponding complete Riemannian manifold. Thus one has three ends in the horizontal direction, $N_0([0,\infty)), M_j([0,\infty))$, $j = 1,2$. Let us choose a large S, and put:

$$\tilde{Y}(S)_0 = \tilde{W}_0^1((-\infty,S]) \cup \tilde{W}_0^2([-S,S]) \cup \ldots,$$
$$Y(S,2)' = \tilde{W}_2 \backslash N_0([S,\infty)) \cup M_1([S,\infty))/N_0(S) \sim M_1(S),$$
$$Y(S,2) = Y(S,2)' \backslash M_2([S,\infty)) \cup \tilde{Y}(S)_0 \backslash N_0([S,\infty)).$$

One may equip a weight function τ on $Y(S,2)$ as before.

Lemma 8.1 *$d_* : W_\tau^{k+1}(Y(S,2);\Lambda^*) \mapsto W_\tau^k(Y(S,2);\Lambda^{*+1})$ has closed range with $H^0(AHS) = H^1(AHS) = 0$.*

Proof : Closedness follows from the excision method used before. Let us take $(W_s = S^1 \times D^3, N_0 = S^1 \times D^2)$, where N_0 represents a generator of $\pi_1(W_s)$. By the construction in $5.A$, one can get a complete Riemannian manifold $\tilde{W}_s$ with one end $N_0([0,\infty))$ along the horizontal direction. Let us put:

$$Y(S,2)(0) = Y(S,2)' \backslash M_2([S,\infty)) \cup \tilde{W}_s \backslash N_0([S,\infty)).$$

Notice that $Y(S,2)(0)$ is diffeomorphic to $Y(S,2)'$, but Riemannian metrics are different. ($Y(S,2)(0)$ has cylindrical end.) By corollary 6.2, the differential of the AHS complex has closed range.

Let $H_i \subset W_\tau^{k+1}(Y(S,2);\Lambda^*)$ be closed subsets with
(1) $H_1 \cup H_2 = W_\tau^{k+1}(Y(S,2);\Lambda^*)$,
(2) $\mathrm{Supp} H_1 \subset Y(S,2)' \backslash M_2([S,\infty))$ and $\mathrm{Supp} H_2 \subset \tilde{Y}(S)_0 \backslash N_0([S-1,\infty))$.

One regards $H_1 \subset W_\tau^{k+1}(Y(S,2)(0);\Lambda^*)$ and $H_2 \subset W_\tau^{k+1}(\tilde{Y}(S);\Lambda^*)$. From this, it follows that $d_*(H_i)$ are both closed subspace of $W_\tau^k(Y(S,2);\Lambda^{*+1})$.

Let us see $\mathrm{Ker}P_w = 0$. $\mathrm{Ker}P_w$ is isomorphic to 1st cohomology $H^1(AHS)$. One can easily check $H^1(Y(S,2) : \mathbf{R}) = \mathbf{R}$. Let us take a non zero element w. Then this has the property $< w, C > \neq 0$ for any loop C representing a generator of $\pi_1(Y(S,2))$. One may choose C sufficiently near to infinity while having a bounded length.

Let us take $u \in H^1(AHS)$, and consider its class $[u] \in H^1(Y(S,2) : \mathbf{R})$. it follows $[u] = 0$, since one can make $< u, C >$ arbitrarily small by choosing C as above. Thus one may express $u = df$ for $f \in C^\infty(Y(S,2))$. One may assume $f \in L^2_w(Y(S,2))$, by subtracting some constant. This shows $u = 0$. Thus we have shown the result.

8.B Some continuity of $H^1(AHS)$: Let us introduce another weight function:

$$\mu : \tilde{Y}(S) \mapsto [0, \infty)$$

by $\mu|\tilde{W}_n(s \leq S)(x) = |n| + t(x)$. We choose another small constant δ' (we take it so that $\delta' << \delta$, where δ is the weight constant for τ). Then we introduce another weight $w = \delta'\mu + \tau$ and weighted Sobolev k spaces W^k_w, where:

$$(|u|W^k_w)^2 = \int_{\tilde{Y}} exp(\delta'\mu + \tau)(\Sigma^k_{l=0}|\nabla^l u|^2).$$

Since one knows $H^*(AHS) = 0$ over $\tilde{Y}(S)$, it follows that there exists a positive constant $C > 0$ such that $|P_w|, |(P_w)^*_w| \geq C$, for any small $\delta' \in [0, \delta'_0]$.

Let Y be a complete Riemannian manifold of bounded geometry. Let us take a family of smooth maps:

$$w(\delta') : Y \mapsto [0, \infty), \quad w(\delta'')(x) \leq w(\delta')(x), \quad \delta'' \leq \delta' \in [0, \delta], \quad x \in Y.$$

Suppose that for all $w(\delta')$, the corresponding AHS complex between weighted Sobolev spaces are Fredholm. In any case it is immediate to see $H^0(AHS) = 0$. Here one has some continuity property as:

Lemma 8.2 *$H^1(AHS) = 0$ for $\delta' = 0$ when $H^1(AHS) = 0$ for $\delta' > 0$.*

Proof : Let us denote $\mathrm{Ker}(d_+)^*_{w(\delta')} = H^2(\delta')$ (the second cohomology $H^2(AHS)$ when it is defined). Let us choose sufficiently small constants:

$$\delta >> \delta' > \delta'' \geq 0.$$

Then one has $H^2(\delta'') \cong H^2(\delta')$. This follows from that by varing these weights from $w(\delta')$ to $w(\delta'')$, one gets a family of Fredholm complexes between the weighted Sobolev spaces (these spaces also vary with respect to the deformation of the weights). For every value of weights, one has $H^0(AHS) = H^1(AHS) = 0$. Thus one gets the above statement.

Sublemma 8.1 *If $H^2(\delta'') \neq 0$, then one has also $H^2(\delta') \neq 0$.*

Proof : Let us denote $w = w(\delta')$ and $w' = w(\delta'')$. Let us take $v \in L^2_{w'}$ with $d^*(exp(w')v) = 0$. Then one puts $u = exp(-w+w')v$. One may assume $u \in L^2_w$. Since u satisfies $d^*(exp(w)u) = 0$, one gets the result.

Proof of lemma: Recall that by putting $w = w(\delta')$ one also has a Fredholm complex $\{W^*_w(Y(S,2);\Lambda^*), d^*\}$. Then by variying a parameter $\delta' \in [0,\delta]$, one has a family of Fredholm complex:

$$\{W^*_{w(\delta')}(Y(S,2);\Lambda^*), d^*\}.$$

From the above proof, one has the inclusion $H^2(\delta'') \subset H^2(\delta')$ for all $0 \leq \delta'' \leq \delta'$. Now let us see $H^2(\delta') = H^2(0)$ and $H^1(0) = 0$. Suppose $H^1(0) \neq 0$. Then by invariance of Fredholm indices, one must have $\dim H^2(0) > \dim H^2(\delta')$. This contradicts to the above. This completes the proof.

8.C Computation of $H^2(AHS)$: To show that $P_\tau : W^{k+1}_\tau(Y(S,2);\Lambda^1) \mapsto W^k_\tau(Y(S,2);\Lambda^0 \oplus \Lambda^2_+)$ is invertible, one uses an asymptotic method. Roughly speaking one approximates $Y(S,2)$ by a family of Rimannian manifolds with cylindrical ends. Then the spectrum of P_τ over each cylindrical manifold has a uniform lower bound on 1 forms. From this one gets a lower bound of $(d_+)^*_\tau$ over $Y(S,2)$. In the presence of $H^2(Y : \mathbf{R})$, one can get a uniform bound of $\dim H^2(AHS)$. The approximation of spaces corresponds to the one of infinite tree by its finite subtrees. This method completely works for higher stages in $8.D$.

Computation of $\dim H^2(AHS)$ uses information of $H^1(AHS)$ on both approximation spaces and the limit space. The former is obtained by proposition 7.3 and the latter by sublemma 6.1. On the other hand for all cases $H^0(AHS) = 0$. Then using this and proposition 7.3, one can apply the asymptotic method in this section to verify $H^1(AHS) = 0$ for the limit space. This will be another method to compute $H^1(AHS)$ without using sublemma 6.1.

Let us take $(W_s = S^1 \times D^3, N_0 = S^1 \times D^2)$ where N_0 represents a generator of $\pi_1(W_s)$. Then by the construction in $5.A$, one gets a complete Riemannian manifold $\tilde{W}_s$ with one end $N_0([0,\infty))$ along horizontal direction. Let (W_0, N_0, M_1)

be the simplest kinky handle as before. Let us put:

$$\begin{aligned}
Y(S)_0(n)' &= \tilde{W}_0^1([-S,S]) \cup_{M_1^1(S) \sim N_0^2(S)} \tilde{W}_0^2([-S,S]) \\
&\quad \cdots \cup_{M_1^{n-1}(S) \sim N_0^n(S)} \tilde{W}_0^n([-S,S]), \\
Y(S)_0(n) &= Y(S)_0(n)' \cup_{M_1^n(S) \sim N_0(S)} \tilde{W}_s \backslash N_0([S,\infty)), \\
Y(S)_0'' &= \ldots \tilde{W}_0^{-n}([-S,S]) \cup_{M_1^{-n}(S) \sim N_0^{-n+1}(S)} \tilde{W}_0^{-n+1}([-S,S]) \ldots \\
&\quad \cup_{M_1^{n-1}(S) \sim N_0^n(S)} \tilde{W}_0^n([-S,S]) \cup_{M_1^n(S) \sim N_0(S)} \tilde{W}_s \backslash N_0([S,\infty)).
\end{aligned}$$

Notice $Y(S)_0(\infty)' = Y(S)_0$. Let us put $Y(S,2)(n)' = Y(S,2)' \cup_{M_2(S) \sim N_0^0(S)} Y(S)_0(n)'$ and:

$$Y(S,2)(n) = Y(S,2)' \cup_{M_2(S) \sim N_0^0(S)} Y(S)_0(n).$$

$Y(S,2)(n)$ is a complete Riemannian manifold with cylindrical end. Let $\tau : Y(S,2)(n) \mapsto [0,\infty)$ be the weight function with weight δ as before.

Proposition 8.1 *One gets isomorphisms:*

$$P_\tau : W_\tau^{k+1}(Y(S,2)(n); \Lambda^1) \cong W_\tau^k(Y(S,2)(n); \Lambda^0 \oplus \Lambda_+^2).$$

Proof : By proposition 7.3, the conclusion is true when one uses a sufficiently small weight constant δ'. On the other hand, by the first part of the proof of lemma 8.1, the differentials of AHS complex have all closed range for all $\delta' \in (0,\delta]$. From this, one gets $H^0(AHS) = H^1(AHS) = 0$ for all δ'. To see $H^2(AHS) = 0$, one can use the same method as proposition 7.2. This completes the proof.

In particular there are constants C_n such that for $\Delta = (P_\tau)_\tau^* \circ P_\tau$, one has the bounds $|u|W_\tau^{k+2}(Y(S,2)(n)) \leq C_n |\Delta(u)| W_\tau^k(Y(S,2)(n))$. Also one has bounds:

$$|u|W_\tau^{k+2}(Y(S,2)) \leq C|\Delta(u)|W_\tau^k(Y(S,2)),$$

$$|u|W_\tau^{k+2}(Y(S)_0'') \leq C|\Delta(u)|W_\tau^k(Y(S)_0'').$$

Sublemma 8.2 *There is an upper bound $C_n \leq C$ for all n.*

Proof : For any small $\epsilon > 0$, there exists a large n with the following property; for any $u \in W_\tau^{k+2}(Y(S,2)(n))$ with $|u|W_\tau^{k+2} = 1$, there is at least one n', $0 \leq n' \leq n$ with $|u|W_\tau^{k+2}(\tilde{W}_0^{n'}([-S,S])) < \epsilon$. Let $\varphi(n')$ be a cut off function on $Y(S,2)(n)$ with $\varphi(n')|(\tilde{W}_0^{n'}([-S,S]))^c \equiv 1$ and $Y(S,2)(n)\backslash$ Supp$d\varphi(n')$ has two components. Then one may express $\varphi(n')u = u_1 + u_2$. One may regard

$u_1 \in W_\tau^{k+2}(Y(S,2))$, $u_2 \in W_\tau^{k+2}(Y(S)_0'')$. In particular one has the estimates:

$$\begin{aligned}|u|W_\tau^{k+2} &\leq |u_1|W_\tau^{k+2} + |u_2|W_\tau^{k+2} + |(1-\varphi(n')u|W_\tau^{k+2} \\ &\leq C\{|\Delta(u_1)|W_\tau^k + |\Delta(u_2)|W_\tau^k\} + \epsilon \leq C|\Delta(u)|W_\tau^k + C\epsilon\end{aligned}$$

where C is independent of ϵ and u. This gives the result.

Now suppose $H^2(AHS)$ is non zero over $Y(S,2)$, and take $u \in \mathrm{Ker}(d_+)_\tau^* \cap W_\tau^{k+1}(Y(S,2);\Lambda_+^2)$ with $|u|W_\tau^{k+1} = 1$. For each n, let us take a cut off function $\varphi(n)$ on $Y(S,2)$ with $\varphi(n)|Y(S,2)(n-1)' \equiv 1$, $\varphi(n)|(Y(S,2)(n)')^c \equiv 0$. Then one may regard $\varphi(n)u \in W_\tau^{k+1}(Y(S,2)(n);\Lambda_+^2)$. Thus there exists $v_n \in W_\tau^{k+2}(Y(S,2)(n);\Lambda^1)$ with $d_\tau^*(v_n) = 0$, $d_+(v_n) = \varphi(n)u$. By the above sublemma, one has the uniform estimates:

$$\begin{aligned}&1-\epsilon \leq |\varphi(n)u|W_\tau^{k+1} = |d_+(v_n)|W_\tau^{k+1} \\ &\leq C|v_n|W_\tau^{k+2} \leq C|d_+(v_n)|W_\tau^{k+1} \leq C|\varphi(n)u|W_\tau^{k+1}. \quad (*)\end{aligned}$$

By the above $(*)$, one has the uniform estimates $C \leq |v_n|W_\tau^{k+2} \leq C'$. Let us take a sequence $1 >> \epsilon_1 >> \epsilon_2 >> \cdots \to 0$. Then for each n, there exists a large $N = N(n)$ and $L(n) >> 0$ such that at least one of $|v_N|W_\tau^{k+2}(\tilde{W}_0^{n'}([-S,S]))$ is less than ϵ_n for $N - L(n) \leq n' \leq N$. Let us take a subsequence $\{v_{N(n)}\}_n$ and denote it by $\{v_n\}_n$. For simplicity of the notation, one may assume $n' = n-1$.

Using this, one can verify $\mathrm{Ker}(d_+)_\tau^* = 0$ over $Y(S,2)$. Let us put $\Delta = dd_\tau^* \oplus (d_+)_\tau^* d_+$ on $W_\tau^{k+2}(Y(S,2);\Lambda^1)$. From above, one has the estimate:

$$\begin{aligned}&|dd_\tau^*(\varphi(n-1)v_n)|W_\tau^k \\ &\quad \leq C\{|dv_n|W_\tau^k(\tilde{W}_0^{n-1}([-S,S])) + |v_n|W_\tau^k(\tilde{W}_0^{n-1}([-S,S]))\} \leq C\epsilon_n\end{aligned}$$

where C is independent of n. Then one has the estimates:

$$\begin{aligned}&|\Delta(\varphi(n-1)v_n)|W_\tau^k \leq C[|(d_+)_\tau^*(\varphi(n-1)\varphi(n)u)|W_\tau^k \\ &\quad + |(d_+)_\tau^*[d_+,\varphi(n-1)]v_n|W_\tau^k(\ \mathrm{Supp} d\varphi(n-1))] + C\epsilon_n \\ &\quad \leq C[|\varphi(n-1)\varphi(n)(d_+)_\tau^*(u)|W_\tau^k \\ &\qquad + |[(d_+)_\tau^*,\varphi(n-1)\varphi(n)]u|W_\tau^k(\cup_{j=n-1,n}\tilde{W}_0^j([-S,S])) \\ &\qquad + |(d_+)_\tau^*[d_+,\varphi(n-1)]v_n|W_\tau^k(\tilde{W}_0^{n-1}([-S,S])) + C\epsilon_n \\ &\quad \leq |[(d_+)_\tau^*,\varphi(n-1)\varphi(n)]u|W_\tau^k + |(d_+)_\tau^*[d_+,\varphi(n-1)]v_n|W_\tau^k + C\epsilon_n.\end{aligned}$$

One may assume that the last term is arbitrarily small. This shows that there is a sequence $\{w_n \in W_\tau^{k+2}(Y(S,2);\Lambda^1)\}$ with $C \leq |w_n|W_\tau^{k+2} \leq C'$

and $|P(w_n)|W^{k+1}_\tau$ converges to zero. This is a contradiction. This shows $\mathrm{Ker}(d_+)^*_\tau = 0$ over $Y(S,2)$.

Let $(W_t, N_0) = (D^4, S^1 \times D^2)$ be the standard disk. Let $Y = \tilde{W}_t \backslash N_0([S,\infty)) \cup \tilde{Y}(S)_0 \backslash N_0([S,\infty))$. In this case, one has $H^0(AHS) = H^1(AHS) = 0$ (notice Y is simply connected). But $H^2(Y : \mathbf{R}) = \mathbf{R}$. In this case, one has the following:

Corollary 8.1 *Suppose Y has non zero $H^2(Y:\mathbf{R})$. Then one has a bound $dim H^2(AHS) \leq 2\ dim H^2(Y:\mathbf{R})$.*

Proof : Suppose $\dim H^2(AHS) \geq 2 \dim H^2(Y:\mathbf{R}) + 1 = b_2 + 1$. Let us take L^2_τ orthogonal vectors $u_1, \dots, u_{b_2+1}, \dots \in L^2_\tau(Y : \Lambda^2_+) \cap \mathrm{Ker}(d_+)^*_\tau$, with $|u_i|L^2_\tau = 1$.

One can make a family of cylindrical manifolds $Y(n)$ by the above method. By a straightforward calculation, one has an upper bound of $\dim H^2(Y(n):\mathbf{R})$ by b_2. By remark 7.2, it follows that $\dim H^2(AHS)$ has also an upper bound over $Y(n)$. Now let $\varphi(n)$ be as above. Then for any small $\epsilon > 0$, there exists a large n_0 such that for all $n \geq n_0$, one has:

$$|(d_+)^*_\tau(\varphi(n)u_i)|L^2_\tau \leq \epsilon, \quad |\varphi(n)u_i|L^2_\tau \geq 1-\epsilon,$$
$$<\varphi(n)u_i, \varphi(n)u_j> |L^2_\tau \leq \epsilon, \quad i = 1, \dots, b_2+1.$$

Let $v_1, \dots, v_l$ be an orthonormal basis of $H^2(AHS)$ over $Y(n)$, $l \leq b_2$. Then one may express $\varphi(n)u_i = \Sigma_j a^i_j v_j + d_+(\alpha_i)$, $a^i_j \in \mathbf{R}$. Let $\bar{a}_i = (a^i_1, \dots, a^i_l) \in \mathbf{R}^l$. Then, the set of vectors $\bar{a}_1, \dots, \bar{a}_{b_2+1}$ would satisfy $|\bar{a}_i| \geq 1-\epsilon$ and $<\bar{a}_i, \bar{a}_{i'}> \leq \epsilon$ with respect to the standard norm in $\mathbf{R}^l$. Since $l < b_2+1$, it is impossible to find such set. This completes the proof.

8.D Fourier Laplace transform on higher stages: Let us put:

$$\tilde{W}_2([-S,S]) = \tilde{W}_2 \backslash N_0([S,\infty)) \cup_{j=1,2} M_j([S,\infty)),$$
$$Y(S,2)_0 = \tilde{W}_2 \cup_{M_2(S) \sim N_0(S)} \tilde{Y}(S)_0 \backslash N_0([S,\infty)),$$
$$\tilde{Y}(S,2) = \dots Y(S,2)^{-n}_0 \cup_{M_1^{-n}(S) \sim N_0^{-n+1}(S)} Y(S,2)^{-n+1}_0 \dots$$
$$\tilde{Y}(S,2)_0 = Y(S,2)^1_0 \cup_{M^1_1(S) \sim N^2_0(S)} Y(S,2)^2_0 \cup_{M^2_1(S) \sim N^3_0(S)} Y(S,2)^3_0 \dots$$

where $(Y(S,2)^n_0, M^n_j(S), N^n_0(S))$ are all the same triple $(Y(S,2)_0, M_j(S), N_0(S))$ as before. Notice that $\tilde{Y}(S,2)$ is diffeomorphic to $CH(T_2)$, where $T_2 = \mathbf{R} \cup_{n \in \mathbf{Z}} \mathbf{R}_+$. Now by the Fourier Laplace transform, one gets the following:

Corollary 8.2 $P_\tau : W^{k+1}_\tau(\tilde{Y}(S,2); \Lambda^1) \mapsto W^k_\tau(\tilde{Y}(S,2); \Lambda^0 \oplus \Lambda^2_+)$ *is invertible.*

Let us put $T^0_2 = \mathbf{R}_+ \cup_{n \in \mathbf{N}} \mathbf{R}_+$. Then one defines:

$$T_3 = \mathbf{R} \cup_{n \in \mathbf{Z}} T^0_2.$$

Similarly one puts $T_3^0 = \mathbf{R}_+ \cup_{n \in \mathbf{N}} T_2^0$, and $T_4 = \mathbf{R} \cup_{n \in \mathbf{Z}} T_3^0$. One inductively defines T_j, $j = 1, 2, \dots$ as:

$$T_{j+1} = \mathbf{R} \cup_{n \in \mathbf{Z}} T_j^0, \quad T_{j+1}^0 = \mathbf{R}_+ \cup_{n \in \mathbf{N}} T_j^0$$

where one puts $T_1 = \mathbf{R}$, $T_1^0 = \mathbf{R}_+$.

One can construct the corresponding spaces. Let (W_2, N_0, M_1, M_2) be a kinky handle with two kinks. One has already defined $Y(S)$, $Y(S, 2)$. Let us define inductively $Y(S, j)$ as follows:

$$\begin{aligned}
&Y(S, j)_0 = \tilde{W}_2([-S, S]) \cup_{M_2(S) \sim N_0(S)} \tilde{Y}(S, j-1)_0, \\
&Y(S, j) = Y(S, j)_0 / N_0(S) \sim M_1(S), \\
&\tilde{Y}(S, j) = \dots Y(S, j)_0^{-n} \cup_{M_1^{-n}(S) \sim N_0^{-n+1}(S)} Y(S, j)_0^{-n+1} \dots \\
&\tilde{Y}(S, j)_0 = Y(S, j)^1 \cup_{M_1^1(S) \sim N_0^2(S)} Y(S, j)^2 \cup_{M_1^2(S) \sim N_0^3(S)} Y(S, j)^3 \dots
\end{aligned}$$

One may express $CH(T_j) = \tilde{Y}(S, j)$ and $CH(T_j^0) = \tilde{Y}(S, j)_0$. The previous method works for all $Y(S, j)$ iteratedly.

Proposition 8.2 *P_τ over $Y(S, j)$ are all Fredholm with $H^*(AHS) = 0$ for $* = 0, 1, 2$ and $j = 1, 2, \dots$.*

In the notation of $5.B$, one expresses $T_j = T_{2,2,\dots,2,1}$ ($(2, \dots, 2)$: $j-1$ times). Let $n_1, \dots, n_k \in \{1, 2, \dots\}$ be a set of positive integers. Then using kinky handles with n_j kinks, one has an natural extension, and gets $T_{n_1, \dots, n_k, 1}$ which is an infinite tree. For Riemannian metric on $CH(T_{n_1, \dots, n_k, 1})$, see $5.B$.

Let $\bar{n} = \{n_1, \dots, n_{l-1}, 1\}$ be a set of positive integers, and denote the corresponding homogeneous tree of finite type by $T_{\bar{n}}$. By the previous method, one gets complete Riemannian metrics and weight functions on every $CH(T_{\bar{n}})$. Let us take $\bar{n} = \{n_1, \dots, n_{l-1}, 1\}$. Recall that one has constructed complete Riemannian metrics and weight functions on $Y(S, \bar{n})$.

Proposition 8.3 *$P_\tau : W_\tau^{k+1}(Y(S, \bar{n}); \Lambda^1) \mapsto W_\tau^k(Y(S, \bar{n}); \Lambda^0 \oplus \Lambda_+^2)$ gives an isomorphism for any $\bar{n} = \{n_1, \dots, n_{l-1}, 1\}$.*

Proof : One has shown the result for $l = 2$. Suppose the result is true for $l \le l_0$. Let us put $\bar{n} = \{n_1, \dots, n_{l_0}, 1\}$. Then by Fourier Laplace transform and excision method used before, one knows $P_\tau : W_\tau^{k+1}(Y(S, T_{\bar{n}}); \Lambda^1) \mapsto W_\tau^{k+1}(Y(S, T_{\bar{n}}); \Lambda^0 \oplus \Lambda_+^2)$ gives a closed operator with $H^*(AHS) = 0$ for $* = 0, 1$. One may follow the same process to see $H^2(AHS) = 0$ as $8.C$. Thus one has shown the result for $l_0 + 1$. This completes the induction step.

For a practical application, one considers open four manifolds with 0 handles attatched with Casson handles. Recall that $S \equiv k(S^2 \times S^2) \setminus$ pt is homotopy equivalent to some wedges of S^2. In particular $H^1_{\mathrm{cp}}(S:\mathbf{R}) = H^3(S:\mathbf{R}) = 0$. Recall also that S has a link picture by k disjoint union of Hopf links with 0 framings. Thus $(W_t, M_1, \dots, M_{2k}) = (D^4, S^1 \times D^2, \dots, S^1 \times D^2)$ expresses the link diagram of $k(S^2 \times S^2)$ (see section 4).

Let $T_1, \dots, T_{2k}$ be signed homogeneous trees of finite type. Let us take $(D^4, S^1 \times D^2, \dots, S^1 \times D^2)$ as above, which represents $k(S^2 \times S^2) \setminus$ pt. Then as before, one may construct a complete Riemannian manifold $\tilde{W}_t$. Then one considers:

$$S \equiv \tilde{W}_t \natural \cup_l CH(T_l).$$

As before one can equip a weight function τ, and the AHS complex over S.

Corollary 8.3 *The differential of AHS complex has closed range over S with $H^0(AHS) = H^1(AHS) = 0$ and $dim H^2(AHS) \leq 2\ dim H^2(S:\mathbf{R})$.*

The diffeomorphism types of S as above, appears very often as closed four manifolds minus a topological disk (see [K]).

Lemma 8.3 (F) *The end of S admits a topological color, $\cong S^3 \times [0, \infty)$.*

Now we have completed to verify that any open four manifold with tree-like end of finite type can admit an admissible pair (g, τ) on it.

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