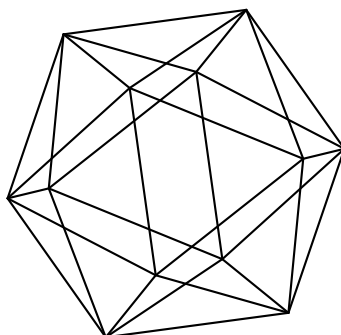


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by

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Abstract: We study the homotopy category $\mathbf{hef}(R, W)$ (and its \mathbb{Z}_2 -graded version $\mathbf{HEF}(R, W)$) of elementary factorizations, where R is a Bézout domain which has prime elements and $W = W_0 W_c$, where $W_0 \in R^\times$ is a square-free element of R and $W_c \in R^\times$ is a finite product of primes with order at least two. In this situation, we give criteria for detecting isomorphisms in $\mathbf{hef}(R, W)$ and $\mathbf{HEF}(R, W)$ and formulas for the number of isomorphism classes of objects. We also study the full subcategory $\mathbf{hef}(R, W)$ of the homotopy category $\mathbf{hmf}(R, W)$ of finite rank matrix factorizations of W which is additively generated by elementary factorizations. We show that $\mathbf{hef}(R, W)$ is Krull-Schmidt and we conjecture that it coincides with $\mathbf{hmf}(R, W)$. Finally, we discuss a few classes of examples.

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Introduction

The study of topological Landau-Ginzburg models [1, 2, 3, 4] often leads to the problem of understanding the triangulated category $\mathbf{hmf}(R, W)$ of finite rank matrix factorizations of an element $W \in R$, where R is a *non-Noetherian* commutative ring. For example, the category of B-type topological D-branes associated to a holomorphic Landau-Ginzburg pair (Σ, W) with Σ a non-compact Riemann surface and $W : \Sigma \rightarrow \mathbb{C}$ a non-constant holomorphic function has this form with $R = \mathcal{O}(\Sigma)$, the non-Noetherian ring of holomorphic functions defined on Σ . When Σ is connected, the ring $\mathcal{O}(\Sigma)$ is a Bézout domain (in fact, an elementary divisor domain). In this situation, this problem can be reduced [5] to the study of the full subcategory $\mathbf{hef}(R, W)$ whose objects are the *elementary factorizations*, defined as those matrix factorizations of W for which

the even and odd components of the underlying supermodule have rank one. In this paper, we study the category $\text{hef}(R, W)$ and the full category $\mathbf{hef}(R, W)$ of $\text{hmf}(R, W)$ which is additively generated by elementary matrix factorizations, for the case when R is a Bézout domain. We say that W is *critically-finite* if it is a product of a square-free element W_0 of R with an element $W_c \in R$ which can be written as a finite product of primes of multiplicities strictly greater than one. When W is critically-finite, the results of this paper provide a detailed description of the categories $\text{hef}(R, W)$ and $\mathbf{hef}(R, W)$, reducing questions about them to the divisibility theory of R .

The paper is organized as follows. In Section 1, we recall some basic facts about finite rank matrix factorizations over unital commutative rings and introduce notation and terminology which will be used later on. In Section 2, we study the category $\text{hef}(R, W)$ and its \mathbb{Z}_2 -graded completion $\text{HEF}(R, W)$ when W is any non-zero element of R , describing these categories in terms of the lattice of divisors of W and giving criteria for deciding when two objects are isomorphic. We also study the behavior of these categories under localization at a multiplicative set as well as their subcategories of primary matrix factorizations. In Section 3, we show that the additive category $\mathbf{hef}(R, W)$ is Krull-Schmidt when R is a Bézout domain and W is a critically-finite element of R and propose a few conjectures about $\text{hmf}(R, W)$. In Section 4, we give a formula for the number of isomorphism classes in the categories $\text{HEF}(R, W)$ and $\text{hef}(R, W)$. Finally, Section 5 discusses a few classes of examples. Appendices A and B collect some information on greatest common divisor (GCD) domains and Bézout domains.

Notations and conventions. The symbols $\hat{0}$ and $\hat{1}$ denote the two elements of the field $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, where $\hat{0}$ is the zero element. Unless otherwise specified, all rings considered are unital and commutative. Given a cancellative Abelian monoid (M, \cdot) , we say that an element $x \in M$ divides $y \in M$ if there exists $q \in M$ such that $y = qx$. In this case, q is uniquely determined by x and y and we denote it by $q = y/x$ or $\frac{y}{x}$.

Let R be a unital commutative ring. The set of non-zero elements of R is denoted by $R^\times \stackrel{\text{def}}{=} R \setminus \{0\}$, while the multiplicative group of units of R is denoted by $U(R)$. The Abelian categories of all R -modules is denoted Mod_R , while the Abelian category of finitely-generated R -modules is denoted mod_R . Let $\text{Mod}_R^{\mathbb{Z}_2}$ denote the category of \mathbb{Z}_2 -graded modules and outer (i.e. even) morphisms of such and $\underline{\text{Mod}}_R^{\mathbb{Z}_2}$ denote the category of \mathbb{Z}_2 -graded modules and inner morphisms of such. By definition, an *R -linear category* is a category enriched in the monoidal category Mod_R while a *\mathbb{Z}_2 -graded R -linear category* is a category enriched in the monoidal category $\underline{\text{Mod}}_R^{\mathbb{Z}_2}$. With this definition, a linear category is pre-additive, but it need not admit finite bi-products (direct sums). For any \mathbb{Z}_2 -graded R -linear category \mathcal{C} , the *even subcategory* $\mathcal{C}^{\hat{0}}$ is the R -linear category obtained from \mathcal{C} by keeping only the even morphisms.

For any unital integral domain R , let \sim denote the equivalence relation defined on R^\times by association in divisibility:

$$x \sim y \text{ iff } \exists \gamma \in U(R) : y = \gamma x \quad .$$

The set of equivalence classes of this relation coincides with the set $R^\times/U(R)$ of orbits for the obvious multiplicative action of $U(R)$. Since R is a commutative domain, the quotient $R^\times/U(R)$ inherits a multiplicative structure of cancellative Abelian monoid. For any $x \in R^\times$, let $(x) \in R^\times/U(R)$ denote the equivalence class of x under \sim . Then for any $x, y \in R^\times$, we have $(xy) = (x)(y)$. The monoid $R^\times/U(R)$ can also be described as follows. Let $G_+(R)$ be the set of non-zero principal ideals of R . If x, y are elements of R^\times , we have $\langle x \rangle \langle y \rangle = \langle xy \rangle$, so the product of principal ideals corresponds to the product of the multiplicative group R^\times and makes $G_+(R)$ into a cancellative Abelian monoid with unit $\langle 1 \rangle = R$. Notice that $G_+(R)$ coincides with the positive cone of the group of divisibility (see Subsection 5.2) $G(R)$ of R , when

the latter is viewed as an Abelian group ordered by reverse inclusion. The monoids $R^\times/U(R)$ and $G_+(R)$ can be identified as follows. For any $x \in R^\times$, let $\langle x \rangle \in G_+(R)$ denote the principal ideal generated by x . Then $\langle x \rangle$ depends only on (x) and will also be denoted by $\langle (x) \rangle$. This gives a group morphism $\langle \cdot \rangle : R^\times/U(R) \rightarrow G_+(R)$. For any non-zero principal ideal $I \in G_+(R)$, the set of all generators x of I is a class in $R^\times/U(R)$ which we denote by (I) ; this gives a group morphism $(\cdot) : G_+(R) \rightarrow R^\times/U(R)$. For all $x \in R^\times$, we have $(\langle x \rangle) = (x)$ and $\langle (x) \rangle = \langle x \rangle$, which implies that $\langle \cdot \rangle$ and (\cdot) are mutually inverse group isomorphisms.

If R is a GCD domain (see Appendix A) and x_1, \dots, x_n are elements of R such that $x_1 \dots x_n \neq 0$, let d be any greatest common divisor (gcd) of x_1, \dots, x_n . Then d is determined by x_1, \dots, x_n up to association in divisibility and we denote its equivalence class by $(x_1, \dots, x_n) \in R^\times/U(R)$. The principal ideal $\langle d \rangle = \langle (x_1, \dots, x_n) \rangle \in G_+(R)$ does not depend on the choice of d . The elements x_1, \dots, x_n also have a least common multiple (lcm) m , which is determined up to association in divisibility and whose equivalence class we denote by $[x_1, \dots, x_n] \in R^\times/U(R)$. For $n = 2$, we have:

$$[x_1, x_2] = \frac{(x_1)(x_2)}{(x_1, x_2)} .$$

If R is a Bézout domain (see Appendix B), then we have $\langle x_1, \dots, x_n \rangle \stackrel{\text{def.}}{=} \langle (x_1, \dots, x_n) \rangle = \langle x_1 \rangle + \dots + \langle x_n \rangle$, so the gcd operation transfers the operation given by taking the finite sum of principal ideals from $G_+(R)$ to $R^\times/U(R)$ through the isomorphism of groups described above. In this case, we have $(x_1, \dots, x_n) = \langle (x_1, \dots, x_n) \rangle$. We also have $\langle [x_1, \dots, x_n] \rangle = \bigcap_{i=1}^n \langle x_i \rangle$ and hence $[x_1, \dots, x_n] = (\bigcap_{i=1}^n \langle x_i \rangle)$. Thus the lcm corresponds to the finite intersection of principal ideals.

1. Matrix factorizations over an integral domain

Let R be an integral domain and $W \in R^\times$ be a non-zero element of R .

1.1. Categories of matrix factorizations. We shall use the following notations:

1. $\text{MF}(R, W)$ denotes the R -linear and \mathbb{Z}_2 -graded differential category of R -valued matrix factorizations of W of finite rank. The objects of this category are pairs $a = (M, D)$, where M is a free \mathbb{Z}_2 -graded R -module of finite rank and D is an odd endomorphism of M such that $D^2 = \text{Wid}_M$. For any objects $a_1 = (M_1, D_1)$ and $a_2 = (M_2, D_2)$ of $\text{MF}(R, W)$, the \mathbb{Z}_2 -graded R -module of morphisms from a_1 to a_2 is given by the inner Hom:

$$\text{Hom}_{\text{MF}(R, W)}(a_1, a_2) = \underline{\text{Hom}}_R(M_1, M_2) = \text{Hom}_{\hat{0}}(M_1, M_2) \oplus \text{Hom}_{\hat{1}}(M_1, M_2) ,$$

endowed with the differential \mathfrak{d}_{a_1, a_2} determined uniquely by the condition:

$$\mathfrak{d}_{a_1, a_2}(f) = D_2 \circ f - (-1)^\kappa f \circ D_1 , \quad \forall f \in \text{Hom}_R^\kappa(M_1, M_2) ,$$

where $\kappa \in \mathbb{Z}_2$.

2. $\text{ZMF}(R, W)$ denotes the R -linear and \mathbb{Z}_2 -graded cocycle category of $\text{MF}(R, W)$. This has the same objects as $\text{MF}(R, W)$ but morphism spaces given by:

$$\text{Hom}_{\text{ZMF}(R, W)}(a_1, a_2) \stackrel{\text{def.}}{=} \{f \in \text{Hom}_{\text{MF}(R, W)}(a_1, a_2) \mid \mathfrak{d}_{a_1, a_2}(f) = 0\} .$$

3. $\text{BMF}(R, W)$ denotes the R -linear and \mathbb{Z}_2 -graded coboundary category of $\text{MF}(R, W)$, which is an ideal in $\text{ZMF}(R, W)$. This has the same objects as $\text{MF}(R, W)$ but morphism spaces given by:

$$\text{Hom}_{\text{BMF}(R, W)}(a_1, a_2) \stackrel{\text{def.}}{=} \{\mathfrak{d}_{a_1, a_2}(f) \mid f \in \text{Hom}_{\text{MF}(R, W)}(a_1, a_2)\} .$$

4. $\text{HMF}(R, W)$ denotes the R -linear and \mathbb{Z}_2 -graded total cohomology category of $\text{MF}(R, W)$. This has the same objects as $\text{MF}(R, W)$ but morphism spaces given by:

$$\text{Hom}_{\text{HMF}(R, W)}(a_1, a_2) \stackrel{\text{def.}}{=} \text{Hom}_{\text{ZMF}(R, W)}(a_1, a_2) / \text{Hom}_{\text{BMF}(R, W)}(a_1, a_2) .$$

5. The subcategories of $\text{MF}(R, W)$, $\text{ZMF}(R, W)$, $\text{BMF}(R, W)$ and $\text{HMF}(R, W)$ obtained by restricting to morphisms of even degree are denoted respectively by $\text{mf}(R, W) \stackrel{\text{def.}}{=} \text{MF}^{\hat{0}}(R, W)$, $\text{zmf}(R, W) \stackrel{\text{def.}}{=} \text{ZMF}^{\hat{0}}(R, W)$, $\text{bmf}(R, W) \stackrel{\text{def.}}{=} \text{BMF}^{\hat{0}}(R, W)$ and $\text{hmf}(R, W) \stackrel{\text{def.}}{=} \text{HMF}^{\hat{0}}(R, W)$.

The categories $\text{MF}(R, W)$, $\text{BMF}(R, W)$ and $\text{ZMF}(R, W)$ admit double direct sums (and hence all finite direct sums of at least two elements) but do not have zero objects. On the other hand, the category $\text{HMF}(R, W)$ is additive, the matrix factorization $\begin{bmatrix} 0 & 1 \\ W & 0 \end{bmatrix}$ being a zero object. Finally, it is well-known that the category $\text{hmf}(R, W)$ is triangulated (see [6] for a detailed treatment).

For later reference, recall that the biproduct (direct sum) of $\text{MF}(R, W)$ is defined as follows:

Definition 1.1 *Given two matrix factorizations $a_i = (M_i, D_i)$, ($i = 1, 2$) of $W \in R$, their direct sum $a_1 \oplus a_2$ is the matrix factorization $a = (M, D)$ of W , where $M \stackrel{\text{def.}}{=} M^{\hat{0}} \oplus M^{\hat{1}}$ and $D \stackrel{\text{def.}}{=} \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, with:*

$$M^\kappa = M_1^\kappa \oplus M_2^\kappa \quad \forall \kappa \in \mathbb{Z}_2 \quad \text{and} \quad u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} .$$

Given a third matrix factorization $a_3 = (M_3, D_3)$ of W and two morphisms $f_i \in \text{Hom}_{\text{MF}(R, W)}(a_i, a_3) = \underline{\text{Hom}}_R(a_i, a_3)$ ($i = 1, 2$) in $\text{MF}(R, W)$, their direct sum of $f_1 \oplus f_2 \in \text{Hom}_{\text{MF}(R, W)}(a_1 \oplus a_2, a_3) = \underline{\text{Hom}}_R(a_1 \oplus a_2, a_3)$ is the ordinary direct sum of the R -module morphisms f_1 and f_2 .

As a consequence, $\text{MF}(R, W)$ admits all finite but non-empty direct sums. The following result is elementary:

Lemma 1.2 *The following statements hold:*

1. *The subcategories $\text{ZMF}(R, W)$ and $\text{BMF}(R, W)$ of $\text{MF}(R, W)$ are closed under finite direct sums (but need not have zero objects).*
2. *The direct sum induces a well-defined biproduct (which is again denoted by \oplus) on the R -linear categories $\text{HMF}(R, W)$ and $\text{hmf}(R, W)$.*
3. *$(\text{HMF}(R, W), \oplus)$ and $(\text{hmf}(R, W), \oplus)$ are additive categories, a zero object in each being given by any of the elementary factorizations e_1 and e_W , which are isomorphic to each other in $\text{hmf}(R, W)$. In particular, any finite direct sum of the elementary factorizations e_1 and e_W is a zero object in $\text{HMF}(R, W)$ and in $\text{hmf}(R, W)$.*

1.2. Reduced rank and matrix description. Let $a = (M, D)$ be an object of $\text{MF}(R, W)$, where $M = M^{\hat{0}} \oplus M^{\hat{1}}$. Taking the supertrace in the equation $D^2 = \text{Wid}_M$ and using the fact that $W \neq 0$ shows that $\text{rk}M^{\hat{0}} = \text{rk}M^{\hat{1}}$. We call this natural number the *reduced rank* of a and denote it by $\rho(a)$; we have $\text{rk}M = 2\rho(a)$. Choosing a homogeneous basis of M (i.e. a basis of $M^{\hat{0}}$ and a basis of $M^{\hat{1}}$) gives an isomorphism of R -supermodules $M \simeq R^{\rho|_{\rho}}$, where $\rho = \rho(a)$ and $R^{\rho|_{\rho}}$ denotes the R -supermodule with \mathbb{Z}_2 -homogeneous components $(R^{\rho|_{\rho}})^{\hat{0}} = (R^{\rho|_{\rho}})^{\hat{1}} = R^{\oplus \rho}$. This isomorphism allows us to identify D with a square matrix of size $2\rho(a)$ which has block off-diagonal form:

$$D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix} ,$$

where u and v are square matrices of size $\rho(a)$ with entries in R . The condition $D^2 = \text{Wid}_M$ amounts to the relations:

$$uv = vu = WI_{\rho} , \quad (1.1)$$

where I_{ρ} denotes the identity matrix of size ρ . Since $W \neq 0$, these conditions imply that the matrices u and v have maximal rank¹:

$$\text{rk}u = \text{rk}v = \rho .$$

Matrix factorizations for which $M = R^{\rho|_{\rho}}$ form a dg subcategory of $\text{MF}(R, W)$ which is *essential* in the sense that it is dg-equivalent with $\text{MF}(R, W)$. Below, we often tacitly identify $\text{MF}(R, W)$ with this essential subcategory and use similar identifications for $\text{ZMF}(R, W)$, $\text{BMF}(R, W)$ and $\text{HMF}(R, W)$.

Given two matrix factorizations $a_1 = (R^{\rho_1|_{\rho_1}}, D_1)$ and $a_2 = (R^{\rho_2|_{\rho_2}}, D_2)$ of W , write $D_i = \begin{bmatrix} 0 & v_i \\ u_i & 0 \end{bmatrix}$, with $u_i, v_i \in \text{Mat}(\rho_i, \rho_i, R)$. Then:

- An even morphism $f \in \text{Hom}_{\text{MF}(R, W)}^{\hat{0}}(a_1, a_2)$ has the matrix form:

$$f = \begin{bmatrix} f_{\hat{0}\hat{0}} & 0 \\ 0 & f_{\hat{1}\hat{1}} \end{bmatrix}$$

with $f_{\hat{0}\hat{0}}, f_{\hat{1}\hat{1}} \in \text{Mat}(\rho_1, \rho_2, R)$ and we have:

$$\mathfrak{d}_{a_1, a_2}(f) = D_2 \circ f - f \circ D_1 = \begin{bmatrix} 0 & v_2 \circ f_{\hat{1}\hat{1}} - f_{\hat{0}\hat{0}} \circ v_1 \\ u_2 \circ f_{\hat{0}\hat{0}} - f_{\hat{1}\hat{1}} \circ u_1 & 0 \end{bmatrix} ;$$

- An odd morphism $g \in \text{Hom}_{\text{MF}(R, W)}^{\hat{1}}(a_1, a_2)$ has the matrix form:

$$g = \begin{bmatrix} 0 & g_{\hat{1}\hat{0}} \\ g_{\hat{0}\hat{1}} & 0 \end{bmatrix}$$

with $g_{\hat{1}\hat{0}}, g_{\hat{0}\hat{1}} \in \text{Mat}(\rho_1, \rho_2, R)$ and we have:

$$\mathfrak{d}_{a_1, a_2}(g) = D_2 \circ g + g \circ D_1 = \begin{bmatrix} v_2 \circ g_{\hat{0}\hat{1}} + g_{\hat{1}\hat{0}} \circ u_1 & 0 \\ 0 & u_2 \circ g_{\hat{1}\hat{0}} + g_{\hat{0}\hat{1}} \circ v_1 \end{bmatrix} .$$

¹ To see this, it suffices to consider equations (1.1) in the field of fractions of R .

Remark 1.1. The cocycle condition $\mathfrak{d}_{a_1, a_2}(f) = 0$ satisfied by an even morphism $f \in \text{Hom}_{\text{ZMF}(R, W)}^{\hat{0}}(a_1, a_2)$ amounts to the system:

$$\begin{cases} v_2 \circ f_{\hat{1}\hat{1}} = f_{\hat{0}\hat{0}} \circ v_1 \\ u_2 \circ f_{\hat{0}\hat{0}} = f_{\hat{1}\hat{1}} \circ u_1 \end{cases} ,$$

which in turn amounts to any of the following equivalent conditions:

$$f_{\hat{1}\hat{1}} = \frac{u_2 \circ f_{\hat{0}\hat{0}} \circ v_1}{W} \iff f_{\hat{0}\hat{0}} = \frac{v_2 \circ f_{\hat{1}\hat{1}} \circ u_1}{W} .$$

Similarly, the cocycle condition $\mathfrak{d}_{a_1, a_2}(g) = 0$ defining an odd morphism $g \in \text{Hom}_{\text{ZMF}(R, W)}^{\hat{1}}(a_1, a_2)$ amounts to the system:

$$\begin{cases} v_2 \circ g_{\hat{0}\hat{1}} + g_{\hat{1}\hat{0}} \circ u_1 = 0 \\ u_2 \circ g_{\hat{1}\hat{0}} + g_{\hat{0}\hat{1}} \circ v_1 = 0 \end{cases} ,$$

which in turn amounts to any of the following equivalent conditions:

$$g_{\hat{1}\hat{0}} = -\frac{v_2 \circ g_{\hat{0}\hat{1}} \circ v_1}{W} \iff g_{\hat{0}\hat{1}} = -\frac{u_2 \circ g_{\hat{1}\hat{0}} \circ u_1}{W} .$$

1.3. Strong isomorphism. Recall that $\text{zmf}(R, W)$ denotes the even subcategory of $\text{ZMF}(R, W)$. This category admits non-empty finite direct sums but does not have a zero object.

Definition 1.3 *Two matrix factorizations a_1 and a_2 of W over R are called strongly isomorphic if they are isomorphic in the category $\text{zmf}(R, W)$.*

It is clear that two strongly isomorphic factorizations are also isomorphic in $\text{hmf}(R, W)$, but the converse need not hold.

Proposition 1.4 *Let $a_1 = (R^{\rho_1 | \rho_1}, D_1)$ and $a_2 = (R^{\rho_2 | \rho_2}, D_2)$ be two matrix factorizations of W over R , where $D_i = \begin{bmatrix} 0 & v_i \\ u_i & 0 \end{bmatrix}$. Then the following statements are equivalent:*

- (a) a_1 and a_2 are strongly isomorphic.
- (b) $\rho_1 = \rho_2$ (a quantity which we denote by ρ) and there exist invertible matrices $A, B \in \text{GL}(\rho, R)$ such that one (and hence both) of the following equivalent conditions is satisfied:

1. $v_2 = Av_1B^{-1}$,
2. $u_2 = Bu_1A^{-1}$.

Proof. a_1 and a_2 are strongly isomorphic iff there exists $U \in \text{Hom}_{\text{zmf}(R, W)}(a_1, a_2)$ which is an isomorphism in $\text{zmf}(R, W)$. Since U is an even morphism in the cocycle category, we have:

$$UD_1 = D_2U . \tag{1.2}$$

The condition that U be even allows us to identify it with a matrix of the form $U = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, while invertibility of U in $\text{zmf}(R, W)$ amounts to invertibility of the matrix U , which in turn means that A and B are square matrices (thus $\rho_1 = \rho_2 = \rho$) belonging to $\text{GL}(\rho, R)$. Thus relation (1.2) reduces to either of conditions 1. or 2., which are equivalent since $v_1u_1 = u_1v_1 = WI_\rho$ and $u_2v_2 = v_2u_2 = WI_\rho$. \square

1.4. Critical divisors and the critical locus of W .

Definition 1.5 A divisor d of W which is not a unit is called critical if $d^2|W$.

Let:

$$\mathfrak{C}(W) \stackrel{\text{def.}}{=} \{d \in R \mid d^2|W\}$$

be the set of all critical divisors of W . The ideal:

$$\mathfrak{I}_W \stackrel{\text{def.}}{=} \bigcap_{d \in \mathfrak{C}(W)} \langle d \rangle \tag{1.3}$$

is called the *critical ideal* of W . Notice that \mathfrak{I}_W consists of those elements of R which are divisible by all critical divisors of W . In particular, we have $(W) \subset \mathfrak{I}_W$ and hence there exists a unital ring epimorphism $R/(W) \rightarrow R/\mathfrak{I}_W$.

Definition 1.6 A critical prime divisor of W is a prime element $p \in R$ such that $p^2|W$. The critical locus of W is the subset of $\text{Spec}(R)$ consisting of the principal prime ideals of R generated by the critical prime divisors of W :

$$\text{Crit}(W) \stackrel{\text{def.}}{=} \{ \langle p \rangle \in \text{Spec}(R) \mid p^2|W \} .$$

1.5. Critically-finite elements. Let R be a Bézout domain. Then R is a GCD domain, hence irreducible elements of R are prime. This implies that any factorizable element² of R has a unique prime factorization up to association in divisibility.

Definition 1.7 A non-zero non-unit W of R is called:

- non-critical, if W has no critical divisors;
- critically-finite if it has a factorization of the form:

$$W = W_0 W_c \quad \text{with} \quad W_c = p_1^{n_1} \dots p_N^{n_N} , \tag{1.4}$$

where $n_j \geq 2$, p_1, \dots, p_N are critical prime divisors of W (with $p_i \not\sim p_j$ for $i \neq j$) and W_0 is non-critical and coprime with W_c .

Notice that the elements W_0 , W_c and p_i in the factorization (1.4) are determined by W up to association, while the integers n_i are uniquely determined by W . The factors W_0 and W_c are called respectively the *non-critical* and *critical* parts of W . The integers $n_i \geq 2$ are called the *orders* of the critical prime divisors p_i .

For a critically-finite element W with decomposition (1.4), we have:

$$\text{Crit}(W) = \{ \langle p_1 \rangle, \dots, \langle p_N \rangle \} \quad \text{and} \quad \mathfrak{I}_W = \langle W_{\text{red}} \rangle ,$$

where³:

$$W_{\text{red}} \stackrel{\text{def.}}{=} p_1^{\lfloor \frac{n_1}{2} \rfloor} \dots p_N^{\lfloor \frac{n_N}{2} \rfloor}$$

is called the *reduction* of W . Notice that W_{red} is determined up to association in divisibility.

² I.e. an element of R which has a finite factorization into irreducibles.

³ The notation $\lfloor x \rfloor \in \mathbb{Z}$ indicates the integral part of a real number $x \in \mathbb{R}$.

1.6. Two-step factorizations of W . Recall that a *two-step factorization* (or two-step multiplicative partition) of W is an ordered pair $(u, v) \in R \times R$ such that $W = uv$. In this case, the divisors u and v are called *W -conjugate*. The *transpose* of (u, v) is the ordered pair (v, u) (which is again a two-step factorization of W), while the *opposite transpose* is the ordered pair $\sigma(u, v) = (-v, -u)$. This defines an involution σ of the set $\text{MP}_2(W)$ of two-step factorizations of W . The two-step factorizations (u, v) and (u', v') are called *similar* (and we write $(u, v) \sim (u', v')$) if there exists $\gamma \in U(R)$ such that $u' = \gamma u$ and $v' = \gamma^{-1}v$. We have $\sigma(u, v) \sim (v, u)$.

Definition 1.8 *The support of a two-step factorization (u, v) of W is the principal ideal $\langle u, v \rangle \in G_+(R)$.*

Let d be a gcd of u and v . Since $W = uv = d^2 u_1 v_1$ (where $u_1 \stackrel{\text{def.}}{=} u/d$, $v_1 \stackrel{\text{def.}}{=} v/d$), it is clear that d is a critical divisor of W . Notice that the opposite transpose of the two step factorization (u, v) has the same support as (u, v) .

1.7. Elementary matrix factorizations.

Definition 1.9 *A matrix factorization $a = (M, D)$ of W over R is called elementary if it has unit reduced rank, i.e. if $\rho(a) = 1$.*

Any elementary factorization is strongly isomorphic to one of the form $e_v \stackrel{\text{def.}}{=} (R^{1|1}, D_v)$, where v is a divisor of W and $D_v \stackrel{\text{def.}}{=} \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, with $u \stackrel{\text{def.}}{=} W/v \in R$. Let $\text{EF}(R, W)$ denote the full subcategory of $\text{MF}(R, W)$ whose objects are the elementary factorizations of W over R . Let $\text{ZEF}(R, W)$ and $\text{HEF}(R, W)$ denote respectively the cocycle and total cohomology categories of $\text{EF}(R, W)$. We also use the notations $\text{zef}(R, W) \stackrel{\text{def.}}{=} \text{ZEF}^{\hat{0}}(R, W)$ and $\text{hef}(R, W) \stackrel{\text{def.}}{=} \text{HEF}^{\hat{0}}(R, W)$. Notice that an elementary factorization is indecomposable in $\text{zmf}(R, W)$, but it need not be indecomposable in the triangulated category $\text{hmf}(R, W)$.

The map $\Phi : \text{ObEF}(M, W) \rightarrow \text{MP}_2(W)$ which sends e_v to the ordered pair (u, v) is a bijection. The suspension of e_v is given by $\Sigma e_v = e_{-u} = (R^{1|1}, D_{-u})$, since:

$$D_{-u} = \begin{bmatrix} 0 & -u \\ -v & 0 \end{bmatrix} .$$

In particular, Σe_v corresponds to the opposite transpose $\sigma(u, v)$ and we have:

$$\Phi \circ \Sigma = \sigma \circ \Phi .$$

Hence Σ preserves the subcategory $\text{EF}(M, W)$ of $\text{MF}(R, W)$ and the subcategories $\text{HEF}(R, W)$ and $\text{hef}(R, W)$ of $\text{HMF}(R, W)$ and $\text{hmf}(R, W)$. This implies that $\text{HEF}(R, W)$ is equivalent with the graded completion $\text{gr}_{\Sigma} \text{hef}(R, W)$. We thus have natural isomorphisms:

$$\begin{aligned} \text{Hom}_{\hat{1}\text{HEF}(R, W)}(e_{v_1}, e_{v_2}) &\simeq_R \text{Hom}_{\text{hef}(R, W)}(e_{v_1}, \Sigma e_{v_2}) = \text{Hom}_{\text{hef}(R, W)}(e_{v_1}, e_{-u_2}) , \\ \text{Hom}_{\hat{1}\text{HEF}(R, W)}(e_{v_1}, e_{v_2}) &\simeq_R \text{Hom}_{\text{hef}(R, W)}(\Sigma e_{v_1}, e_{v_2}) = \text{Hom}_{\text{hef}(R, W)}(e_{-u_1}, e_{v_2}) , \end{aligned} \quad (1.5)$$

for any divisors v_1, v_2 of W , where $u_1 = W/v_1$ and $u_2 = W/v_2$.

Definition 1.10 *The support of an elementary matrix factorization e_v is the ideal of R defined through:*

$$\text{supp}(e_v) \stackrel{\text{def.}}{=} \text{supp}(\Phi(e_v)) = \langle v, W/v \rangle .$$

Notice that this ideal is generated by any gcd d of v and W/v and that d is a critical divisor of W .

We will see later that an elementary factorization is trivial iff its support equals R .

Definition 1.11 *Two elementary matrix factorizations e_{v_1} and e_{v_2} of W are called similar if $v_1 \sim v_2$ or equivalently $u_1 \sim u_2$. This amounts to existence of a unit $\gamma \in U(R)$ such that $v_2 = \gamma v_1$ and $u_2 = \gamma^{-1} u_1$.*

Proposition 1.12 *Two elementary factorizations e_{v_1} and e_{v_2} are strongly isomorphic iff they are similar. In particular, strong isomorphism classes of elementary factorization are in bijection with the set of those principal ideals of R which contain W .*

Proof. Suppose that e_{v_1} and e_{v_2} are strongly isomorphic. By Proposition 1.4, there exist units $x, y \in U(R)$ such that $v_2 = xv_1y^{-1}$ and $u_2 = yu_1x^{-1}$, where $u_i \stackrel{\text{def.}}{=} W/v_i$. Setting $\gamma \stackrel{\text{def.}}{=} xy^{-1}$ gives $v_2 = \gamma v_1$ and $u_2 = \gamma^{-1} u_1$, hence e_{v_1} and e_{v_2} are similar. Conversely, suppose that $e_{v_1} \sim e_{v_2}$. Then there exists a unit $\gamma \in U(R)$ such that $v_2 = \gamma v_1$ and $u_2 = \gamma^{-1} u_1$. Setting $x = \gamma$ and $y = 1$ gives $v_2 = xv_1y^{-1}$ and $u_2 = yu_1x^{-1}$, which shows that e_{v_1} and e_{v_2} are strongly isomorphic upon using Proposition 1.4. The map which sends the strong isomorphism class of e_v to the principal ideal (v) gives the bijection stated. \square

It is clear that e_{v_1} and e_{v_2} are similar iff the corresponding two-step factorizations (v_1, u_1) and (v_2, u_2) of W are similar. Since any strong isomorphism induces an isomorphism in $\text{hef}(R, W)$, it follows that similar elementary factorizations are isomorphic in $\text{hef}(R, W)$.

1.8. The categories $\mathbf{HEF}(R, W)$ and $\mathbf{hef}(R, W)$. Let $\mathbf{EF}(R, W)$ denote the smallest full R -linear subcategory of $\mathbf{MF}(R, W)$ which contains all objects of $\mathbf{EF}(R, W)$ and is closed under finite direct sums. It is clear that $\mathbf{EF}(R, W)$ is a full dg subcategory of $\mathbf{MF}(R, W)$. Let $\mathbf{HEF}(R, W)$ denote the total cohomology category of $\mathbf{EF}(R, W)$. Let $\mathbf{hef}(R, W) \stackrel{\text{def.}}{=} \mathbf{HEF}^{\hat{0}}(R, W)$ denote the subcategory obtained from $\mathbf{HEF}(R, W)$ by keeping only the even morphisms. Notice that $\mathbf{hef}(R, W)$ coincides with the smallest full subcategory of $\mathbf{hmf}(R, W)$ which contains all elementary factorizations of W .

2. Elementary matrix factorizations over a Bézout domain

Throughout this section, let R be a Bézout domain and W be a non-zero element of R .

2.1. The subcategory of elementary factorizations. Let v_1, v_2 be divisors of W and $e_1 := e_{v_1}$, $e_2 := e_{v_2}$ be the corresponding elementary matrix factorizations of W . Let $u_1 \stackrel{\text{def.}}{=} W/v_1$, $u_2 = W/v_2$. Let a be a gcd of v_1 and v_2 . Define:

$$b \stackrel{\text{def.}}{=} v_1/a , c \stackrel{\text{def.}}{=} v_2/a , d \stackrel{\text{def.}}{=} \frac{W}{abc} , a' \stackrel{\text{def.}}{=} a/s , d' \stackrel{\text{def.}}{=} d/s , \quad (2.1)$$

where s is a gcd of a and d . Then $a = a's$ and $d = d's$ with $(a', d') = (1) = (b, c)$ and $W = abcd = s^2 a' b c d'$. In particular, s is a critical divisor of W . Moreover:

$$v_1 = ab = sa'b \quad , \quad v_2 = ac = sa'c \quad , \quad u_1 = cd = scd' \quad , \quad u_2 = bd = sbd' \quad (2.2)$$

and we have:

$$(d) = (u_1, u_2) \quad , \quad (s) = (v_1, v_2, u_1, u_2) \quad . \quad (2.3)$$

Notice the following relations in the cancellative monoid $R^\times/U(R)$:

$$\begin{aligned} (v_1, v_2) &= (sa') \quad , \quad (u_1, u_2) = (s)(d') \quad , \quad (u_1, v_1) = (s)(a', c)(b, d') \quad , \\ (u_1, v_2) &= (s)(c) \quad , \quad (u_2, v_1) = (s)(b) \quad , \quad (u_2, v_2) = (s)(a', b)(c, d') \quad . \end{aligned} \quad (2.4)$$

In this notation:

$$D_{v_1} = \begin{bmatrix} 0 & v_1 \\ u_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ cd & 0 \end{bmatrix} = s \begin{bmatrix} 0 & a'b \\ cd' & 0 \end{bmatrix} \quad \text{and} \quad D_{v_2} = \begin{bmatrix} 0 & v_2 \\ u_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ac \\ bd & 0 \end{bmatrix} = s \begin{bmatrix} 0 & a'c \\ bd' & 0 \end{bmatrix} \quad .$$

For $f \in \text{Hom}_{\text{MF}(R,W)}^{\hat{0}}(e_1, e_2) = \text{Hom}_R^{\hat{0}}(R^{1|1}, R^{1|1})$ and $g \in \text{Hom}_{\text{MF}(R,W)}^{\hat{1}}(e_1, e_2) = \text{Hom}_R^{\hat{1}}(R^{1|1}, R^{1|1})$, we have:

$$\mathfrak{d}_{e_1, e_2}(f) = (cf_{\hat{1}\hat{1}} - f_{\hat{0}\hat{0}}b) \begin{bmatrix} 0 & a \\ -d & 0 \end{bmatrix} \quad \text{and} \quad \mathfrak{d}_{e_1, e_2}(g) = (ag_{\hat{0}\hat{1}} + g_{\hat{1}\hat{0}}d) \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \quad . \quad (2.5)$$

Remark 2.1. Relations (2.1) and (2.4) imply the following equalities in the cancellative monoid $R^\times/U(R)$:

$$(s) = \frac{(u_1, v_2)}{\left((u_1, v_2), \frac{v_2}{(v_1, v_2)}\right)} = \frac{(u_2, v_1)}{\left((u_2, v_1), \frac{v_1}{(v_1, v_2)}\right)} \quad . \quad (2.6)$$

2.1.1. Morphisms in HEF(R, W). Let $\text{Mat}(n, R^\times/U(R))$ denote the set of square matrices of size n with entries from the multiplicative semigroup $R/U(R)$. Any matrix $S \in \text{Mat}(n, R^\times/U(R))$ can be viewed as an equivalence class of matrices $A \in \text{Mat}(n, R^\times)$ under the equivalence relation:

$$A \sim_n B \quad \text{iff} \quad \forall i, j \in \{1, \dots, n\} : \exists q_{ij} \in U(R) \text{ such that } B_{ij} = q_{ij} A_{ij} \quad . \quad (2.7)$$

Proposition 2.1 *With the notations above, we have:*

1. $\text{Hom}_{\text{ZMF}(R,W)}^{\hat{0}}(e_1, e_2)$ is the free R -module of rank one generated by the matrix:

$$\epsilon_{\hat{0}}(v_1, v_2) \stackrel{\text{def.}}{=} \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \in \left[\begin{array}{cc} \frac{(v_2)}{(v_1, v_2)} & 0 \\ 0 & \frac{(v_1)}{(v_1, v_2)} \end{array} \right] \stackrel{\text{def.}}{=} \epsilon_{\hat{0}}(v_1, v_2) \quad ,$$

where the matrix $\epsilon_{\hat{0}}(v_1, v_2) \in \text{Mat}(2, R/U(R))$ in the right hand side is viewed as an equivalence class under the relation (2.7).

2. $\text{Hom}_{\text{ZMF}(R,W)}^{\hat{1}}(e_1, e_2)$ is the free R -module of rank one generated by the matrix:

$$\epsilon_{\hat{1}}(v_1, v_2; W) \stackrel{\text{def.}}{=} \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} \in \left[\begin{array}{cc} 0 & \frac{(v_2)}{(u_1, v_2)} \\ -\frac{(u_1)}{(u_1, v_2)} & 0 \end{array} \right] \stackrel{\text{def.}}{=} \epsilon_{\hat{1}}(v_1, v_2; W)$$

and we have $\epsilon_{\hat{1}}(v_1, v_2; W) = \epsilon_{\hat{1}}(v_2, v_1; W)$ in $\text{Mat}(2, R/U(R))$.

Proof. Relations (2.4) imply:

$$\frac{(v_2)}{(v_1, v_2)} = (c), \quad \frac{(v_1)}{(v_1, v_2)} = (b), \quad \frac{(v_2)}{(u_1, v_2)} = \frac{(v_1)}{(u_2, v_1)} = (a'), \quad \frac{(u_1)}{(u_1, v_2)} = \frac{(u_2)}{(u_2, v_1)} = (d'). \quad (2.8)$$

These relations show that $\epsilon_{\hat{0}}(v_1, v_2)$ and $\epsilon_{\hat{1}}(v_1, v_2; W)$ belong to the equivalence classes $\epsilon_{\hat{0}}(v_1, v_2)$ and $\epsilon_{\hat{1}}(v_1, v_2; W)$ and that we have $\epsilon_{\hat{1}}(v_1, v_2; W) = \epsilon_{\hat{1}}(v_2, v_1; W)$.

For an even morphism $f : e_1 \rightarrow e_2$ in $\text{MF}(R, W)$, the first equation in (2.5) shows that the condition $\mathfrak{d}_{e_1, e_2}(f) = 0$ amounts to:

$$f_{\hat{1}\hat{1}}c - f_{\hat{0}\hat{0}}b = 0 \quad .$$

Since b and c are coprime, this condition is equivalent with the existence of an element $\gamma \in R$ such that $f_{\hat{0}\hat{0}} = \gamma c$ and $f_{\hat{1}\hat{1}} = \gamma b$. Thus:

$$f = \gamma \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} = \gamma \epsilon_{\hat{0}}(v_1, v_2) \quad . \quad (2.9)$$

On the other hand, the second equation in (2.5) shows that an odd morphism $g : e_1 \rightarrow e_2$ in $\text{MF}(R, W)$ satisfies $\mathfrak{d}_{e_1, e_2}(g) = 0$ iff:

$$ag_{\hat{0}\hat{1}} + dg_{\hat{1}\hat{0}} = 0 \quad .$$

Since a' and d' are coprime, this condition is equivalent with the existence of an element $\gamma \in R$ such that $g_{\hat{1}\hat{0}} = \gamma a'$ and $g_{\hat{1}\hat{1}} = -\gamma d'$. Thus:

$$g = \gamma \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} = \gamma \epsilon_{\hat{1}}(v_1, v_2; W) \quad . \quad (2.10)$$

□

Proposition 2.2 *Let v_i be as in Proposition 2.1. Then $\text{Hom}_{\text{HMF}(R, W)}^{\hat{0}}(e_1, e_2)$ and $\text{Hom}_{\text{HMF}(R, W)}^{\hat{1}}(e_1, e_2)$ are cyclically presented cyclic R -modules generated respectively by the matrices $\epsilon_{\hat{0}}(v_1, v_2)$ and $\epsilon_{\hat{1}}(v_1, v_2; W)$, whose annihilators are equal to each other and coincide with the following principal ideal of R :*

$$\alpha_W(v_1, v_2) \stackrel{\text{def.}}{=} \langle v_1, u_1, v_2, u_2 \rangle = \langle s \rangle \quad .$$

Proof. Let $f \in \text{Hom}_{\text{ZMF}(R, W)}^{\hat{0}}(e_1, e_2)$. Then f is exact iff there exists an odd morphism $g \in \text{Hom}_{\text{MF}(R, W)}^{\hat{1}}(e_1, e_2)$ such that:

$$f = \mathfrak{d}_{e_1, e_2}(g) = (ag_{\hat{0}\hat{1}} + g_{\hat{1}\hat{0}}d) \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \quad .$$

Comparing this with (2.9), we find that f is exact if and only if $s \in (a, d)$ divides γ . This implies that the principal ideal generated by the element:

$$s \in ((v_1, v_2), (u_1, u_2)) = (v_1, u_1, v_2, u_2)$$

is the annihilator of $\text{Hom}_{\text{ZMF}(R, W)}^{\hat{0}}(e_1, e_2)$.

On the other hand, an odd morphism $g \in \text{Hom}_{\text{ZMF}(R, W)}^{\hat{1}}(e_1, e_2)$ is exact iff there exists an even morphism $f \in \text{Hom}_{\text{MF}(R, W)}^{\hat{0}}(e_1, e_2)$ such that:

$$g = \mathfrak{d}_{e_1, e_2}(f) = (f_{\hat{1}\hat{1}}c - f_{\hat{0}\hat{0}}b) \begin{bmatrix} 0 & a \\ -d & 0 \end{bmatrix} \quad .$$

Comparing with (2.10) and recalling that $(b, c) = (1)$, we find that g is exact iff $(a, d) | \gamma$. Hence the annihilator of $\text{Hom}_{\text{HMF}(R, W)}^{\hat{1}}(e_1, e_2)$ coincides with that of $\text{Hom}_{\text{HMF}(R, W)}^{\hat{0}}(e_1, e_2)$. □

Remark 2.2. Since s is a critical divisor of W , we have $\mathfrak{I}_W \text{Hom}_{\text{HEF}(R,W)}(e_1, e_2) = 0$, where \mathfrak{I}_W denotes the critical ideal of W defined in (1.3). In particular, $\text{HEF}(R, W)$ can be viewed as an R/\mathfrak{I}_W -linear category.

Let $\mathfrak{Div}(W) \stackrel{\text{def.}}{=} \{d \in R \mid d|W\}$ and consider the function $\alpha_W : \mathfrak{Div}(W) \times \mathfrak{Div}(W) \rightarrow G_+(R)$ defined in Proposition 2.2. This function is symmetric since $\alpha_W(v_1, v_2) = \alpha_W(v_2, v_1)$. Let $1_{G(R)} = \langle 1 \rangle = R$ denote the neutral element of the group of divisibility $G(R)$, whose group operation we write multiplicatively.

Proposition 2.3 *The symmetric function $\alpha_W(v_1, v_2)$ is multiplicative with respect to each of its arguments in the following sense:*

- For any two relatively prime elements v_2 and \tilde{v}_2 of R such that $v_2\tilde{v}_2$ is a divisor of W , we have:

$$\alpha_W(v_1, v_2\tilde{v}_2) = \alpha_W(v_1, v_2)\alpha_W(v_1, \tilde{v}_2) \quad (2.11)$$

and $\alpha_W(v_1, v_2) + \alpha_W(v_1, \tilde{v}_2) = 1_{G(R)}$, where $+$ denotes the sum of ideals of R .

- For any two relatively prime elements v_1 and \tilde{v}_1 of R such that $v_1\tilde{v}_1$ is a divisor of W , we have:

$$\alpha_W(v_1\tilde{v}_1, v_2) = \alpha_W(v_1, v_2)\alpha_W(\tilde{v}_1, v_2) \quad (2.12)$$

and $\alpha_W(v_1, v_2) + \alpha_W(\tilde{v}_1, v_2) = 1_{G(R)}$, where $+$ denotes the sum of ideals of R .

Proof. To prove the first statement, we start from relation (2.6), which allows us to write:

$$\alpha_W(v_1, v_2\tilde{v}_2) = \left\langle \frac{(u_1, v_2\tilde{v}_2)}{\left((u_1, v_2\tilde{v}_2), \frac{v_2\tilde{v}_2}{(v_1, v_2\tilde{v}_2)}\right)} \right\rangle, \quad (2.13)$$

where $u_1 = W/v_1$. Recall that the function $(-, r)$ is multiplicative on relatively prime elements for any $r \in R^\times$, i.e. $(xy, r) = (x, r)(y, r)$. Thus:

$$(u_1, v_2\tilde{v}_2) = (u_1, v_2)(u_1, \tilde{v}_2), \quad (v_1, v_2\tilde{v}_2) = (v_1, v_2)(v_1, \tilde{v}_2). \quad (2.14)$$

The second of these relations gives $\frac{(v_2\tilde{v}_2)}{(v_1, v_2\tilde{v}_2)} = \frac{(v_2)}{(v_1, v_2)} \frac{(\tilde{v}_2)}{(v_1, \tilde{v}_2)}$. Notice that $(\frac{(v_2)}{(v_1, v_2)}, \frac{(\tilde{v}_2)}{(v_1, \tilde{v}_2)}) = (1)$ since v_2 and \tilde{v}_2 are coprime. Hence:

$$\begin{aligned} \left((u_1, v_2\tilde{v}_2), \frac{(v_2)(\tilde{v}_2)}{(v_1, v_2\tilde{v}_2)} \right) &= \left((u_1, v_2\tilde{v}_2), \frac{(v_2)}{(v_1, v_2)} \right) \left((u_1, v_2\tilde{v}_2), \frac{(\tilde{v}_2)}{(v_1, \tilde{v}_2)} \right) = \\ &= \left((u_1, v_2), \frac{(v_2)}{(v_1, v_2)} \right) \left((u_1, \tilde{v}_2), \frac{(v_2)}{(v_1, v_2)} \right) \left((u_1, v_2), \frac{(\tilde{v}_2)}{(v_1, \tilde{v}_2)} \right) \left((u_1, \tilde{v}_2), \frac{(\tilde{v}_2)}{(v_1, \tilde{v}_2)} \right), \end{aligned}$$

where in the last equality we used the first relation in (2.14) and noticed that (u_1, v_2) and (u_1, \tilde{v}_2) are coprime (since $(v_2, \tilde{v}_2) = (1)$), which allows us to use similar-multiplicativity of the function $(-, r)$ for $(r) = \frac{(v_2)}{(v_1, v_2)}$ and for $(r) = \frac{(\tilde{v}_2)}{(v_1, \tilde{v}_2)}$. Since $(v_2, \tilde{v}_2) = (1)$, we have $\left((u_1, \tilde{v}_2), \frac{(v_2)}{(v_1, v_2)} \right) = \left((u_1, v_2), \frac{(\tilde{v}_2)}{(v_1, \tilde{v}_2)} \right) = (1)$. Thus:

$$\left((u_1, v_2\tilde{v}_2), \frac{(v_2)(\tilde{v}_2)}{(v_1, v_2\tilde{v}_2)} \right) = \left((u_1, v_2), \frac{(v_2)}{(v_1, v_2)} \right) \left((u_1, \tilde{v}_2), \frac{(\tilde{v}_2)}{(v_1, \tilde{v}_2)} \right).$$

Using this and the first equation of (2.14) in the expression (2.13) gives relation (2.11). The second statement now follows from the first by symmetry of α_W . \square

2.1.2. Isomorphisms in $\text{HEF}(R, W)$.

We start with a few lemmas.

Lemma 2.1. *Let s, x, y, z be four elements of R . Then the equation:*

$$s(g_1x + g_2y) + g_3z = 1 \quad (2.15)$$

has a solution $(g_1, g_2, g_3) \in R^3$ iff $(s(x, y), z) = (1)$.

Proof. Let t be a gcd of x and y . We treat each implication in turn:

1. Assume that $(g_1, g_2, g_3) \in R^3$ is a solution. Then t divides $g_1x + g_2y$, so there exists $g_4 \in R$ such that $g_1x + g_2y = g_4t$. Multiplying both sides with s and using (2.15), this gives $stg_4 + g_3z = 1$, which implies $(st, z) = (1)$.
2. Assume that $(st, z) = (1)$. Then there exist $g_3, g_4 \in R$ such that:

$$stg_4 + g_3z = 1 \quad (2.16)$$

Since $(t) = (x, y)$, the Bézout identity shows that there exist $\tilde{g}_1, \tilde{g}_2 \in R$ such that $\tilde{g}_1x + \tilde{g}_2y = t$. Substituting this into (2.16) shows that (g_1, g_2, g_3) satisfies (2.15), where $g_1 \stackrel{\text{def.}}{=} \tilde{g}_1g_4$ and $g_2 \stackrel{\text{def.}}{=} \tilde{g}_2g_4$. \square

Lemma 2.2. *Let s, a', b, c, d' be five elements of R such that $(a', d') = (1)$. Then the system of equations:*

$$\begin{cases} bcg - s(a'bg_1 + cd'h_2) = 1 \\ bcg - s(a'ch_1 + bd'h_2) = 1 \end{cases} \quad (2.17)$$

has a solution $(g, g_1, g_2, h_1, h_2) \in R^5$ iff a', b, c, d' are pairwise coprime and $(bc, s) = (1)$.

Proof. Consider the two implications in turn.

1. Assume that (2.17) has a solution $(g, g_1, g_2, h_1, h_2) \in R^5$. By Lemma 2.1, we must have $(bc, s(a'b, cd')) = (1)$ and $(bc, s(a'c, bd')) = (1)$. This implies $(bc, s) = (1)$ and $(b, c) = (1)$. If a prime element $p \in R$ divides $(a'b, cd')$, then it divides both $a'b$ and cd' , hence $p|c$ or $p|b$ since $(a', d') = (1)$. Thus $p|bc$, which contradicts the fact that bc and $s(a'b, cd')$ are coprime. It follows that we must have $(a'b, cd') = (1)$. Similarly, the second equation implies that we must have $(a'c, bd') = (1)$. Since $(a', d') = (1)$ and $(b, c) = (1)$, the last two conditions imply that a', b, c, d' must be pairwise coprime.
2. Conversely, assume that a', b, c, d' are pairwise coprime and $(bc, s) = (1)$. Following the strategy and notations of the previous lemma, we first solve the equation $bcg - sg_4 = 1$ for g and g_4 using the Bézout identity. Using the same identity, we solve the system:

$$\begin{cases} a'b\tilde{g}_1 + cd'\tilde{g}_2 = 1 \\ a'c\tilde{h}_1 + bd'\tilde{h}_2 = 1 \end{cases}, \quad (2.18)$$

obtaining the solution $(g, g_4\tilde{g}_1, g_4\tilde{g}_2, g_4\tilde{h}_1, g_4\tilde{h}_2)$ of (2.17). \square

Proposition 2.4 *With the notations (2.1), we have:*

1. e_1 and e_2 are isomorphic in $\text{hef}(R, W)$ iff a', b, c, d' are pairwise coprime and $(bc, s) = (1)$.
2. An odd isomorphism between e_1 and e_2 in $\text{HEF}(R, W)$ exists iff a', b, c, d' are pairwise coprime and $(a'd', s) = (1)$.

Proof. 1. Proposition 2.1 gives:

$$\text{Hom}_{\text{zef}(R, W)}(e_1, e_2) = R \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \quad \text{and} \quad \text{Hom}_{\text{zef}(R, W)}(e_2, e_1) = R \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} .$$

Two non-zero morphisms $f_{12} = \alpha \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \in \text{Hom}_{\text{zef}(R, W)}(e_1, e_2)$ and $f_{21} = \beta \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \in \text{Hom}_{\text{zef}(R, W)}(e_2, e_1)$ (where $\alpha, \beta \in R^\times$) induce mutually inverse isomorphisms in $\text{hef}(R, W)$ iff:

$$f_{21}f_{12} = 1 + \mathfrak{d}_{e_1, e_1}(g) \quad , \quad f_{12}f_{21} = 1 + \mathfrak{d}_{e_2, e_2}(h)$$

for some $g, h \in \text{End}_R^{\hat{1}}(R^{1|1})$. These conditions read:

$$\begin{aligned} \alpha\beta \begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (abg_{\hat{0}\hat{1}} + g_{\hat{1}\hat{0}}cd) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s(a'bg_{\hat{0}\hat{1}} + g_{\hat{1}\hat{0}}cd') \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha\beta \begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (ach_{\hat{0}\hat{1}} + h_{\hat{1}\hat{0}}bd) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s(a'ch_{\hat{0}\hat{1}} + h_{\hat{1}\hat{0}}bd') \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and hence amount to the following system of equations for $\alpha\beta$ and g, h :

$$\begin{cases} \alpha\beta bc - s(a'bg_{\hat{0}\hat{1}} + g_{\hat{1}\hat{0}}cd') = 1 \\ \alpha\beta bc - s(a'ch_{\hat{0}\hat{1}} + h_{\hat{1}\hat{0}}bd') = 1 \end{cases} . \quad (2.19)$$

Since this system has the form (2.17), Lemma 2.2 shows that it has solutions iff a', b, c, d' are pairwise coprime and $(bc, s) = (1)$.

2. Proposition 2.1 gives $\text{Hom}_{\text{ZEF}(R, W)}^{\hat{1}}(e_1, e_2) = \text{Hom}_{\text{ZEF}(R, W)}^{\hat{1}}(e_2, e_1) = R \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix}$. Two non-zero odd morphisms $g_{12} = \alpha \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} \in \text{Hom}_{\text{ZEF}(R, W)}^{\hat{1}}(e_1, e_2)$ and $g_{21} = \beta \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} \in \text{Hom}_{\text{ZEF}(R, W)}^{\hat{1}}(e_2, e_1)$ (with $\alpha, \beta \in R^\times$) induce mutually inverse isomorphisms in $\text{HEF}(R, W)$ iff:

$$g_{21}g_{12} = 1 + \mathfrak{d}_{e_1, e_1}(f) \quad , \quad g_{12}g_{21} = 1 + \mathfrak{d}_{e_2, e_2}(q)$$

for some $f, q \in \text{End}_R^{\hat{0}}(R^{1|1})$. This gives the equations:

$$\begin{aligned} \alpha\beta \begin{bmatrix} a'd' & 0 \\ 0 & a'd' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (acf_{\hat{0}\hat{0}} + f_{\hat{1}\hat{1}}bd) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \\ \alpha\beta \begin{bmatrix} a'd' & 0 \\ 0 & a'd' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (abq_{\hat{0}\hat{0}} + q_{\hat{1}\hat{1}}cd) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

which amount to the system:

$$\begin{cases} \alpha\beta a'd' - s(a'cf_{\hat{0}\hat{0}} + f_{\hat{1}\hat{1}}bd) = 1 \\ \alpha\beta a'd' - s(a'bq_{\hat{0}\hat{0}} + q_{\hat{1}\hat{1}}cd) = 1 \end{cases} . \quad (2.20)$$

This system again has the form (2.17), as can be seen by the substitution of the quadruples $(b, c, a', d') := (a', d', b, c)$. As a consequence, it has a solution iff a', b, c, d' are pairwise coprime and $(a'd', s) = (1)$. \square

Corollary 2.5 *Similar elementary matrix factorizations of W are isomorphic in $\text{hef}(R, W)$.*

Proof. The statement follows immediately from Proposition 2.4 by taking $a = v\gamma$, $b = \gamma^{-1}$, $c = 1$, $d = u$ (where $\gamma \in U(R)$), since the gcd in R is defined modulo $U(R)$. \square

Proposition 2.6 *Any elementary matrix factorization of W is odd-isomorphic in $\text{HEF}(R, W)$ to its suspension:*

$$e_v \simeq_{\text{HEF}(R, W)} \Sigma e_v = e_{-u} \quad ,$$

where $u = W/v$.

Proof. Let $s \in (u, v)$. The isomorphism follows from Proposition 2.4 for $a' = 1 = d'$, $b = -v/s$, $c = -u/s$:

$$\begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & bs \\ cs & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & cs \\ bs & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix} \quad ,$$

since $(ad, s) = (1)$. \square

Remark 2.3. An odd isomorphism in $\text{HEF}(R, W)$ between e_v and $\Sigma e_v = e_{-u}$ can also be obtained more abstractly by transporting the identity endomorphism of e_v through the isomorphism of R -modules $\text{Hom}^{\hat{1}}(e_v, e_{-u}) = \text{Hom}^{\hat{1}}(e_v, \Sigma e_v) \simeq \text{Hom}_{\text{hmf}(R, W)}(e_v, e_v)$ which results by taking $v_1 = v$ and $v_2 = -u$ in the first line of (1.5). Since e_{-u} is similar to e_u , the Proposition implies that e_v and e_u are oddly isomorphic. When $v = 1$, both $e_v = e_1$ and $e_u = e_W$ are zero objects and we have $\text{Hom}_{\text{HMF}(R, W)}^{\hat{0}}(e_1, e_W) = \text{Hom}_{\text{HMF}(R, W)}^{\hat{1}}(e_1, e_W) = \{0\}$, so the odd isomorphism is the zero morphism.

Proposition 2.7 *Let $W = W_1W_2$ with $(W_1, W_2) = (1)$ and let v be a divisor of W_1 . Then $e_{W_2v} \simeq_{\text{hmf}(R, W)} e_v$.*

Proof. Let $u_0 \stackrel{\text{def.}}{=} \frac{W_1}{v}$. Setting $v_1 = v$, $u_1 = \frac{W}{v} = W_2u_0$, $v_2 = W_2v$ and $u_2 = \frac{W}{v_2} = u_0$, we compute:

$$\begin{aligned} a \in (v_1, v_2) &= (v) \quad , \quad b = \frac{v_1}{a} \in (1) \quad , \quad c = \frac{v_2}{a} \in (W_2) \quad , \quad d \in \frac{W}{[v_1, v_2]} = \left(\frac{W_1}{v} \right) \\ s \in (u_1, v_1, u_2, v_2) &= (u_0, v) \quad , \quad a' = a/s \in \frac{v}{(u_0, v)} \quad , \quad d' = d/s \in \frac{(W_1)}{(v)(u_0, v)} = \frac{(u_0)}{(u_0, v)} \quad . \end{aligned}$$

It is clear that a', b, c, d' are mutually coprime and that $(s, bc) = (1)$. \square

2.1.3. The composition of morphisms in $\text{HEF}(R, W)$.

Proposition 2.8 *Given three divisors v_1, v_2 and v_3 of W , we have the following relations:*

$$\begin{aligned} \epsilon_{\hat{0}}(v_2, v_3)\epsilon_{\hat{0}}(v_1, v_2) &= \frac{(v_2)(v_1, v_3)}{(v_1, v_2)(v_2, v_3)}\epsilon_{\hat{0}}(v_1, v_3) \\ \epsilon_{\hat{0}}(v_2, v_3)\epsilon_{\hat{1}}(v_1, v_2; W) &= \frac{(v_2)(u_1, v_3)}{(u_1, v_2)(v_2, v_3)}\epsilon_{\hat{1}}(v_1, v_3; W) \\ \epsilon_{\hat{1}}(v_2, v_3; W)\epsilon_{\hat{0}}(v_1, v_2) &= \frac{(v_3)(v_1, u_3)}{(v_1, v_2)(u_2, v_3)}\epsilon_{\hat{1}}(v_3, v_1; W) \\ \epsilon_{\hat{1}}(v_2, v_3; W)\epsilon_{\hat{1}}(v_1, v_2; W) &= -\frac{(u_1)(v_1, v_3)}{(u_2, v_3)(u_1, v_2)}\epsilon_{\hat{0}}(v_1, v_3) \quad . \end{aligned} \tag{2.21}$$

Proof. Given three divisors v_1, v_2 and v_3 of W , we have:

$$\epsilon_{\hat{0}}(v_2, v_3)\epsilon_{\hat{0}}(v_1, v_2) = \begin{bmatrix} \frac{v_2 v_3}{(v_1, v_2)(v_2, v_3)} & 0 \\ 0 & \frac{v_1 v_2}{(v_1, v_2)(v_2, v_3)} \end{bmatrix} = \frac{v_2(v_1, v_3)}{(v_1, v_2)(v_2, v_3)}\epsilon_{\hat{0}}(v_1, v_3) = \frac{[v_1, v_2, v_3](v_1, v_3)}{(v_1, v_2, v_3)[v_1, v_3]}\epsilon_{\hat{0}}(v_1, v_3)$$

where we used the identity:

$$[a, b, c](a, b)(b, c)(c, a) = (a)(b)(c)(a, b, c) \quad . \quad (2.22)$$

This establishes the first of equations (2.21). The remaining equations follow similarly \square .

Corollary 2.9 *Let v be a divisor of W and $u = W/v$. Then:*

1. *The R -algebra $\text{End}_{\text{zmf}(R, W)}(e_v)$ is isomorphic with R .*
2. *We have an isomorphism of \mathbb{Z}_2 -graded R -algebras:*

$$\text{End}_{\text{ZMF}(R, W)}(e_v) \simeq \frac{R[\omega]}{\langle u^2 + t \rangle} \quad ,$$

where ω is an odd generator and $t \in \frac{[u, v]}{(u, v)}$. In particular, $\text{End}_{\text{ZMF}(R, W)}(e_v)$ is a commutative \mathbb{Z}_2 -graded ring.

Proof. For $v_1 = v_2 = v$, we have $\alpha_W(v, v) = \langle u, v \rangle$. Proposition 2.1 gives:

$$\epsilon_{\hat{0}}(v, v) \stackrel{\text{def.}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \epsilon_{\hat{1}}(v, v) \stackrel{\text{def.}}{=} \begin{bmatrix} 0 & \frac{(v)}{(u, v)} \\ -\frac{(u)}{(u, v)} & 0 \end{bmatrix}$$

and we have:

$$\begin{aligned} \epsilon_{\hat{0}}(v, v)^2 &= \epsilon_{\hat{0}}(v, v) \\ \epsilon_{\hat{0}}(v, v)\epsilon_{\hat{1}}(v, v; W) &= \epsilon_{\hat{1}}(v, v; W)\epsilon_{\hat{0}}(v, v) = \epsilon_{\hat{1}}(v, v) \\ \epsilon_{\hat{1}}(v, v; W)^2 &= -\frac{[u, v]}{(u, v)}\epsilon_{\hat{0}}(v, v) \quad , \end{aligned}$$

which also follows from Proposition 2.8. Setting $\omega = \epsilon_{\hat{1}}(v, v; W)$, these relations imply the desired statements upon using Proposition 2.1. \square

Corollary 2.10 *Let v be a divisor of W and $u = W/v$. Then:*

1. *The R -algebra $\text{End}_{\text{hmf}(R, W)}(e_v)$ is isomorphic with $R/\langle d \rangle = R/\langle u, v \rangle$, where $d \in (u, v)$.*
2. *We have an isomorphism of \mathbb{Z}_2 -graded R -algebras:*

$$\text{End}_{\text{HMF}(R, W)}(e_v) \simeq \frac{(R/\langle d \rangle)[\omega]}{\langle u^2 + t \rangle} \quad ,$$

where ω is an odd generator, $d \in (u, v)$ and $t \in \frac{[u, v]}{(u, v)}$. In particular, $\text{End}_{\text{ZMF}(R, W)}(e_v)$ is a supercommutative \mathbb{Z}_2 -graded ring.

Proof. The same relations as in the previous Corollary imply the conclusion upon using Proposition 2.2. \square

Corollary 2.11 *An elementary matrix factorization e_v is a zero object of $\text{hmf}(R, W)$ iff $(u, v) = (1)$, where $u = W/v$.*

Proof. The R -algebra $\text{End}_{\text{HMF}(R, W)}^{\hat{0}}(e_v) \simeq R/t$ vanishes iff $(u, v) = (1)$. \square

2.2. Localizations. Let $S \subset R$ be a multiplicative subset of R containing the identity $1 \in R$ and $\lambda_S : R \rightarrow R_S$ denote the natural ring morphism from R to the localization $R_S = S^{-1}R$ of R at S . For any $r \in R$, let $r_S \stackrel{\text{def.}}{=} \lambda_S(r) = \frac{r}{1} \in R_S$ denote its extension. For any R -module N , let $N_S = S^{-1}N = N \otimes_R R_S$ denote the localization of N at S . For any morphism of R -modules $f : N \rightarrow N'$, let $f_S \stackrel{\text{def.}}{=} f \otimes_R \text{id}_{R_S} : N_S \rightarrow N'_S$ denote the localization of f at S . For any \mathbb{Z}_2 -graded R -module $M = M^{\hat{0}} \oplus M^{\hat{1}}$, we have $M_S = M_S^{\hat{0}} \oplus M_S^{\hat{1}}$, since the localization functor is exact. In particular, localization at S induces a functor from the category of \mathbb{Z}_2 -graded R -modules to the category of \mathbb{Z}_2 -graded R_S -modules.

Let $a = (M, D)$ be a matrix factorization of W . The *localization of a at S* is the following matrix factorization of W_S over the ring R_S :

$$a_S \stackrel{\text{def.}}{=} (M_S, D_S) \in \text{MF}(R_S, W_S) \quad .$$

It is clear that this extends to an even dg functor $\text{loc}_S : \text{MF}(R, W) \rightarrow \text{MF}(R_S, W_S)$, which is R -linear and preserves direct sums. In turn, this induces dg functors $\text{ZMF}(R, W) \rightarrow \text{ZMF}(R_S, W_S)$, $\text{BMF}(R, W) \rightarrow \text{BMF}(R_S, W_S)$, $\text{HMF}(R, W) \rightarrow \text{HMF}(R_S, W_S)$ and $\text{hmf}(R, W) \rightarrow \text{hmf}(R_S, W_S)$, which we again denote by loc_S . We have $\text{loc}_S(a) = a_S$ for any matrix factorization a of W over R .

Proposition 2.12 *The functor $\text{loc}_S : \text{hmf}(R, W) \rightarrow \text{hmf}(R_S, W_S)$ is a triangulated functor. Moreover, the strictly full subcategory of $\text{hmf}(R, W)$ defined through:*

$$K_S \stackrel{\text{def.}}{=} \{a \in \text{Ob}[\text{hmf}(R, W)] \mid a_S \simeq_{\text{hmf}(R_S, W_S)} 0\}$$

is a triangulated subcategory of $\text{hmf}(R, W)$.

Proof. It is clear that loc_S commutes with the cone construction (see [6] for a detailed account of the latter). It is also clear that the subcategory K_S is closed under shifts. Since any distinguished triangle in which two objects vanish has the property that its third object also vanishes, K_S is also closed under forming triangles. \square

Proposition 2.13 *For any matrix factorizations a, b of W , there exists a natural isomorphism of \mathbb{Z}_2 -graded R_S -modules:*

$$\text{Hom}_{\text{HMF}(R_S, W_S)}(a_S, b_S) \simeq_{R_S} \text{Hom}_{\text{HMF}(R, W)}(a, b)_S \quad .$$

Proof. Follows immediately from the fact that localization at S is an exact functor from Mod_R to Mod_{R_S} . \square

2.3. Behavior of $\text{hef}(R, W)$ under localization.

Lemma 2.14 *The following statements are equivalent for any elements s, r of R :*

1. $(s, r) = (1)$
2. *The class of s modulo the ideal $\langle r \rangle$ is a unit of the ring $R/\langle r \rangle$.*

Proof. We have $(s, r) = (1)$ iff there exist elements $a, b \in R$ such that $as + br = 1$. In turn, this is equivalent with the condition $\bar{a}\bar{s} = \bar{1}$ in the ring $R/\langle r \rangle$, where $\bar{x} = x + \langle r \rangle$ denotes the equivalence class of an element $x \in R$ modulo the ideal $\langle r \rangle$. \square

Consider the multiplicative set:

$$S_W \stackrel{\text{def.}}{=} \{s \in R \mid (s, W) = (1)\} .$$

Since $0 \notin S_W$, the localization $R_S = S^{-1}R$ of R at any multiplicative set $S \subset S_W$ is a sub-ring of the field of fractions K of R :

$$R_S = \left\{ \frac{r}{s} \mid r \in R, s \in S \right\} \subset K .$$

In particular, R_S is an integral domain.

Proposition 2.15 *Let S be any multiplicative subset of R such that $S \subset S_W$. Then the localization functor $\text{loc}_S : \text{hmf}(R, W) \rightarrow \text{hmf}(R_S, W_S)$ restricts to an R -linear equivalence of categories between $\text{hef}(R, W)$ and $\text{hef}(R_S, W_S)$.*

Proof. Since loc_S preserves the reduced rank of matrix factorizations, it is clear that it restricts to a functor from $\text{hef}(R, W)$ to $\text{hef}(R_S, W_S)$. Given two elementary factorizations $e_{v_1}, e_{v_2} \in \text{Ob}[\text{hef}(R, W)]$, let $r \in (v_1, v_2, W/v_1, W/v_2)$. By Proposition 2.13, we have:

$$\text{Hom}_{\text{hef}(R_S, W_S)}((e_{v_1})_S, (e_{v_2})_S) \simeq_R \text{Hom}_{\text{hef}(R, W)}(e_{v_1}, e_{v_2})_S . \quad (2.23)$$

Let s be any element of S . Since S is a subset of S_W , we have $(s, W) = (1)$ and hence $(s, r) = (1)$ since r is a divisor of W . By Lemma 2.14, the image $\bar{s} = s + \langle r \rangle$ is a unit of the quotient ring $R/\langle r \rangle$, hence the operator of multiplication with s is an isomorphism of the cyclic R -module $\text{Hom}_{\text{HMF}(R, W)}(e_{v_1}, e_{v_2}) \simeq R/\langle r \rangle$. Thus every element of S acts as an automorphism of this module, which implies that the localization map $\text{Hom}_{\text{hmf}(R, W)}(e_{v_1}, e_{v_2}) \rightarrow \text{Hom}_{\text{hmf}(R, W)}(e_{v_1}, e_{v_2})_S$ is an isomorphism of R -modules (where $\text{Hom}_{\text{hmf}(R, W)}(e_{v_1}, e_{v_2})_S$ is viewed as an R -module by the extension of scalars $R \rightarrow R_S$). Combining this with (2.23) shows that the restriction $\text{loc}_S : \text{hef}(R, W) \rightarrow \text{hef}(R_S, W_S)$ is a full and faithful functor.

Now let e_x be an elementary factorization of W_S corresponding to the divisor x of $W_S = W/1$ in the ring R_S . Let $y = W_S/x \in R_S$. Write $x = v/s$ and $y = u/t$ with $x, y \in R$ and $s, t \in S$ chosen such that $(v, s) = (u, t) = (1)$. Then the relation $xy = W_S$ amounts to $uv = stW$. Since S is a subset of S_W , we have $(s, W) = (t, W) = (1)$. Thus $st|uv$, which implies $s|v$ and $t|u$ since $(v, s) = (u, t) = (1)$. Thus $v = v_1t$ and $u = u_1s$ with $u_1, v_1 \in R$ and we have $u_1v_1 = W$. This gives $x = \gamma v_1$ and $y = \gamma^{-1}u_1$, where $\gamma \stackrel{\text{def.}}{=} t/s$ is a unit of R_S . It follows that e_x is similar to the elementary matrix factorization e_{v_1} of W_S over R_S , and hence isomorphic to the latter in the category $\text{hef}(R_S, W_S)$ by Proposition 2.5. Since u_1 and v_1 are divisors of W satisfying $u_1v_1 = W$, we can view e_{v_1} as an elementary factorization of W over R (it lies in the image of the functor loc_S). This shows that any object of $\text{hef}(R_S, W_S)$ is even-isomorphic with an object lying in the image of the restricted localization functor, hence the latter is essentially surjective. \square

2.4. Behavior of $\text{HEF}(R, W)$ under multiplicative partition of W . For any divisor W_1 of W , let $\text{HEF}_{W_1}(R, W)$ denote the full subcategory of $\text{HEF}(R, W)$ whose objects are those elementary factorization e_v of W for which v is a divisor of W_1 .

Proposition 2.16 *Let e_1 and e_2 be as above. Consider elements of R chosen as follows:*

$$\begin{aligned} s_1 \in (u_1, v_1) = (s)(a', c)(b, d') , \quad s_2 \in (u_2, v_2) = (s)(a', b)(c, d') , \\ u'_1 \stackrel{\text{def.}}{=} u_1/s = cd' , \quad u'_2 \stackrel{\text{def.}}{=} u_2/s = bd' , \quad v'_1 \stackrel{\text{def.}}{=} v_1/s = a'b , \quad v'_2 \stackrel{\text{def.}}{=} v_2/s = a'c , \\ x(e_1) \in (s, v'_1) = (s, a'b) , \quad y(e_1) \in (s, u'_1) = (s, cd') , \\ x(e_2) \in (s, v'_2) = (s, a'c) , \quad y(e_2) \in (s, u'_2) = (s, bd') . \end{aligned} \quad (2.24)$$

Then:

1. e_1 and e_2 are isomorphic in $\text{hef}(R, W)$ iff:

$$\begin{aligned} (i) \quad & (s_1) = (s_2) \quad \text{and} \\ (ii) \quad & ((x(e_1)), (y(e_1))) = ((x(e_2)), (y(e_2))) \quad \text{as ordered pairs of elements in } R^\times/U(R). \end{aligned} \quad (2.25)$$

2. e_1 and e_2 are isomorphic in $\text{HEF}(R, W)$ iff:

$$\begin{aligned} (i) \quad & (s_1) = (s_2) \quad \text{and} \\ (ii) \quad & \{(x(e_1)), (y(e_1))\} = \{(x(e_2)), (y(e_2))\} \quad \text{as unordered pairs of elements in } R^\times/U(R). \end{aligned} \quad (2.26)$$

Notice that $(s_1) = (s_2)$ implies $(s_1) = (s) = (s_2)$, with s defined in (2.3).

Proof.

1. Assume that $e_1 \simeq_{\text{hmf}(R, W)} e_2$. By Proposition 2.4, part 1, to such pair of elementary factorizations we can associate four pairwise coprime divisors a', b, c, d' of W such that $v_1 = a'bs$, $u_1 = d'cs$, $v_2 = a'cs$, $u_2 = d'bs$ and together with the equality $(bc, s) = (1)$. Thus $(s_1) = (v_1, u_1) = (a'bs, d'cs) = (s)$, since $a'b$ and $d'c$ are coprime. Similarly, $(s_2) = (s)$. The equality $(bc, s) = (1)$ is equivalent to $(b, s) = (1)$ and $(c, s) = (1)$. Using this, we compute:

$$(x(e_1)) = (s, v'_1) = (s, a'b) = (s, a') = (s, a'c) = (s, v'_2) = (x(e_2)) \quad (2.27)$$

Acting similarly, we also find $(y(e_1)) = (s, d') = (y(e_2))$. Thus (2.25) holds.

Now assume that (2.25) is satisfied for two elementary factorizations e_1 and e_2 . Let $s \in (s_1) = (s_2)$ and define a', b, c, d' as before, following (2.1). By the very construction, $(b, c) = (1)$ and $(a', d') = (1)$. We first show that $s_1 \sim s_2$ all a', b, c, d' are pairwise coprime. Indeed, if we assume that $p|(a', b)$ then $s_1 \sim s_2$ implies:

$$(s)(a', c)(b, d') = (s_1) = (s_2) = (s)(a', b)(c, d') \quad .$$

Since p divides the right hand side, it should divide $(a', c)(b, d')$ and $(c, b) = (1)$ and $(a', d') = (1)$. Thus $p \in U(R)$. In much the same way we prove that other pairs from a', b, c, d' are coprime.

Condition (ii) in (2.25) reads:

$$(s, a'b) = (x(e_1)) = (x(e_2)) = (s, a'c) \quad .$$

If $p|b$ and $p|s$ then $p|(s, b)$ and thus $p|(s, a'b)$. By the equality above, we also have $p|(s, a'c)$ and hence $p|a'c$. But b is coprime with both a' and c , thus $p \in U(R)$. Similarly, $p|c$ and $p|s$ implies $p \in U(R)$. Thus $(bc, s) = (1)$. Note that $(y(e_1)) = (y(e_2))$ is now automatically satisfied. Proposition 2.4, part 1 implies that $e_1 \simeq_{\text{hef}(R, W)} e_2$.

2. Assume $e_1 \simeq_{\text{HEF}(R, W)} e_2$. If the isomorphism is even, then it comes from the isomorphism in $\text{hef}(R, W)$ and part 1 above already proves that (2.25) and thus also (2.26). Thus we can assume that the isomorphism is odd. We will prove that $(s_1) = (s_2)$ and $(x(e_1)) = (y(e_2))$, $(x(e_2)) = (y(e_1))$. Applying Proposition 2.4, part 2, we obtain a', b, c, d' pairwise coprime and s such that $(s, a'd') = (1)$. Then $(s_1) = (s) = (s_2)$ similarly to part 1 above. Using $(s, a'd') = (1)$, we also compute:

$$(x(e_1)) = (s, v'_1) = (s, a'b) = (s, b) = (s, d'b) = (s, u'_2) = (y(e_2))$$

and also $(x(e_2)) = (y(e_1))$. Thus (2.26).

Assume now that (2.26) is satisfied. Since the statement for even morphisms is covered by (2.25), we only need to consider the situation $(x(e_1)) = (y(e_2))$ and $(x(e_2)) = (y(e_1))$. As in part 1, $(s_1) = (s_2)$ implies that a', b, c, d' are pairwise coprime. Condition (ii) reads:

$$(s, a'b) = (x(e_1)) = (y(e_2)) = (s, d'b) .$$

If we assume that $p|a'$ and $p|s$ then $p|(s, a')$ and $p|(s, a'b)$. The equality implies $(p|d'b)$. Since a' is coprime with both d' and b , we obtain $p \in U(R)$. Similarly $(d', s) = (1)$ and thus $(a'd', s) = (1)$. Proposition 2.4, part 2 implies that $e_1 \simeq_{\text{HEF}(E, W)} e_2$ by an odd isomorphism. \square

Proposition 2.17 *Let W_1 and W_2 be divisors of W such that $W = W_1W_2$ and $(W_1, W_2) = (1)$. Then there exist equivalences of R -linear \mathbb{Z}_2 -graded categories:*

$$\text{HEF}(R, W_1) \simeq \text{HEF}_{W_1}(R, W) \quad , \quad \text{HEF}(R, W_2) \simeq \text{HEF}_{W_2}(R, W) \quad .$$

which are bijective on objects.

Proof. For any divisor v of W_1 , let $e'_v = (R^{1|1}, D'_v)$ and $e_v = (R^{1|1}, D_v)$ be the corresponding elementary factorizations of W_1 and W , where:

$$D'_v = \begin{bmatrix} 0 & v \\ W_1/v & 0 \end{bmatrix} \quad , \quad D_v = \begin{bmatrix} 0 & v \\ W/v & 0 \end{bmatrix} \quad .$$

For any two divisors v_1, v_2 of W_1 and any $\kappa \in \mathbb{Z}_2$, we have $W/v_i = W_2 \frac{W_1}{v_i}$ and $(v_i, W_2) = (1)$. Thus $(v_1, v_2, W_1/v_1, W_1/v_2) = (v_1, v_2, W/v_1, W/v_2)$. By Proposition 2.2, this gives:

$$\text{Ann}(\text{Hom}_{\text{HEF}(R, W_1)}^\kappa(e'_{v_1}, e'_{v_2})) = \text{Ann}(\text{Hom}_{\text{HEF}(R, W)}^\kappa(e_{v_1}, e_{v_2})) \quad , \quad \forall \kappa \in \mathbb{Z}_2 \quad .$$

On the other hand, the modules $\text{Hom}_{\text{HEF}(R, W_1)}^{\hat{0}}(e'_{v_1}, e'_{v_2})$ and $\text{Hom}_{\text{HEF}(R, W_1)}^{\hat{0}}(e_{v_1}, e_{v_2})$ are generated by the same element $\epsilon_{\hat{0}}(v_1, v_2)$ while $\text{Hom}_{\text{HEF}(R, W_1)}^{\hat{1}}(e'_{v_1}, e'_{v_2})$ and $\text{Hom}_{\text{HEF}(R, W_1)}^{\hat{1}}(e_{v_1}, e_{v_2})$ are generated by the elements $\epsilon_{\hat{1}}(v_1, v_2; W_1)$ and $\epsilon_{\hat{1}}(v_1, v_2; W)$, respectively. Hence the functor which maps e'_v to e_v for any divisor v of W_1 and takes $\epsilon_{\hat{0}}(v_1, v_2)$ to $\epsilon_{\hat{0}}(v_1, v_2)$ and $\epsilon_{\hat{0}}(v_1, v_2; W_1)$ to $\epsilon_{\hat{1}}(v_1, v_2; W)$ for any two divisors v_1, v_2 of W is an R -linear equivalence from $\text{HEF}(R, W_1)$ to $\text{HEF}_{W_1}(R, W)$. A similar argument establishes the equivalence $\text{HEF}(R, W_2) \simeq \text{HEF}_{W_2}(R, W)$. \square

2.5. Primary matrix factorizations. Recall that an element of R is called primary if it is a power of a prime element.

Definition 2.18 *An elementary factorization e_v of W is called primary if v is a primary divisor of W .*

Let $\text{HEF}_0(R, W)$ denote the full subcategory of $\text{HEF}(R, W)$ whose objects are the primary factorizations of W .

Proposition 2.19 *Let $W = W_1W_2$ be a factorization of W , where W_1 and W_2 are coprime elements of R . Then there exists an equivalence of R -linear \mathbb{Z}_2 -graded categories:*

$$\mathrm{HEF}_0(R, W) \simeq \mathrm{HEF}_0(R, W_1) \vee \mathrm{HEF}_0(R, W_2) \quad ,$$

where \vee denotes the coproduct of Mod_R -enriched categories.

Proof. Let $\mathrm{HEF}_{0, W_i}(R, W)$ denote the full subcategory of $\mathrm{HEF}_0(R, W)$ whose objects are the primary factorizations e_v of W for which v is a (primary) divisor of W_i . Since $W = W_1W_2$ and $(W_1, W_2) = (1)$, a primary element $v \in R$ is a divisor of W iff it is either a divisor of W_1 or a divisor of W_2 . Hence $\mathrm{Ob}\mathrm{HEF}_0(R, W) = \mathrm{Ob}\mathrm{HEF}_{0, W_1}(R, W) \sqcup \mathrm{Ob}\mathrm{HEF}_{0, W_2}(R, W)$. For any primary divisors v_1 and v_2 of W and any $\kappa \in \mathbb{Z}_2$, we have:

$$\mathrm{Hom}_{\mathrm{HEF}(R, W)}^\kappa(e_{v_1}, e_{v_2}) \simeq R/\langle d \rangle \simeq \begin{cases} \mathrm{Hom}_{\mathrm{HEF}(R, W_1)}^\kappa(e_{v_1}, e_{v_2}) & \text{if } v_1|W_1 \text{ \& } v_2|W_1 \\ \mathrm{Hom}_{\mathrm{HEF}(R, W_2)}^\kappa(e_{v_1}, e_{v_2}) & \text{if } v_1|W_2 \text{ \& } v_2|W_2 \\ 0 & \text{if } v_1|W_1 \text{ \& } v_2|W_2 \text{ or } v_1|W_2 \text{ \& } v_2|W_1 \end{cases} \quad ,$$

where $d \in (v_1, v_2, W/v_1, W/v_2)$ and in the third case we used the fact that $v_1|W_1$ and $v_2|W_2$ or $v_1|W_2$ and $v_2|W_1$ implies $(v_1, v_2) = (1)$ since W_1 and W_2 are coprime. This shows that $\mathrm{HEF}_0(R, W) = \mathrm{HEF}_{0, W_1}(R, W) \vee \mathrm{HEF}_{0, W_2}(R, W)$. By Proposition 2.17, we have R -linear equivalences $\mathrm{HEF}_{0, W_i}(R, W) \simeq \mathrm{HEF}_0(R, W_i)$ which are bijective on objects. This implies the conclusion. \square

Definition 2.20 *A reduced multiplicative partition of W is a factorization:*

$$W = W_1W_2 \dots W_n$$

where W_1, \dots, W_n are mutually coprime elements of R .

Corollary 2.21 *Let $W = W_1 \dots W_n$ be a reduced multiplicative partition of W . Then there exists a natural equivalence of R -linear categories:*

$$\mathrm{HEF}_0(R, W) \simeq \bigvee_{i=1}^n \mathrm{HEF}_0(R, W_i) \quad .$$

Proof. Follows immediately from Proposition 2.19. \square

Let e_v be a primary matrix factorization of W . Then $v = p^i$ for some prime divisor p of W and some integer $i \in \{0, \dots, n\}$, where n is the order of p as a divisor of W . We have $W = p^n W_1$ for some element $W_1 \in R$ such that p does not divide W_1 and $u = p^{n-i} W_1$. Thus $(u, v) = (p^{\min(i, n-i)})$.

Definition 2.22 *The prime divisor p of W is called the prime locus of e_v . The order n of p is called the order of e_v while the integer $i \in \{0, \dots, n\}$ is called the size of e_v .*

Let R be a Bézout domain and $p \in R$ be a prime element. Fix an integer $n \geq 2$ and consider the quotient ring:

$$A_n(p) \stackrel{\mathrm{def.}}{=} R/\langle p^n \rangle \quad .$$

Let $\mathbf{m}_n(p) = pA_n(p) = \langle p \rangle / \langle p^n \rangle$ and $\mathbf{k}_p = R/\langle p \rangle$.

Lemma 2.23 *The following statements hold:*

1. The principal ideal $\langle p \rangle$ generated by p is maximal.
2. The primary ideal $\langle p^n \rangle$ is contained in a unique maximal ideal of R .
3. The quotient $A_n(p)$ is a quasi-local ring with maximal ideal $\mathfrak{m}_n(p)$ and residue field \mathbf{k}_p .
4. $A_n(p)$ is a generalized valuation ring.

Proof.

1. Let I be any ideal containing $\langle p \rangle$. If $\langle p \rangle \neq I$, then take any element $x \in I \setminus \langle p \rangle$. Then we have the proper inclusion $\langle p \rangle \subsetneq \langle p, x \rangle$. Since R is a Bézout domain, the ideal $\langle p, x \rangle$ is generated by a single element y . We have $y|p$, so y is a unit of R since p is prime. Since y belongs to I , this gives $I = R$. Thus $\langle p \rangle$ is a maximal ideal.
2. Let m be a maximal ideal of R containing $\langle p^n \rangle$. Then $p^n \in m$, which implies $p \in m$ since m is prime. Thus $\langle p \rangle \subset m$, which implies $m = \langle p \rangle$ since $\langle p \rangle$ is maximal by point 1. This shows that $R/\langle p^n \rangle$ has a unique maximal ideal, namely $\langle p \rangle/\langle p^n \rangle$.
3. Since R is Bézout and $\langle p^n \rangle$ is finitely-generated, the quotient $R/\langle p^n \rangle$ is a Bézout ring (which has divisors of zero when $n \geq 2$). By point 2. above, $R/\langle p^n \rangle$ is also a quasi-local ring.
4. Follows from [7, Lemma 1.3 (b)] since R is a valuation ring. \square

Recall that an object of an additive category is called *indecomposable* if it is not isomorphic with a direct sum of two non-zero objects.

Proposition 2.24 *Let e_v be a primary factorization of W with prime locus p , order n and size i . Then e_v is an indecomposable object of $\text{hmf}(R, W)$ whose endomorphism ring $\text{End}_{\text{hmf}(R, W)}(e_v)$ is a quasi-local ring isomorphic with $A_{\min(i, n-i)}(p)$.*

Proof. We have $\text{End}_{\text{hmf}(R, W)}(e_v) = R/\langle u, v \rangle = R/\langle p^{\min(i, n-i)} \rangle$ by Corollary 2.10. This ring is quasi-local by Lemma 2.23. Since quasi-local rings have no nontrivial idempotents, it follows that e_v is an indecomposable object of $\text{hmf}(R, W)$. \square

Lemma 2.25 *Let v_1 and v_2 be two divisors of W which are mutually coprime. Then $\text{Hom}_{\text{hmf}(R, W)}(e_{v_1}, e_{v_2}) = 0$.*

Proof. Let $u_i := W/v_i$. Then $(v_1, v_2, u_1, u_2) = (1)$ since $(v_1, v_2) = (1)$. Thus $\langle v_1, v_2, u_1, u_2 \rangle = R$ and the statement follows from Proposition 2.2. \square

Proposition 2.26 *Let p be a prime divisor of W of order n and $i \in \{1, \dots, n\}$. Then:*

$$\Sigma e_{p^i} \simeq_{\text{hmf}(R, W)} e_{p^{n-i}} \quad .$$

Proof. Let $W_1 \stackrel{\text{def.}}{=} p^n$, $W_2 \stackrel{\text{def.}}{=} W/p^n$ and $v \stackrel{\text{def.}}{=} p^i$, $u \stackrel{\text{def.}}{=} W/v = p^{n-i}W_2$. We have $\Sigma e_{p^i} = \Sigma e_v = e_{-u} \simeq_{\text{hmf}(R, W)} e_u$. Since p^{n-i} is a divisor of W_1 and $(W_1, W_2) = 1$, Proposition 2.7 gives $e_u = e_{p^{n-i}W_2} \simeq_{\text{hmf}(R, W)} e_{p^{n-i}}$. \square

3. The additive category $\mathbf{hef}(R, W)$ for a Bézout domain and critically-finite W

Let R be a Bézout domain and W be a critically-finite element of R .

Proposition 3.1 *Let e_v be an elementary factorization of W over R such that $v = \prod_{i=1}^n v_i$, where $v_i \in R$ are mutually coprime divisors of W . Then there exists a natural isomorphism in $\mathbf{hmf}(R, W)$:*

$$e_v \simeq_{\mathbf{hmf}(R, W)} \bigoplus_{i=1}^n e_{v_i}$$

In particular, an elementary factorization e_v for which v is finitely-factorizable divisor of W is isomorphic in $\mathbf{hmf}(R, W)$ with a direct sum of primary factorizations.

Proof. Let d be any divisor of W . By Proposition 2.2, we have isomorphisms of R -modules:

$$\mathrm{Hom}_{\mathbf{hmf}(R, W)}(e_d, e_{v_i}) \simeq_R R/\alpha_W(d, v_i) \quad \text{and} \quad \mathrm{Hom}_{\mathbf{hmf}(R, W)}(e_d, e_v) \simeq_R R/\alpha_W(d, v) .$$

Since v_i are mutually coprime, Proposition 2.3 gives $\alpha_W(d, v) = \prod_{i=1}^n \alpha_W(d, v_i)$, where $\alpha_W(d, v_i)$ are principal ideals generated by mutually coprime elements. The Chinese remainder theorem gives an isomorphism of R -modules:

$$R/\alpha_W(d, v) \simeq_R \bigoplus_{i=1}^n [R/\alpha_W(d, v_i)]$$

Combining the above, we conclude that there exist natural isomorphisms of R -modules:

$$\varphi_d : \mathrm{Hom}_{\mathbf{hmf}(R, W)}(e_d, e_v) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{hmf}(R, W)}(e_d, \bigoplus_{i=1}^n e_{v_i}) , \quad (3.1)$$

where we used the fact that $\mathrm{Hom}_{\mathbf{hmf}(R, W)}(e_d, \bigoplus_{i=1}^n e_{v_i}) \simeq_R \bigoplus_{i=1}^n \mathrm{Hom}_{\mathbf{hmf}(R, W)}(e_d, e_{v_i})$. This implies that the functors $\mathrm{Hom}_{\mathbf{hef}(R, W)}(-, e_v)$ and $\mathrm{Hom}_{\mathbf{hef}(R, W)}(-, \bigoplus_{i=1}^n e_{v_i})$ are isomorphic. By the Yoneda lemma, we conclude that there exists a natural isomorphism $e_v \simeq_{\mathbf{hef}(R, W)} \bigoplus_{i=1}^n e_{v_i}$. \square

Recall that a *Krull-Schmidt category* is an additive category for which every object decomposes into a finite direct sum of objects having quasi-local endomorphism rings.

Theorem 3.2 *The additive category $\mathbf{hef}(R, W)$ is Krull-Schmidt and its non-zero indecomposable objects are the non-trivial primary matrix factorizations of W . In particular, $\mathbf{hef}(R, W)$ is additively generated by $\mathbf{hef}_0(R, W)$.*

Proof. Suppose that W has the decomposition (1.4). Any elementary factorization e_v of W corresponds to a divisor v of W , which must have the form $v = v_0 p_{s_1}^{l_1} \dots p_{s_m}^{l_m}$, where $1 \leq s_1 < \dots < s_m \leq N$ and $1 \leq l_i \leq n_{s_i}$, while v_0 is a divisor of W_0 . Applying Proposition 3.1 with $v_i = p_{s_i}^{l_i}$ for $i \in \{1, \dots, m\}$, we find $e_v \simeq_{\mathbf{hmf}(R, W)} \bigoplus_{i=0}^m e_{v_i}$, where we defined $u_0 = W/v_0 = \frac{W_0}{v_0} p_1^{n_1} \dots p_N^{n_N}$ and $u_i = W/v_i = W_0 p_1^{n_1} \dots p_{s_i}^{n_{s_i} - l_i} \dots p_N^{n_N}$ for $i \in \{1, \dots, m\}$. We have $(u_0, v_0) = (u_0, W_0/u_0)$. Since $W_0 = u_0 v_0$, it follows that $(u_0, v_0)^2 | (W_0)$. Since W_0 has no critical divisors, we must have $(u_0, v_0) = (1)$ and hence $e_{v_0} \simeq_{\mathbf{hmf}(R, W)} 0$. For $i \in \{1, \dots, m\}$, we have $(u_i, v_i) = p_{s_i}^{l_i}$, where

$\mu_i \stackrel{\text{def.}}{=} \min(l_i, n_{s_i} - l_i)$. Thus e_{v_i} is primary of order μ_{s_i} when $\mu_{s_i} \geq 1$ and trivial when $\mu_i = 0$. This gives a direct sum decomposition:

$$e_v \simeq_{\text{hmf}(R,W)} e_{v_0} \oplus_{i \in \{1, \dots, m \mid l_i \leq n_{s_i} - 1\}} e_{p_{s_i}^{l_i}} \simeq \oplus_{i \in \{1, \dots, m \mid l_i < n_{s_i}\}} e_{p_{s_i}^{l_i}} \quad ,$$

where all matrix factorizations in the direct sum are primary except for e_{v_0} . If $l_i = n_{s_i}$ for all $i \in \{1, \dots, m\}$, then the sum in the right hand side is the zero object of $\text{hmf}(R, W)$. We conclude that any elementary matrix factorization decomposes into a finite direct sum of primary matrix factorizations. On the other hand, any matrix factorization of W decomposes as a finite direct sum of elementary factorizations and hence also as a finite direct sum of primary factorizations whose prime supports are the prime divisors of W . By Proposition 2.24, every primary matrix factorization has a quasi-local endomorphism ring. \square

Theorem 3.3 *Suppose that R is a Bézout domain and W has the decomposition (1.4). Then there exists an equivalence of categories:*

$$\mathbf{hef}(R, W) \simeq \vee_{i=1}^N \mathbf{hef}(R, p_i^{n_i}) \quad ,$$

where \vee denotes the coproduct of additive categories.

Proof. Theorem 3.2 and Proposition 3.1 imply that $\mathbf{hef}(R, W)$ is additively generated by the additive subcategories $\mathbf{hef}_{p_i^{n_i}}(R, W) \simeq \mathbf{hef}(R, p_i^{n_i})$, where we used Proposition 2.17. These categories are mutually orthogonal by Lemma 2.25. \square

3.1. A conjecture. Consider the inclusion functor:

$$\iota : \mathbf{hef}(R, W) \rightarrow \text{hmf}(R, W)$$

Conjecture 3.4 *The inclusion functor ι is an equivalence of R -linear categories.*

Conjecture 3.4 and Theorem 3.2 imply:

Conjecture 3.5 *Let R be a Bézout domain and W be a critically-finite element of R . Then $\text{hmf}(R, W)$ is a Krull-Schmidt category.*

In [5], we establish Conjecture 3.4 for the case when R is an elementary divisor domain. This shows that Conjecture 3.4 is implied by the still unsolved conjecture [8] that any Bézout domain is an elementary divisor domain. Some recent work on that conjecture can be found in [9].

4. Counting elementary factorizations

In this section, we give formulas for the number of isomorphism classes of objects in the categories $\mathbf{HEF}(R, W)$ and $\mathbf{hef}(R, W)$ when W is critically-finite.

4.1. *Counting isomorphism classes in $\text{HEF}(R, W)$.* Let $W = W_0 W_c$ be a critically-finite element of R , where $W_0 \in R$ is non-critical and $W_c = p_1^{n_1} \dots p_r^{n_r}$ with prime $p_j \in R$ and $n_j \geq 2$ (see Definition 1.7). Let $\mathcal{H}ef(R, W)$ denote the set of isomorphism classes of objects in the category $\text{HEF}(R, W)$. We are interested in the cardinality:

$$N(R, W) \stackrel{\text{def.}}{=} |\mathcal{H}ef(R, W)|$$

of this set. In this subsection, we derive a formula for $N(R, W)$ as a function of the orders n_i of the prime elements p_i arising in the prime decomposition of W_c . The main result of this subsection is Theorem 4.12 below.

Lemma 4.1 *The cardinality $N(R, W)$ depends only on the critical part W_c of W .*

Proof. Let $W = puv$ with a divisor p coprime with both u and v . Taking $b = p$ and $c = 1$ in Proposition 2.4 gives:

$$\begin{bmatrix} 0 & pv \\ u & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & v \\ pu & 0 \end{bmatrix} . \quad (4.1)$$

Together with Corollary 2.5, this implies $N(R, W) = N(R, W_c)$. \square

From now on, we will assume that $W \in R$ is fixed and is of the form:

$$W = W_c = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r} . \quad (4.2)$$

To simplify notations, we will omit to indicate the dependence of some quantities on W .

Definition 4.2 *Let T be a non-empty set. A map $f : \text{ObEF}(R, W) \rightarrow T$ is called an elementary invariant if $f(e_1) = f(e_2)$ for any $e_1, e_2 \in \text{ObEF}(R, W)$ such that $e_1 \simeq_{\text{HEF}(R, W)} e_2$. An elementary invariant f is called complete if the map $\underline{f} : \mathcal{H}ef(R, W) \rightarrow T$ induced by f is injective.*

To determine $N(R, W)$, we will construct a complete elementary invariant. Let:

$$I \stackrel{\text{def.}}{=} \{1, \dots, r\} ,$$

where r is the number of non-associated prime factors of W , up to association in divisibility.

Similarity classes of elementary factorizations and normalized divisors of W . Let $\text{HEF}_{\text{sim}}(R, W)$ be the groupoid having the same objects as $\text{HEF}(R, W)$ and morphisms given by similarity transformations of elementary factorizations and let $\mathcal{H}ef_{\text{sim}}(R, W)$ be its set of isomorphism classes. Since the similarity class of an elementary factorization e_v is uniquely determined by the principal ideal $\langle v \rangle$ generated by the divisor v of W , the map $e_v \rightarrow \langle v \rangle$ induces a bijection:

$$\mathcal{H}ef_{\text{sim}}(R, W) \simeq \text{Div}(W) ,$$

where:

$$\text{Div}(W) \stackrel{\text{def.}}{=} \{ \langle v \rangle, |v|W \} = \{ \langle v \rangle \mid v \in R : W \in \langle v \rangle \}$$

is the set of principal ideals of R containing W . Let:

$$\text{Div}_1(W) \stackrel{\text{def.}}{=} \left\{ \prod_{i \in I} p_i^{k_i} \mid \forall i \in I : k_i \in \{0, \dots, n_i\} \right\} , \quad (4.3)$$

be the set of *normalized divisors* of W . The map $v \rightarrow \langle v \rangle$ induces a bijection between $\text{Div}_1(W)$ and $\text{Div}(W)$. Indeed, any principal ideal of R which contains W has a unique generator which belongs to $\text{Div}_1(W)$, called its *normalized generator*. Given any divisor v of W , its *normalization* v_0 is the unique normalized divisor $v_0 \in \text{Div}_1(W)$ such that $\langle v \rangle = \langle v_0 \rangle$. Given two divisors t, s of W , their *normalized greatest common divisor* is the unique normalized divisor $(t, s)_1$ of W which generates the ideal $Rt + Rs$. The *set of exponent vectors* of W is defined through:

$$A_W \stackrel{\text{def.}}{=} \prod_{i=1}^r \{0, \dots, n_i\} .$$

The map $A_W \ni \mathbf{k} = (k_1, \dots, k_r) \mapsto \prod_{i \in I} p_i^{k_i} \in \text{Div}_1(W)$ is bijective, with inverse $\boldsymbol{\mu} : \text{Div}_1(W) \rightarrow A_W$ given by:

$$\boldsymbol{\mu}(v) = (\text{ord}_{p_1}(v), \dots, \text{ord}_{p_r}(v)) = (k_1, \dots, k_r) \quad \text{for } v = \prod_{i \in I} p_i^{k_i} \in \text{Div}_1(W) .$$

Combining everything, we have natural bijections:

$$\mathcal{H}ef_{\text{sim}}(R, W) \simeq \text{Div}(R, W) \simeq \text{Div}_1(R, W) \simeq A_W .$$

Remark 4.1. In Proposition 2.16, the quantity s_1 was an arbitrary element of the class (u_1, v_1) for e_{v_1} . For a fixed critically-finite W , we have a canonical choice for this quantity, namely the normalized gcd of u_1 and v_1 . Thus we define $s(e_1) = (u_1, v_1)_1$. The two definitions are connected by the relation $(s_1) = (s(e_1))$. Below we introduce “normalized” quantities $x(e), y(e)$ which belong to the same classes in $R^\times/U(R)$ as the quantities x and y defined in Section 2. The results of Section 2 hold automatically for these normalized choices.

Given $t \in \text{Div}(W)$, its *index set* is the subset of I given by:

$$I(t) \stackrel{\text{def.}}{=} \text{supp } \boldsymbol{\mu}(t_0) = \{i \in I \mid p_i | t\} .$$

Notice that $I(t)$ depends only on the principal ideal $\langle t \rangle$, that in turn depends only on the class $(t) \in R^\times/U(R)$. This gives a map from $\text{Div}(W)$ to the power set $\mathcal{P}(I)$ of I . Note that $(t) = (1)$ iff $I(t) = \emptyset$.

The essence and divisorial invariant of an elementary factorization. Consider an elementary factorization e of W and let $v(e) \in \text{Div}_1(W)$ be the unique normalized divisor of W for which e is similar to e_v . Let $u = u(e) \stackrel{\text{def.}}{=} W/v \in \text{Div}_1(W)$ and let $s(e) \stackrel{\text{def.}}{=} (v, u)_1 \in \text{Div}_1(W)$ be the normalized greatest common denominator of v and u . Let $I_s(e) \stackrel{\text{def.}}{=} I(s(e))$. Let $m_i(e) \stackrel{\text{def.}}{=} \text{ord}_{p_i}(s(e))$ and $\mathbf{m}(e) = \boldsymbol{\mu}(s(e)) = (m_1(e), \dots, m_n(e))$. Then $I_s(e) = \text{supp } \mathbf{m}(e)$ and:

$$s(e) = \prod_{i \in I_s(e)} p_i^{m_i(e)} . \tag{4.4}$$

Let $v'(e) \stackrel{\text{def.}}{=} v/s(e)$ and $u'(e) \stackrel{\text{def.}}{=} u/s(e)$. Then $(v'(e), u'(e)) = (1)$ and $W = v(e)u(e) = v'(e)u'(e)s(e)^2$. Define:

$$x(e) \stackrel{\text{def.}}{=} (s(e), v'(e))_1 , \quad y(e) \stackrel{\text{def.}}{=} (s(e), u'(e))_1 \tag{4.5}$$

and:

$$I_x(e) \stackrel{\text{def.}}{=} I(x(e)) , \quad I_y(e) \stackrel{\text{def.}}{=} I(y(e)) .$$

Notice that $(x(e), y(e))_1 = 1$, thus $I_x(e) \cap I_y(e) = \emptyset$. Defining $v''(e) \stackrel{\text{def.}}{=} v'(e)/x(e)$ and $u''(e) \stackrel{\text{def.}}{=} u'(e)/y(e)$, we have:

$$W = x(e)y(e)v''(e)u''(e)s(e)^2 \quad ,$$

where $v''(e), u''(e)$ and $s(e)$ are mutually coprime. Moreover, we have:

$$\text{ord}_{p_i}(v'(e)) = n_i - 2m_i(e) \quad \forall i \in I_x(e) \quad \text{and} \quad \text{ord}_{p_i}(u'(e)) = n_i - 2m_i(e) \quad \forall i \in I_y(e) \quad ,$$

which implies:

$$\begin{aligned} \text{ord}_{p_i}x(e) &= \max(m_i(e), n_i - 2m_i(e)) \quad \text{for } i \in I_x(e) \quad , \\ \text{ord}_{p_i}y(e) &= \max(m_i(e), n_i - 2m_i(e)) \quad \text{for } i \in I_y(e) \quad . \end{aligned} \quad (4.6)$$

Notice that $\text{ord}_{p_i}x(e)y(e) \equiv n_i \pmod{2}$ for $i \in I_x \cup I_y$ if $3m_i < n_i$.

Definition 4.3 *The essence $z := z(e)$ of an elementary factorization e of W is the normalized divisor of W defined through:*

$$z(e) \stackrel{\text{def.}}{=} \prod_{I_z(e)} p_i^{m_i(e)} \quad , \quad \text{where} \quad I_z(e) \stackrel{\text{def.}}{=} I(s) \setminus (I_x(e) \cup I_y(e)) \quad . \quad (4.7)$$

An elementary factorization e is called *essential* if $z(e) = 1$, i.e. if $I_z(e) = \emptyset$.

The divisor s defines x, y and z uniquely by (4.6) and (4.7). These 3 divisors in turn also define s uniquely that can be seen by inverting the max functions above:

$$\text{ord}_{p_i}s = m_i = \begin{cases} \text{ord}_{p_i}x(e) & \text{if } i \in I_x(e) \text{ and } 3\text{ord}_{p_i}x(e) \geq n_i \\ (n_i - \text{ord}_{p_i}x(e))/2 & \text{if } i \in I_x(e) \text{ and } 3\text{ord}_{p_i}x(e) < n_i \\ \text{ord}_{p_i}y(e) & \text{if } i \in I_y(e) \text{ and } 3\text{ord}_{p_i}y(e) \geq n_i \\ (n_i - \text{ord}_{p_i}y(e))/2 & \text{if } i \in I_y(e) \text{ and } 3\text{ord}_{p_i}y(e) < n_i \\ \text{ord}_{p_i}z(e), & \text{if } i \in I_z(e) \quad . \end{cases} \quad (4.8)$$

The fundamental property of an essential factorization of e is the equality of sets $I_s(e) = I_x(e) \sqcup I_y(e)$, which will allow us to compute the number $N_\emptyset(R, W)$ of isomorphism classes of such factorizations (see Proposition 4.11 below). Then $N(R, W)$ will be determined by relating it to N_\emptyset for various reductions of the potential W .

Notice that the essence $z(e)$ is a critical divisor of W and that we have $(z(e), v'(e))_1 = (z(e), u'(e))_1 = 1$. Since $W = v(e)u(e) = v'(e)u'(e)s(e)^2$, this gives:

$$\text{ord}_{p_i}W = n_i = 2m_i(e) = 2\text{ord}_{p_i}z(e) \quad \text{for any } i \in I_z(e) \quad . \quad (4.9)$$

Definition 4.4 *The divisorial invariant of an elementary factorization e of W is the element $h(e)$ of the set $\text{Div}_1(W) \times \text{Sym}^2(\mathcal{P}(I))$ defined through:*

$$h(e) = (s(e), \{I_x(e), I_y(e)\}) \quad .$$

This gives a map $h : \text{EF}(R, W) \rightarrow \text{Div}_1(W) \times \text{Sym}^2(\mathcal{P}(I))$.

We have already given a criterion for two elementary factorizations of W to be isomorphic in Proposition 2.16. There exists another way to characterize when two objects of $\text{HEF}(R, W)$ (and also of $\text{hef}(R, W)$) are isomorphic, which will be convenient for our purpose.

Proposition 4.5 *Consider two elementary factorizations of W . The following statements are equivalent:*

1. *The two factorizations are isomorphic in $\text{HEF}(R, W)$ (respectively in $\text{hef}(R, W)$).*
2. *The two factorizations have the same $(s, \{x, y\})$ (respectively same (s, x, y)).*
3. *The two factorizations have the same divisorial invariant $(s, \{I_x, I_y\})$ (respectively same (s, I_x, I_y)).*

In particular, the divisorial invariant $h : \text{ObEF}(R, W) \rightarrow \text{Div}_1(W) \times \text{Sym}^2(\mathcal{P}(I))$ is a complete elementary invariant.

Proof. The equivalence between 1. and 2. follows from Proposition 2.16. Indeed, the proposition shows that for two isomorphic factorizations e_1 and e_2 the corresponding s_1 and s_2 are similar: $(s_1) = (s_2)$ in the notations of Section 2. We compute $(s_2) = (u_2, v_2) = ((u_2)_1, (v_2)_1) = ((u(e_2), v(e_2))_1) = (s(e_2))$ with the last $s(e_2)$ defined in Div_1 by (4.4). Similarly $(s_1) = (s(e_1))$. By the very definition of Div_1 we have $(s(e_1)) = (s(e_2))$ implies $s(e_1) = s(e_2)$.

The implication 2. \Rightarrow 3. is obvious. Thus it suffices to prove that 3. implies 2. For this, let e_1 and e_2 be the two elementary factorizations of W . Assume that $s(e_1) = s(e_2)$ and $\{I_x(e_1), I_y(e_1)\} = \{I_x(e_2), I_y(e_2)\}$ and let $s := s(e_1) = s(e_2) = \prod_{i \in I(s)} p_i^{m_i}$. Consider the case $I_x(e_1) = I_x(e_2)$ and $I_y(e_1) = I_y(e_2)$. Applying (4.6) to $v = v(e_1)$ and $v = v(e_2)$ and using the relations $m_i(e_1) = \text{ord}_{p_i} s = m_i(e_2)$ gives:

$$x(e_1) = x(e_2) \quad \text{and} \quad y(e_1) = y(e_2) \quad .$$

When $I_x(e_1) = I_y(e_2)$ and $I_x(e_2) = I_y(e_1)$, a similar argument gives $x(e_1) = y(e_2)$ and $y(e_1) = x(e_2)$. \square

Proposition 4.6 *The map $z : \text{ObEF}(R, W) \rightarrow \text{Div}_1(W)$ which gives the essence of an elementary factorization is an elementary invariant.*

Proof. Let e_1 and e_2 be two factorizations of W which are isomorphic in $\text{HEF}(R, W)$. By Proposition 4.5, we have $s(e_1) = s(e_2)$ and $\{I_x(e_1), I_y(e_1)\} = \{I_x(e_2), I_y(e_2)\}$. Hence:

$$I(z(e_1)) = I(s(e_1)) \setminus (I_x(e_1) \cup I_y(e_1)) = I(s(e_2)) \setminus (I_x(e_2) \cup I_y(e_2)) = I(z(e_2)) \quad . \quad (4.10)$$

Applying (4.9) for $e = e_1$ and $e = e_2$ gives $\text{ord}_{p_i} z(e_1) = \text{ord}_{p_i} z(e_2)$ for any $i \in I(z(e_1)) = I(z(e_2))$. Thus $z(e_1) = z(e_2)$. \square

The essential reduction of an elementary factorization. For any normalized critical divisor z of W , let $\text{HEF}_z(R, W)$ denote the full subcategory of $\text{HEF}(R, W)$ consisting of those elementary factorizations whose essence equals z and let $\mathcal{H}ef_z(R, W)$ be its set of isomorphism classes. Then $\mathcal{H}ef_1(R, W)$ consists of the isomorphism classes of essential factorizations.

Definition 4.7 *The essential reduction of an elementary factorization $e := e_v$ of W is the essential elementary factorization of $W/z(e)^2$ defined through:*

$$\text{essred}(e) \stackrel{\text{def.}}{=} e_{v/z(e)} .$$

This gives a map $\text{essred} : \text{ObEF}(R, W) \rightarrow \text{ObHEF}_1(R, W/z(e)^2)$.

To see that essred is well-defined, consider the elementary factorization $\tilde{e} = e_{v/z(e)}$:

$$\widetilde{W} \stackrel{\text{def.}}{=} W/z(e)^2 = u(\tilde{e})v(\tilde{e}) , \quad (4.11)$$

where $v(\tilde{e}) = v(e)/z(e)$ and $u(\tilde{e}) = u(e)/z(e)$. We compute:

$$s(\tilde{e}) \stackrel{\text{def.}}{=} (v(\tilde{e}), u(\tilde{e}))_1 = (v(e)/z(e), u(e)/z(e))_1 = s(e)/z(e)$$

and $v'(\tilde{e}) \stackrel{\text{def.}}{=} v(\tilde{e})/s(\tilde{e}) = v'(e)$, $u'(\tilde{e}) \stackrel{\text{def.}}{=} u(\tilde{e})/s(\tilde{e}) = u'(e)$. Thus $x(\tilde{e}) \stackrel{\text{def.}}{=} (s(\tilde{e}), v'(\tilde{e}))_1 = x(e)$ and $y(\tilde{e}) \stackrel{\text{def.}}{=} (s(\tilde{e}), u'(\tilde{e}))_1 = y(e)$. By (4.7) applied to \tilde{e} and e , we derive $I_z(\tilde{e}) = I_s(\tilde{e}) \setminus (I_x(\tilde{e}) \cup I_y(\tilde{e})) = I_z(e) \cup I_s(e) \setminus (I_x(e) \cup I_y(e)) = \emptyset$, which implies $z(\tilde{e}) = 1$. Hence $\text{essred}(e)$ is an essential elementary matrix factorization of \widetilde{W} . Also notice the relation:

$$(z(e), W/z(e)^2) = (1) ,$$

which follows from the fact that $z(e)$ is coprime with $v'(e)$ and $u'(e)$.

Lemma 4.8 *For any critical divisor z of W such that $(z, W/z^2) \sim 1$, the map essred induces a well-defined bijection $\text{essred}_z : \mathcal{Hef}_z(R, W) \xrightarrow{\sim} \mathcal{Hef}_1(R, W/z^2)$.*

Proof. We perform the proof in two steps:

1. Let $e_1 := e_{v_1}$ and $e_2 := e_{v_2}$ be two elementary factorizations of W such that $z(e_1) = z(e_2) = z$ and let $v_1 = v(e_1)$, $v_2 = v(e_2)$. Define also $v_3 \stackrel{\text{def.}}{=} v_1/z$ and $v_4 \stackrel{\text{def.}}{=} v_2/z$. To show that essred_z is well-defined, we have to show that $e_1 \simeq_{\text{HEF}(R, W)} e_2$ implies that the two essential elementary factorizations $e_3 := e_{v_3}$ and $e_4 := e_{v_4}$ of $\widetilde{W} \stackrel{\text{def.}}{=} W/z^2$ are isomorphic in $\text{HEF}(R, \widetilde{W})$. For this, we compute:

$$s(e_1) \stackrel{\text{def.}}{=} (v_1, u_1)_1 = (z(e_1) \cdot v_3, z(e_1) \cdot u_3)_1 = z(e_1) \cdot (v_3, u_3)_1 = z(e_1) \cdot s(e_3) .$$

Thus:

$$x(e_1) \stackrel{\text{def.}}{=} (s(e_1), v'_1)_1 = (z(e_1) \cdot s(e_3), z(e_1)v_3/z(e_1)s(e_3))_1 = (s(e_3), v'_3)_1 = x(e_3).$$

The third equality above holds since $(z(e_1), v'_1)_1 = 1$ and thus $(z(e_1), v_3)_1 = 1$. Similarly, we have $s(e_2) = z(e_2) \cdot s(e_4)$ and we find $y(e_1) = y(e_3)$ as well as $x(e_2) = x(e_4)$ and $y(e_2) = y(e_4)$. By Proposition 4.5, the condition $e_1 \simeq_{\text{HEF}(R, W)} e_2$ implies $s(e_1) = s(e_2) = s$ and $I(s(e_1)) = I(s(e_2))$, thus $z(e_1) = z(e_2) = z$. If $(s(e_1), \{x(e_1), y(e_1)\}) = (s(e_2), \{x(e_2), y(e_2)\})$, then $(s(e_3), \{x(e_3), y(e_3)\}) = (s(e_4), \{x(e_4), y(e_4)\})$. Thus $e_3 \simeq e_4$.

2. Let z be a critical divisor of W such that $(z, W/z^2) = (1)$. For any essential elementary factorization e_v of \widetilde{W} , the elementary factorization e_{zv} of W is an object of $\text{HEF}_z(R, W)$ and we have $\text{essred}(e_{zv}) = e_v$. This shows that essred_z is surjective. Now let e_3 and e_4 be two essential elementary factorizations of $\widetilde{W} \stackrel{\text{def.}}{=} W/z^2$ which are isomorphic in $\text{HEF}(R, \widetilde{W})$.

Let $v_1 \stackrel{\text{def.}}{=} zv_3$ and $v_2 \stackrel{\text{def.}}{=} zv_4$. To show that essred_z is injective, we have to show that the two elementary factorizations $e_1 := e_{v_1}$ and $e_2 := e_{v_2}$ of W are isomorphic in $\text{HEF}(R, W)$. For this, notice that $(s(e_3), \{x(e_3), y(e_3)\}) = (s(e_4), \{x(e_4), y(e_4)\})$ by Proposition 4.5. This implies $(s(e_1), \{x(e_1), y(e_1)\}) = (s(e_2), \{x(e_2), y(e_2)\})$, with $z(e_1) = z(e_2) = z$. Hence e_1 and e_2 are isomorphic in $\text{HEF}(R, W)$ by the same proposition. \square

A formula for $N(R, W)$ in terms of essential reductions. Let:

$$S \stackrel{\text{def.}}{=} \text{im } h \subset \text{Div}_1(W) \times \text{Sym}^2(\mathcal{P}(I)) \quad . \quad (4.12)$$

The degrees of the prime factors p_i in the decomposition (4.2) of W define on $I = \{1, \dots, r\}$ a \mathbb{Z}_2 -grading given by:

$$I^{\hat{0}} \stackrel{\text{def.}}{=} \{i \in I \mid n_i \text{ is even}\} \quad , \quad I^{\hat{1}} \stackrel{\text{def.}}{=} \{i \in I \mid n_i \text{ is odd}\} \quad . \quad (4.13)$$

Let:

$$r^{\hat{0}} \stackrel{\text{def.}}{=} |I^{\hat{0}}| \quad , \quad r^{\hat{1}} \stackrel{\text{def.}}{=} |I^{\hat{1}}| \quad .$$

Since $I = I^{\hat{0}} \sqcup I^{\hat{1}}$, we have $r = r^{\hat{0}} + r^{\hat{1}}$. Any non-empty subset $K \subset I$ is endowed with the \mathbb{Z}_2 -grading induced from I . For any critical divisor z of W , we have $z^2|W$, which implies $I(z) \subset I^{\hat{0}}$. For any subset $J \subset I^{\hat{0}}$, define:

$$z_J \stackrel{\text{def.}}{=} \prod_{i \in J} p_i^{n_i/2} \quad , \quad (4.14)$$

which is a normalized critical divisor of W satisfying $(z_J, W/z_J^2)_1 = 1$. Also define:

$$S_J \stackrel{\text{def.}}{=} h(\mathcal{H}ef_{z_J}(R, W)) \subset S \quad .$$

and:

$$N_J(R, W) \stackrel{\text{def.}}{=} |S_J| \quad . \quad (4.15)$$

Since h is a complete elementary invariant, we have $N_{\emptyset}(R, W) = |h(\mathcal{H}ef_1(R, W))| = |\mathcal{H}ef_1(R, W)|$. Moreover, Lemma 4.8 gives:

$$N_J(R, W) = N_{\emptyset}(R, W/z_J^2) \quad . \quad (4.16)$$

Proposition 4.9 *We have:*

$$N(R, W) = \sum_{J \subset I^{\hat{0}}} N_{\emptyset}(R, W/z_J^2) \quad . \quad (4.17)$$

Proof. Follows immediately from Lemma 4.8 and the remarks above. \square .

Computation of $N_{\emptyset}(R, W)$. Notice that $z_{\emptyset} = 1$. Since h is a complete elementary invariant, we have $N_{\emptyset}(R, W) = |S_{\emptyset}|$, where:

$$S_{\emptyset} = \{h(e) \mid e \in \text{ObHEF}(R, W) : z(e) = 1\} \quad .$$

We will first determine the cardinality of the set:

$$S_{\emptyset, k} \stackrel{\text{def.}}{=} \{h(e) \mid e \in \text{ObHEF}(R, W) : z(e) = 1 \text{ and } |I_s(e)| = k\} \quad .$$

We have:

$$S_{\emptyset} = \sqcup_{k=1}^r S_{\emptyset, k} \quad .$$

Lemma 4.10 For $k \geq 1$, we have:

$$|S_{\emptyset,k}| = 2^{k-1} \cdot \sum_{\substack{K \subset I, \\ |K|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor . \quad (4.18)$$

Proof. Consider a subset $K \subset I(s)$ of cardinality $|K| = k$. Since $s^2|W$, we have:

$$1 \leq m_i(e_v) \leq \lfloor (n_i - 1)/2 \rfloor \quad \forall i \in I(s) .$$

There are $\prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor$ different possibilities for s such that $I(s) = K$. We can have several elements $(s, \{x, y\})$ of $S_{1,k}$ with the same s since x and y can vary. This is where the coefficient 2^{k-1} in front of (4.18) comes from, as we now explain. Fixing the set $I(s)$ with $|I(s)| = k$, we have a set $\mathcal{P}(I(s))$ of 2^k partitions $I(s) = I_x \sqcup I_y$ as disjoint union of 2 sets. These can be parameterized by the single subset $I_v \subset I(s)$ since $I_u = I(s) \setminus I_v$. Define:

$$S_{\emptyset,k,s} = \{h(e) \mid e \in \text{ObHEF}(R, W) : z(e) = 1 \text{ and } |I_s(e)| = k \text{ and } s(e) = s\} .$$

Consider the surjective map

$$\Psi : \mathcal{P}(I(s)) \rightarrow S_{\emptyset,k,s}$$

which sends a partition $\beta = (I_1, I_2)$ of $I(s)$ to the element $\alpha = (s, \{I_1, I_2\})$. The preimage $\Psi^{-1}(h(e))$ of an element $h(e) \in S_{\emptyset,k,s}$ consist of two elements : (I_1, I_2) and (I_2, I_1) . Thus the map is 2:1. This holds for every K with $|K| = k$. Comparing the cardinalities of $\mathcal{P}(I(s))$ and $S_{\emptyset,k,s}$, we find:

$$|S_{\emptyset,k,s}| = |\mathcal{P}(I(s))|/2 = 2^k/2 = 2^{k-1} .$$

This holds for any s with $I(s) = K$, where $K \subset I$ has cardinality k . Since $S_{\emptyset,k} = \sqcup_s S_{\emptyset,k,s}$ and since the cardinality $|S_{\emptyset,k,s}|$ does not depend on s , we find:

$$|S_{\emptyset,k}| = \sum_s |S_{\emptyset,k,s}| = 2^{k-1} \sum_{\substack{K \subset I \\ |K|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor .$$

□

Proposition 4.11 With the definitions above, we have:

$$N_{\emptyset}(R, W) = |S_{\emptyset}| = 1 + \sum_{k=1}^r 2^{k-1} \sum_{\substack{K \subset I \\ |K|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor . \quad (4.19)$$

Proof. Since $S_{\emptyset} = \sqcup_{k=1}^r S_{\emptyset,k}$, the the previous lemma gives:

$$|S_{\emptyset}| = 1 + \sum_{k=1}^r |S_{\emptyset,k}| = 1 + \sum_{k=1}^r 2^{k-1} \sum_{\substack{K \subset I \\ |K|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor . \quad (4.20)$$

The term 1 in front corresponds to the unique element $(1, \{\emptyset, \emptyset\})$ of S . □

Computation of $N(R, W)$. The main result of this subsection is the following:

Theorem 4.12 *The number of isomorphism classes of $\text{HEF}(R, W)$ for a critically-finite W as in (4.2) is given by:*

$$N(R, W) = 2^{r^{\hat{0}}} + \sum_{k=0}^{r^{\hat{1}}} 2^{r^{\hat{0}}+k-1} \sum_{\substack{K \subseteq I \\ |K^{\hat{1}}|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor . \quad (4.21)$$

Proof. Combining Proposition 4.9 and Proposition 4.11, we have:

$$N(R, W) = \sum_{J \subset I^{\hat{0}}} \left(1 + \sum_{k=1}^{r-j} 2^{k-1} \sum_{\substack{K \subset I \setminus J \\ |K|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor \right) , \quad (4.22)$$

where $j \stackrel{\text{def.}}{=} |J|$. We will simplify this expression by changing the summation signs and applying the binomial formula.

Since $r_0 = |I_0|$ and $J \subset I^{\hat{0}}$, we have $j \leq r_0$. For fixed j , we have $\binom{r_0}{j}$ different subsets $J \subset I^{\hat{0}}$ of this cardinality. The contribution to $N_{\emptyset}(R, \widetilde{W})$ of any such J has the free coefficient 1. Then the free coefficient of $N(R, W)$ is:

$$\sum_{j=0}^{r^{\hat{0}}} \binom{r^{\hat{0}}}{j} = 2^{r^{\hat{0}}} . \quad (4.23)$$

For the other coefficients of (4.22), we consider a subset $K \subset I$ as in (4.19). Such an index set $K = K^{\hat{1}} \sqcup K^{\hat{0}} \subset I$ of cardinality $k = k^{\hat{1}} + k^{\hat{0}}$ appears in $N_{\emptyset}(R, \widetilde{W})$ if $K^{\hat{0}} \subset I^{\hat{0}} \setminus J$. The coefficient of $\prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor$ in $N(R, \widetilde{W})$ is 2^{k-1} for any of $\binom{r^{\hat{0}} - k^{\hat{0}}}{j}$ choices of J . It follows that this coefficient in $N(R, W)$ is:

$$\sum_{j=0}^{r^{\hat{0}} - k^{\hat{0}}} 2^{k-1} \binom{r^{\hat{0}} - k^{\hat{0}}}{j} = 2^{k^{\hat{0}} + k^{\hat{1}} - 1} 2^{r^{\hat{0}} - k^{\hat{0}}} = 2^{r^{\hat{0}} + k^{\hat{1}} - 1} . \quad (4.24)$$

Together with (4.23) and (4.24), relation (4.22) gives:

$$N(R, W) = 2^{r^{\hat{0}}} + \sum_{\substack{\emptyset \subset K \subseteq I \\ |K^{\hat{1}}|=k^{\hat{1}}}} 2^{r^{\hat{0}} + k^{\hat{1}} - 1} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor ,$$

which is equivalent to (4.21). \square

The two examples below illustrate how the coefficients behave for $r = 2$ and $r = 3$.

Example 4.1. Let $W = p_1^n p_2^m$ for prime elements $p_1, p_2 \in R$ such that $(p_1) \neq (p_2)$ and $n, m \geq 2$ with odd n and m . Then:

$$N(R, W) = 1 + \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{m-1}{2} \rfloor .$$

Example 4.2. Consider $W = p_1^n p_2^m p_3^k$ for primes $p_1, p_2, p_3 \in R$ which are mutually non-associated in divisibility and orders $n, m, k \geq 2$ subject to the condition that n and m are even while k is

odd. Then:

$$N(R, W) = 2 + 2 \lfloor \frac{n-1}{2} \rfloor + 2 \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{k-1}{2} \rfloor + 4 \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 2 \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{k-1}{2} \rfloor + 2 \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor . \quad (4.25)$$

4.2. Counting isomorphism classes in $\text{hef}(R, W)$. We next derive a formula for the number of isomorphism classes in the category $\text{hef}(R, W)$ for a critically-finite W (see Theorem 4.21 below). Since the morphisms of $\text{hef}(R, W)$ coincide with the even morphisms of $\text{HEF}(R, W)$, the number of isomorphism classes of $\text{hef}(R, W)$ is larger than $N(R, W)$. The simplest difference between the two cases arises from the fact that suspension does not preserve the isomorphism class of an elementary factorization in the category $\text{hef}(R, W)$. Let $\check{\mathcal{H}}ef(R, W)$ be the set of isomorphism classes of objects in $\text{hef}(R, W)$ and:

$$\check{N}(R, W) = |\check{\mathcal{H}}ef(R, W)| .$$

Lemma 4.13 *The cardinality $\check{N}(R, W)$ depends only on the critical part W_c of W .*

Proof. The proof is identical to that of Lemma 4.1.

Definition 4.14 *Let T be a non-empty set. A map $f : \text{ObEF}(R, W) \rightarrow T$ is called an even elementary invariant if $f(e_1) = f(e_2)$ for any $e_1, e_2 \in \text{ObEF}(R, W)$ such that $e_1 \simeq_{\text{hef}(R, W)} e_2$. An even elementary invariant f is called complete if the map $\underline{f} : \check{\mathcal{H}}ef(R, W) \rightarrow T$ induced by f is injective.*

As in the previous subsection, we will compute $\check{N}(R, W)$ by constructing an even complete elementary invariant.

Definition 4.15 *The even divisorial invariant of an elementary factorization e of W is the element $\check{h}(e)$ of the set $\text{Div}_1(W) \times \mathcal{P}(I)^2$ defined through:*

$$\check{h}(e) = (s(e), I_x(e), I_y(e)) .$$

This gives a map $\check{h} : \text{EF}(R, W) \rightarrow \text{Div}_1(W) \times \mathcal{P}(I)^2$.

Lemma 4.16 *The even divisorial invariant $\check{h} : \text{ObEF}(R, W) \rightarrow \text{Div}_1(W) \times \mathcal{P}(I)^2$ is a complete even elementary invariant.*

Proof. By Proposition 4.5, two elementary factorizations of W are isomorphic in $\text{hef}(R, W)$ iff they have the same (s, x, y) , which in turn is equivalent with coincidence of their even elementary invariants. \square

Using the essence $z(e)$ defined in (4.7), each elementary factorization e_v of W determines an elementary factorization $\text{essred}(e) \stackrel{\text{def.}}{=} e_{v/z(e)}$ of $\widetilde{W} \stackrel{\text{def.}}{=} W/z(e)^2$ (see Definition 4.7). For any normalized critical divisor z of W , let $\text{hef}_z(R, W)$ denote the full subcategory of $\text{hef}(R, W)$ consisting of those elementary factorizations whose essence equals z and let $\check{\mathcal{H}}ef_z(R, W)$ be its set of isomorphism classes.

Lemma 4.17 *For any critical divisor z of W such that $(z, W/z^2) = (1)$, the map essred induces a well-defined bijection $\text{essred}_z : \check{\mathcal{H}}ef_z(R, W) \xrightarrow{\sim} \check{\mathcal{H}}ef_1(R, W/z^2)$.*

Proof. The proof is almost identical to that of Lemma 4.8, but taking into account that in $\text{hef}(R, W)$ we deal only with the even morphisms of $\text{HEF}(R, W)$. \square

Let:

$$\check{S} \stackrel{\text{def.}}{=} \text{im } h \subset \text{Div}_1(W) \times \mathcal{P}(I)^2 . \quad (4.26)$$

For a subset $J \subset I^{\hat{0}}$, let:

$$\check{S}_J \stackrel{\text{def.}}{=} \check{h}(\check{\mathcal{H}}ef_{z_J}(R, W)) \subset \check{S} ,$$

where z_J was defined in (4.14). Define $\check{N}_J(R, W) \stackrel{\text{def.}}{=} |\check{S}_J|$ and $\check{N}_{\emptyset}(R, W) = |\check{h}(\check{\mathcal{H}}ef_1(R, W))| = |\check{\mathcal{H}}ef_1(R, W)|$, where the last equality holds since \check{h} is a complete even elementary invariant. We can again compute $\check{N}(R, W)$ in terms of $\check{N}_{\emptyset}(R, W)$:

Proposition 4.18 *We have:*

$$\check{N}(R, W) = \sum_{J \subset I^{\hat{0}}} \check{N}_{\emptyset}(R, W/z_J^2) . \quad (4.27)$$

Proof. Follows from Lemma 4.17. \square

Define:

$$\check{S}_{\emptyset} = \{ \check{h}(e) \mid e \in \text{ObHEF}(R, W) \text{ and } z(e) = 1 \}$$

and:

$$\check{S}_{\emptyset, k} \stackrel{\text{def.}}{=} \{ \check{h}(e) \mid e \in \text{ObHEF}(R, W), z(e) = 1 \text{ and } |I_s(e)| = k \} .$$

Lemma 4.19 *For $k \geq 1$, we have:*

$$|\check{S}_{\emptyset, k}| = 2^k \cdot \sum_{\substack{K \subset I, \\ |K|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor . \quad (4.28)$$

Proof. The proof is similar to that of Lemma 4.10. Consider a subset $K \subset I(s)$ of cardinality k . As in Lemma 4.10, there are $\prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor$ different possibilities for s such that $I(s) = K$. Fixing the set $I(s)$ with $|I(s)| = k$, we have a set $\mathcal{P}(I(s))$ of 2^k partitions $I(s) = I_x \sqcup I_y$. Define:

$$\check{S}_{\emptyset, k, s} = \{ \check{h}(e) \mid e \in \text{Obhef}(R, W), z(e) = 1, |I_s(e)| = k \text{ and } s(e) = s \} .$$

The map which sends a partition $\beta = (I_1, I_2)$ of $I(s)$ to the element $\alpha = (s, I_1, I_2)$ is a bijection. We compute:

$$|\check{S}_{\emptyset, k, s}| = |\mathcal{P}(I(s))| = 2^k .$$

This holds for any s with $I(s) = K$, where $K \subset I$ has cardinality k . Since $\check{S}_{\emptyset, k} = \sqcup_s \check{S}_{\emptyset, k, s}$ and since the cardinality $|\check{S}_{\emptyset, k, s}|$ does not depend on s , we find:

$$|\check{S}_{\emptyset, k}| = \sum_s |\check{S}_{\emptyset, k, s}| = 2^k \cdot \sum_{K \subset I: |K|=k} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor .$$

\square

An immediate consequence is the following:

Proposition 4.20 *With the definitions above, we have:*

$$\check{N}_\emptyset(R, W) = |\check{S}_\emptyset| = \sum_{k=0}^r 2^k \sum_{\substack{K \subset I \\ |K|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor . \quad (4.29)$$

We are now ready to compute $\check{N}(R, W)$.

Theorem 4.21 *The number of isomorphism classes of the category $\text{hef}(R, W)$ for a critically-finite W as in (4.2) is given by:*

$$\check{N}(R, W) = \sum_{k=0}^{r^{\hat{1}}} \sum_{\substack{K \subsetneq I, \\ |K^{\hat{1}}|=k}} 2^{r^{\hat{0}}+k} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor , \quad (4.30)$$

Proof. Using Proposition 4.18 and Proposition 4.20, we write:

$$\check{N}(R, W) = \sum_{J \subset I^{\hat{0}}} \left(\sum_{k=0}^{r-j} 2^k \sum_{\substack{K \subset I \setminus J \\ |K|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor \right) , \quad (4.31)$$

where $j \stackrel{\text{def.}}{=} |J|$. Consider a subset $K \subset I$ as in (4.29). Such an index set $K = K^{\hat{1}} \sqcup K^{\hat{0}} \subset I$ of cardinality $k = k^{\hat{1}} + k^{\hat{0}}$ appears in $\check{N}_\emptyset(R, \widetilde{W})$ if $K^{\hat{0}} \subset I^{\hat{0}} \setminus J$. The coefficient of $\prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor$ in $\check{N}(R, \widetilde{W})$ is 2^k for any of $\binom{r^{\hat{0}} - k^{\hat{0}}}{j}$ choices of J . It follows that this coefficient in $\check{N}(R, W)$ is:

$$\sum_{j=0}^{r^{\hat{0}} - k^{\hat{0}}} 2^k \binom{r^{\hat{0}} - k^{\hat{0}}}{j} = 2^{k^{\hat{0}} + k^{\hat{1}}} 2^{r^{\hat{0}} - k^{\hat{0}}} = 2^{r^{\hat{0}} + k^{\hat{1}}} . \quad (4.32)$$

Together with (4.32), relation (4.31) yields (4.30). \square

5. Some examples

In this section, we discuss a few classes of examples to which the results of the previous sections apply. Subsection 5.1 considers the ring of complex-valued holomorphic functions defined on a smooth, non-compact and connected Riemann surface, which will be discussed in more detail in a separate paper. Subsection 5.2 considers rings arising through the Krull-Kaplansky-Jaffard-Ohm construction, which associates to any lattice-ordered Abelian group a Bézout domain having that ordered group as its group of divisibility. Subsection 5.3 discusses Bézout domains with a specified spectral poset, examples of which can be produced by a construction due to Lewis.

5.1. Elementary holomorphic factorizations over a non-compact Riemann surface. Let Σ be any non-compact connected Riemann surface (notice that such a surface need not be algebraic and that it may have infinite genus and an infinite number of ends). It is known that the cardinal Krull dimension of $O(\Sigma)$ is independent of Σ and is greater than or equal to 2^{\aleph_1} (see [10,11]). The following classical result (see [12,13]) shows that the \mathbb{C} -algebra of holomorphic functions entirely determines the complex geometry of Σ .

Theorem 5.1 (Bers) *Let Σ_1 and Σ_2 be two connected non-compact Riemann surfaces. Then Σ_1 and Σ_2 are biholomorphic iff $O(\Sigma_1)$ and $O(\Sigma_2)$ are isomorphic as \mathbb{C} -algebras.*

A Bézout domain R is called *adequate* if for any $a \in R^\times$ and any $b \in R$, there exist $r, s \in R$ such that $a = rs$, $(r, b) = (1)$ and any non-unit divisor s' of s satisfies $(s', b) \neq (1)$. It is known that any adequate Bézout domain R is an elementary divisor domain, i.e. any matrix with elements from R admits a Hermite normal form. The following result provides a large class of examples of non-Noetherian adequate Bézout domains:

Theorem 5.2 *For any smooth and connected non-compact Riemann surface Σ , the ring $O(\Sigma)$ is an adequate Bézout domain and hence an elementary divisor domain.*

Proof. The case $\Sigma = \mathbb{C}$ was established in [8,14]. This generalizes to any Riemann surface using [11,12]. Since $O(\Sigma)$ is an adequate Bézout domain, it is also a PM^* ring⁴ [8] and hence [15] an elementary divisor domain. The fact that $O(\Sigma)$ is an elementary divisor domain can also be seen as follows. Guralnick [16] proved that $O(\Sigma)$ is a Bézout domain of stable range one. By [17,18], this implies that $O(\Sigma)$ is an elementary divisor domain. \square

The prime elements of $O(\Sigma)$ are those holomorphic functions $f : \Sigma \rightarrow \mathbb{C}$ which have a single simple zero on Σ . This follows, for example, from the Weierstrass factorization theorem on non-compact Riemann surfaces (see [19, Theorem 26.7]). A critically-finite element $W \in O(\Sigma)$ has the form $W = W_0 W_c$, where $W_0 : \Sigma \rightarrow \mathbb{C}$ is a holomorphic function with (possibly infinite) number of simple zeroes and no multiple zeroes while W_c is a holomorphic function which has only a finite number of zeroes, all of which have multiplicity at least two. All results of this paper apply to this situation, allowing one to determine the homotopy category $\text{hef}(R, W)$ of elementary D-branes (and to count the isomorphism classes of such) in the corresponding *holomorphic* Landau-Ginzburg model [3,4] defined by (Σ, W) .

5.2. Constructions through the group of divisibility. Recall that the *group of divisibility* $G(R)$ of an integral domain R is the quotient group $K^\times/U(R)$, where K is the quotient field of R and $U(R)$ is the group of units of R . This is a partially-ordered Abelian group when endowed with the order induced by the R -divisibility relation, whose positive cone equals $R^\times/U(R)$. Equivalently, $G(R)$ is the group of principal non-zero fractional ideals of R , ordered by reverse inclusion. Since the positive cone generates $G(R)$, a theorem due to Clifford implies that $G(R)$ is a directed group (see [20, par. 4.3]). It is an open question to characterize those directed Abelian groups which arise as groups of divisibility of integral domains. It is known that $G(R)$ is totally-ordered iff R is a valuation domain, in which case $G(R)$ is order-isomorphic with the value group of R and the natural surjection of K^\times to $G(R)$ gives the corresponding valuation. Moreover, a theorem due to Krull [21] states that any totally-ordered Abelian group arises as

⁴ A PM^* -ring is a unital commutative ring R which has the property that any non-zero prime ideal of R is contained in a *unique* maximal ideal of R .

the group of divisibility of some valuation domain. It is also known⁵ that R is a UFD iff $G(R)$ is order-isomorphic with a (generally infinite) direct sum of copies of \mathbb{Z} endowed with the product order (see [20, Theorem 4.2.2]).

An ordered group (G, \leq) is called *lattice-ordered* if the partially ordered set (G, \leq) is a lattice, i.e. any two element subset $\{x, y\} \subset G$ has an infimum $\inf(x, y)$ and a supremum $\sup(x, y)$ (these two conditions are in fact equivalent for a group order); in particular, any totally-ordered Abelian group is lattice-ordered. Any lattice-ordered Abelian group is torsion-free (see [22, p. 10] or [23, 15.7]). The divisibility group $G(R)$ of an integral domain R is lattice-ordered iff R is a GCD domain [20]. In particular, the group of divisibility of a Bézout domain is a lattice-ordered group.

When R is a Bézout domain, the prime elements of R are detected by the lattice-order of $G(R)$ as follows. Given any Abelian lattice-ordered group (G, \leq) and any $x \in G$, let $\uparrow x \stackrel{\text{def.}}{=} \{y \in G \mid x \leq y\}$ and $\downarrow x \stackrel{\text{def.}}{=} \{y \in G \mid y \leq x\}$ denote the up and down sets determined by x . A *positive filter* of (G, \leq) is a filter of the lattice (G_+, \leq) , i.e. a proper subset $F \subset G_+$ having the following two properties:

1. F is upward-closed, i.e. $x \in F$ implies $\uparrow x \subset F$
2. F is closed under finite meets, i.e. $x, y \in F$ implies $\inf(x, y) \in F$.

Notice that $\uparrow x$ is a positive filter for any $x \in G_+$. A positive filter F of (G, \leq) is called:

- (a) *prime*, if $G_+ \setminus F$ is a semigroup, i.e. if $x, y \in G_+ \setminus F$ implies $x + y \in G_+ \setminus F$.
- (b) *principal*, if there exists $x \in F$ such that $F = \uparrow x$.

If R is a Bézout domain with field of fractions K and group of divisibility $G = K^\times / U(R)$, then the natural projection $\pi : K^\times \rightarrow G$ induces a bijection between the set of proper ideals of R and the set of positive filters of G , taking a proper ideal I to the positive filter $\pi(I \setminus \{0\})$ and a positive filter F to the proper ideal $\{0\} \cup \pi^{-1}(F)$ (see [24, 25]). This correspondence maps prime ideals to prime positive filters and non-zero principal ideals to principal positive filters. In particular, the prime elements of R correspond to the principal prime positive filters of G .

The following result shows (see [7, Theorem 5.3, p. 113]) that any lattice-ordered Abelian group is the group of divisibility of some Bézout domain, thus allowing one to construct a very large class of examples of such domains using the theory of lattice-ordered groups:

Theorem 5.3 (*Krull-Kaplansky-Jaffard-Ohm*) *If (G, \leq) is a lattice-ordered Abelian group, then there exists a Bézout domain R whose group of divisibility is order-isomorphic to (G, \leq) .*

For any totally ordered group G_0 , the result of Krull mentioned above gives a valuation ring whose divisibility group is order-isomorphic to G_0 . This valuation ring can be taken to be the group ring $k[G_0]$, where k is a field together with the following valuation on the field of fractions $k(G_0)$:

$$v \left(\frac{\sum_{i=1}^m a_i X_{g_i}}{\sum_{j=1}^n b_j X_{h_j}} \right) = \inf(g_1, \dots, g_m) - \inf(h_1, \dots, h_n) \quad ,$$

where it is assumed that all coefficients appearing in the expression are non-zero. Lorenzen [26] proved that every lattice-ordered group can be embedded into a direct product of totally ordered groups with the product ordering. This embedding is used by Kaplansky and Jaffard to construct the valuation domain R of Theorem 5.3. By the result of Lorenzen, there exists

⁵ Notice that a UFD is a Bézout domain iff it is Noetherian iff it is a PID (see Appendix B).

a lattice embedding $f : G \rightarrow H \stackrel{\text{def.}}{=} \prod_{\gamma \in \Gamma} G_\gamma$, where G_γ is a totally ordered group for all $\gamma \in \Gamma$ and H has the product ordering. Let $Q = k(\{X_g : g \in G\})$ be the group field with coefficients in a field k with the set of formal variables X_g indexed by elements of G . There is a valuation $\varphi : Q^\times \rightarrow H$. The integral domain R is the domain defined by this valuation, i.e. $R \stackrel{\text{def.}}{=} \{0\} \cup \{x \in Q^\times : \varphi(x) \geq 0\}$. It is proved by Ohm that the divisibility group of R is order-isomorphic to G . Combining this with the results of the previous sections, we have:

Proposition 5.4 *Let G be any lattice-ordered group and consider the ring R constructed by Theorem 5.3. Assume further that W is a critically-finite element of R . Then statements of Theorems 3.2, 3.3 and 4.12 hold.*

The simplest situation is when the lattice-ordered group G is totally ordered, in which case R is a valuation domain. Then a proper subset $F \subset G_+$ is a positive filter iff it is upward-closed, in which case the complement $G \setminus F$ is non-empty and downward-closed. If $G \setminus F$ has a greatest element m , then $G \setminus F = \downarrow m$ and $F = (\uparrow m) \setminus \{m\}$. If $G \setminus F$ does not have a greatest element, then $(G \setminus F, F)$ is a Dedekind cut of the totally-ordered set (G, \leq) . For example we can take $G \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ with the natural total order:

- When $G = \mathbb{Z}$, the Bézout domain R is a discrete valuation domain and thus a PID with a unique non-zero prime ideal and hence with a prime element $p \in R$ which is unique up to association in divisibility. In this case, any positive filter of \mathbb{Z} is principal and there is only one prime filter, namely $\uparrow 1 = \mathbb{Z}_+ \setminus \{0\}$. A critically-finite potential has the form $W = W_0 p^k$, where $k \geq 2$ and W_0 is a unit of R .
- When $G = \mathbb{Q}$, there are two types of positive filters. The first have the form $F = (\uparrow q) \setminus \{q\} = (q, +\infty) \cap \mathbb{Q}$ with $q \in \mathbb{Q}_{\geq 0}$, while the second correspond to Dedekind cuts and have the form $F = [a, +\infty) \cap \mathbb{Q}$ with $a \in \mathbb{R}_{>0}$. In particular, a positive filter is principal iff it has the form $F = [q, +\infty) \cap \mathbb{Q}$ with $q \in \mathbb{Q}_{>0}$. A principal positive filter can never be prime, since $\mathbb{Q}_+ \setminus F = [0, q) \cap \mathbb{Q}$ is not closed under addition for $q > 0$. Hence the Bézout domain R has no prime elements when $G = \mathbb{Q}$.
- When $G = \mathbb{R}$, any proper subset of \mathbb{R}_+ has an infimum and hence positive filters have the form $F = (a, +\infty)$ or $F = [a, +\infty)$ with $a \in \mathbb{R}_{>0}$, the latter being the principal positive filters. No principal positive filters can be prime, so the corresponding Bézout domain has no prime elements.

We can construct more interesting examples as follows. Let $(G_i, \leq_i)_{i \in I}$ be any family of lattice-ordered Abelian groups, where the non-empty index set I is arbitrary. Then the direct product group $G \stackrel{\text{def.}}{=} \prod_{i \in I} G_i$ is a lattice-ordered Abelian group when endowed with the product order \leq :

$$(g_i)_{i \in I} \leq (g'_i)_{i \in I} \quad \text{iff} \quad \forall i \in I : g_i \leq_i g'_i .$$

Let $\text{supp}(g) \stackrel{\text{def.}}{=} \{i \in I \mid g_i \neq 0\}$. The direct sum $G^0 \stackrel{\text{def.}}{=} \bigoplus_{i \in I} G_i = \{g = (g_i) \in G \mid |\text{supp}(g)| < \infty\}$ is a subgroup of G which becomes a lattice-ordered Abelian group when endowed with the order induced by \leq . For any $x = (x_i)_{i \in I} \in G$, we have $\uparrow_G x = \prod_{i \in I} \uparrow x_i$ while for any $x^0 \in G^0$, we have $\uparrow_{G^0} x^0 = \bigoplus_{i \in I} \uparrow x_i^0$, where \uparrow_G and \uparrow_{G^0} denote respectively the upper sets computed in G and G^0 . Hence:

I. The principal positive filters of G have the form $F = \prod_{i \in I} F_i$, where:

1. each F_i is a non-empty subset of G_{i+} which either coincides with $\uparrow 0_i$ or is a principal positive filter of (G_i, \leq_i)

2. at least one of F_i is a principal positive filter of (G_i, \leq_i) .

Such a principal positive filter F of G is prime iff the set $G_{i+} \setminus F_i$ is empty or a semigroup for all $i \in I$. In particular, the principal prime ideals of the Bézout domain R associated to G by the construction of Theorem 5.3 are in bijection with families (indexed by I) of principal prime ideals of the Bézout domains R_i associated to G_i by the same construction. The *non-zero* principal prime ideals of R are in bijection with families $(J_i)_{i \in S}$, where S is a non-empty subset of I and J_i is a non-zero principal prime ideal of R_i for each $i \in S$.

- II. The principal positive filters of G^0 have the form $F^0 = \bigoplus_{i \in I} F_i$, where $(F_i)_{i \in I}$ is a family of subsets of $F_i \subseteq G_{i+}$ such that the set $\text{supp } F \stackrel{\text{def.}}{=} \{i \in I \mid F_i \neq \uparrow_{G_i} 0_i\}$ is finite and non-empty and such that F_i is a principal positive filter of (G_i, \leq_i) for any $i \in \text{supp } F$. Such a principal positive filter F^0 of G^0 is prime iff F_i is prime in (G_i, \leq_i) for any $i \in \text{supp } F$. In particular, the non-zero principal prime ideals of the Bézout domain R^0 associated to G^0 by the construction of Theorem 5.3 are in bijection with finite families of the form J_{i_1}, \dots, J_{i_n} ($n \geq 1$), where i_1, \dots, i_n are distinct elements of I and J_{i_k} is a non-zero principal prime ideal of the Bézout domain R_{i_k} associated to G_{i_k} by the same construction.

It is clear that this construction produces a very large class of Bézout domains which have prime elements and hence to which Proposition 5.4 applies. For example, consider the direct power $G = \mathbb{Z}^I$ and the direct sum $G = \mathbb{Z}^{(I)}$, endowed with the product order. Then the Bézout domain R^0 associated to $\mathbb{Z}^{(I)}$ is a UFD whose non-zero principal prime ideals are indexed by the *finite* non-empty subsets of I . On the other hand, the non-zero principal prime ideals of the Bézout domain R associated to \mathbb{Z}^I are indexed by *all* non-empty subsets of I . Notice that R and R^0 coincide when I is a finite set.

5.3. Constructions through the spectral poset. Given a unital commutative ring R , its *spectral poset* is the prime spectrum $\text{Spec}(R)$ of R viewed as a partially-ordered set with respect to the order relation \leq given by inclusion between prime ideals. Given a poset (X, \leq) and two elements $x, y \in X$, we write $x \ll y$ if $x < y$ and x is an immediate neighbor of y , i.e. if there does not exist any element $z \in X$ such that $x < z < y$. It was shown in [27] that the spectral poset of any unital commutative ring satisfies the following two conditions, known as *Kaplansky's conditions*:

- I. Every non-empty totally-ordered subset of $(\text{Spec}(R), \leq)$ has a supremum and an infimum (in particular, \leq is a lattice order).
- II. Given any elements $x, y \in \text{Spec}(R)$ such that $x < y$, there exist distinct elements x_1, y_1 of $\text{Spec}(R)$ such that $x \leq x_1 < y_1 \leq y$ and such that $x_1 \ll y_1$.

It is known [28, 29] that these conditions are not sufficient to characterize spectral posets. It was shown in [30] that a poset (X, \leq) is order-isomorphic with the spectral poset of a unital commutative ring iff (X, \leq) is profinite, i.e. iff (X, \leq) is an inverse limit of finite posets; in particular, any finite poset is order-isomorphic with a spectral poset [29].

A partially ordered set (X, \leq) is called a *tree* if for every $x \in X$, the lower set $\downarrow x = \{y \in X \mid y \leq x\}$ is totally ordered. The following result was proved by Lewis:

Theorem 5.5 [29] *Let (X, \leq) be a partially-ordered set. Then the following statements are equivalent:*

- (a) (X, \leq) is a tree which has a unique minimal element $\theta \in X$ and satisfies Kaplansky's conditions I. and II.

(b) (X, \leq) is isomorphic with the spectral poset of a Bézout domain.

Moreover, R is a valuation domain iff (X, \leq) is a totally-ordered set.

An explicit Bézout domain R whose spectral poset is order-isomorphic with a tree (X, \leq) satisfying condition (a) of Theorem 5.5 is found by first constructing a lattice-ordered Abelian group G associated to (X, \leq) and then constructing R from G as in Theorem 5.3. The lattice-ordered group G is given by [29]:

$$G = \{f : X^* \rightarrow \mathbb{Z} \mid |\text{supp}(f)| < \infty\} \quad ,$$

where $X^* \stackrel{\text{def.}}{=} \{x \in X \mid \exists y \in X : y \ll x\}$ and $\text{supp } f \stackrel{\text{def.}}{=} \{x \in X^* \mid f(x) \neq 0\}$ is a tree when endowed with the order induced from X . The group operation is given by pointwise addition. The lattice order on G is defined by the positive cone:

$$G_+ \stackrel{\text{def.}}{=} \{f \in G \mid f(x) > 0 \ \forall x \in \text{minsupp}(f)\} = \{f \in G \mid f(x) \geq 1 \ \forall x \in \text{minsupp}(f)\} \quad , \quad (5.1)$$

where the order on \mathbb{Z} is the natural order and the *minimal support* of $f \in G$ is defined through:

$$\text{minsupp}(f) \stackrel{\text{def.}}{=} \{x \in \text{supp}(f) \mid \forall y \in X^* \text{ such that } y < x : f(y) = 0\} \quad . \quad (5.2)$$

Notice that $f \in G_+$ if $\text{minsupp}(f) = \emptyset$ (in particular, we have $0 \in G_+$). The lattice-ordered Abelian group G has the property that the set of its prime positive filters⁶ (ordered by inclusion) is order-isomorphic with the tree obtained from (X, \leq) by removing the minimal element θ (which corresponds to the zero ideal of R). Explicitly, the positive prime filter F_x associated to an element $x \in X \setminus \{\theta\}$ is defined through [29, p. 432]:

$$F_x \stackrel{\text{def.}}{=} \{f \in G_+ \mid \exists y \in \text{minsupp}(f) : y \leq x\} = \{f \in G_+ \mid \text{minsupp}(f) \cap (\downarrow x) \neq \emptyset\} \quad . \quad (5.3)$$

By Lemma 2.23, a principal prime ideal of a Bézout domain is necessarily maximal. This implies that the prime elements of R (considered up to association in divisibility) correspond to certain maximal elements of the tree (X, \leq) . Notice, however, that a Bézout domain can have maximal ideals which are not principal (for example, the so-called “free maximal ideals” of the ring of complex-valued holomorphic functions defined on a non-compact Riemann surface Σ [11]). For any maximal element x of X which belongs to X^* , let $1_x \in G$ be the element defined by the characteristic function of the set $\{x\}$ in X^* :

$$1_x(y) \stackrel{\text{def.}}{=} \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \in X^* \setminus \{x\} \end{cases} \quad .$$

Then $\text{supp}(1_x) = \text{minsupp}(1_x) = \{x\}$ and $1_x \in G_+ \setminus \{0\}$. Notice that $1_x \in F_x$.

Proposition 5.6 *Let (X, \leq) be a tree which has a unique minimal element and satisfies Kaplansky’s conditions I. and II. and let R be the Bézout domain determined by (X, \leq) as explained above.*

(a) *For each maximal element x of X which belongs to X^* , the principal positive filter $\uparrow 1_x$ is prime and hence corresponds to a principal prime ideal of R . Moreover, we have:*

$$\uparrow 1_x = \{f \in G_+ \mid \text{supp}(f) \cap \downarrow x \neq \emptyset\} \quad (5.4)$$

⁶ Called “prime V-segments” in [29].

and:

$$F_x = \{f \in \uparrow 1_x \mid \inf S_f(x) \in S_f(x)\} = \{f \in \uparrow 1_x \mid \exists \min S_f(x)\} \quad , \quad (5.5)$$

where:

$$S_f(x) \stackrel{\text{def.}}{=} \text{supp}(f) \cap \downarrow x \quad .$$

(b) Let W be a critically-finite element of R . Then the statements of Theorem 3.2, 3.3 and 4.12 hold.

Proof. The second statement follows from the results of the previous sections. To prove the first statement, let x be a maximal element of X which belongs to X^* . We have:

$$\uparrow 1_x = \{f \in G_+ \mid f - 1_x \in G_+\} = \{f \in G_+ \mid f(y) > 1_x(y) \ \forall y \in \text{minsupp}(f - 1_x)\} \quad . \quad (5.6)$$

On the other hand, we have $\text{minsupp}(f - 1_x) = \{y \in X^* \mid f(y) \neq 1_x(y) \ \& \ \forall z \in X^* \text{ such that } z < y : f(z) = 1_x(z)\}$. Since x is maximal, any element $z \in X^*$ for which there exists $y \in X^*$ such that $z < y$ satisfies $z \neq x$ and hence $1_x(z) = 0$. This gives:

$$\begin{aligned} \text{minsupp}(f - 1_x) &= \{y \in X^* \mid f(y) \neq 1_x(y) \ \& \ \forall z \in X^* \text{ such that } z < y : f(z) = 0\} \\ &= \begin{cases} \text{minsupp}(f) \cup \{x\} & \text{if } f \in Q_x \\ \text{minsupp}(f) \setminus \{x\} & \text{if } f \in G_+ \setminus Q_x \end{cases} \quad , \end{aligned}$$

where:

$$Q_x \stackrel{\text{def.}}{=} \{f \in G_+ \mid f(x) \neq 1 \ \& \ \forall z \in X^* \text{ such that } z < x : f(z) = 0\} = A_x \sqcup B_x \quad ,$$

with:

$$\begin{aligned} A_x &\stackrel{\text{def.}}{=} \{f \in G_+ \mid \forall z \in X^* \text{ such that } z \leq x : f(z) = 0\} = \{f \in G_+ \mid \text{supp}(f) \cap \downarrow x = \emptyset\} \\ B_x &\stackrel{\text{def.}}{=} \{f \in G_+ \mid x \in \text{minsupp}(f) \ \& \ f(x) > 1\} \subset F_x \subset G_+ \setminus A_x \quad . \end{aligned}$$

This gives:

$$\uparrow 1_x = (G_+ \setminus Q_x) \cup B_x = (G_+ \setminus A_x) \cup B_x = G_+ \setminus A_x = \{f \in G_+ \mid \text{supp}(f) \cap \downarrow x \neq \emptyset\} \quad , \quad (5.7)$$

which establishes (5.4). Notice that $G_+ \setminus (\uparrow 1_x) = A_x$ is a semigroup, so $\uparrow 1_x$ is a prime principal positive filter and hence it corresponds to a principal prime ideal of R . Also notice that $F_x \subset \uparrow 1_x$.

Consider an element $f \in \uparrow 1_x$. Then the non-empty set $S_f(x) \stackrel{\text{def.}}{=} \text{supp}(f) \cap \downarrow x$ is totally ordered (since X is a tree and hence $\downarrow x$ is totally ordered). By Kaplansky's condition I., this set has an infimum which we denote by $x_f = \inf S_f(x)$; notice that $x_f \in \downarrow x$. For any $y \in X^*$ with $y < x_f$, we have $y \notin S_f(x)$ and hence $f(y) = 0$. Hence if x_f belongs to $S_f(x)$ (i.e. if $S_f(x)$ has a minimum), then $x_f = \min S_f(x)$ is an element of $\text{minsupp}(f) \cap \downarrow x$ and in this case we have $f \in F_x$. Conversely, given any element $f \in F_x$, it is easy to see that the totally-ordered set $\text{minsupp}(f) \cap \downarrow x$ must be a singleton, hence $\text{minsupp}(f) \cap \downarrow x = \{x_f\}$ for a unique element $x_f \in S_f(x)$. This element must be a minimum (and hence an infimum) of the totally-ordered set $S_f(x)$, since x_f belongs to $\text{minsupp}(f)$. We conclude that (5.5) holds. \square

Remark 5.1. Statement (a) of Proposition 5.6 allows us to construct particular critically-finite elements of R as follows. For each maximal element of X which belongs to X^* , let p_x be prime element of R which generates the principal prime ideal corresponding to the principal prime positive filter $\uparrow 1_x$ (notice that p_x is determined up to association in divisibility). For any finite collection x_1, \dots, x_N ($N \geq 1$) of maximal elements of X which belong to X^* and any integers n_1, \dots, n_N such that $n_j \geq 2$ for each $j \in \{1, \dots, N\}$, the element $W = \prod_{j=1}^N p_{x_j}^{n_j} \in R$ is critically-finite.

The following statement will be used in the construction of some examples below:

Proposition 5.7 *Let (S, \leq) be a well-ordered set. Then (S, \leq) is a tree with a unique minimal element. Moreover, (S, \leq) satisfies Kaplansky's conditions I. and II. iff S has a maximum.*

Proof. Since S is well-ordered, it is totally ordered and has a minimum, therefore it is a tree with a unique minimal element. Given $x, y \in S$ such that $x < y$, we have $x \leq x_1 \ll y_1 \leq y$, where $x_1 \stackrel{\text{def.}}{=} \min\{s \mid x < s \leq y\}$ and $y_1 \stackrel{\text{def.}}{=} \min\{s \mid x_1 < s \leq y\}$. Thus S satisfies Kaplansky's condition I. Any non-empty totally-ordered subset $A \subset S$ has a minimum since S is well-ordered. Moreover, A has a supremum (namely $\min\{s \in S \mid \forall x \in A : x \leq s\}$) iff it has an upper bound. Hence S satisfies Kaplansky's condition II. iff every non-empty subset of S has an upper bound, which amounts to the condition that S has a greatest element. \square

Remark 5.2. Every element of S (except a possible greatest element) has an immediate successor (upper neighbor). In particular, S has a maximal element iff it has a maximum M , which in turn happens iff the order type α of S is a successor ordinal. In this case, M has a predecessor iff α is a double successor ordinal, i.e. iff there exists an ordinal β such that $\alpha = \beta + 2$.

Example 5.8 *Consider the tree T whose underlying set is the set $\mathbb{N} = \mathbb{Z}_{\geq 0}$ of non-negative integers together with the following partial order: $0 < n$ for every $n \in \mathbb{N}$ and there is no further strict inequality; notice that any maximal vertex $n \in \mathbb{N}^* = \mathbb{Z}_{> 0}$ has an immediate lower neighbor, namely 0. This corresponds to a countable corolla, i.e. a tree rooted at 0 and with an edge connecting the root to n for every $n \in \mathbb{N}^*$ (and no other edges). By Proposition 5.6, each maximal vertex $n \in \mathbb{N}^*$ corresponds to a principal prime ideal of the associated Bézout domain.*

Example 5.9 *We can make the previous example more interesting by replacing the edges of T with a tree. For each $x \in \mathbb{N}^*$, consider a tree T_x with a unique root (minimal element) $r_x \in T_x$ and which satisfies Kaplansky's conditions I. and II. Consider the tree \mathcal{T} obtained by connecting 0 to r_x for $x \in \mathbb{N}^*$. Then T has a unique minimal element (namely 0) and satisfies Kaplansky's conditions I. and II. By Proposition 5.6, those maximal elements of each of the trees T_x which have an immediate lower neighbor correspond to prime elements of the associated Bézout domain R . We obtain many examples of Bézout domains by varying the trees T_x :*

1. *Assume that for every $x \in \mathbb{N}^*$, the tree T_x is reduced to the single point $r_x = x$. Then we recover Example 5.8.*
2. *For any element $x \in \mathbb{N}^*$, consider a finite tree T_x and let Σ_x be the set of maximal elements of T_x . Then $\mathcal{T}^* = \mathcal{T} \setminus \{0\}$ and any maximal element of \mathcal{T} different from 0 has an immediate lower neighbor. The corresponding Bézout domain R has a principal prime ideal for every element of the set $\cup_{x \in \mathbb{N}^*} \Sigma_x$.*
3. *For each $x \in \mathbb{N}^*$, consider a well-ordered set S_x which has a maximum m_x and denote the minimum element of S_x by r_x . By Proposition 5.7, we can take $T_x = S_x$ in the general construction above, thus obtaining a tree \mathcal{T} and a corresponding Bézout domain R . Let $U \subset \mathbb{N}^*$ be the set of those $x \in \mathbb{N}^*$ for which S_x is a double successor ordinal. Then each element of U corresponds to a principal prime ideal of R .*

A. GCD domains

Let R be an integral domain and $U(R)$ its multiplicative group of units. For any finite sequence of elements $f_1, \dots, f_n \in R$, let $\langle f_1, \dots, f_n \rangle$ denote the ideal generated by the set $\{f_1, \dots, f_n\}$. An element $u \in R$ is a unit iff $\langle u \rangle = R$. Two elements $f, g \in R$ are called *associated in divisibility* (we write $f \sim g$) if there exists $u \in U(R)$ such that $g = uf$. This is equivalent with the condition $\langle f \rangle = \langle g \rangle$. The association relation is an equivalence relation on R .

Definition A.1 *An integral domain R is called a GCD domain if any two elements f, g admit a greatest common divisor (gcd).*

Let R be a GCD domain. In this case, the gcd of two elements f, g is determined up to association and the corresponding equivalence class is denoted by (f, g) . Any two elements f, g of R also admit a least common multiple (l.c.m.), which is determined up to association and whose equivalence class is denoted by $[f, g]$. By induction, any finite collection of elements f_1, \dots, f_n admits a gcd and and lcm, both of which are determined up to association and whose equivalence classes are denoted by:

$$(f_1, \dots, f_n) \text{ and } [f_1, \dots, f_n] \text{ .}$$

Remark A.1. Any irreducible element of a GCD domain is prime, hence primes and irreducibles coincide in a GCD domain. In particular, any element of a GCD domain which can be factored into primes has unique prime factorization, up to permutation and association of the prime factors.

B. Bézout domains

Let R be a GCD domain. We say that the *Bézout identity* holds for two elements f and g of R if for one (equivalently, for any) gcd d of f and g , there exist $a, b \in R$ such that $d = af + bg$. This amounts to the condition that the ideal $\langle f, g \rangle$ is principal, namely we have $\langle f, g \rangle = \langle d \rangle$.

B.1. Definition and basic properties.

Definition B.1 *An integral domain R is called a Bézout domain if any (and hence all) of the following equivalent conditions hold:*

- *R is a GCD domain and the Bézout identity holds for any two non-zero elements $f, g \in R$.*
- *The ideal generated by any two elements of R is principal.*
- *Any finitely-generated ideal of R is principal.*

More generally, a *Bézout ring* is a unital commutative ring R which has the property that its finitely-generated ideals are principal. Hence a Bézout domain is a Bézout ring which is an integral domain. The following well-known statement shows that the Bézout property is preserved under quotienting by principal ideals:

Proposition B.2 *Let R be a Bézout ring and I be a finitely-generated (hence principal) ideal of R . Then R/I is a Bézout ring.*

If R is a Bézout domain and $f_1, \dots, f_n \in R$, then we have $\langle f_1, \dots, f_n \rangle = \langle d \rangle$ for any $d \in (f_1, \dots, f_n)$ and there exist $a_1, \dots, a_n \in R$ such that $d = a_1 f_1 + \dots + a_n f_n$. The elements f_1, \dots, f_n

are called *coprime* if $(f_1, \dots, f_n) = (1)$, which amounts to the condition $\langle f_1, \dots, f_n \rangle = R$. This happens iff there exist elements $a_1, \dots, a_n \in R$ such that $a_1 f_1 + \dots + a_n f_n = 1$. Notice that every Bézout domain is integrally closed [31].

Remark B.1. Bézout domains coincide with those Prüfer domains which are GCD domains. Since any Prüfer domain is coherent, it follows that any Bézout domain is a coherent ring.

The following result characterizes finitely-generated projective modules over Bézout domains:

Proposition B.3 [7] *Every finitely-generated projective module over a Bézout domain is free.*

In particular, finitely-generated projective factorizations over a Bézout domain coincide with finite-rank matrix factorizations.

B.2. Examples of Bézout domains. The following rings are Bézout domains:

- Principal ideal domains (PIDs) coincide with the Noetherian Bézout domains. Other characterizations of PIDs among Bézout domains are given below.
- Any generalized valuation domain is a Bézout domain.
- The ring $O(\Sigma)$ of holomorphic complex-valued functions defined on any⁷ smooth connected non-compact Riemann surface Σ is a non-Noetherian Bézout domain. In particular, the ring $O(\mathbb{C})$ of entire functions is a non-Noetherian Bézout domain.
- The ring \mathbb{A} of all algebraic integers (the integral closure of \mathbb{Z} inside \mathbb{C}) is a non-Noetherian Bézout domain which has no prime elements.

B.3. The Noetherian case. The following is well-known:

Proposition B.4 *Let R be a Bézout domain. Then the following statements are equivalent:*

- R is Noetherian
- R is a principal ideal domain (PID)
- R is a unique factorization domain (UFD)
- R satisfies the ascending chain condition on principal ideals (ACCP)
- R is an atomic domain.

B.4. Characterizations of Bézout domains.

Definition B.5 *Let R be a commutative ring. The Bass stable rank $\text{bsr}(R)$ of R is the smallest integer n , such that for any collection $\{a_0, a_1, \dots, a_n\}$ of generators of the unit ideal, there exists a collection $\{\lambda_1, \dots, \lambda_n\}$ in R such that the collection $\{a_i - \lambda_i a_0 : 1 \leq i \leq n\}$ also generate the unit ideal. If no such n exists, then $\text{bsr}(R) \stackrel{\text{def.}}{=} \infty$.*

⁷ Notice that Σ need not be algebraic. In particular, Σ can have infinite genus and an infinite number of ends.

Definition B.6 *A unital commutative ring R is called a Hermite ring (in the sense of Kaplansky) if every matrix A over R is equivalent with an upper or a lower triangular matrix.*

The following result is proved in [32, Theorem 8.1]

Theorem B.7 [32] *Let R be a Bézout domain. Then $\text{bsr}(R) \leq 2$. Moreover, R is a Hermite ring.*

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