Inverting Reid's exact plurigenera formula.

A. R. Fletcher.

Max-Planck-Institut für Mathematik,
Gottfried-Claren-Straße 26,
5300 Bonn 3,
West Germany.

The Mathematics Institute,
Unversity of Warwick,
Coventry CV4 7AL,
England.

MPI/88- 66

Inverting Reid's exact plurigenera formula.

A. R. Fletcher,
Max-Planck-Institut für Mathematik,
Gottfried-Claren-Straße 26,
5300 Bonn 3,
West Germany.

1 Introduction.

The main theorem of this paper is the following:

1.1 Theorem. *Let $P : \mathbf{N} \to \mathbf{Z}$ be an arithmetic function such that $P(n) = \chi(\mathcal{O}_X(nK_X))$ for some projective 3-fold X with at worst canonical singularities. Then the record (i.e. K_X^3, $\chi(\mathcal{O}_X)$, the global index R and the basket $\mathcal{B}$ of singularities; see [R, section 10]) of X are uniquely determined by P.*

For canonical 3-folds this arithmetic function corresponds to the plurigenera and for $\mathbf{Q}$-Fano 3-folds to the anti-plurigenera. This theorem shows that the minimal model of X has a unique record.

In section 3 we recap the relevent definitions and theorems from [F1] and [R, Chapter III]. Section 4 contains 2 technical lemmas and section 5 contains the proof of Theorem 1.1. In section 6 we discuss the practicalities of deducing the record from an arithmetic function P.

The contents of this paper were first presented in my Ph.D. thesis [F2] and follows on from the work in [F1].

2 Acknowledgements.

I would like to thank Miles Reid at the University of Warwick, England for many useful conversations. I would also thank all those at the Mathematics Institute, University of Warwick and the mathematics department of the University of Leicester, especially Maria Iano. I am grateful to Prof. Hirzebruch and the Institut for the invitation and their kind hospitality. The work was mainly funded by the University of Leicester, England and this paper was written at the Max-Planck-Institut für Mathematik, Bonn.

3 Reid's exact plurigenera formula.

Throughout this paper we use the notation of [F1] and [R, Chapter III] and assume that X is a projective 3-fold with at worst canonical singularities.

3.1 Definition. Let $r > 0$, a_1, a_2 and a_3 be integers and suppose that $\mathbf{Z}_r$ act on $\mathbf{A}^3$ via:

$$x \mapsto \epsilon^{a_1} x$$
$$y \mapsto \epsilon^{a_2} y$$
$$z \mapsto \epsilon^{a_3} z$$

where x, y and z are coordinates on $\mathbf{A}^3$ and ϵ a primitive r^{th} root of unity. A singularity $Q \in X$ is of *type* $\frac{1}{r}(a_1, a_2, a_3)$ if (X, Q) is isomorphic to an analytic neighbourhood to $(\mathbf{A}^3, 0)/\mathbf{Z}_r$.

3.2 Note. Let $\overline{x}$ denote the least non-negative residue of $x \in \mathbf{Z}$ modulo r. A singularity Q of type $\frac{1}{r}(a_1, a_2, a_3)$ is canonical (respectively terminal) if and only if

$$\sum_i \overline{ba_i} \geq r$$

(respectively $> r$) for all $b \in \mathbf{Z}_r^*$ (see [R, section 4.11]).

So the terminal cyclic quotient singularities are of type $\frac{1}{r}(1, -1, b)$ for some $r > 0$ and some b coprime to r.

3.3 Definition. A *basket* is a list of types $\frac{1}{r}(1, -1, b)$ of terminal cyclic quotient 3-fold singularities. A *record* is a collection

$$\{K^3 \in \mathbf{Q}, p_g \in \mathbf{Z}, \chi \in \mathbf{Z}, R \in \mathbf{N}, \text{ and a basket } \mathcal{B} \text{ of singularities}\}$$

where R is the lowest common multiple of the indexes r of the types in $\mathcal{B}$.

3.4 Definition. For a singularity Q of type $\frac{1}{r}(1, -1, b)$ define:

$$l(Q, n) = \begin{cases} 0 & \text{if } n = 0, 1 \\ \sum_{k=1}^{n-1} \frac{\overline{bk}(r - \overline{bk})}{2r} & \text{if } n \geq 2 \end{cases}$$

This is extended to negative integers via:

$$l(-n) = -l(n+1)$$

for all $n \geq 0$. This is consistant with Serre duality. For a basket $\mathcal{B}$ of singularities define:

$$l(n) = \sum_{Q \in \mathcal{B}} l(q, n)$$

for all $n \in \mathbf{Z}$.

From [F1, Theorem 2.5, equation (4)] (see also [R, Chapter III]) we have the following:

3.5 Theorem. *For any projective 3-fold X with at worst canonical singularities there exists a basket $\mathcal{B}$ of singularities such that*

$$\chi(\mathcal{O}_X(nK_X)) = \frac{(2n-1)n(n-1)}{12r} K_X^3 - (2n-1)\chi(\mathcal{O}_X) + l(n)$$

for all $n \in \mathbf{Z}$.

In the case when X is a canonical 3-fold this formula is Reid's exact plurigenera formula. We shall call the above formula the *plurigenera formula*.

3.6 Definition. The *record corresponding to* X is the record

$$\{K_X^3, p_g(X), \chi(\mathcal{O}_X), R, \mathcal{B}\},$$

where R is the global index of X and $\mathcal{B}$ is the basket corresponding to X.

3.7 Note. The types of singularity in $\mathcal{B}$ are not necessarily the types of singularity on X. If X has canonical but nonterminal singularities then the basket will contain extra types corresponding to these singularities (see [R, section 8.2]).

For example, the weighted projective space $X = \mathbf{P}(1,2,3,4)$ has an isolated terminal singularity of type $\frac{1}{3}(1,-1,1)$ and a line of canonical singularities of type $\frac{1}{2}(1,1,0)$, upon which there is a canonical singularity of type $\frac{1}{4}(1,2,3)$. The basket corresponding to $\mathbf{P}$ is:

$$\{1 \times \frac{1}{3}(1,-1,1), 2 \times \frac{1}{2}(1,1,1)\}.$$

3.8 A natural question. The previous theorem gives a formula for calculating $\chi(\mathcal{O}_X(nK_X))$ from a record. Can this process be reversed? i.e. can the record be determined by the list $\{\chi(\mathcal{O}_X(nK_X)) : n = 0, 1, ...\}$? Is this record unique? These questions are answered by the Theorem 1.1.

4 Technical lemmas.

First we need 2 technical lemmas.

4.1 Lemma. *Consider a periodic function* $f : \mathbf{Z} \to \mathbf{Q}$ *with exact period* r. *Consider the differenced function* $\delta f(n) = f(n+1) - f(n)$. *Then* δf *is periodic with exact period* r.

Proof. $\delta f(n+r) = f(n+r+1) - f(n+r) = \delta f(n)$. So δf is periodic, with period dividing r.

Conversely suppose δf is periodic with period s.

$$f(n) = f(0) + \sum_{m=0}^{n-1} \delta f(m)$$

for $n \geq 0$. Thus

$$\begin{aligned} f(n+s) &= f(0) + \sum_{m=0}^{n+s-1} \delta f(m) \\ &= f(n) + \sum_{m=n}^{n+s-1} \delta f(m) \\ &= f(n) + \sum_{m=0}^{s-1} \delta f(m) \end{aligned}$$

However $f(r) = f(0)$ and so $\sum_{m=0}^{r-1} \delta f(m) = 0$. Hence $\frac{r}{s} \sum_{m=0}^{s-1} \delta f(m) = 0$ and so $\sum_{m=0}^{s-1} \delta f(m) = 0$. Thus $f(n+s) = f(n)$ for all $n \geq 0$. Therefore $r = s$.

□

4.2 Lemma. *Let $\mathcal{B}$ be a basket of isolated terminal 3-fold singularities. Then*

$$\delta^3 l(n) = l(n+3) - 3l(n+2) + 3l(n+1) - l(n)$$

for this basket has exact period R, the global period of $\mathcal{B}$.

Proof. Now

$$\delta l(n) = \sum_{Q \in \mathcal{B}} \frac{\overline{nb_Q}(r_Q - \overline{nb_Q})}{2r_Q}$$

has exact period $R = \text{lcm}_{Q \in \mathcal{B}}(r_Q)$. By Lemma 4.1 $\delta^3 l(n)$ has exact period R.

□

5 The proof of Theorem 1.1.

Theorem 1.1 follows immediately from Theorems 5.1 and 5.2 below. The following theorem and its proof show how to calculate the global index.

5.1 Theorem. *Let $P : \mathbf{N} \to \mathbf{Z}$ be an arithmetic function which corresponds to a list of $\{\chi(\mathcal{O}_X(nK_X))\}$ of some projective 3-fold X with at worst canonical singularities. Then K_X^3, $\chi(\mathcal{O}_X)$, the global index R, and the correction function $l(n)$ can be determined uniquely.*

Proof. By the plurigenera formula, $\delta^3 P(n) = K_X^3 + \delta^3 l(n)$. By Lemma 4.2, this is of exact period R and so determines R. Now $\delta l(mR) = 0$ for all m and so

$$Q_R = \delta P(R) = \frac{1}{2} R^2 K_X^3 - 2\chi$$

and

$$Q_{2R} = \delta P(2R) = 2R^2 K_X^3 - 2\chi$$

allowing both K_X^3 and χ to be determined. Hence $l(n)$ can also be determined.

□

So Theorem 1.1 has been reduced to decoding the correction function $l(n)$. This is done using the next theorem.

5.2 Theorem. *The functions $l(Q, n)$ for each type of terminal quotient 3-fold singularity Q, with index dividing some global index R, are linearly independent.*

5.3 Origin of the idea of proof. The proof of this theorem follows a similar proof due to Reid (see [R, appendix to section 5]). However Reid deals with the linear functions $\overline{bk} - r/2$ and odd characters arise; whereas this section deals with the quadratic functions $\overline{bk}(R - \overline{bk})$ (modulo R) and hence with even characters.

Like Reid I shall start with a slightly easier problem (compare [R, Proposition 5.9]).

5.4 Lemma. *For fixed index r, the functions $l(Q, n)$ for each type of terminal quotient singularity $Q = \frac{1}{r}(1, -1, a)$, with $\text{hcf}(r, a) = 1$ and $a \leq r/2$, are linearly independent.*

5.5 Well-known Results. Let $\phi(r)$ be Euler's function (i.e. the order of $\mathbf{Z}_r^*$). We have the following from [H&W, section V.5.5]:

(1) $\phi(1) = 1 = \phi(2)$,

(2) $2 \mid \phi(r)$ for all $r \geq 3$ (see [H&W, Theorem 62]),

(3) $R = \sum_{r|R} \phi(r)$ (see [H&W, Theorem 63]).

The following is deduced from the above 3 facts.

$$\left\lfloor \frac{R}{2} \right\rfloor = \sum_{r|R, r\geq 2} \left\lfloor \frac{\phi(r)+1}{2} \right\rfloor.$$

Notice that for a fixed index $r > 2$ there are $\phi(r)/2$ types of singularity since the types $\frac{1}{r}(1,-1,a)$ and $\frac{1}{r}(1,-1,r-a)$ are equivalent.

5.6 Proof of Lemma 5.4. If $r = 1$ or $r = 2$ the result is trivial. Without loss of generality assume that $r \geq 3$. There are $\phi(r)/2$ such types of singularity of index r and

$$l(Q,n) = l(\frac{1}{r}(1,-1,b),n) = \sum_{k=0}^{n-1} \frac{\overline{nb}\,(r-\overline{nb})}{2r}$$

with b coprime to r. Clearly these correction functions are linearly independent if and only if the functions

$$2r\delta l(Q,n) = \overline{nb}(r-\overline{nb})$$

are. In fact we need only consider the $\phi(r)$ vectors in $\mathbf{Z}^{r-1}$:

$$T_b = (\overline{kb}(r-\overline{kb}))_{k=1,\ldots,r-1}$$

for b coprime to r. Let $\mathcal{V}$ be the $\mathbf{C}$-vector space spanned by these vectors. Note that

$$(T_a)_k = (T_{r-a})_k = (T_a)_{r-k} = (T_{r-a})_{r-k}$$

and so $\dim_{\mathbf{C}}\mathcal{V} \leq \phi(r)/2$.

Let G be the the group of Dirichlet characters:

$$\chi : \mathbf{Z}_r^* \to \mathbf{C}^*$$

and let G_{even} be the even characters (i.e. those characters χ such that $\chi(-1) = 1$). By [Wash, Lemma 3.1], $|G| = \phi(r)$ and $|G_{even}| = \phi(r)/2$.

For each character χ define

$$W_\chi = \sum_{a\in\mathbf{Z}_r^*} \chi(a)T_a$$

Note that $W_\chi = 0$ for odd χ. In comparison with [R, appendix to section 5] Reid finds in his case that $W_\chi = 0$ for even χ.

Let 1 be the trivial character.

$$(W_1)_1 = \sum_{a \in \mathbf{Z}_r^*} (T_a)_1$$
$$= \sum_{a \in \mathbf{Z}_r^*} a(r-a) \neq 0$$

So $W_1 \neq 0$.

Consider a non-trivial character χ with conductor $f \neq 1$. The following commutes:

$$\begin{array}{ccc} \mathbf{Z}_r^* & \xrightarrow{\chi} & \mathbf{C}^* \\ & \searrow \sigma \quad \nearrow \chi' & \\ & \mathbf{Z}_f^* & \end{array}$$

where σ is the projection mod f. So $1 = \chi(-1) = \chi'\sigma(-1) = \chi'(-1)$. Hence $\chi' : \mathbf{Z}_f^* \to \mathbf{C}^*$ is an even character. Let $q = r/f$.

$$(W_\chi)_q = \sum_{a \in \mathbf{Z}_r^*} \chi(a)(T_a)_q$$
$$= \sum_{a \in \mathbf{Z}_r^*} \chi(a)\,\overline{aq}(r - \overline{aq})$$

But $(T_a)_q = (T_a + f)_q$, and so depends only on a mod f. Thus

$$(W_\chi)_q = \frac{\phi(r)}{\phi(f)} \sum_{a' \in \mathbf{Z}_f^*} \chi'(a')\,a'q(r - a'q)$$
$$= \frac{\phi(r)}{\phi(f)} q^2 \sum_{a' \in \mathbf{Z}_f^*} \chi'(a')\,a'(f - a')$$
$$= \frac{\phi(r)}{\phi(f)} q^2 \left[\frac{f^2}{6} \sum_{a \in \mathbf{Z}_f^*} \chi'(a) - \sum_{a \in \mathbf{Z}_f^*} \chi'(a)\,(a^2 - fa + f^2/6) \right]$$

Let $B_{n,\chi}$ be the generalized Bernoulli numbers as defined in [Wash, p. 30]. Then

$$B_{2,\chi'} = \sum_{a \in \mathbf{Z}_f^*} \chi'(a) \frac{(a^2 - af + f^2/6)}{f}.$$

Also $\sum \chi'(a) f^2/6 = 0$ since $f^2/6$ is a constant. So

$$(W_\chi)_q = -\frac{\phi(r)}{\phi(f)}\, q^2 f\, B_{2,\chi'} = -\frac{\phi(r)}{\phi(f)}\, qr B_{2,\chi'}.$$

See also [Wash, Exercise 4.2 (a)]. By [Wash, Theorem 4.2 and p. 30],

$$B_{2,\chi'} = -2L(-1,\chi') \neq 0$$

for even χ. Thus $W_\chi \neq 0$ for non-trivial χ.

Let $\mathbf{Z}_r^*$ act on $\mathbf{C}^{\phi(r)}$ by permuting the coordinates. Let b be a generator. Then $(bx)_k = (x)_{bk}$ for all $b \in \mathbf{Z}_r^*$. So

$$bT_a = T_{ab}$$

and

$$bW_\chi = \chi(b)^{-1} W_\chi$$

for all even characters χ. As the characters χ are distinct and the $\{W_\chi\}$ are non-zero, then the vectors $\{W_\chi\}$ are eigenvectors of the action of $\mathbf{Z}_r^*$ on $\mathcal{V}$. Therefore they are linearly independent. Hence $\dim_{\mathbf{C}}\mathcal{V} \geq \phi(r)/2$ and so $\dim_{\mathbf{C}}\mathcal{V} = \phi(r)/2$. Thus the original vectors $\{T_a\}$ are linearly independent.

□

5.7 The proof of Theorem 5.2.

The above proof does not generalise to Theorem 5.2 since there are not enough characters. The proof of Theorem 5.2 will be done in a number of stages, and involves 2 changes of 'basis'. The main steps in the proof are Theorems 5.12 and 5.13, and sections 5.14 and 5.21.

There are 3 sets of bases T_a, $W_\chi(a)$ and $V_\chi(a)$ used, defined in Definitions 5.8, 5.9 and 5.17 respectively. Lemmas 5.11, 5.18, 5.19 and 5.20 are technical results on the vanishing and non-vanishing of certain coordinates of $W_\chi(a)$ and $V_\chi(a)$.

As before in the proof of Lemma 5.4, we consider the following vector space.

5.8 Definition. Let $\overline{x}$ denote residue of $x \in \mathbf{Z}$ modulo R and let $T_a = (\overline{ak}(R - \overline{ak}))_{k=1,\ldots,R-1}$. Define $\mathcal{V}$ to be the subspace of $\mathbf{C}^{R-1}$ spanned by these vectors.

As in the previous proof, $\dim_{\mathbf{C}}\mathcal{V} \leq \lfloor R/2 \rfloor$. Clearly the vectors $\{T_a : a = 1, \ldots, \lfloor R/2 \rfloor\}$ are linearly independent if and only if Theorem 5.2 is true.

5.9 Definition. Let $\chi : \mathbf{Z}_R^* \to \mathbf{C}^*$ be an even Dirichlet character with conductor $f_\chi = f$ and $q = R/f$. Define

$$W_\chi(a) = \sum_{b \in \mathbf{Z}_R^*} \chi(b) T_{ab} \in \mathcal{V}$$

$W_\chi(1)$ corresponds to W_χ in Proof 5.6.

5.10 Note. The multiplicative action of $\mathbf{Z}_R^*$ partitions $\mathbf{Z}_R$ into equivalence sets; each set consists of all elements of $\mathbf{Z}_R$ with a given highest common factor with R. For example

$$\mathbf{Z}_8 = \{1,3,5,7\} \cup \{2,6\} \cup \{4\}$$

under the action of $\mathbf{Z}_8^*$.

Let $a \in \mathbf{Z}_R$. There exists $\beta \in \mathbf{Z}_R^*$ such that $a = \beta\,\mathrm{hcf}(a,R)$ and therefore

$$\begin{aligned} W_\chi(a) &= \sum \chi(b\beta^{-1}) T_{ab\beta^{-1}} \\ &= \chi(\beta)^{-1} \sum \chi(b) T_{ab\beta^{-1}} \\ &= \chi(\beta)^{-1} W_\chi(a\beta^{-1}) \\ &= \chi(\beta)^{-1} W_\chi(\mathrm{hcf}(a,R)). \end{aligned}$$

5.11 Lemma.

(i) The c^th^ *coordinate* $(\dot{W}_\chi(a))_c$ *depends only on* ac *mod* R

(ii) If $ac = q$ *then* $(W_\chi(a))_c = -\frac{\phi(R)}{\phi(f)} R q B_{2,\chi} \neq 0$.

(iii) If $\beta \in \mathbf{Z}_r^*$ *then* $W_\chi(\beta a) = \chi(\beta)^{-1} W_\chi(a)$

(iv) If $a \mid R$ *and* $\mathrm{hcf}(ac, R) \not| \, q$ *then* $(W_\chi(a))_c = 0$.

Proof.

(i) $(W_\chi(a))_c = \sum_{b \in \mathbf{Z}_R^*} \chi(b) \overline{acb}(R - \overline{acb})$, which depends only on ac mod R.

(ii)

$$\begin{aligned}(W_\chi(a))_c &= (W_\chi(1))_{ac} \\ &= (W_\chi(1))_q \\ &= -\frac{\phi(R)}{\phi(f)} R q B_{2,\chi} \neq 0\end{aligned}$$

(Compare with the proof of Lemma 5.4.)

(iii)

$$\begin{aligned}W_\chi(\beta a) &= \sum_{b \in \mathbf{Z}_R^*} \chi(b) T_{a\beta b} \\ &= \chi(\beta)^{-1} \sum_b \chi(\beta b) T_{a\beta b} \\ &= \chi(\beta)^{-1} W_\chi(a).\end{aligned}$$

(iv) Let $q' = \mathrm{hcf}(ac, R)$ and $f' = R/q'$. Then there exists $\beta \in \mathbf{Z}_R^*$ (see Note 5.10) such that

$$\begin{aligned}(W_\chi(a))_c &= (W_\chi(ac))_1 \\ &= \left(\chi(\beta)^{-1} W_\chi(\mathrm{hcf}(ac, R))\right)_1 \\ &= \chi(\beta)^{-1} (W_\chi(q'))_1 \\ &= \chi(\beta)^{-1} \sum_b \chi(b) (T_{bq'})_1 \\ &= \chi(\beta)^{-1} \sum_b \chi(b) (T_b)_{q'} \\ &= \chi(\beta)^{-1} \sum_b \chi(b) Q(bq'),\end{aligned}$$

where $Q(x) = \overline{x}(R - \overline{x})$. The function $b \mapsto Q(bq')$ depends only on b mod f' and so

$$Q(kbq') = Q(bq')$$

for all $k \in K = \mathrm{Ker}\ (\mathbf{Z}_R^* \to \mathbf{Z}_{f'}^*)$ (i.e. $k \in \mathbf{Z}_R^*$ such that $f' \mid k - 1$).

Assume that χ is trivial on K. Then K is contained in $\mathrm{Ker}\ (\mathbf{Z}_R^* \to \mathbf{Z}_f^*)$ (since f is the conductor). So $f \mid f' \mid R$ (i.e. $1 \mid q' \mid q$). But $q' = \mathrm{hcf}(ac, R) \not| \, q$, a contradiction.

Thus χ is not trivial on K. So

$$\begin{aligned}(W_\chi(a))_c &= \chi(\beta^{-1})\sum_{b}\chi(b)Q(bq') \\ &= \chi(\beta^{-1})\sum_{k\in K}\sum_{b'\in \mathbf{Z}^*_{f'}}\chi(kb')Q(b'q') \\ &= \chi(\beta^{-1})\sum_{b'\in \mathbf{Z}^*_{f'}}Q(b'q')\sum_{k\in K}\chi(kb') = 0.\end{aligned}$$

□

5.12 Theorem. *The subspace of* $\mathbf{C}^{R-1}$ *generated by the set*

$$\left\{W_\chi(a) : \chi \text{ even characters of } \mathbf{Z}^*_R \text{ and } a \in \mathbf{Z}_R \text{ such that } a \mid \frac{R}{f_\chi}\right\}$$

lies in $\mathcal{V}$ *and splits into* $\phi(R)$ *distinct eigenspaces, one for each* χ. *There are* $\lfloor \frac{R}{2} \rfloor$ *vectors in the above set.*

Proof. By Lemma 5.11(ii), each vector in the above set is a non-zero sum of the vectors $\{T_a\}$. Each $W_\chi(a)$ is an eigenvector with eigenvalue χ^{-1} under the action of the group $\mathbf{Z}^*_R$.

□

Fix the character χ once and for all and consider the χ^{-1}-eigenspace.

5.13 Theorem. *For a fixed character* χ *the vectors* $\{W_\chi(a) : a \mid q\}$ *are linearly independent, where* $q = R/f_\chi$.

This will be proved after some preliminary work.

5.14 Proof of Theorem 5.2. Theorems 5.12 and 5.13 imply that $\{W_\chi(a)\}$ are linearly independent and hence so are vectors $\{T_a\}$. This proves Theorem 5.2, subject to proving Theorem 5.13.

□

To prove Theorem 5.13 the following definition and another change of basis is required.

5.15 Definition. Let $\mathcal{P}$ be the set of primes which divide q but not f. For each $p \in \mathcal{P}$ define β_p by

$$\begin{aligned}\beta_p &\equiv p \bmod R/p^\alpha \\ \beta_p &\equiv 1 \bmod p^\alpha\end{aligned}$$

where p^α is the highest power of p dividing R. These 2 equations have a unique common solution modulo R.

Extend this definition to the set $\mathcal{D}$ of products of distinct primes in $\mathcal{P}$

$$\beta_d = \prod \beta_{p_i} \in \mathbf{Z}_R$$

where $d = \prod p_i \in \mathcal{D}$.

5.16 Note.

(i) $\beta_p \in \mathbf{Z}_R^*$ since $\mathrm{hcf}(\beta_p, p^\alpha) = 1$ and $\mathrm{hcf}(\beta_p, R) \mid p$.

(ii) $\beta_p x \equiv px \bmod R$ whenever $p^{\alpha+1} \mid x$.

We now make the second change of basis.

5.17 Definition. For all $a \mid q$, define

$$V_\chi(a) = \sum_{d\in\mathcal{D}:d|a} \mu(d)\chi^{-1}(\beta_d)W_\chi(a/d)$$
$$= \sum_{d\in\mathcal{D}:d|a} \mu(d)W_\chi(\beta_d a/d)$$

where $\mu(d)$ is the Möbius function (i.e. $\mu(d) = (-1)^m$, where d is a product of m distinct primes, and $\mu(d) = 0$ if $p^2 \mid d$ for some prime p).

5.18 Lemma. *Let $a \mid q$ and $c \mid q$ but $ac \not\mid q$. Then $(V_\chi(a))_c = 0$.*

Proof. As $ac \not\mid q$ there is a prime p such that $p^\gamma \mid ac$ but $p^\gamma \not\mid q$. There are 2 cases:

(i) $p \notin \mathcal{P}$ (i.e. $p \mid f$). Then $p^\gamma \mid \mathrm{hcf}(ac, R)$ and so $\mathrm{hcf}(ac, R) \not\mid q$. By Lemma 5.11$(iv)$, $(W_\chi(a))_c = 0$. Similarly $(W_\chi(\beta_d a/d))_c = \chi^{-1}(\beta_d)(W_\chi(a/d))_c$ and $p^\gamma \mid ac/d$ (since $p \notin \mathcal{P}$). Thus $(W_\chi(\beta_d a/d))_c = 0$ and so $(V_\chi(a))_c = 0$.

(ii) $p \in \mathcal{P}$ (i.e. $p \not\mid f$). By the careful grouping of terms,

$$(V_\chi(a))_c = \sum_{d|a,p\not\mid d} \mu(d)(W_\chi(\beta_d a/d))_c + \mu(pd)(W_\chi(\beta_d\beta_p a/pd))_c$$
$$= \sum_{d|a,p\not\mid d} \mu(d)\left[(W_\chi(\beta_d a/d))_c - (W_\chi(\beta_d\beta_p a/pd))_c\right]$$

Notice that $p^\alpha \mid ac$ but $p^{\alpha+1} \not\mid ac$. By Note 5.16(ii),

$$\beta_p\frac{ac}{pd} \equiv p\frac{ac}{pd} \equiv \frac{ac}{d} \bmod R$$

So each pair of terms cancels out to give $(V_\chi(a))_c = 0$.

□

5.19 Lemma. *Let $ac = q$ and $d \mid a$ for some $d \in \mathcal{D}$. Then*

$$(W_\chi(\beta_d a/d))_c = -\frac{\phi(R)}{\phi(f)}qRB_{2,\chi'}\prod_{p|d}\frac{\chi(p)^{-1} - p}{p(p-1)}.$$

Proof. Define $Q(x) = \overline{x}(R - \overline{x})$. By definition

$$(W_\chi(\beta_d a/d))_c = (W_\chi(1))_{\beta_d q/d}$$
$$= \sum_{b\in\mathbf{Z}_f^*} \chi(b)Q(b\beta_d q/d)$$
$$= \frac{\phi(R)}{\phi(f)}\sum_{b'\in\mathbf{Z}_f^*}\chi(b')\sum_{\substack{b\mapsto b'\\ b\in\mathbf{Z}_{df}^*}} Q(b\beta_d q/d)$$

Let $t(d') = \sum_{\substack{b \mapsto b' \\ b \in \mathbf{Z}^*_{df}}} Q(b\beta_d q/d)$, where $d' \mid d$. We have

$$\mathbf{Z}_{df} = \mathbf{Z}^*_{df} \bigcup_{\substack{d' \mid d \\ d'1}} d'\mathbf{Z}_{df} \bigcup_{\substack{\alpha \mid f \\ \alpha \geq 0}} \alpha \mathbf{Z}_{df}.$$

Notice that $\{b \in \alpha\mathbf{Z}_{df} : b \mapsto b'\} = \emptyset$ for all $\alpha \mid f$ and $\alpha \geq 0$. The reason is the following. An element b of this set is of the form:

$$\alpha \mid b = b' + nf$$

and so $\alpha \mid b'$, i.e. $\alpha \mid \text{hcf}(b', f)$. But $b' \in \mathbf{Z}^*_f$, a contradiction.

This simplifies the sum:

$$\begin{aligned}
\sum_{\substack{b \mapsto b' \\ b \in \mathbf{Z}^*_{df}}} Q(b\beta_d q/d) &= \sum_{\substack{b \mapsto b' \\ b \in \mathbf{Z}_{df}}} Q(b\beta_d q/d) - \sum_{\text{prime } p \mid d} \sum_{\substack{b \mapsto b' \\ b \in p\mathbf{Z}_{df}}} Q(b\beta_d q/d) \\
&\quad + \sum_{p_1 p_2 \mid d} \sum_{\substack{b \mapsto b' \\ b \in p_1 p_2 \mathbf{Z}_{df}}} Q(b\beta_d q/d) - \ldots \\
&= \sum_{d' \mid d} \mu(d') \sum_{\substack{b \mapsto b' \\ b \in d'\mathbf{Z}_{df}}} Q(b\beta_d q/d) \\
&= \sum_{d' \mid d} \mu(d') t(d').
\end{aligned}$$

Consider the sum $t(d')$. As d and f are coprime there is a unique integer $i_0 < d$ such that $dx = b' + i_0 f$ for some x. Since the sum $t(d')$ involves only $b \in d'\mathbf{Z}_{df}$ and $Q(\beta_d bq/d)$ depends only on b mod f then

$$t(d') = \sum_{j=0}^{d''-1} Q((b' + i_0 f + jd'f)\beta_d q/d)$$

where $d'' = d/d'$. However $(b' + i_0 f)\beta_d q/d \equiv b'q \bmod R$ (by the definition of β_d and Note 5.16(ii)). So

$$\begin{aligned}
t(d') &= \sum_{j=0}^{d''-1} Q(b'q + jd'\beta_d fq/d) \\
&= \sum_{j=0}^{d''-1} Q(b'q + jd'R/d) \\
&= \sum_{j=0}^{d''-1} Q(b'q + jR/d'')
\end{aligned}$$

The numbers $\{\overline{b'q + jR/d''} : j = 0, ..., d'' - 1\}$ take their smallest value of $(\overline{\overline{b'd''}} \cdot q/d'')$ where $\overline{\overline{h}}$ denotes least positive residue mod f. Thus the range of summation can be rewritten;

$$t(d') = \sum_{i=0}^{d''-1} Q(\overline{\overline{b'd''}} \cdot q/d'' + iR/d'')$$

Notice that $Q(a+b) = Q(a) + Q(b) - 2ab$ for $a + b < R$. So

$$\begin{aligned} t(d'') &= \sum_{i=0}^{d''-1} \left[Q(\overline{\overline{b'd''}} \cdot q/d'') + Q(iR/d'') - 2 \cdot \overline{\overline{b'd''}} \cdot iRq/d''^2 \right] \\ &= d''Q(\overline{\overline{b'd''}} \cdot q/d'') + \sum_{i=0}^{d''-1} Q(iR/d'') - 2\frac{qR}{d''^2}\binom{d''}{2} \cdot \overline{\overline{b'd''}}. \end{aligned}$$

The calculation of $\sum_{b' \in \mathbf{Z}_f^*} \chi(b') \sum_{d'|d} \mu(d')t(d')$ consists of 3 parts:

(i)

$$\sum_{d'|d} \sum_{b' \in \mathbf{Z}_f^*} \chi(b')\mu(d') \left(\sum_{j=0}^{d''-1} Q(iR/d'') \right) = 0,$$

since the summand is independent of b'.

(ii)

$$\sum_{d'|d} \sum_{b' \in \mathbf{Z}_f^*} \chi(b')\mu(d')\binom{d''}{2}\frac{2qR}{d''^2} \cdot \overline{\overline{b'd''}} = \sum_{d'|d} \sum_{b' \in \mathbf{Z}_f^*} \chi(b')\mu(d')\binom{d''}{2}\frac{2qR}{d''^2} \cdot \overline{\overline{b'}} = 0,$$

since d'' and f are coprime.

(iii)

$$\begin{aligned} &\sum_{d'|d} \sum_{b' \in \mathbf{Z}_f^*} \chi(b')\mu(d')d''Q(\overline{\overline{b'd''}} \cdot q/d'') \\ &= \sum_{d'|d} \sum_{b' \in \mathbf{Z}_f^*} \chi(b')\mu(d')d''\frac{q}{d''} \cdot \overline{\overline{b'd''}} \cdot (R - \frac{q}{d''} \cdot \overline{\overline{b'd''}}) \\ &= \sum_{d'|d} \frac{q^2}{d''}\mu(d') \sum_{b' \in \mathbf{Z}_f^*} \chi(b') \cdot \overline{\overline{b'd''}} \cdot (fd'' - \overline{\overline{b'd''}}) \\ &= \sum_{d'|d} \frac{q^2}{d''}\mu(d')\chi(d'')^{-1} \sum_{b' \in \mathbf{Z}_f^*} \chi(b')b'(fd'' - b') \\ &= \sum_{d'|d} \frac{q^2}{d''}\mu(d')\chi(d'')^{-1} \sum_{b' \in \mathbf{Z}_f^*} \chi(b')\left[b'(f - b') + b'f(d'' - 1)\right] \\ &= -\sum_{d'|d} \frac{q^2}{d''}\mu(d')\chi(d'')^{-1} fB_{2,\chi'}. \end{aligned}$$

So $\sum_{b'\in \mathbf{Z}_f^*} \chi(b') \sum_{d'|d} \mu(d')t(d')$ is the sum of these 3 parts. Therefore

$$\begin{aligned}(W_\chi(\beta dq/d))_c &= -\frac{\phi(R)}{\phi(df)} RqB_{2,\chi'} \sum_{d'|d} \frac{\mu(d')}{d''}\chi(d'')^{-1} \\ &= -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi'} \frac{\chi(d)^{-1}}{d\phi(d)} \sum_{d'|d} d'\mu(d')\chi(d'')^{-1} \\ &= -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi'} \frac{\chi(d)^{-1}}{d\phi(d)} \prod_{p|d} [1 - p\chi(p)] \\ &= -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi'} \prod_{p|d} \left[\frac{\chi(p)^{-1} - p}{p(p-1)}\right] \\ &= -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi'} \prod_{p|d} \left(1 - \frac{p^2 - \chi(p)^{-1}}{p(p-1)}\right)\end{aligned}$$

□

5.20 Lemma. *Let $ac = q$. Then*

$$(V_\chi(a))_c = -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi'} \prod_{p|a} \frac{p^2 - \chi(p)^{-1}}{p(p-1)} \neq 0.$$

Proof.

$$\begin{aligned}(V_\chi(a))_c &= \sum_{d|a} (W_\chi(\beta_d a/d))_c \\ &= -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi'} \sum_{d|a} \left[\mu(d) \prod_{p|d} \left(1 - \frac{p^2 - \chi(p)^{-1}}{p(p-1)}\right)\right]\end{aligned}$$

But

$$\begin{aligned}\prod_{p|a} \frac{p^2 - \chi(p)^{-1}}{p(p-1)} &= \prod_{p|a} \left(1 - \frac{\chi(p)^{-1} - p}{p(p-1)}\right) \\ &= \sum_{d|a} \mu(d) \prod_{p|d} \frac{\chi(p)^{-1} - p}{p(p-1)}\end{aligned}$$

Therefore

$$(V_\chi(a))_c = -\frac{\phi(R)}{\phi(f)} RqB_{2,\chi'} \prod_{p|a} \frac{p^2 - \chi(p)^{-1}}{p(p-1)}.$$

As $\chi(p)^{-1} \neq p^2$ we have $(V_\chi(a))_c \neq 0$.

□

5.21 Proof of Theorem 5.13. Clearly the vectors $\{V_\chi(a) : a \mid q\}$ lie in the subspace spanned by $\{W_\chi(a) : a \mid q\}$. Let $\{a_i : i = 1, ..., n\}$ be the set of $a \mid q$, ordered such that $a_i \not| \, a_j$ for $i > j$. Let $c_i = q/a_i$. Let M be the matrix with entries

$$M_{i,j} = (V_\chi(a_i))_{c_j}$$

Suppose $i > j$. Then $a_i \mid q$, $c_j \mid q$ and $a_i c_j \not| \, q$. By Lemma 5.18, $M_{i,j} = 0$. So M is an upper triangular matrix. By Lemma 5.20, the diagonal enties $M_{i,i}$ are non-zero.

Thus M has maximal rank and the vectors $\{V_\chi(a) : a \mid q\}$ are linearly independent. So the vectors $\{W_\chi(a) : a \mid q\}$ also are independent. This completes the proof of Theorem 5.13.

□

6 Practicalities.

The proof of Theorem 5.1 is constructive and can be used to find K^3, χ, R and the function $l(n)$ provided some way of limiting the global index R can be found. Without this limit the period of an infinite list of integers must be found.

In the case of the example $X = \mathrm{P}(1,2,3,4)$ we have $R < 1.2.3.4 = 24$ and so this technique can be used.

7 References.

[F1] A. R. Fletcher: *Contributions to Riemann-Roch on projective 3-folds with only canonical singularities and applications*, Proc. Symposia in Pure Maths **46** (1987) Vol 1, p. 221 - 231.

[F2] A. R. Fletcher: *Plurigenera of 3-folds and weighted hypersurfaces*, Thesis submitted to the University of Warwick for the degree of Ph.D. (1988).

[H&W] G. H. Hardy, E. M. Wright: *An Introduction to the Theory of Numbers*, Fifth Edition (1979) Oxford University Press.

[R] M. Reid: *Young Person's Guide to Canonical Singularities*, Proc. Symposia in Pure Maths **46** (1987) Vol 1, p. 345 - 414.

[Wash] L. Washington: *Introduction to Cyclotomic Fields*, Grad. Texts in Mathematics **83**, Springer Verlag 1982.